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Mikołaj Piniewski (Warszawa)

ASYMPTOTIC DYNAMICS IN DOUBLE-DIFFUSIVE CONVECTION

Abstract. We consider the double-diffusive convection phenomenon and analyze the governing equations. A system of partial differential equations describing the convective flow arising when a layer of fluid with a dissolved solute is heated from below is considered. The problem is placed in a functional analytic setting in order to prove a theorem on existence, uniqueness and continuous dependence on initial data of weak solutions in the class $\mathcal{C}([0,\infty);H) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+;V)$. This theorem enables us to show that the infinite-dimensional dynamical system generated by the double-diffusive convection equations has a global attractor on which the long-term dynamics of solutions is focused.

1. Introduction. The double-diffusive convection (DDC) has become a phenomenon of considerable scientific interest since Stern rediscovered it in 1960 [11]. It had been Jevons [4] who performed first experiments one century before but he was not able to explain them. Since the early 1960s most of the research has been devoted to applications in oceanology [9], and because heat and salt are then the relevant properties, the process has been called *thermohaline convection*. So far, DDC has been recognized in fields as diverse as astrophysics, metallurgy and geology so the name *double-diffusive convection* has been chosen to encompass the wider range of phenomena. An interesting physical insight into DDC is given in two monographs: of Joseph [5] and Turner [13]. The inspiration to deal with the governing equations for DDC in this paper came from Renardy [7].

Contemporary research on DDC includes both theoretical [3, 6], numerical [15], laboratory [14] and expedition studies [10]. In this paper we propose

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a rather abstract mathematical approach, which belongs to the first group of studies. We hope it will be interesting not only for applied mathematicians but also for ocean physicists and engineers. Two mathematical theories are used to study DDC: the theory of nonlinear partial differential equations and the theory of infinite-dimensional dynamical systems. The first one gives tools (theorem on existence and uniqueness of solutions) to develop the second one. The methods used in this paper (in Sections 3 and 4) come from the books of Temam [12] and Robinson [8] as well as a paper of Foiaş et al. [2].

This paper is organized as follows. In Section 2 the main physical motivations coming primarily from oceanology are presented. Two cases of hydrodynamic instabilities are particularly interesting: the so-called salt fingers and diffusive convection. The basic governing equations are given in Section 3. The major part of that section is devoted to the proof of existence and uniqueness of weak solutions. In Section 4 we prove the existence of the absorbing set and, most importantly, of the global attractor. Finally, in Section 5 we attempt to give a physical interpretation in terms of the attractor's dimension and dimensionless parameters appearing in the model. We also propose some potential directions of further research.

Notations. In order to avoid any misunderstandings, we have gathered here some useful notations that we use in this paper. We write scalars in normal font and vectors in bold,

$$u, \psi, \qquad \boldsymbol{u}, \boldsymbol{\psi}.$$

The problem is set in space dimension 2 where x stands for the horizontal axis and z for the vertical axis,

$$\boldsymbol{x} = (x, z),$$

and the versors from the canonical basis in $\mathbb{R}^2$ are

$$\boldsymbol{e}_x = (1, 0), \qquad \boldsymbol{e}_z = (0, 1).$$

The partial derivatives in $\mathbb{R}^2$ are denoted by ∂_x, ∂_z, and

$$\nabla = (\partial_x, \partial_z).$$

We write the scalar product in $\mathbb{R}^n$ as

$$\boldsymbol{x} \cdot \boldsymbol{y},$$

while the divergence of a vector $\boldsymbol{u}$ is denoted by

$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u}.$$

If H is a Hilbert space, its dual is denoted by H^*. In numerous estimations a lot of constants appear and we usually denote them by

$$C, C_1, C_2, \ldots$$

Sometimes our notations may be ambiguous, e.g. $|\cdot|$ usually stands for the norm in H,

$$\|\boldsymbol{u}\|_H = |\boldsymbol{u}|,$$

but sometimes it may denote the absolute value,

$$|(L\boldsymbol{u}, \boldsymbol{\psi})|,$$

or the measure of Ω,

$$|\Omega|.$$

The notations used for numerous function spaces, e.g. $\mathcal{C}^k(\Omega)$, $L^p(\Omega)$ or $H^k(\Omega)$, as well as the spaces of functions with values in a Banach space X, e.g. $\mathcal{C}([a,b];X)$ or $L^p(a,b;X)$, are standard. In case of doubt we refer to [8].

2. Physical motivations. Before we start our strictly mathematical treatment we shall mention some physical motivations lying behind the model. DDC related to oceanology is often called "thermohaline convection". It has large influence on vertical and horizontal mixing of ocean water and thus on global ocean circulation and climate change [9]. Thermohaline convection is no longer an "oceanographic curiosity", as it was in the early 60s.

2.1. *Mechanisms of hydrodynamic instabilities in oceanology.* Our starting point is the question: what influences the ocean density distribution? Roughly speaking, the density ϱ of sea water is determined by its temperature T and salinity S. We are primarily interested in vertical changes of these variables so we have $T = T(z)$, $S = S(z)$ and so $\varrho = \varrho(z)$. The equation of state is then

$$\varrho = \varrho_0[1 - \alpha(T - T_0) + \beta(S - S_0)], \tag{1}$$

where ϱ_0, T_0 and S_0 are fixed values related to the level $z = z_0$, and α and β are positive constants denoting the thermal and salt expansion coefficients respectively.

A particularly interesting issue in ocean physics is stability of the vertical density distribution. We say that the density distribution is *stable* when the water density grows with depth, *unstable* when it diminishes with depth, and *neutral* when it is constant. Differentiation of (1) with respect to z gives

$$\partial_z \varrho = \varrho_0(\beta \partial_z S - \alpha \partial_z T), \tag{2}$$

so the stability of the water density distribution depends on the sign of the bracketed expression.

Let us consider the distribution of water temperature, salinity and density in an ocean layer between the planes $z = z_0$ and $z = z_1$, where $z_0 < z_1$. As boundary values we set

$$T(z_0) = T_0, \quad S(z_0) = S_0, \quad T(z_1) = T_1, \quad S(z_1) = S_1.$$

Furthermore, let us assume that T and S are linear between the planes z_0 and z_1 and constant above z_1 and below z_0 (Fig. 1), which is a frequent case in oceans. We shall see in (9) that linearity of T and S is not really an assumption, but a conclusion that comes from the equations.

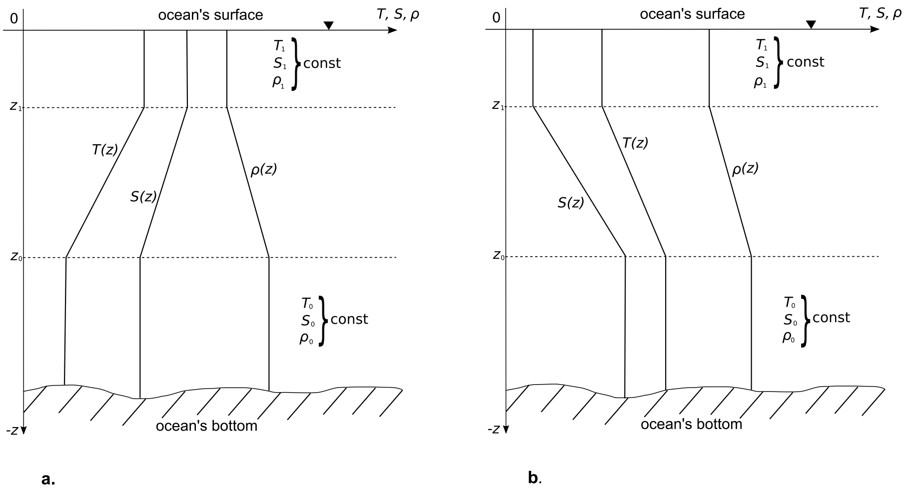

Fig. 1. Vertical distribution of physical water characteristics: a. generating the "salt fingers", b. generating the mechanism of diffusive convection

There are four distinct cases resulting from the signs of $T_0 - T_1$ and $S_0 - S_1$. Two of them are particularly interesting from the stability point of view:

1. $T_0 < T_1$ and $S_0 < S_1$ in such a way (i.e. for proper values of α and β) that $\partial_z \varrho > 0$ according to (2), which means a natural (stabilizing) fall of temperature with depth and an inverse (destabilizing) distribution of salinity causing a relatively stable density distribution (Fig. 1a).
2. $T_0 > T_1$ and $S_0 > S_1$ in such a way that $\partial_z \varrho > 0$ according to (2), which means an inverse (destabilizing) distribution of temperature and a natural (stabilizing) rise of salinity with depth causing a relatively stable density distribution (Fig. 1b).

