APPLICATIONES MATHEMATICAE
28,1 (2001), pp. 17–30

Jaume Giné and Xavier Santallusia (Lleida)

ON THE POINCARÉ–LYAPUNOV CONSTANTS AND THE POINCARÉ SERIES

Abstract. For an arbitrary analytic system which has a linear center at the origin we compute recursively all its Poincaré–Lyapunov constants in terms of the coefficients of the system, giving an answer to the classical center problem. We also compute the coefficients of the Poincaré series in terms of the same coefficients. The algorithm for these computations has an easy implementation. Our method does not need the computation of any definite or indefinite integral. We apply the algorithm to some polynomial differential systems.

1. Introduction. Many models of natural phenomena use systems of differential equations in the plane and the qualitative theory of differential equations, introduced by Poincaré, can be used to describe the behavior of such systems in most cases. One of the problems here is to distinguish between a focus and a center (the *center problem*). The resolution of this problem requires the computation of the so-called Poincaré–Lyapunov constants. Therefore to have a fast and easy method for this computation is of great importance. Another important problem is to determine systems that have centers at some singular points due to the fact that perturbations of such systems give rich bifurcations of limit cycles.

In the last years many papers have been published giving different methods to compute the Poincaré–Lyapunov constants. In this work we compute them recursively in terms of the coefficients of the system for an arbitrary analytic system which has a linear center at the origin, thus answering the classical center problem. We also compute the coefficients of the Poincaré series in terms of the same coefficients. Our method does not need the com-

2000 *Mathematics Subject Classification*: Primary 34A05; Secondary 34C05, 34C25.
Key words and phrases: center-focus problem, nonlinear differential equations.

putation of any definite or indefinite integral, and is easy to implement on a computer.

Consider two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = -y + X(x,y), \qquad \dot{y} = x + Y(x,y), \tag{1}$$

where the nonlinearities are

$$X(x,y) = \sum_{s=2}^{\infty} X_s(x,y) \quad \text{and} \quad Y(x,y) = \sum_{s=2}^{\infty} Y_s(x,y)$$

with $X_s(x,y) = \sum_{k=0}^{s} a_k^s x^k y^{s-k}$ and $Y_s(x,y) = \sum_{k=0}^{s} b_k^s x^k y^{s-k}$ and a_k^s and b_k^s are arbitrary real coefficients.

For such systems Poincaré [20] developed an important technique that consists in finding a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{2}$$

where $H_2(x,y) = (x^2+y^2)/2$, and for each n, $H_n(x,y) = \sum_{k=0}^{n} C_k^n x^k y^{n-k}$, such that the derivative of H along the solutions of system (1) satisfies

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k, \tag{3}$$

where V_{2k} are called the *Poincaré–Lyapunov constants.*

In order to solve the problem of the stability of system (1) at the origin, it is sufficient to consider the sign of the first Poincaré–Lyapunov constant different from zero. If it is positive we have asymptotic stability for negative times, and if it is negative we have asymptotic stability for positive times. If all Poincaré–Lyapunov constants are zero, then the origin is stable for all times, but there is no asymptotic stability for any time (see for instance [2]). In this last case, we have a center at the origin, i.e. there is an open neighborhood of the origin where all orbits are periodic, except of course the origin. The origin is said to be a *fine focus* of order k if V_{2k+2} is the first nonzero Poincaré–Lyapunov constant. In this case at most k limit cycles can bifurcate from this fine focus [4]; they are called *small-amplitude limit cycles.* Therefore to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus. It is known that this maximum number is three for quadratic systems [3] and it has been shown recently that it is greater than or equal to eleven for cubic systems [24].

In this work we are going to see that we can always determine C_k^n and V_{2k} from a_k^s and b_k^s, but the C_k^n are not unique and in consequence neither are the V_{2k}. Therefore, the Poincaré formal series is not unique. Poincaré [20] proved, by delimitation, that for polynomial systems there exists one which is convergent, and Lyapunov [17] generalized Poincaré's theorem to analytic systems. In [6] Chazy, using the theorem of analytical dependence on initial parameters, demonstrated that there exists one which is convergent, by a suitable choice of parameters that appear in the construction of Poincaré series. For polynomial systems we have uniqueness of the V_{2k} in the sense of the following theorem due to Shi Songling [22].

THEOREM 1. *Let* $\mathbf{A}$ *be the ring of real polynomials whose variables are the coefficients of a polynomial differential system. Given a set of Poincaré–Lyapunov constants* $V_1, \ldots, V_i$, *let* $\mathbf{J}_{k-1}$ *be the ideal of* $\mathbf{A}$ *generated by* $V_1, \ldots, V_{k-1}$. *If* $V_1', \ldots, V_i'$ *is another set of Poincaré–Lyapunov constants, then* $V_k \equiv V_k' \pmod{\mathbf{J}_{k-1}}$.

