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ON THE POINCARÉ–LYAPUNOV CONSTANTS
AND THE POINCARÉ SERIES

Abstract. For an arbitrary analytic system which has a linear center at
the origin we compute recursively all its Poincaré–Lyapunov constants in
terms of the coefficients of the system, giving an answer to the classical
center problem. We also compute the coefficients of the Poincaré series in
terms of the same coefficients. The algorithm for these computations has an
easy implementation. Our method does not need the computation of any
definite or indefinite integral. We apply the algorithm to some polynomial
differential systems.

1. Introduction. Many models of natural phenomena use systems of
differential equations in the plane and the qualitative theory of differential
equations, introduced by Poincaré, can be used to describe the behavior
of such systems in most cases. One of the problems here is to distinguish
between a focus and a center (the center problem). The resolution of this
problem requires the computation of the so-called Poincaré–Lyapunov con-
stants. Therefore to have a fast and easy method for this computation is of
great importance. Another important problem is to determine systems that
have centers at some singular points due to the fact that perturbations of
such systems give rich bifurcations of limit cycles.

In the last years many papers have been published giving different meth-
ods to compute the Poincaré–Lyapunov constants. In this work we compute
them recursively in terms of the coefficients of the system for an arbitrary
analytic system which has a linear center at the origin, thus answering the
classical center problem. We also compute the coefficients of the Poincaré
series in terms of the same coefficients. Our method does not need the com-
putation of any definite or indefinite integral, and is easy to implement on a computer.

Consider two-dimensional autonomous systems of differential equations of the form

\[ \begin{align*}
\dot{x} &= -y + X(x, y), \\
\dot{y} &= x + Y(x, y),
\end{align*} \]

where the nonlinearities are

\[ X(x, y) = \sum_{s=2}^{\infty} X_s(x, y) \quad \text{and} \quad Y(x, y) = \sum_{s=2}^{\infty} Y_s(x, y) \]

with \( X_s(x, y) = \sum_{k=0}^{s} a_k x^k y^{s-k} \) and \( Y_s(x, y) = \sum_{k=0}^{s} b_k x^k y^{s-k} \), and \( a_k \) and \( b_k \) are arbitrary real coefficients.

For such systems Poincaré [20] developed an important technique that consists in finding a formal power series of the form

\[ H(x, y) = \sum_{n=2}^{\infty} H_n(x, y), \]

where \( H_2(x, y) = (x^2 + y^2)/2 \), and for each \( n \), \( H_n(x, y) = \sum_{k=0}^{n} C_k^n x^k y^{n-k} \), such that the derivative of \( H \) along the solutions of system (1) satisfies

\[ \dot{H} = \sum_{k=2}^{\infty} V_{2k} (x^2 + y^2)^k, \]

where \( V_{2k} \) are called the Poincaré–Lyapunov constants.

In order to solve the problem of the stability of system (1) at the origin, it is sufficient to consider the sign of the first Poincaré–Lyapunov constant different from zero. If it is positive we have asymptotic stability for negative times, and if it is negative we have asymptotic stability for positive times. If all Poincaré–Lyapunov constants are zero, then the origin is stable for all times, but there is no asymptotic stability for any time (see for instance [2]). In this last case, we have a center at the origin, i.e. there is an open neighborhood of the origin where all orbits are periodic, except of course the origin. The origin is said to be a fine focus of order \( k \) if \( V_{2k+2} \) is the first nonzero Poincaré–Lyapunov constant. In this case at most \( k \) limit cycles can bifurcate from this fine focus [4]; they are called small-amplitude limit cycles. Therefore to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus. It is known that this maximum number is three for quadratic systems [3] and it has been shown recently that it is greater than or equal to eleven for cubic systems [24].

In this work we are going to see that we can always determine \( C_k^n \) and \( V_{2k} \) from \( a_k \) and \( b_k \), but the \( C_k^n \) are not unique and in consequence neither
are the $V_{2k}$. Therefore, the Poincaré formal series is not unique. Poincaré [20] proved, by delimitation, that for polynomial systems there exists one which is convergent, and Lyapunov [17] generalized Poincaré’s theorem to analytic systems. In [6] Chazy, using the theorem of analytical dependence on initial parameters, demonstrated that there exists one which is convergent, by a suitable choice of parameters that appear in the construction of Poincaré series. For polynomial systems we have uniqueness of the $V_{2k}$ in the sense of the following theorem due to Shi Songling [22].