Two other cases lead to *absolutely* stable ($T_0 < T_1$ and $S_0 > S_1$) or unstable ($T_0 > T_1$ and $S_0 < S_1$) density distribution and in this sense they are less interesting.

2.2. *Salt fingers and oscillatory convection.* First of all let us consider case 1. Suppose a parcel of water in the upper (warm and salty) layer shifts slightly downwards as a result of small perturbations. Now, as heat diffuses much faster than salt, which is in fact the key issue in DDC, the parcel of water will quickly transmit its heat surplus to the surrounding colder water. Therefore it will become heavier than the surrounding water (as it contains more salt) and will start to sink. At the same time another parcel of water in lower (colder and fresher) layer shifts slightly upwards as a result of small perturbations. Again, due to faster diffusion of heat, the surrounding warm water will transmit heat to the parcel of water. After some time buoyant forces will cause it to rise. A checkerboard of channels (called fingers) will thus be created: rising fresh fingers vs. sinking salt fingers (Fig. 2). This kind of phenomenon is not only understood theoretically, but also confirmed in laboratory and numerical experiments [1, 13].

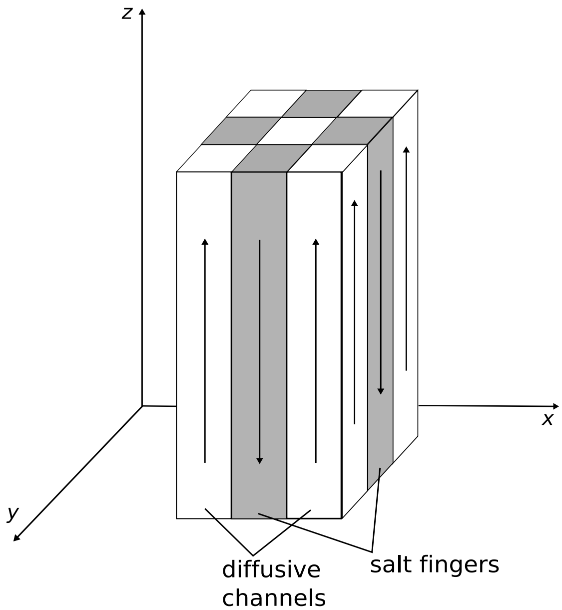

Fig. 2. Salt fingering mechanism

In the opposite case (of warm salty water underneath cold fresh water) consider a parcel of water which is slightly shifted due to small perturbations. Then diffusion of heat will make it equalize its temeperature with surrounding water and afterwards buoyancy forces will drive it back to its initial position. In this way an oscillatory motion will be produced with stabilizing effect of the difference in diffusivities and competing effect of thermal instablity leading to free convection known from the classical Bénard problem [2, 13]. Depending on the value of the Rayleigh number (to be defined later), laminar convection may be transformed into turbulent convection. However, the oscillatory convection phenomenon is less understood than salt fingers.

3. Existence and uniqueness of solutions

3.1. *Governing equations.* Consider the following set of PDEs:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\varrho_0}\nabla p - \nu\Delta\boldsymbol{u} = [\alpha(T-T_0) - \beta(S-S_0) - 1]g\boldsymbol{e_z}, \tag{3}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4}$$

$$\partial_t T + (\boldsymbol{u}\cdot\nabla)T = \kappa_T\Delta T, \tag{5}$$

$$\partial_t S + (\boldsymbol{u}\cdot\nabla)S = \kappa_S\Delta S, \tag{6}$$

where $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure, T is the temperature and S the salinity (to fix ideas; it could be the concentration of any solute). The constants $\varrho_0, \nu, g, \alpha, \beta, \kappa_T$ and κ_S denote respectively: the fluid density, fluid kinematic viscosity, standard gravity, heat and salt expansion coefficients, heat and salt diffusion coefficients. Equation (3) is a modified form of the classical Navier–Stokes equation with the RHS term, body force $\boldsymbol{f}$, depending on the temperature T and salinity S. Such a force may be interpreted as the buoyancy force. The continuity equation (4) expresses the mass conservation rule for the incompressible fluid. Equations (5)–(6) are nonlinear convection-diffusion equations.

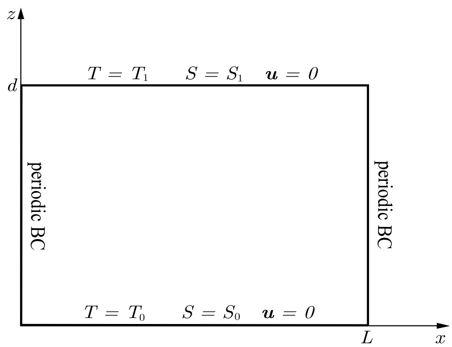

Fig. 3. The model setting

The problem (3)–(6) is set in a rectangle $\Omega = [0,L]\times[0,d]$ (Fig. 3). We impose periodic boundary conditions in the x direction for mathematical convenience:

$$\phi_{|x=L} = \phi_{|x=0}, \qquad \partial_x\phi_{|x=L} = \partial_x\phi_{|x=0} \qquad \text{for } \phi = \boldsymbol{u}, T, S. \tag{7}$$

At $z = 0$ and $z = d$ we impose Dirichlet boundary conditions

$$\boldsymbol{u}_{|z=0} = \boldsymbol{u}_{|z=d} = 0,\ T_{|z=0} = T_0,\ T_{|z=d} = T_1,\ S_{|z=0} = S_0,\ S_{|z=d} = S_1. \tag{8}$$

We could have set the problem in space dimension 3 as well, which would be more physically relevant. However, in that case we would not be able to prove existence of solutions. Problems of this kind always arise when the model is based on highly nonlinear PDEs, such as Navier–Stokes equations.

Some authors [5] refer to the equations (3)–(6) as the Oberbeck–Boussinesq equations. We shall refer to them however as the *DDC equations*.

One can easily check that the stationary solution of the problem (3)–(6) together with the boundary conditions (7)–(8) is

$$\widetilde{\boldsymbol{u}} = 0, \quad \widetilde{T}(z) = -\frac{T_0 - T_1}{d} z + T_0, \quad \widetilde{S}(z) = -\frac{S_0 - S_1}{d} z + S_0. \tag{9}$$

These solutions correspond to the linear vertical distributions of temperature and salinity described in the previous section (Fig. 1).