As mentioned above, the origin is a center if and only if all the V_i's are zero. Let $\mathbf{J} = (V_1, V_2, \ldots)$ be the ideal of $\mathbf{A}$ generated by all the V_i's. For polynomial systems, by Hilbert's basis theorem, $\mathbf{J}$ is finitely generated, i.e. there exist $B_1, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = (B_1, \ldots, B_q)$. Such a set of generators is called a basis of $\mathbf{J}$.

There exist various algorithms to compute the Poincaré–Lyapunov constants. The technique used by Bautin [3] is based on computing the derivatives of the return map from a nonlinear system of recursive differential equations. There is another algorithm which involves the solution of a system of linear equations for the coefficients of H_n in terms of the coefficients of X_s, Y_s and H_k for $k = 2, \ldots, n-1$ (see for instance [16] and [19]). Another method is to construct a Poincaré formal power series in polar coordinates and the Poincaré–Lyapunov constants can be computed from recursive linear formulas as definite integrals of trigonometric polynomials (see for example [1] and [5]). In [10] the authors give a survey of different ways to compute the Poincaré–Lyapunov constants.

Modifying the standard techniques explained in [2] for obtaining the Poincaré–Lyapunov constants, in [7] the first and second Poincaré–Lyapunov constants are computed for an arbitrary analytic system using the return function and some algebraic properties of the Poincaré–Lyapunov constants. In [15] taking advantage of the complex structure that simplifies their effective computation, V_3 and V_5 have been found by hand. A development of the method presented in [15] is used in [13] to obtain V_7 for an arbitrary analytic system. Using the Runge–Kutta–Fehlberg methods and Richardson's extrapolation, an analytic-numerical method of computation of the Poincaré–Lyapunov constants is given in [14]. Another algorithm is developed in [12] and [11], based on the calculation of successive derivatives of the first return map associated with the perturbations of some planar Hamiltonian systems. An important generalization of this last method is given in [23].

The paper is organized as follows. In the next section we present a formula for the Poincaré–Lyapunov constants (see Theorem 2) and we describe the algorithm that we have developed. As a particular case the formula is applied to quadratic systems. Section 3 is devoted to the center problem for some particular systems as an application of the method.

2. The main result. We present a formula for the Poincaré–Lyapunov constants and Poincaré series for general systems (1) in recurrent form following the ideas of Shi Songling [21] who found the same expression for the Poincaré–Lyapunov constants, but he did not find the recursive relation with the Poincaré series, which establishes a method to compute them.

The advantages of this method are:

(a) In the whole process we only calculate products and sums without definite or indefinite integrals as in the majority of other methods.

(b) As a consequence of (a) the process is easy to implement on a computer.

(c) The method gives the Poincaré–Lyapunov constants and Poincaré series at the same time. This allows us to find systems with a polynomial first integral by requiring the Poincaré series to have a finite number of terms.

THEOREM 2. *The Poincaré–Lyapunov constants of system* (1) *are*

$$V_n = \frac{\sum_{l=0}^{n/2}(n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{\sum_{l=0}^{n/2}(n-(2l+1))!!\,(2l-1)!!\binom{n/2}{l}}, \qquad n = 4,6,8,\ldots,$$

where $d_k^n = \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}(la_{k-l+1}^{n-m} + (m+1-l)b_{k-l}^{n-m})C_l^{m+1}$, $n \geq 3$, $k = 0,\ldots,n$, *with* $a_k^s = b_k^s = 0$ *for* $k<0$ *or* $k>s$, $C_0^2 = C_2^2 = 1/2$ *and* $C_1^2 = 0$, *and for* $n \geq 3$,

$$C_k^n = \begin{cases} \dfrac{\sum_{l=0}^{(k-1)/2}(n-(2l+1))!!\,(2l-1)!!\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-k)!!\,k!!}, & \\ & k = 1,3,5,\ldots, \\ -\dfrac{\sum_{l=k/2}^{[(n-1)/2]}(n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n + \lambda_n}{(n-k)!!\,k!!}, & k = 0,2,4,\ldots \end{cases}$$

where λ_n *are arbitrary constants and* V_n *and* λ_n *are zero for* n *odd.*

Proof. From the evaluation of the derivative of $H(x,y)$ along the solutions of system (1) we have