**Theorem 1.** Let $\mathbf{A}$ be the ring of real polynomials whose variables are the coefficients of a polynomial differential system. Given a set of Poincaré–Lyapunov constants $V_1, \ldots, V_i$, let $\mathbf{J}_{k-1}$ be the ideal of $\mathbf{A}$ generated by $V_1, \ldots, V_{k-1}$. If $V_1’, \ldots, V_i’$ is another set of Poincaré–Lyapunov constants, then $V_k \equiv V_k’ \pmod{\mathbf{J}_{k-1}}$.

As mentioned above, the origin is a center if and only if all the $V_i$’s are zero. Let $\mathbf{J} = (V_1, V_2, \ldots)$ be the ideal of $\mathbf{A}$ generated by all the $V_i$’s. For polynomial systems, by Hilbert’s basis theorem, $\mathbf{J}$ is finitely generated, i.e. there exist $B_1, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = (B_1, \ldots, B_q)$. Such a set of generators is called a basis of $\mathbf{J}$.

There exist various algorithms to compute the Poincaré–Lyapunov constants. The technique used by Bautin [3] is based on computing the derivatives of the return map from a nonlinear system of recursive differential equations. There is another algorithm which involves the solution of a system of linear equations for the coefficients of $H_n$ in terms of the coefficients of $X_s, Y_s$ and $H_k$ for $k = 2, \ldots, n - 1$ (see for instance [16] and [19]). Another method is to construct a Poincaré formal power series in polar coordinates and the Poincaré–Lyapunov constants can be computed from recursive linear formulas as definite integrals of trigonometric polynomials (see for example [1] and [5]). In [10] the authors give a survey of different ways to compute the Poincaré–Lyapunov constants.

Modifying the standard techniques explained in [2] for obtaining the Poincaré–Lyapunov constants, in [7] the first and second Poincaré–Lyapunov constants are computed for an arbitrary analytic system using the return function and some algebraic properties of the Poincaré–Lyapunov constants. In [15] taking advantage of the complex structure that simplifies their effective computation, $V_3$ and $V_5$ have been found by hand. A development of the method presented in [15] is used in [13] to obtain $V_7$ for an arbitrary analytic system. Using the Runge–Kutta–Fehlberg methods and Richardson’s extrapolation, an analytic-numerical method of computation of the Poincaré–Lyapunov constants is given in [14]. Another algorithm is developed in [12] and [11], based on the calculation of successive derivatives of the first return map associated with the perturbations of some planar Hamil-
tonian systems. An important generalization of this last method is given in [23].

The paper is organized as follows. In the next section we present a formula for the Poincaré–Lyapunov constants (see Theorem 2) and we describe the algorithm that we have developed. As a particular case the formula is applied to quadratic systems. Section 3 is devoted to the center problem for some particular systems as an application of the method.

2. The main result. We present a formula for the Poincaré–Lyapunov constants and Poincaré series for general systems (1) in recurrent form following the ideas of Shi Songling [21] who found the same expression for the Poincaré–Lyapunov constants, but he did not find the recursive relation with the Poincaré series, which establishes a method to compute them.

The advantages of this method are:

(a) In the whole process we only calculate products and sums without definite or indefinite integrals as in the majority of other methods.

(b) As a consequence of (a) the process is easy to implement on a computer.

(c) The method gives the Poincaré–Lyapunov constants and Poincaré series at the same time. This allows us to find systems with a polynomial first integral by requiring the Poincaré series to have a finite number of terms.

**Theorem 2.** The Poincaré–Lyapunov constants of system (1) are

\[
V_n = \frac{\sum_{l=0}^{n/2} (n - (2l + 1))!! (2l - 1)!! d_{2l}^n}{\sum_{l=0}^{n/2} (n - (2l + 1))!! (2l - 1)!! \left(\frac{n}{2}\right)_l^2}, \quad n = 4, 6, 8, \ldots,
\]

where \(d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (la_{k-l+1} + (m + 1 - l)b_{k-l}^n)C_{l+1}^{m+1} \right), \ n \geq 3, \ k = 0, \ldots, n, \) with \(a_k^n = b_k^n = 0 \) for \(k < 0\) or \(k > s, \) \(C_0^2 = C_2^2 = 1/2\) and \(C_1^2 = 0,\) and for \(n \geq 3,\)