We shall perform the standard dimensional analysis in order to diminish the number of parameters involved. The variable change according to

$$\boldsymbol{x}' = \frac{1}{d}\boldsymbol{x}, \quad t' = \frac{\kappa_T}{d^2} t, \quad \boldsymbol{u}' = \frac{d}{\kappa_T}\boldsymbol{u}, \quad T' = \frac{T - T_0}{T_0 - T_1},$$
$$S' = \frac{S - S_0}{S_0 - S_1}, \quad p' = \frac{d^2}{\varrho_0 \kappa_T^2} p,$$

leads to the new system of equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \frac{\nu}{\kappa_T}\Delta\boldsymbol{u} + \frac{d^3 g}{\kappa_T^2}[1 - \alpha(T_0 - T_1)T + \beta(S_0 - S_1)S]\boldsymbol{e_z} = 0, \tag{10}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{11}$$

$$\partial_t T + (\boldsymbol{u}\cdot\nabla)T = \Delta T, \tag{12}$$

$$\partial_t S + (\boldsymbol{u}\cdot\nabla)S = \frac{\kappa_S}{\kappa_T}\Delta S, \tag{13}$$

where we have omitted the primes for convenience. In dimensionless variables the stationary solution (9) is

$$\widetilde{\boldsymbol{u}} = 0, \quad \widetilde{T}(z) = -z, \quad \widetilde{S}(z) = -z. \tag{14}$$

Instead of working with the variables $\boldsymbol{u}$, p, T and S, we shall decompose each variable into a sum of the stationary solution and its perturbation, according to $\omega = \widetilde{\omega} + \widehat{\omega}$ for $\omega = \boldsymbol{u}, p, T, S$. The system (10)–(13) may be reformulated into

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - P\Delta\boldsymbol{u} + (P\widetilde{R}S - PRT)\boldsymbol{e_z} = 0, \tag{15}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{16}$$

$$\partial_t T + (\boldsymbol{u}\cdot\nabla)T = \Delta T + u_2, \tag{17}$$

$$\partial_t S + (\boldsymbol{u}\cdot\nabla)S = \tau\Delta S + u_2. \tag{18}$$

Again, we have omitted the hats on the variables $\boldsymbol{u}, p, T, S$. The dimensionless parameters appearing in (15)–(18) are: P, the Prandtl number; $\widetilde{R}$, the salinity Rayleigh number; R, the Rayleigh number; and γ, the Lewis number. They read

$$(19)\quad P=\frac{\nu}{\kappa_T},\qquad R=\frac{g\alpha d^3(T_0-T_1)}{\nu\kappa_T},\qquad \widetilde{R}=\frac{g\beta d^3(S_0-S_1)}{\nu\kappa_T},\qquad \tau=\frac{\kappa_S}{\kappa_T}.$$

After all these modifications the set Ω and the boundary conditions have also changed. We have $\Omega=[0,l]\times[0,1]$, where $l=L/d$, whereas the conditions (7)–(8) are now

$$(20)\qquad \phi_{|x=l}=\phi_{|x=0},\qquad \partial_x\phi_{|x=l}=\partial_x\phi_{|x=0},$$

$$(21)\qquad \phi_{|z=0}=\phi_{|z=1}=0,$$

for $\phi=\boldsymbol{u},T,S$.

3.2. *Variational formulation.* Prior to formulating the existence theorem, we need to define the proper function spaces.

DEFINITION 3.1. Let $H=H_1\times H_2\times H_2$, where

$$H_1=\{\boldsymbol{f}\in L^2(\Omega)^2\colon \nabla\cdot\boldsymbol{f}=0 \text{ and } \boldsymbol{f} \text{ satisfies (20)–(21)}\},$$
$$H_2=\{f\in L^2(\Omega)\colon f \text{ satisfies (20)–(21)}\}.$$

We have thus incorporated the divergence-free condition and the boundary condition into the definition of the function spaces, which is a standard technique. H is certainly a Hilbert space with the scalar product $(\cdot,\cdot)$ induced from L^2. The respective norm will be denoted by $|\cdot|$. The same notations will be used for H_1 and H_2, which should not lead to any confusion.

DEFINITION 3.2. Let $V=V_1\times V_2\times V_2$, where

$$V_1=\{\boldsymbol{f}\in H^1(\Omega)^2\colon \nabla\cdot\boldsymbol{f}=0 \text{ and } \boldsymbol{f} \text{ satisfies (20)–(21)}\},$$
$$V_2=\{f\in H^1(\Omega)\colon f \text{ satisfies (20)–(21)}\}.$$

The reader should distinguish the space H_1, which has appeared in Definition 3.1, from the standard Sobolev space H^1. The spaces V_1,V_2 and V are Hilbert spaces with the scalar product induced from H^1,

$$(f,g)_{H^1}=(f,g)+(\nabla f,\nabla g).$$

Since Ω is bounded and the boundary conditions (20)–(21) are periodic in the x direction and uniform in z direction, the Poincaré [1] inequality holds:

$$(22)\qquad |f|\le C_P|\nabla f|,$$

so V_2 can also be equipped with the inner product

$$((f,g))=\int_\Omega \nabla f\cdot\nabla g\,dx$$

[1] From now on the constant C_P will always denote the constant from the Poincaré inequality.

and the respective (equivalent) norm, which will be denoted by $\|\cdot\|$. The same notations will be used for V_1 and V, which should not lead to any ambiguities.

We proceed to the variational formulation of the problem (15)–(18) with the boundary conditions (20)–(21). Let $\boldsymbol{v} = (\boldsymbol{u}, T, S)$ and $\boldsymbol{\psi} = (\boldsymbol{\phi}, \theta, \eta) \in V$. We now take the inner product of (15), (17), (18) with $\boldsymbol{\phi}, \theta, \eta$ respectively and integrate the appropriate terms by parts to obtain

$$\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{\phi}) + \int_{\Omega} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\,dx + P[((\boldsymbol{u}, \boldsymbol{\phi})) + \widetilde{R}(S, \phi_2) - R(T, \phi_2)] = 0, \tag{23}$$

$$\frac{d}{dt}(T, \theta) + \int_{\Omega} (\boldsymbol{u}\cdot\nabla)T\theta\,dx + ((T, \theta)) - (u_2, \theta) = 0, \tag{24}$$

$$\frac{d}{dt}(S, \eta) + \int_{\Omega} (\boldsymbol{u}\cdot\nabla)S\eta\,dx + \tau((S, \eta)) - (u_2, \eta) = 0. \tag{25}$$

The pressure term has dropped out, since the vectors from V_1 are divergence-free. Let us now consider the bilinear form

$$a(\boldsymbol{v}, \boldsymbol{\psi}) = P((\boldsymbol{u}, \boldsymbol{\phi})) + ((T, \theta)) + \tau((S, \eta)), \tag{26}$$

and the associated linear operator

$$(A\boldsymbol{v}, \boldsymbol{\psi}) = a(\boldsymbol{v}, \boldsymbol{\psi}). \tag{27}$$

Then A is bounded from V into V^* and from $D(A)$ into H, where $D(A)$ is defined as

$$D(A) = \{\boldsymbol{v} \in V : \partial_x \boldsymbol{v}_{|x=0} = \partial_x \boldsymbol{v}_{|x=l}\} \cap H^2(\Omega)^4. \tag{28}$$

Let $\boldsymbol{v}_i = (\boldsymbol{u}_i, T_i, S_i) \in V$ for $i = 1, 2, 3$. Consider the following trilinear form b in V:

$$b(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \int_{\Omega} (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_2\cdot\boldsymbol{u}_3\,dx + \int_{\Omega} (\boldsymbol{u}_1\cdot\nabla)T_2T_3\,dx + \int_{\Omega} (\boldsymbol{u}_1\cdot\nabla)S_2S_3\,dx. \tag{29}$$

Then b is continuous in V and we may associate with it a continuous bilinear operator $B\colon V\times V \to V^*$ in the following way:

$$(B(\boldsymbol{v}_1, \boldsymbol{v}_2), \boldsymbol{v}_3) = b(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3).$$

Finally, let us define the bounded linear operator L in H by

$$L\colon \boldsymbol{v} = \{\boldsymbol{u}, T, S\} \mapsto L\boldsymbol{v} = \{(P\widetilde{R}S - PRT)\boldsymbol{e}_z, -u_2, -u_2\}. \tag{30}$$

Adding the equations (23)–(25) we obtain

$$\frac{d}{dt}(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) + a(\boldsymbol{v}, \boldsymbol{\psi}) + (L\boldsymbol{v}, \boldsymbol{\psi}) = 0, \qquad \forall \boldsymbol{\psi} \in V. \tag{31}$$

Putting $B(\boldsymbol{v}, \boldsymbol{v}) = B(\boldsymbol{v})$ we have

$$\frac{d\boldsymbol{v}}{dt} + A\boldsymbol{v} + B(\boldsymbol{v}) + L\boldsymbol{v} = 0. \tag{32}$$

Now we shall deal with some elementary but useful properties of the operators A, B and L.