$$
\begin{aligned}
\dot{H} &= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \\
&= \Big(\sum_{n=2}^{\infty} \frac{\partial H_n}{\partial x}\Big)\Big(-y + \sum_{s=2}^{\infty} X_s\Big) + \Big(\sum_{n=2}^{\infty} \frac{\partial H_n}{\partial y}\Big)\Big(x + \sum_{s=2}^{\infty} Y_s\Big) \\
&= \sum_{n=2}^{\infty}\Big(-y\frac{\partial H_n}{\partial x} + x\frac{\partial H_n}{\partial y}\Big) + \sum_{s=2}^{\infty} X_s \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial x} + \sum_{s=2}^{\infty} Y_s \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial y} \\
&\quad = \sum_{n=3}^{\infty}\Big(-y\frac{\partial H_n}{\partial x} + x\frac{\partial H_n}{\partial y} + \sum_{m=1}^{n-2}\Big(X_{n-m}\frac{\partial H_{m+1}}{\partial x} + Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\Big)\Big).
\end{aligned}
$$

Comparing with (3) we have

$$
\begin{aligned}
(4)\qquad -y\frac{\partial H_n}{\partial x} + x\frac{\partial H_n}{\partial y} + \sum_{m=1}^{n-2}\Big(X_{n-m}\frac{\partial H_{m+1}}{\partial x} + Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\Big) \\
= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ V_n(x^2+y^2)^{n/2} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

For the second term on the left hand side of (4) we have

$$
\begin{aligned}
&\sum_{m=1}^{n-2}\Big(X_{n-m}\frac{\partial H_{m+1}}{\partial x} + Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\Big) \\
&\quad = \sum_{m=1}^{n-2}\Big(\Big(\sum_{k=0}^{n-m} a_k^{n-m} x^k y^{n-m-k}\Big)\Big(\sum_{l=0}^{m+1} l C_l^{m+1} x^{l-1} y^{m+1-l}\Big)\Big) \\
&\qquad + \sum_{m=1}^{n-2}\Big(\Big(\sum_{k=0}^{n-m} b_k^{n-m} x^k y^{n-m-k}\Big)\Big(\sum_{l=0}^{m+1} (m+1-l) C_l^{m+1} x^{l} y^{m-l}\Big)\Big) \\
&\quad = \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=0}^{n-m} l a_k^{n-m} C_l^{m+1} x^{k+l-1} y^{n+1-l-k} \\
&\qquad + \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=0}^{n-m} (m+1-l) b_k^{n-m} C_l^{m+1} x^{k+l} y^{n-l-k} \\
&\quad = \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=l-1}^{n-m+l-1} l a_{k-l+1}^{n-m} C_l^{m+1} x^{k} y^{n-k} \\
&\qquad + \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=l}^{n-m+l} (m+1-l) b_{k-l}^{n-m} C_l^{m+1} x^{k} y^{n-k}.
\end{aligned}
$$

Since $a_k^s = b_k^s = 0$ for $k < 0$ or $k > s$ this last expression takes the form

$$\sum_{m=1}^{n-2} \sum_{l=0}^{m+1} \sum_{k=0}^{n} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1} x^k y^{n-k}$$
$$= \sum_{k=0}^{n} \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1} x^k y^{n-k}.$$

Define $d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1}$ for $k = 0, \ldots, n$. We remark that the computation of d_k^n involves a_k^s, b_k^s and C_k^s for $s = 2, 3, \ldots, n-1$. Therefore we obtain

$$\sum_{m=1}^{n-2} \left(X_{n-m} \frac{\partial H_{m+1}}{\partial x} + Y_{n-m} \frac{\partial H_{m+1}}{\partial y} \right) = \sum_{k=0}^{n} d_k^n x^k y^{n-k}.$$

For the first term of the left hand side of (4) we have

$$y \frac{\partial H_n}{\partial x} - x \frac{\partial H_n}{\partial x} = y \frac{\partial}{\partial x} \sum_{k=0}^{n} C_k^n x^k y^{n-k} - x \frac{\partial}{\partial y} \sum_{k=0}^{n} C_k^n x^k y^{n-k}$$
$$= \sum_{k=0}^{n} k C_k^n x^{k-1} y^{n-k+1} - \sum_{k=0}^{n} (n-k) C_k^n x^{k+1} y^{n-k-1}$$
$$= \sum_{k=0}^{n-1} (k+1) C_{k+1}^n x^k y^{n-k} - \sum_{k=1}^{n} (n-k+1) C_{k-1}^n x^k y^{n-k}$$
$$= C_1^n y^n + \sum_{k=1}^{n-1} ((k+1) C_{k+1}^n - (n-k+1) C_{k-1}^n) x^k y^{n-k} - C_{n-1}^n x^n.$$

On the other hand,

$$V_n (x^2+y^2)^{n/2} = V_n \sum_{k=0}^{n/2} \binom{n/2}{k} x^{2k} y^{n-2k} = \sum_{\substack{k=0 \\ 2|k}} \binom{n/2}{k/2} V_n x^k y^{n-k}.$$