\[
C_k^n = \begin{cases} 
\frac{\sum_{l=0}^{(k-1)/2} (n - (2l + 1))!! (2l - 1)!! d_{2l}^n - \left(\frac{n}{2}\right)_l^2 V_n}{(n-k)!! k!!}, & k = 1, 3, 5, \ldots, \\
-\frac{\sum_{l=k/2}^{(n-1)/2} (n - (2l + 2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k)!! k!!}, & k = 0, 2, 4, \ldots 
\end{cases}
\]

where \(\lambda_n\) are arbitrary constants and \(V_n\) and \(\lambda_n\) are zero for \(n\) odd.

**Proof.** From the evaluation of the derivative of \(H(x, y)\) along the solutions of system (1) we have
\[
\dot{H} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} \\
= \left( \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial x} \right) (-y + \sum_{s=2}^{\infty} X_s) + \left( \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial y} \right) \left( x + \sum_{s=2}^{\infty} Y_s \right) \\
= \sum_{n=2}^{\infty} \left( -y \frac{\partial H_n}{\partial x} + x \frac{\partial H_n}{\partial y} \right) + \sum_{s=2}^{\infty} X_s \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial x} \quad + \quad \sum_{s=2}^{\infty} Y_s \sum_{n=2}^{\infty} \frac{\partial H_n}{\partial y} \\
= \sum_{n=3}^{\infty} \left( -y \frac{\partial H_n}{\partial x} + x \frac{\partial H_n}{\partial y} \right) + \sum_{m=1}^{n-2} \left( X_{n-m} \frac{\partial H_{m+1}}{\partial x} + Y_{n-m} \frac{\partial H_{m+1}}{\partial y} \right).
\]

Comparing with (3) we have

\begin{align*}
(4) \quad -y \frac{\partial H_n}{\partial x} + x \frac{\partial H_n}{\partial y} + \sum_{m=1}^{n-2} \left( X_{n-m} \frac{\partial H_{m+1}}{\partial x} + Y_{n-m} \frac{\partial H_{m+1}}{\partial y} \right) \\
&= \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
V_n(x^2 + y^2)^{n/2} & \text{if } n \text{ is even}.
\end{cases}
\end{align*}

For the second term on the left hand side of (4) we have

\begin{align*}
\sum_{m=1}^{n-2} \left( X_{n-m} \frac{\partial H_{m+1}}{\partial x} + Y_{n-m} \frac{\partial H_{m+1}}{\partial y} \right) \\
&= \sum_{m=1}^{n-2} \left( \left( \sum_{k=0}^{n-m} a_k^{n-m} x^k y^{n-m-k} \right) \left( \sum_{l=0}^{m+1} l C_l^{m+1} x^{l-1} y^{m+1-l} \right) \right) \\
&\quad + \sum_{m=1}^{n-2} \left( \left( \sum_{k=0}^{n-m} b_k^{n-m} x^k y^{n-m-k} \right) \left( \sum_{l=0}^{m+1} (m+1-l) C_l^{m+1} x^l y^{m-l} \right) \right) \\
&= \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} l a_k^{n-m} C_l^{m+1} x^{k+l-1} y^{n+1-l-k} \\
&\quad + \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (m+1-l) b_k^{n-m} C_l^{m+1} x^{k+l} y^{n-l-k} \\
&= \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} l a_{k-l+1}^{n-m} C_l^{m+1} x^k y^{n-k} \\
&\quad + \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (m+1-l) b_{k-l}^{n-m} C_l^{m+1} x^k y^{n-k}.
\end{align*}
Since \( a_k^s = b_k^s = 0 \) for \( k < 0 \) or \( k > s \) this last expression takes the form
\[
\sum_{m=1}^{n-2} \sum_{l=0}^{m} \sum_{k=0}^{m+1} (l a_{k-l+1}^{n-m} + (m + 1 - l) b_{k-l}^{n-m}) C_l^{m+1} x^k y^{n-k}
= \sum_{k=0}^{n-2} \sum_{m=1}^{m+1} \sum_{l=0}^{m} (l a_{k-l+1}^{n-m} + (m + 1 - l) b_{k-l}^{n-m}) C_l^{m+1} x^k y^{n-k}.
\]