Lemma 3.1. *The form a defined in* (26) *is bilinear, continuous and coercive in V. The associated operator A is linear and continuous from V into V^*, and unbounded self-adjoint from $D(A)$ into H.*

Lemma 3.2. *Let $\boldsymbol{u} \in H$ and $\boldsymbol{v}, \boldsymbol{w} \in V$. The trilinear form b defined in* (29) *is antisymmetric*:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}),$$

and in particular

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0. \tag{33}$$

Lemma 3.3. *If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ then there exists a constant k such that*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le k|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|\, |\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}.$$

If $\boldsymbol{u} \in V$, $\boldsymbol{v} \in D(A)$ and $\boldsymbol{w} \in H$ then there exists a constant $\widetilde{k}$ such that

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \widetilde{k}|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\|^{1/2}|\boldsymbol{v}|^{1/2}|A\boldsymbol{v}|^{1/2}|\boldsymbol{w}|. \tag{34}$$

Lemma 3.4. *The operator L defined in* (30) *is linear and continuous from V into V^*. Furthermore, there exists a positive constant C such that*

$$|(L\boldsymbol{v}, \boldsymbol{\psi})| \le C|\boldsymbol{v}|\, |\boldsymbol{\psi}|, \qquad \forall \boldsymbol{v}, \boldsymbol{\psi} \in V. \tag{35}$$

The proofs of all these lemmas are analogous to the proofs of the properties of the operators appearing in the classical Navier–Stokes equations [8, 12]. This is undoubtedly an advantage coming from the simplicity of the functional analytic problem setting.

3.3. *The existence and uniqueness theorem*

Definition 3.3. Let $\boldsymbol{v}_0 \in H$. A *weak* (*variational*) solution to the problem (32) with initial condition $\boldsymbol{v}_0$ is a function $\boldsymbol{v} \in \mathcal{C}([0,\infty); H) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; V)$ such that

1. for all $T > 0$ the equality (31) holds in $L^2(0, T; V^*)$, i.e.

$$\int_0^T \left(\frac{d\boldsymbol{v}}{dt}, \boldsymbol{\psi}\right) dt + \int_0^T b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi})\, dt + \int_0^T a(\boldsymbol{v}, \boldsymbol{\psi})\, dt + \int_0^T (L\boldsymbol{v}, \boldsymbol{\psi})\, dt = 0 \tag{36}$$

   for all $\boldsymbol{\psi}$ in $L^2(0, T; V)$,
2. $\boldsymbol{v}(0) = \boldsymbol{v}_0$.

It is not difficult to check that the equality (36) holds also in V for almost every $t \in [0, T]$, which is probably more intuitive than the above statement. We refer the interested reader to [8]. Let us now formulate the main theorem of this section.

THEOREM 3.5. *For a given $\boldsymbol{v}_0$ in H there exists a weak solution to the problem in question. This solution depends continuously on the initial condition $\boldsymbol{v}_0$, and in particular is unique. Furthermore, for all $t > 0$ the map $\boldsymbol{v}_0 \mapsto \boldsymbol{v}(t)$ is continuous in H.*

Proof. We shall apply the Galerkin method in the following steps:

1. Existence of approximate solutions in a finite-dimensional subspace of H.
2. Uniform boundedness of approximate solutions in V.
3. Existence of solution to the initial problem by passing to the limit in the identities defining approximate solutions.
4. Raising regularity: continuity into H.
5. Uniqueness of the weak solution and continuous dependence on the initial condition $\boldsymbol{v}_0$.

Let $V_m = \operatorname{lin}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be a subspace of V such that the family $\{\boldsymbol{w}_j\}$ is orthonormal in H and orthogonal in V. Let $\boldsymbol{v} = \sum_{j=1}^{\infty} c_j(t)\boldsymbol{w}_j(\boldsymbol{x})$. Then we may define a projection of $\boldsymbol{v} \in V$ onto the subspace V_m as

$$P_m\boldsymbol{v} = \sum_{j=1}^{m} c_j(t)\boldsymbol{w}_j(\boldsymbol{x}), \qquad \boldsymbol{w}_j \in V.$$

We set $\boldsymbol{v}_m = P_m\boldsymbol{v}$. For all m we call $\boldsymbol{v}_m$ fulfilling

$$\frac{d}{dt}(\boldsymbol{v}_m, \boldsymbol{w}_i) + b(\boldsymbol{v}_m, \boldsymbol{v}_m, \boldsymbol{w}_i) + a(\boldsymbol{v}_m, \boldsymbol{w}_i) + (L\boldsymbol{v}_m, \boldsymbol{w}_i) = 0, \qquad i = 1, \ldots, m, \tag{37}$$

and the initial condition $\boldsymbol{v}_m(0) = P_m\boldsymbol{v}_0$ the *approximate solution* to (36). We rewrite (37) as

$$\frac{d}{dt}c_m(t) + \sum_{j,k=1}^{m} c_j(t)c_k(t)b(\boldsymbol{w}_j, \boldsymbol{w}_k, \boldsymbol{w}_i) + \lambda_i c_i(t) + \sum_{j=1}^{m} c_j(t)(L\boldsymbol{w}_j, \boldsymbol{w}_i) = 0,$$

hence we obtain a set of m ODEs, which has a solution on a finite time interval $[0, T_m)$. Existence on $[0, \infty)$ and uniform boundedness are more subtle issues resulting from *a priori* estimates and the maximum principle. We take $\boldsymbol{v}_m$ in place of $\boldsymbol{w}_i$ in (37) and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}_m|^2 + b(\boldsymbol{v}_m, \boldsymbol{v}_m, \boldsymbol{v}_m) + a(\boldsymbol{v}_m, \boldsymbol{v}_m) = -(L\boldsymbol{v}_m, \boldsymbol{v}_m), \tag{38}$$

where the nonlinear term disappears thanks to Lemma 3.2. Coerciveness of

a (Lemma 3.1) together with the estimate (35) from Lemma 3.4 gives

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}_m|^2 + C_1\|\boldsymbol{v}_m\|^2 \le C_2|\boldsymbol{v}_m|^2. \tag{39}$$

In order to obtain the existence of solutions $\boldsymbol{v}_m(t)$ on $[0,T)$ for all $T>0$ we use the Poincaré inequality (see (22) to deduce that

$$\frac{d}{dt}|\boldsymbol{v}_m|^2 \le \left(2C_2 - \frac{2C_1}{C_P^2}\right)|\boldsymbol{v}_m|^2.$$

From the classical Gronwall lemma we get

$$|\boldsymbol{v}_m(t)|^2 \le |\boldsymbol{v}_m(0)|^2\exp(C_3 t),$$

where $C_3 = 2C_2 - 2C_1/C_P^2$ is usually a positive constant. This very rough estimate is sufficient to prolong the solutions $\boldsymbol{v}_m$ on arbitrarily long time intervals. In order to obtain a stronger result, we need to improve the bounds on RHS in (38). One way to do that is to assume the boundedness of the temperature and salinity at $t=0$, which is quite a strong hypothesis, though physically obvious. We shall call the lemma below the *maximum principle.* It is formulated in terms of T,S satisfying the initial equations (5)–(6).