Substituting in (4) we obtain

$$\begin{aligned} C_1^n + \binom{n/2}{0} V_n &= d_0^n, \\ (k+1) C_{k+1}^n - (n-k+1) C_{k-1}^n + \binom{n/2}{k/2} V_n &= d_k^n, \qquad k = 1, \ldots, n-1, \\ -C_{n-1}^n + \binom{n/2}{n/2} V_n &= d_n^n, \end{aligned} \tag{5}$$

where the term $\binom{n/2}{k/2} V_n$, for $k = 0, \ldots, n$, is different from zero only for n

and k even. We can rewrite (5) as follows:

$$
\begin{aligned}
&C_1^n = d_0^n - \binom{n/2}{0} V_n, \qquad C_{n-1}^n = \binom{n/2}{n/2} V_n - d_n^n,\\
(6)\quad &C_k^n = \frac{1}{k}\left(d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n + (n-k+2) C_{k-2}^n\right), \qquad k = 2, \dots, n,\\
&C_k^n = -\frac{1}{n-k}\left(d_{k+1}^n - \binom{n/2}{(k+1)/2} V_n - (k+2) C_{k+2}^n\right),\\
&\qquad\qquad k = 0, \dots, n-2.
\end{aligned}
$$

For n odd and k odd this yields

$$C_1^n = d_0^n, \qquad C_k^n = \frac{1}{k}(d_{k-1}^n + (n-k+2) C_{k-2}^n), \qquad k = 3, 5, \dots, n.$$

In this case we claim that

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{(n-k)!!\,k!!} \qquad \text{for } k = 1, 3, 5, \dots, n.$$

We prove the claim by induction. It is easy to see that it is true for $k = 1$. Now, suppose that it is true for $k-2$, that is,

$$C_{k-2}^n = \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{(n-k+2)!!\,(k-2)!!}.$$

Then

$$
\begin{aligned}
C_k^n &= \frac{1}{k}\left(d_{k-1}^n + (n-k+2)\frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{(n-k+2)!!\,(k-2)!!}\right)\\
&= \frac{d_{k-1}^n}{k} + \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{(n-k)!!\,k!!}\\
&= \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!\,(2l-1)!!\,d_{2l}^n}{(n-k)!!\,k!!}.
\end{aligned}
$$

For n odd and k even, (6) gives

$$C_{n-1}^n = -d_n^n, \qquad C_k^n = \frac{1}{n-k}(-d_{k+1}^n + (k+2) C_{k+2}^n), \qquad k = 0, 2, 4, \dots, n-3.$$

In this case we claim that

$$C_k^n = -\frac{\sum_{l=k/2}^{(n-1)/2} (n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n}{(n-k)!!\,k!!} \qquad \text{for } k = 0, 2, 4, \dots, n-1.$$

It is easy to see that it is true for $k = n-1$. Suppose that it is true for $k+2$, that is,

$$C_{k+2}^n = -\frac{\sum_{l=(k+2)/2}^{(n-1)/2} (n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n}{(n-k-2)!!\,(k+2)!!}.$$

Then

$$C_k^n = \frac{1}{n-k}\left(-d_{k+1}^n + (k+2)\frac{-\sum_{l=(k+2)/2}^{(n-1)/2}(n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n}{(n-k-2)!!\,(k+2)!!}\right)$$
$$= -\frac{d_{k+1}^n}{n-k} - \frac{\sum_{l=(k+2)/2}^{(n-1)/2}(n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n}{(n-k)!!\,k!!}$$
$$= -\frac{\sum_{l=k/2}^{(n-1)/2}(n-(2l+2))!!\,(2l)!!\,d_{2l+1}^n}{(n-k)!!\,k!!}.$$

For n even and k odd, (6) gives

$$C_1^n = d_0^n - \binom{n/2}{0}V_n, \qquad C_{n-1}^n = \binom{n/2}{n/2}V_n - d_n^n,$$
$$C_k^n = \frac{1}{k}\left(d_{k-1}^n - \binom{n/2}{(k-1)/2}V_n + (n-k+2)C_{k-2}^n\right), \qquad k = 3, 5, \ldots, n-1.$$

In this case we claim that

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2}(n-(2l+1))!!\,(2l-1)!!\,\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-k)!!\,k!!} \qquad \text{for } k = 1, 3, 5, \ldots, n-1.$$

It is easy to see that this is true for $k=1$. Suppose that it is true for $k-2$, that is,

$$C_{k-2}^n = \frac{\sum_{l=0}^{(k-3)/2}(n-(2l+1))!!\,(2l-1)!!\,\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-k+2)!!\,(k-2)!!}.$$