Define \( d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m} (l a_{k-l+1}^{n-m} + (m + 1 - l) b_{k-l}^{n-m}) C_l^{m+1} \) for \( k = 0, \ldots, n \).
We remark that the computation of \( d_k^n \) involves \( a_k^s, \ b_k^s \) and \( C_k^s \) for \( s = 2, 3, \ldots, n-1 \). Therefore we obtain
\[
\sum_{m=1}^{n-2} \left( X_{n-m} \frac{\partial H_{m+1}}{\partial x} + Y_{n-m} \frac{\partial H_{m+1}}{\partial y} \right) = \sum_{k=0}^{n} d_k^n x^k y^{n-k}.
\]

For the first term of the left hand side of (4) we have
\[
y \frac{\partial H_n}{\partial x} - x \frac{\partial H_n}{\partial x} = y \frac{\partial}{\partial x} \sum_{k=0}^{n} C_k^n x^k y^{n-k} - x \frac{\partial}{\partial y} \sum_{k=0}^{n} C_k^n x^k y^{n-k}
= \sum_{k=0}^{n} k C_k^n x^{k-1} y^{n-k+1} - \sum_{k=0}^{n} (n-k) C_k^n x^{k+1} y^{n-k-1}
= \sum_{k=0}^{n-1} (k+1) C_{k+1}^n x^k y^{n-k} - \sum_{k=1}^{n} (n-k+1) C_{k-1}^n x^k y^{n-k}
= C_1^n y^n + \sum_{k=1}^{n-1} ((k+1) C_{k+1}^n - (n-k+1) C_{k-1}^n) x^k y^{n-k} - C_{n-1}^n x^n.
\]

On the other hand,
\[
V_n (x^2 + y^2)^{n/2} = V_n \sum_{k=0}^{n/2} \left( \frac{n/2}{k} \right) x^{2k} y^{n-2k} = \sum_{k=0}^{n/2} \left( \frac{n/2}{k} \right) V_n x^k y^{n-k}.
\]

Substituting in (4) we obtain
\[
C_1^n + \left( \frac{n/2}{0} \right) V_n = d_0^n,
\]
\[
(k+1) C_{k+1}^n - (n-k+1) C_{k-1}^n + \left( \frac{n/2}{k/2} \right) V_n = d_k^n, \quad k = 1, \ldots, n-1,
\]
\[
-C_{n-1}^n + \left( \frac{n/2}{n/2} \right) V_n = d_n^n,
\]
where the term \( \left( \frac{n/2}{k/2} \right) V_n \), for \( k = 0, \ldots, n \), is different from zero only for \( n \).
and $k$ even. We can rewrite (5) as follows:

\[ C^n_n = d^n_0 - \binom{n/2}{0} V_n, \quad C^n_{n-1} = \binom{n/2}{n/2} V_n - d^n_n, \]

\[ C^n_k = \frac{1}{k} \left( d^n_{k-1} - \binom{n/2}{(k-1)/2} V_n + (n-k+2) C^n_{k-2} \right), \quad k = 2, \ldots, n, \]

\[ C^n_k = -\frac{1}{n-k} \left( d^n_{k+1} - \binom{n/2}{(k+1)/2} V_n - (k+2) C^n_{k+2} \right), \quad k = 0, \ldots, n-2. \]

For $n$ odd and $k$ odd this yields

\[ C^n_1 = d^n_0, \quad C^n_k = \frac{1}{k} (d^n_{k-1} + (n-k+2) C^n_{k-2}), \quad k = 3, 5, \ldots, n. \]

In this case we claim that

\[ C^n_k = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!! (2l-1)!! d^n_{2l}}{(n-k)!! k!!} \quad \text{for } k = 1, 3, 5, \ldots, n. \]

We prove the claim by induction. It is easy to see that it is true for $k = 1$. Now, suppose that it is true for $k-2$, that is,

\[ C^n_{k-2} = \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! (2l-1)!! d^n_{2l}}{(n-k+2)!! (k-2)!!}. \]

Then

\[ C^n_k = \frac{1}{k} \left( d^n_{k-1} + (n-k+2) \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! (2l-1)!! d^n_{2l}}{(n-k+2)!! (k-2)!!} \right) \]

\[ = \frac{d^n_{k-1}}{k} + \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! (2l-1)!! d^n_{2l}}{(n-k)!! k!!} \]

\[ = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!! (2l-1)!! d^n_{2l}}{(n-k)!! k!!}. \]