LEMMA 3.6. *Let* $\boldsymbol{v}=(\boldsymbol{u},T,S)$ *be a solution to* (3)–(8) *and let* T *and* S *satisfy*

$$T_1 \le T(\boldsymbol{x},0) \le T_0, \qquad S_1 \le S(\boldsymbol{x},0)\le S_0 \qquad \forall \boldsymbol{x}\in\Omega. \tag{40}$$

Then

$$T_1 \le T(\boldsymbol{x},t) \le T_0, \qquad S_1 \le S(\boldsymbol{x},t)\le S_0 \qquad \forall \boldsymbol{x}\in\Omega,\ t\ge 0. \tag{41}$$

Proof. We will show that $T_1 \le T(\boldsymbol{x},t)$ (proof of the remaining inequalities is analogous). Define

$$\gamma(\boldsymbol{x},t) := \max\{(T-T_0)(\boldsymbol{x},t),0\}. \tag{42}$$

We want to show that $\gamma(\boldsymbol{x},t)=0$ for all $t\ge0$, supposing that $\gamma(\boldsymbol{x},0)=0$. Since T is a solution to (5) and γ is of the same regularity as T, we have $\gamma\in L^2(0,s;H^1(\Omega))$ for all $s>0$. Furthermore, γ vanishes at $z=0$ and $z=d$ and is periodic along the x-axis, hence it satisfies the Poincaré inequality. We multiply (5) by γ and integrate over Ω to obtain

$$\int_\Omega \partial_t T\gamma\,dx + \int_\Omega(\boldsymbol{u}\cdot\nabla)T\gamma\,dx - \kappa_T\int_\Omega \Delta T\gamma\,dx = 0.$$

The definition of γ in (42) and integration of the last term by parts give

$$\int_\Omega \gamma\partial_t\gamma\,dx + \int_\Omega(\boldsymbol{u}\cdot\nabla)\gamma\gamma\,dx + \kappa_T\int_\Omega |\nabla\gamma|^2\,dx = 0.$$

Thanks to Lemma 3.2 we may rewrite it equivalently as

$$\frac{1}{2}\frac{d}{dt}|\gamma(t)|^2 + C_P\kappa_T|\gamma(t)|^2 \le 0,$$

where we have used the Poincaré inequality. Hence γ is a nonincreasing positive function and since $\gamma(0) = 0$ we have $\gamma(t) = 0$ for all $t > 0$. ■

We may reformulate the maximum principle in terms of T and S being dimensionless perturbations of the stationary solution. In place of (40)–(41) we have respectively

$$-1 \le T(\boldsymbol{x}, 0), S(\boldsymbol{x}, 0) \le 1 \quad \forall \boldsymbol{x} \in \Omega, \tag{43}$$

$$-1 \le T(\boldsymbol{x}, t), S(\boldsymbol{x}, t) \le 1 \quad \forall \boldsymbol{x} \in \Omega,\ t > 0. \tag{44}$$

Now we shall find uniform bounds on the approximate solutions. By the definition of the operator L and the Schwarz inequality, the equality (38) yields

$$|(L\boldsymbol{v}_m, \boldsymbol{v}_m)| \le C|u_{m2}|(|T_m| + |S_m|),$$

where C depends on P, R and $\widetilde{R}$. Next, thanks to the Poincaré inequality and the maximum principle we obtain

$$|(L\boldsymbol{v}_m, \boldsymbol{v}_m)| \le CC_P\|\boldsymbol{v}_m\|(|\Omega|^{1/2} + |\Omega|^{1/2}) = \widetilde{C}\|\boldsymbol{v}_m\|,$$

and by Young's inequality with $\varepsilon = C_1$, where C_1 is the coerciveness constant of the bilinear form a,

$$|(L\boldsymbol{v}_m, \boldsymbol{v}_m)| \le \frac{C_1\|\boldsymbol{v}_m\|^2}{2} + \frac{\widetilde{C}^2}{2C_1}.$$

Using the last inequality in (38) as well as Lemma 3.1, we obtain the uniform (with respect to m) bound on the LHS,

$$\frac{d}{dt}|\boldsymbol{v}_m|^2 + C_1\|\boldsymbol{v}_m\|^2 \le \frac{\widetilde{C}^2}{C_1}. \tag{45}$$

Hence we conclude (using the Gronwall lemma and Poincaré inequality once again) that

$$\boldsymbol{v}_m \in L^\infty(0, T; H) \cap L^2(0, T; V) \tag{46}$$

for all $T > 0$ uniformly with respect to m.

Now we shall find a bound on $d\boldsymbol{v}_m/dt$. We rewrite (37) as

$$\frac{d\boldsymbol{v}_m}{dt} = -A\boldsymbol{v}_m - P_mB(\boldsymbol{v}_m, \boldsymbol{v}_m) - L\boldsymbol{v}_m. \tag{47}$$

From (46) and the continuity of A and L we deduce that

$$A\boldsymbol{v}_m, L\boldsymbol{v}_m \in L^2(0, T; V^*).$$

In order to find a bound on the nonlinear term in (47) we observe that

$$\int_0^T \|P_mB(\boldsymbol{v}_m(s), \boldsymbol{v}_m(s))\|_*^2\, ds \le \int_0^T \|B(\boldsymbol{v}_{\boldsymbol{m}}(s), \boldsymbol{v}_m(s))\|_*^2\, ds.$$

From the contnuity of the trilinear form b and its antisymmetry (Lemmas 3.2 and 3.3) we deduce that

$$\|B(\boldsymbol{u},\boldsymbol{u})\|_* \le c|\boldsymbol{u}|\,\|\boldsymbol{u}\|.$$

In the end we obtain

$$\begin{aligned}\|P_m B(\boldsymbol{v}_m,\boldsymbol{v}_m)\|^2_{L^2(0,T;V^*)} &\le c\int_0^T |\boldsymbol{v}_m(s)|^2\|\boldsymbol{v}_m(s)\|^2\,ds \\ &\le c\|\boldsymbol{v}_m\|^2_{L^\infty(0,T;H)}\|\boldsymbol{v}_m\|^2_{L^2(0,T;V)}.\end{aligned}$$

By (46) we have shown that $P_m B(\boldsymbol{v}_m,\boldsymbol{v}_m)$, and hence $d\boldsymbol{v}_m/dt$, are uniformly bounded in $L^2(0,T;V^*)$.

Now we move to the next step of the proof, which is the passage to the limit in the equality (47). We begin with the definition of weak* convergence.

Definition 3.4. Let X be a Banach space. A sequence $f_n \in X^*$ *converges weak** to f, written

$$f_n \rightharpoonup^* f,$$

if $f_n(x) \to f(x)$ for every $x \in X$.

Notation. We denote the standard convergence by $\to$, weak convergence by $\rightharpoonup$, and weak* convergence by $\rightharpoonup^*$.