Then

$$C_k^n = \frac{1}{k}\left(d_{k-1}^n - \binom{n/2}{(k-1)/2}V_n + (n-k+2)C_{k-2}^n\right)$$
$$= \frac{d_{k-1}^n - \binom{n/2}{(k-1)/2}V_n}{k} + \frac{\sum_{l=0}^{(k-3)/2}(n-(2l+1))!!\,(2l-1)!!\,\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-k)!!\,k!!}$$
$$= \frac{\sum_{l=0}^{(k-1)/2}(n-(2l+1))!!\,(2l-1)!!\,\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-k)!!\,k!!}.$$

Hence

$$C_{n-1}^n = \frac{\sum_{l=0}^{(n-2)/2}(n-(2l+1))!!\,(2l-1)!!\,\left(d_{2l}^n - \binom{n/2}{l}V_n\right)}{(n-1)!!};$$

but we know that $C^n_{n-1} = \binom{n/2}{n/2} V_n - d^n_n$, so

$$\sum_{l=0}^{(n-2)/2} (n-(2l+1))!!\,(2l-1)!! \left(d^n_{2l} - \binom{n/2}{l} V_n \right) + (n-1)!! \left(d^n_n - \binom{n/2}{n/2} V_n \right) = 0,$$

which is equivalent to

$$\sum_{l=0}^{n/2} (n-(2l+1))!!\,(2l-1)!! \left(d^n_{2l} - \binom{n/2}{l} V_n \right) = 0. \tag{7}$$

From (7) we obtain

$$V_n = \frac{\sum_{l=0}^{n/2} (n-(2l+1))!!\,(2l-1)!!\, d^n_{2l}}{\sum_{l=0}^{n/2} (n-(2l+1))!!\,(2l-1)!! \binom{n/2}{l}}, \qquad n = 4, 6, 8, \ldots$$

Finally, for n even and k even, (6) gives

$$C^n_k = \frac{1}{n-k}(-d^n_{k+1} + (k+2)C^n_{k+2}), \qquad k = 0, 2, 4, \ldots, n-2.$$

Now we have only a recurrence between C^n_n, C^n_{n-2}, C^n_{n-4}, $\ldots$, C^n_4, C^n_2, C^n_0, and one of them is arbitrary. If we choose $C^n_n = -\lambda_n/n!!$, with λ_n arbitrary, then

$$C^n_k = -\frac{\sum_{l=k/2}^{n/2-1} (n-(2l+2))!!\,(2l)!!\, d^n_{2l+1} + \lambda_n}{(n-k)!!\,k!!} \qquad \text{for } k = 0, 2, 4, \ldots, n.$$

Indeed, it is easy to see that this is true for $k = n$. Suppose that it is true for $k+2$, that is,

$$C^n_{k+2} = -\frac{\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!!\,(2l)!!\, d^n_{2l+1} + \lambda_n}{(n-k-2)!!\,(k+2)!!}.$$

Then

$$\begin{aligned} C^n_k &= \frac{1}{n-k}\left(-d^n_{k+1} + (k+2) \frac{-\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!!\,(2l)!!\, d^n_{2l+1} + \lambda_n}{(n-k-2)!!\,(k+2)!!} \right) \\ &= -\frac{d^n_{k+1}}{n-k} - \frac{\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!!\,(2l)!!\, d^n_{2l+1} + \lambda_n}{(n-k)!!\,k!!} \\ &= -\frac{\sum_{l=k/2}^{n/2-1} (n-(2l+2))!!\,(2l)!!\, d^n_{2l+1} + \lambda_n}{(n-k)!!\,k!!}, \end{aligned}$$

which completes the proof of the theorem. ■

The method works as follows. From the first terms of the Poincaré series (2), i.e. $C_0^2 = C_2^2 = 1/2$ and $C_1^2 = 0$ one calculates d_k^3 for $k = 0,1,2,3$ and hence C_k^3 for $k = 0,1,2,3$. The next step is calculating d_k^4 for $k = 0,1,2,3,4$ and finally we obtain V_4 and C_k^4 for $k = 0,1,2,3,4$. The process continues in an analogous way. The method has been implemented in Mathematica 2.2.

A particular case: quadratic systems. We apply the above expressions to quadratic systems. In this case all a_k^s and b_k^s are zero except a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2. Therefore in the expression

$$d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1},$$

we have $n - m = 2$, i.e. $m = n-2$, and the expression takes the form

$$d_k^n = \sum_{l=0}^{n-1} (l a_{k-l+1}^2 + (n-1-l) b_{k-l}^2)\ C_l^{n-1},$$

where we can omit the upper index of a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2 because it is always 2. Taking into account that the subscripts $k-l+1$ and $k-l$ must be $0,1,2$, we have respectively $l = k+1$, $l = k$, $l = k-1$ and $l = k$, $l = k-1$, $l = k-2$ with $0 \le l \le n-1$. Then

$$\begin{aligned} d_k^n = {} & (k+1) a_0 C_{k+1}^{n-1} + (k a_1 + (n-1-k) b_0) C_k^{n-1} + \\ & + ((k-1) a_2 + (n-k) b_1) C_{k-1}^{n-1} + (n+1-k) b_2 C_{k-2}^{n-1}, \end{aligned}$$

and $C_l^{n-1} = 0$ if the restriction $0 \le l \le n-1$ is not satisfied. Then the Poincaré–Lyapunov constants for quadratic systems are

$$V_n = \frac{\sum_{l=0}^{n/2} (n-(2l+1))!!\,(2l-1)!!\, d_{2l}^n}{\sum_{l=0}^{n/2} (n-(2l+1))!!\,(2l-1)!!\binom{n/2}{l}}, \qquad n = 4,6,8,\ldots,$$

where

$$\begin{aligned} d_{2l}^n = {} & (2l+1) a_0 C_{2l+1}^{n-1} + (2l a_1 + (n-1-2l) b_0) C_{2l}^{n-1} \\ & + ((2l-1) a_2 (n-2l) b_1) C_{2l-1}^{n-1} + (n+1-2l) b_2 C_{2l-2}^{n-1}. \end{aligned}$$

Application to more general systems is based on finding the expression d_k^n and it is easy to see that the contributions to d_k^n of each homogeneous term of the system are independent.

3. Applications. When we apply our method to particular cases of system (1) we can determine the Poincaré–Lyapunov constants more explicitly. The system of Proposition 1 was studied in [7] and [13] with $a_4 = b_4 = 0$ and in [23] with $b_2 = a_2$. Here we present the following result.

PROPOSITION 1. *Consider the system*

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4, \\ \dot{y} = x + b_2y^2 + b_3y^3 + b_4y^4, \end{cases} \tag{8}$$

where a_i and b_i are real numbers. Then the origin is a center if and only if one of the following conditions holds: $a_2 - b_2 = a_3 + b_3 = a_4 - b_4 = 0$, $a_2 + b_2 = a_3 + b_3 = a_4 + b_4 = 0$, $a_2 = a_3 = a_4 = b_3 = 0$ *and* $b_2 = b_3 = b_4 = a_3 = 0$.

Proof. (a) *Sufficiency.* Every group of conditions gives the necessary symmetries to show that system (8) is reversible and then the origin is a center (the symmetry principle, see [18], p. 135).

(b) *Necessity.* The first Poincaré–Lyapunov constant is $V_4 = a_3 + b_3$ so taking $b_3 = -a_3$ we see that the second Poincaré–Lyapunov constant takes the form

$$V_6 = 5a_2^2a_3 - 6a_2^3b_2 - 22a_4b_2 - 5a_3b_2^2 + 6a_2b_2^3 + 22a_2b_4.$$

If a_2 is different from zero we can express b_4 in terms of the other parameters. In this case the third Poincaré–Lyapunov constant is

$$\begin{aligned} V_8 = \frac{1}{a_2}&(b_2 + a_2)(b_2 - a_2)(-235a_2^3a_3 - 1254a_3a_4 + 84a_2^4b_2 \\ &- 285a_3^2b_2 + 1100a_2a_4b_2 + 357a_2a_3b_2^2 - 216a_2^2b_2^3), \end{aligned}$$

and the vanishing of the second factor, that is, $b_2 = a_2$ gives the first condition of Proposition 1. Next, $b_2 = -a_2$ corresponds to the second condition. From the last factor of V_8 we can isolate a_4 if $57a_3 - 50a_2b_2$ is different from zero, and the vanishing of the next Poincaré–Lyapunov constants implies $a_2 = b_2 = 0$. If $57a_3 - 50a_2b_2$ is zero, that is, $a_3 = 50a_2b_2/57$, the last factor of V_8 takes the form $a_2b_2(a_2^2 + b_2^2)$, which implies $b_2 = 0$, and we obtain the fourth condition of Proposition 1.