Theorem 3.7 (Alaoglu, [8]). *Let X be a reflexive Banach space and f_n a bounded sequence in X^*. Then f_n has a subsequence which converges weak* in X^*.*

From (46) and the Alaoglu theorem we deduce that $\boldsymbol{v}_m$ has a subsequence (not relabelled) converging weakly in $L^2(0,T;V)$ and weak* in $L^\infty(0,T;H)$ to a function $\boldsymbol{v}$ which satisfies

$$\boldsymbol{v} \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{48}$$

Next, since $d\boldsymbol{v}_m/dt$ is bounded in $L^2(0,T;V^*)$, we may extract a subsequence (not relabelled) which converges weakly to some $\boldsymbol{h}$,

$$\frac{d\boldsymbol{v}_m}{dt} \rightharpoonup \boldsymbol{h} \quad \text{in } L^2(0,T;V^*), \tag{49}$$

and integrating by parts we deduce that $\boldsymbol{h} = d\boldsymbol{v}/dt$. In particular, in (49) we also have weak* convergence. One can also easily check that

$$A\boldsymbol{v}_m \rightharpoonup^* A\boldsymbol{v}, \quad L\boldsymbol{v}_m \rightharpoonup^* L\boldsymbol{v} \quad \text{in } L^2(0,T;V^*).$$

There remains the convergence of the nonlinear term. The proof is not difficult but a little messy. First we formulate a following compactness theorem:

Theorem 3.8 ([8]). *Let $X \subset\subset H \subset Y$ be Banach spaces and let X be reflexive. Suppose that a sequence u_n is uniformly bounded in $L^2(0,T;X)$*

and du_n/dt in $L^p(0,T;Y)$ for some $p>1$. Then there exists a subsequence u_m which converges strongly in $L^2(0,T;H)$.

In our case of $X=V$, $Y=V^*$ and $p=2$, the theorem states that $\boldsymbol{v}_m$ is strongly convergent in $L^2(0,T;H)$. We want to show that

$$P_m B(\boldsymbol{v}_m,\boldsymbol{v}_m) \rightharpoonup^* B(\boldsymbol{v},\boldsymbol{v}) \quad \text{in } L^2(0,T;V^*). \tag{50}$$

In the first step we will show that

$$B(\boldsymbol{v}_m,\boldsymbol{v}_m) \rightharpoonup^* B(\boldsymbol{v},\boldsymbol{v}) \quad \text{in } L^2(0,T;V^*). \tag{51}$$

Let $\boldsymbol{\psi}=(\boldsymbol{\phi},\theta,\eta)\in L^2(0,T;\mathcal{V})$, where $\mathcal{V}=V\cap\mathcal{C}_0^\infty(\Omega)^4$. We introduce this space in order to show the necessary convergence. From the definition of the trilinear form b (29) and its antisymmetry we have

$$\begin{aligned} b(\boldsymbol{v}_m,\boldsymbol{v}_m,\boldsymbol{\psi}) &= -b(\boldsymbol{v}_m,\boldsymbol{\psi},\boldsymbol{v}_m)\\ &= -\int_\Omega(\boldsymbol{u}_m\cdot\nabla)\boldsymbol{\phi}\cdot\boldsymbol{u}_m\,dx-\int_\Omega(\boldsymbol{u}_m\cdot\nabla)\theta T_m\,dx-\int_\Omega(\boldsymbol{u}_m\cdot\nabla)\eta S_m\,dx. \end{aligned}$$

Hence (we recall that $\boldsymbol{v}=(u_1,u_2,T,S)$)

$$\begin{aligned} &\int_0^T b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\psi})-b(\boldsymbol{v}_m,\boldsymbol{v}_m,\boldsymbol{\psi})\,dt \\ &\quad=\sum_{i,j=1}^2\int_0^T\int_\Omega\{[(u_m)_i-u_i](D_i\phi_j)u_j+(u_m)_i(D_i\phi_j)[(u_m)_j-u_j]\\ &\qquad+[(u_m)_i-u_i](D_i\theta)T+(u_m)_i(D_i\theta)(T_m-T)\\ &\qquad+[(u_m)_i-u_i](D_i\eta)S+(u_m)_i(D_i\eta)(S_m-S)\}\,dx\,dt. \end{aligned} \tag{52}$$

There are two types of terms in the above equality: type 1,

$$\int_0^T\int_\Omega[(u_m)_i-u_i](D_i\theta)T\,dx\,dt,$$

and type 2,

$$\int_0^T\int_\Omega(u_m)_i(D_i\theta)(T_m-T)\,dx\,dt.$$

Both decay as $m\to\infty$. From the Schwarz inequality,

$$\Big|\int_0^T\int_\Omega[(u_m)_i-u_i](D_i\theta)T\,dx\,dt\Big|\le\|(u_m)_i-u_i\|_{L^2(0,T;H)}\|(D_i\theta)T\|_{L^2(0,T;H)}$$

and

$$\Big|\int_0^T\int_\Omega(u_m)_i(D_i\theta)(T_m-T)\,dx\,dt\Big|\le\|T_m-T\|_{L^2(0,T;H)}\|(u_m)_iD_i\theta\|_{L^2(0,T;H)}.$$

By the regularity of θ, the uniform boundedness of $(u_m)_i$ in $L^2(0,T;V)$ and Theorem 3.8, both RHSs tend to zero. Now, from the density of $L^2(0,T;\mathcal{V})$ in $L^2(0,T;V)$ we have (51).

We leave the remaining proof of (50) to the reader. One way to obtain this property is to consider the family of functions

$$\boldsymbol{\psi} = \sum_{j=1}^{k} \boldsymbol{\psi}_j \alpha_j(t), \qquad \boldsymbol{\psi}_j \in V,\ \alpha_j \in L^2(0,T;\mathbb{R}),$$

which is dense in $L^2(0,T;V)$ and use the fact that V has an orthogonal basis.

We have thus shown the weak* convergence in $L^2(0,T;V^*)$ of all the terms in (47) for every $T>0$, i.e. there exists a function $\boldsymbol{v} \in L^2_{\mathrm{loc}}(\mathbb{R}^+;V) \cap L^\infty(\mathbb{R}^+;H)$ such that the following equality holds in $L^2(0,T;V^*)$:

$$\int_0^T \left(\frac{d\boldsymbol{v}}{dt}, \boldsymbol{\psi}\right) dt + \int_0^T b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\psi})\, dt + \int_0^T a(\boldsymbol{v},\boldsymbol{\psi})\, dt + \int_0^T (L\boldsymbol{v},\boldsymbol{\psi})\, dt = 0 \tag{53}$$

for every $\boldsymbol{\psi} \in L^2(0,T;V)$ and $T>0$.

We already have the existence of solutions, but their regularity is not satisfactory. We emphasize that what have been done so far is, after small changes, also valid in the 3D case. The difference between the 2D and 3D cases lies in continuity and uniqueness of solutions. This is a consequence of weaker estimates of the trilinear form b in the 3D case. However, for $n=2$ we have the following lemma:

LEMMA 3.9 ([8]). *Let V and H be Hilbert spaces satisfying*

$$V \subset\subset H \subset V^*.$$

If

$$\boldsymbol{v} \in L^2(0,T;V) \quad \textit{and} \quad \frac{d\boldsymbol{v}}{dt} \in L^2(0,T;V^*),$$

then

$$\boldsymbol{v} \in \mathcal{C}(0,T;H).$$

All the assumptions of this lemma are satisfied in our case, so the statement is true. Since $T>0$ may be chosen arbitrarily, we have the continuity of solutions to the problem (32) from $[0,\infty)$ into H as stated in Definition 3.3.