If a_2 is zero then the second Poincaré–Lyapunov constant is $V_6 = b_2(22a_4 + 5a_3b_2)$. Let $22a_4 + 5a_3b_2$ be zero with $b_2 \neq 0$, that is, $a_4 = -5a_3b_2/22$; in this case $V_8 = a_3b_2(235b_2^2 + 1254b_4)$. The case $a_3 = 0$ corresponds to the third condition of Proposition 1. In the case $b_4 = -235b_2^3/1254$ the next Poincaré–Lyapunov constants imply $a_3 = b_2 = 0$. Finally, if $b_2 = 0$ the Poincaré–Lyapunov constant V_8 is zero and $V_{10} = a_3(a_4 - b_4)(a_4 + b_4)$. The vanishing of the factors $a_4 - b_4$ and $a_4 + b_4$ corresponds to particular cases of the first and second conditions respectively. When $a_3 = 0$ with $(a_4 - b_4)(a_4 + b_4) \neq 0$ we have $V_{12} = 0$ and $V_{14} = a_4b_4(a_4 - b_4)(a_4 + b_4)$. The cases $a_4 = 0$ and $b_4 = 0$ correspond to particular cases of the third and fourth conditions respectively. ∎

Consider the system $\dot{x} = -y + xf(x, y)$, $\dot{y} = x + yf(x, y)$ with $f(x, y) = \sum_{i=1}^{3} f_i(x, y)$ where $f_i(x, y)$ are homogeneous polynomials of degree i. Any center at the origin of this type of systems is necessarily isochronous (all the closed orbits around the center have the same period), since in polar coordinates (r, φ) the angle φ satisfies the equation $\dot{\varphi} = 1$. This type of isochronous centers are called *uniformly isochronous centers* (see [9]). If $f_2 = f_3 = 0$ the origin is automatically a center because the system has $R(x, y) = (1 - a_2 x + a_1 y)^{-3}$ as integrating factor. These classes of systems have been studied in [8] with $f_3 = 0$. Here we present the center conditions for $f_2 = 0$ and $f_3 \neq 0$.

PROPOSITION 2. *Consider the system*

$$\begin{cases} \dot{x} = -y + x(a_1 x + a_2 y + a_6 x^3 + a_7 x^2 y + a_8 x y^2 + a_9 y^3), \\ \dot{y} = x + y(a_1 x + a_2 y + a_6 x^3 + a_7 x^2 y + a_8 x y^2 + a_9 y^3), \end{cases} \tag{9}$$

where a_i are real numbers. Then the origin is a center if and only if

$$a_1(a_7 + 3a_9) - a_2(a_8 + 3a_6) = 0 \quad \text{and}$$
$$(3a_1 a_2^2 - 3a_2^3)a_6 + (3a_1 a_2^2 - a_1^3)a_7 - 2a_1^2 a_2 a_8 = 0.$$

Proof. (a) *Sufficiency.* Suppose that the two conditions of Proposition 2 hold. If $a_1 = a_2 = 0$ then the system has an integrating factor of the form

$$R(x, y) = (1 - (a_7 + 2a_9)x^3 + 3a_6 x^2 y - 3a_9 x y^2 + (2a_6 + a_8)y^3)^{-5/3},$$

which is defined at the origin and therefore the origin is a center. If $a_1 = 0$ with $a_2 \neq 0$ the first condition of Proposition 2 reads $a_2(3a_6 + a_8) = 0$, which implies $a_8 = -3a_6$. In this case the second condition is $a_2^3 a_6 = 0$ and therefore $a_6 = 0$. System (9) with $a_1 = a_6 = a_8 = 0$ is invariant under the change of variables $(x, y, t) \mapsto (x, -y, -t)$ and so the origin is a center. If $a_2 = 0$ with $a_1 \neq 0$ the first condition of Proposition 2 reads $a_1(3a_9 + a_7) = 0$, which implies $a_7 = -3a_9$. In this case the second condition is $a_1^3 a_9 = 0$ and therefore $a_9 = 0$. System (9) with $a_2 = a_7 = a_9 = 0$ is invariant under the change of variables $(x, y, t) \mapsto (-x, y, -t)$ and so the origin is also a center. Finally if $a_1 a_2 \neq 0$ from the first condition of Proposition 2 we can isolate a_9, and from the second condition of Proposition 2 we obtain a_8 in terms of the other parameters. In this case we make a rotation with $\tan\alpha = -a_2/a_1$; in the new variables (X, Y), system (9) is invariant under the change of variables $(X, Y, t) \mapsto (-X, Y, -t)$ and therefore has a center at the origin.

(b) *Necessity.* The first Poincaré–Lyapunov constant V_4 is zero. The second and third Poincaré–Lyapunov constants are the two conditions of Proposition 2. ■

To find the maximum number of small-amplitude limit cycles which can bifurcate from the origin, the method is to find a fine focus of maximum order. From our calculations it is easy to see that if $a_2 = b_2 = a_3 = b_3 = 0$ then $V_4 = V_6 = V_8 = V_{10} = V_{12} = 0$ and $V_{14} = a_4 b_4 (a_4 - b_4)(a_4 + b_4)$, which is diferent from zero if $a_4 b_4 \neq 0$ and $a_4 \neq b_4$, and $a_4 \neq -b_4$, and therefore we obtain a fine focus of order six for system (8). In the same way if $a_1 = 0$ and $a_8 = -3a_6$ then $V_4 = V_6 = 0$ and $V_8 = a_2^3 a_6$, which is diferent from zero if a_2 and a_6 are diferent from zero, and therefore we obtain a fine focus of order three for system (9). Therefore we obtain the following result

PROPOSITION 3. *The maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least six for system* (8) *and three for system* (9)*.*

The Poincaré–Lyapunov constants of systems (8) and (9) are available from the e-mail address gine@eup.udl.es.

Acknowledgements. The authors were partially supported by University of Lleida Project P98-207.

References

[1] M. A. M. Alwash, *Computing the Poincaré–Liapunov constants*, Differential Equations Dynam. Systems 6 (1998), 349–361.

[2] A. A. Andronov, E. A. Leontovich, I. I. Gordon and A. G. Maier, *Theory of Bifurcations of Dynamic Systems on a Plane*, Wiley, New York, 1967.

[3] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb. 30 (72) (1952), 181–196 (in Russian); English transl.: Amer. Math. Soc. Transl. 100 (1954), 397–413.

[4] T. R. Blows and N. G. Lloyd, *The number of limit cycles of certain polynomial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A 98 (1984), 215–239.

[5] J. Chavarriga, *Integrable systems in the plane with a center type linear part*, Appl. Math. (Warsaw) 22 (1994), 285–309.

[6] J. Chazy, *Sur la théorie des centres*, C. R. Acad. Sci. Paris 221 (1947), 7–10.

[7] A. Cima, A. Gasull, V. Mañosa and F. Mañosas, *Algebraic properties of the Liapunov and period constants*, Rocky Mountain J. Math. 27 (1997), 471–501.

[8] C. B. Collins, *Conditions for a centre in a simple class of cubic systems*, Differential Integral Equations 10 (1997), 333–356.

[9] R. Conti, *Uniformly isochronous centers of polynomials systems in* $\mathbb{R}^2$, in: Lecture Notes in Pure and Appl. Math. 152, Dekker, 1994, New York, 21–31.

[10] W. W. Farr, C. Li, I. S. Labouriau and W. F. Langford, *Degenerate Hopf-bifurcation formulas and Hilbert's 16th problem*, SIAM J. Math. Anal. 20 (1989), 13–29.

[11] J. P. Françoise, *Successive derivatives of a first return map*, Ergodic Theory Dynam. Systems 16 (1996), 87–96.

[12] J. P. Françoise and R. Pons, *Computer algebra methods and the stability of differential systems*, Random Comput. Dynamics 3 (1995), 265–287.

[13] A. Gasull, A. Guillamon and V. Mañosa, *An explicit expression of the first Liapunov and period constants with applications*, J. Math. Anal. Appl. 211 (1997), 190–202.

[14] —, —, —, *An analytic-numerical method of computation of the Liapunov and period constants derived from their algebraic structure*, SIAM J. Numer. Anal. 36 (1999), 1030–1043.

[15] A. Gasull and R. Prohens, *Effective computation of the first Lyapunov quantities for a planar differential equation*, Appl. Math. (Warsaw) 24 (1997), 243–250.
[16] N. G. Lloyd and J. M. Pearson, *REDUCE and the bifurcation of limit cycles*, J. Symbolic Comput. 9 (1990), 215–224.
[17] M. A. Liapunov [M. A. Lyapunov], *Problème général de la stabilité du mouvement*, Ann. of Math. Stud. 17, Princeton Univ. Press, 1947.
[18] V. V. Nemytskiĭ and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Dover Publ., New York, 1989.
[19] J. M. Pearson, N. G. Lloyd and C. J. Christopher, *Algorithmic derivation of centre conditions*, SIAM Rev. 38 (1996), 619–636.
[20] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, J. de Mathématiques 37 (1881), 375–422; 8 (1882), 251–296; Oeuvres de Henri Poincaré, Vol. I, Gauthier-Villars, Paris, 1951, 3–84.
[21] S. L. Shi, *A method of constructing cycles without contact around a weak focus*, J. Differential Equations 41 (1981), 301–312.
[22] — *On the structure of Poincaré–Lyapunov constants for the weak focus of polynomial vector fields*, ibid. 52 (1984), 52–57.
[23] J. Torregrosa, *Punts singulars i òrbites periòdiques per a camps vectorials*, Ph.D., Universitat Autònoma de Barcelona, 1998 (in Catalan).
[24] H. Żołądek, *Eleven small limit cycles in a cubic vector field*, Nonlinearity 8 (1995), 843–860.

Departament de Matemàtica
Universitat de Lleida
Avda. Jaume II, 69
25001 Lleida, Spain
E-mail: gine@eup.udl.es

Received on 2.6.2000 (1541)