To finish the proof of Theorem 3.5, there remains the uniqueness. The procedure is standard so we limit ourselves to a sketch. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are two different solutions to (32). We set $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and take the scalar product of the equation for $\boldsymbol{w}$ with $\boldsymbol{w}$ to obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{w}|^2 + b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w}) + b(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w}) + a(\boldsymbol{w},\boldsymbol{w}) + (L\boldsymbol{w},\boldsymbol{w}) = 0.$$

Then we use Lemmas 3.1–3.4 to estimate the appropriate terms. In the end, from Young's inequality we deduce that

$$\frac{d}{dt}|\boldsymbol{w}|^2 \le |\boldsymbol{w}|^2\left(\frac{C_2^2}{C_1}\|\boldsymbol{u}\|^2 + C_3\right),$$

and since the bracketed expression is integrable with respect to time, from the Gronwall lemma, if only $\boldsymbol{w}(0) = 0$ then $\boldsymbol{w}(t) = 0$ for every $t > 0$. Furthermore, if we take $\boldsymbol{w}(0) \neq 0$ we have the continuous dependence on the initial condition, as stated in Theorem 3.5, which terminates the proof. ■

4. Asymptotic dynamics. In this section we focus on the asymptotic (as $t \to \infty$) dynamics of solutions to the DDC equations (3)–(8), which we reformulated in the variational form (47). In dynamical systems language we have the problem

$$\frac{d\boldsymbol{v}}{dt} = F(\boldsymbol{v}(t)) \tag{54}$$

with $\boldsymbol{v} = (\boldsymbol{u}, T, S)$ and with the initial condition

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{55}$$

The problem (54)–(55) is well-posed for all $t > 0$, which is an immediate consequence of Theorem 3.5. Thus we may define the corresponding semigroup $\{\mathcal{S}(t)\}_{t\ge 0}$, i.e. a family of operators

$$\mathcal{S}(t)\colon \boldsymbol{v}_0 \in H \mapsto \boldsymbol{v}(t) \in H,$$

with the usual semigroup properties:

$$\mathcal{S}(0) = I, \qquad \mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(s)\mathcal{S}(t) = \mathcal{S}(s+t),$$

and $\mathcal{S}(t)\boldsymbol{v}_0$ is continuous with respect to $\boldsymbol{v}_0$ and t.

4.1. *Absorbing set.* One of the basic ideas in studying asymptotic dynamics is the notion of an *absorbing set.*

Definition 4.1. A set $\mathcal{B} \subset H$ is an *absorbing set* in H if for every bounded set $X \subset H$ there exists a time t_0 for which

$$\mathcal{S}(t)X \subset \mathcal{B} \quad \forall t \ge t_0.$$

A semigroup $\mathcal{S}(t)$ is *dissipative* if it has a compact absorbing set.

Now we will show the existence of an absorbing set in the spaces H and V for the semigroup generated by the DDC equations. Much work has already been done; we will use the *a priori* estimates developed in the previous section once again.

Proposition 4.1. *For the two-dimensional DDC equations there exists an absorbing set that is bounded in H.*

Proof. In the first step of the proof of Theorem 3.5 we have established an estimate (45) on the approximate solutions $\boldsymbol{v}_m$, which is also valid for the solution $\boldsymbol{v}$:

$$\frac{d}{dt}|\boldsymbol{v}|^2 + C_1\|\boldsymbol{v}\|^2 \le \frac{\widetilde{C}^2}{C_1}.$$

Using the Poincaré inequality and Gronwall lemma we obtain

$$|\boldsymbol{v}(t)|^2 \le |\boldsymbol{v}(0)|^2 \exp\Big(-\frac{C_1}{C_P}t\Big) + \frac{\widetilde{C}^2 C_P}{C_1^2}, \tag{56}$$

where C_P is the constant arising in the Poincaré inequality (depending only on $|\Omega|$), $C_1 = \min\{1, P, \kappa\}$, and $\widetilde{C}$ depends on $|\Omega|$, P and the rest of parameters appearing in the DDC equations: R and $\widetilde{R}$. From (56) we deduce that for every $\boldsymbol{v}_0 \in H$ and $\varepsilon > 0$ there exists a time $t_0(\boldsymbol{v}_0, \varepsilon)$ such that for $t \ge t_0$,

$$|\boldsymbol{v}(t)| \le \frac{\widetilde{C}^2 C_P}{C_1^2} + \varepsilon =: \varrho_\varepsilon. \tag{57}$$

We notice that for $\boldsymbol{v}_0$ in a bounded set B_0 the time t_0 may be chosen uniformly. Hence according to Definition 4.1, the ball

$$\mathcal{B}_1 = \{\boldsymbol{v} \in H : |\boldsymbol{v}| \le \varrho_\varepsilon\}$$

is absorbing in H. ■

An analogous statement is also valid in the phase space V.

PROPOSITION 4.2. *For the two-dimensional DDC equations there exists an absorbing set that is bounded in V.*

Proof. We take the scalar product in H of (32) with $A\boldsymbol{v}$,

$$\Big(\frac{d\boldsymbol{v}}{dt}, A\boldsymbol{v}\Big) + |A\boldsymbol{v}|^2 + b(\boldsymbol{v}, \boldsymbol{v}, A\boldsymbol{v}) + (L\boldsymbol{v}, A\boldsymbol{v}) = 0. \tag{58}$$

From the estimate (34), Schwarz inequality and coerciveness of a we deduce that

$$C_1 \frac{d}{dt}\|\boldsymbol{v}\|^2 + |A\boldsymbol{v}|^2 \le \widetilde{k}|\boldsymbol{v}|^{1/2}\|\boldsymbol{v}\||A\boldsymbol{v}|^{3/2} + C_2|A\boldsymbol{v}|\,|\boldsymbol{v}|$$

for some positive constants C_1 and C_2. Now we use Young's inequality twice on the RHS to get

$$C_1 \frac{d}{dt}\|\boldsymbol{v}\|^2 + |A\boldsymbol{v}|^2 \le \frac{|A\boldsymbol{v}|^2}{4} + 2C_2|\boldsymbol{v}|^2 + \frac{|A\boldsymbol{v}|^2}{4} + \frac{27}{4}\widetilde{k}|\boldsymbol{v}|^2\|\boldsymbol{v}\|^4;$$

hence, after relabelling constants, we obtain

$$\frac{d}{dt}\|\boldsymbol{v}\|^2 + C_3|A\boldsymbol{v}|^2 \le C_4|\boldsymbol{v}|^2 + C_5|\boldsymbol{v}|^2\|\boldsymbol{v}\|^4. \tag{59}$$

Since we only assume that $\boldsymbol{v}_0$ is in H and not necessarily in V, we cannot use the classical Gronwall lemma. We need to apply the uniform Gronwall lemma, which we state below.

Lemma 4.3 (Uniform Gronwall lemma, [12]). *Let x, a, b be positive functions on (t_0,∞) such that*

$$\frac{dx}{dt} \le ax + b,$$

and

$$\int_t^{t+r} x(s)\,ds \le X, \qquad \int_t^{t+r} a(s)\,ds \le A, \qquad \int_t^{t+r} b(s)\,ds \le B,$$

for some $X, A, B, r > 0$ and for every $t \ge t_0$. Moreover, let x be absolutely continuous on (t_0,∞). Then for $t \ge t_0 + r$ we have

$$x(t) \le \left(\frac{X}{r} + B\right)\exp(A).$$

We may drop the second term of the LHS of (59) so in our case $x = \|\boldsymbol{v}\|^2$, $a = C_5|\boldsymbol{v}|^2\|\boldsymbol{v}\|^2$ and $b = C_4|\boldsymbol{v}|^2$. Now we check the remaining three assumptions of the lemma. We return to the inequality (39), which was established for the approximate solutions $\boldsymbol{v}_m$, but is also valid for the solution $\boldsymbol{v}$. We integrate it over the interval $(t, t+1)$ to obtain

$$C_1 \int_t^{t+1} \|\boldsymbol{v}(s)\|^2\,ds \le \frac{1}{2}\,|\boldsymbol{v}(t)|^2 + C_2 \int_t^{t+1} |\boldsymbol{v}(s)|^2\,ds.$$

From (57) we deduce that there exists t_0 such that for $t \ge t_0$ we have

$$\int_t^{t+1} \|\boldsymbol{v}(s)\|^2 ds \le \frac{\varrho_\varepsilon^2}{2C_1}\,(1 + 2C_2) =: X.$$

We find the constants A and B in a similar way:

$$\int_t^{t+1} C_5|\boldsymbol{v}(s)|^2\|\boldsymbol{v}(s)\|^2\,ds \le \varrho_\varepsilon^2 C_5 \int_t^{t+1} \|\boldsymbol{v}(s)\|^2\,ds \le \varrho_\varepsilon^2 C_5 X =: A,$$

$$\int_t^{t+1} C_4|\boldsymbol{v}(s)|^2\,ds \le C_4\varrho_\varepsilon^2 =: B.$$

The uniform Gronwall lemma gives

$$\|\boldsymbol{v}(t)\|^2 \le (X + B)\exp(A) = r_\varepsilon^2$$

for $t \ge t_0 + 1$. Hence, according to Definition 4.1, the ball

$$\mathcal{B}_2 = \{\boldsymbol{v} \in V : \|\boldsymbol{v}\| \le r_\varepsilon\}$$

is absorbing in V. ∎

4.2. *Global attractor.* We begin with the definition of a global attractor.

DEFINITION 4.2. Let $\{\mathcal{S}(t)\}_{t\geq 0}$ be a semigroup in a Banach space X. A set $\mathcal{A}\subset X$ is a *global attractor* for the semigroup $\mathcal{S}(t)$ if $\mathcal{A}$ is compact, invariant and attracts all bounded sets, i.e. for every bounded set B in X,

$$\operatorname{dist}(\mathcal{S}(t)B,\mathcal{A})\to 0 \quad \text{as } t\to\infty. \tag{60}$$

The dist in (60) is a semidistance defined in the following way:

$$\operatorname{dist}(X,Y)=\sup_{x\in X}\inf_{y\in Y}|x-y|.$$

The convergence in (60) is equivalent to the following one: for every $\varepsilon>0$ there exists t_ε such that for $t\geq t_\varepsilon$, $\mathcal{S}(t)B$ is included in U_ε, the ε-neighbourhood of $\mathcal{A}$. Furthermore, it is straightforward to check the following properties of the global attractor: $\mathcal{A}$ is unique, it is a maximal bounded invariant set, and it is a minimal closed set attracting all the bounded sets.

In order to prove the existence of a global attractor for the semigroup associated with the DDC equations we will use the general theorem on existence of a global attractor. First we have to introduce the notion of uniform compactness of a semigroup.

DEFINITION 4.3. A semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ in X is *uniformly compact* if for every bounded set $B\subset X$ there exists a time $t_0=t_0(B)$ such that the set

$$\overline{\bigcup_{t\geq t_0}\mathcal{S}(t)B}$$

is compact.

THEOREM 4.4 ([7, 12]). *Let $\{\mathcal{S}(t)\}_{t\geq 0}$ be a continuous semigroup in a Banach space X. Assume there exists a bounded absorbing set $\mathcal{B}$ in X and the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ is uniformly compact. Then the ω-limit set of $\mathcal{B}$,*

$$\omega(\mathcal{B})=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\mathcal{S}(t)\mathcal{B}},$$

is a global attractor.

Equipped with this theorem it is straightforward to prove the existence of a global attractor in our case.

THEOREM 4.5. *The dynamical system on H generated by the two-dimensional DDC equations has a global attractor.*

Proof. The continuity of the semigroup and the existence of an absorbing set $\mathcal{B}_1$ in H are already proven. We are left only with the asymptotic compactness of the semigroup. We have shown in Proposition 4.2 that there exists an absorbing set $\mathcal{B}_2$ in V. Now, since V is compactly embedded in H

we deduce that for every bounded set B in V there exists a time t_0 such that the set

$$\bigcup_{t \geq t_0} \mathcal{S}(t)B$$

is bounded in V and, after taking closure, compact in H. Thus Theorem 4.4 implies that $\mathcal{A} = \omega(\mathcal{B}_1)$ is a global attractor. ■

5. Conclusions. In this final section we give some remarks with an emphasis on potentially interesting future research.

In infinite-dimensional dynamical systems governed by nonlinear PDEs the question about existence of absorbing sets and global attractors is far more complicated than in finite-dimensional dynamical systems governed by ODEs. With some effort, especially in the proof of existence of weak solutions, we have shown the existence of a global attractor in the phase space H for the problem of double-diffusive convection. However, we cannot say much about the structure of this theoretical object, which would be interesting from the physical point of view. One of the most popular approaches in studying the structure of a global attractor is via the notions of fractal and Hausdorff dimension [8, 12].

C. Foiaş et al. [2] have shown, in a paper thematically close to ours, that the global attractor's fractal dimension in the Bénard problem satisfies

$$\dim(\mathcal{A}) \leq cG(1+P)^2,$$

where c is a constant dependent only on the flow geometry, G is the Grashof number, and P the Prandtl number. The Grashof number is a dimensionless parameter that may be defined as

$$G = \frac{R}{P},$$

where R is the Rayleigh number introduced in (19). The natural and equally challenging question is whether the attractor's dimension in the two-dimensional DDC equations is also finite and what are its bounds, if any. One could suspect, taking into consideration the similarity to the Bénard problem, that the attractor's fractal dimension is finite and satisfies

$$\dim(\mathcal{A}) \leq c\,\frac{R+\widetilde{R}}{P}\,(1+P)^2, \tag{61}$$

where $\widetilde{R}$ is the salinity Rayleigh number, also defined in (19). Though it is only a hypothesis, let us take (61) for granted and try to find its physical interpretation. Assuming constant flow geometry and fluid viscosity, the Rayleigh numbers R and $\widetilde{R}$ are controlled mainly by the thermal and salinity gradients, $T_0 - T_1$ and $S_0 - S_1$ respectively. The Prandtl number P expresses the ratio of momentum diffusity and thermal diffusity, and may be considered constant for a "fixed" fluid. Hence, under some simplification, the attractor's fractal dimension is controlled by the values of $T_0 - T_1$ and $S_0 - S_1$. The bigger these gradients are, the worse is the estimate. This purely hypothetical result is in agreement with intuition just as in the Bénard problem, in which the temperature gradient controls the transformation from laminar into turbulent convection. We have described this mechanism in Section 2.

We therefore consider studies on the attractor's fractal (or Hausdorff) dimension in the DDC problem as the most important (and interesting) issue. Some other directions of further research include:

1. Raising regularity of weak solutions. Undoubtedly we have not reached optimal results in this matter—we have just done what was necessary to define the dynamical system. In [12] there are theorems which could help raise the regularity of solutions.
2. Defining the dynamical system on the phase space V. With appropriate theorems it would be interesting to compare the global attractors on H and V.
3. Studying the three-dimensional DDC equations. This is problematic, since at present we cannot show that the 3D Navier–Stokes equations generate unique weak solutions. Hence consideration of the dynamical system in the 3D case is somehow unjustified. On the other hand, the topics pertaining to turbulence are currently extremely interesting, so the 3D case is worth studying.

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Institute of Applied Mathematics
and Mechanics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland

Current address:
Department of Hydraulic Engineering
and Environmental Restoration
Warsaw University of Life Sciences
Nowoursynowska 159
02-787 Warszawa, Poland
E-mail: mpiniewski@levis.sggw.pl

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