For $n$ odd and $k$ even, (6) gives

\[ C^n_{n-1} = -d^n_n, \quad C^n_k = \frac{1}{n-k} (-d^n_{k+1} + (k+2) C^n_{k+2}), \quad k = 0, 2, 4, \ldots, n-3. \]

In this case we claim that

\[ C^n_k = -\frac{\sum_{l=k/2}^{(n-1)/2} (n-(2l+2))!! (2l)!! d^n_{2l+1}}{(n-k)!! k!!} \quad \text{for } k = 0, 2, 4, \ldots, n-1. \]

It is easy to see that it is true for $k = n-1$. Suppose that it is true for $k+2$, that is,

\[ C^n_{k+2} = -\frac{\sum_{l=(k+2)/2}^{(n-1)/2} (n-(2l+2))!! (2l)!! d^n_{2l+1}}{(n-k-2)!! (k+2)!!}. \]
Then
\[
C_k^n = \frac{1}{n-k} \left( -d_{k+1}^n + (k+2) - \frac{\sum_{l=(k+2)/2}^{(n-1)/2} (n-(2l+2))!! (2l)!! d_{2l+1}^n}{(n-k-2)!! (k+2)!!} \right)
\]
\[
= - \frac{d_{k+1}^n}{n-k} - \frac{\sum_{l=(k+2)/2}^{(n-1)/2} (n-(2l+2))!! (2l)!! d_{2l+1}^n}{(n-k)!! k!!}
\]
\[
= - \frac{\sum_{l=k/2}^{(n-1)/2} (n-(2l+2))!! (2l)!! d_{2l+1}^n}{(n-k)!! k!!}.
\]

For \(n\) even and \(k\) odd, (6) gives
\[
C_{k-2}^n = d_0^n - \binom{n/2}{0} V_n, \quad C_{n-1}^n = \binom{n/2}{n/2} V_n - d_n^n,
\]
\[
C_k^n = \frac{1}{k} \left( d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n + (n-k+2)C_{k-2}^n \right), \quad k = 3, 5, \ldots, n-1.
\]

In this case we claim that
\[
C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right)}{(n-k)!! k!!}
\]
for \(k = 1, 3, 5, \ldots, n-1\).

It is easy to see that this is true for \(k = 1\). Suppose that it is true for \(k-2\), that is,
\[
C_{k-2}^n = \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right)}{(n-k+2)!! (k-2)!!}.
\]

Then
\[
C_k^n = \frac{1}{k} \left( d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n + (n-k+2)C_{k-2}^n \right)
\]
\[
= \frac{d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n}{k}
\]
\[
+ \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right)}{(n-k)!! k!!}
\]
\[
= \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right)}{(n-k)!! k!!}.
\]

Hence
\[
C_{n-1}^n = \frac{\sum_{l=0}^{(n-2)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right)}{(n-1)!!};
\]
but we know that $C_{n-1}^n = \binom{n/2}{n/2} V_n - d_n^n$, so

$$\sum_{l=0}^{(n-2)/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right) + (n-1)!! \left( d_n^n - \binom{n/2}{n/2} V_n \right) = 0,$$

which is equivalent to

$$(7) \sum_{l=0}^{n/2} (n-(2l+1))!! (2l-1)!! \left( d_{2l}^n - \binom{n/2}{l} V_n \right) = 0.$$  

From (7) we obtain

$$V_n = \frac{\sum_{l=0}^{n/2} (n-(2l+1))!! (2l-1)!! d_{2l}^n}{\sum_{l=0}^{n/2} (n-(2l+1))!! (2l-1)!! \binom{n/2}{l}}, \quad n = 4, 6, 8, \ldots$$

Finally, for $n$ even and $k$ even, (6) gives

$$C_k^n = \frac{1}{n-k} \left( -d_{k+1}^n + (k+2)C_{k+2}^n \right), \quad k = 0, 2, 4, \ldots, n-2.$$ 

Now we have only a recurrence between $C_n^0, C_n^2, C_n^4, \ldots, C_n^n, C_0^n$, and one of them is arbitrary. If we choose $C_n^n = -\lambda_n/n!!$, with $\lambda_n$ arbitrary, then

$$C_k^n = -\frac{\sum_{l=k/2}^{n/2-1} (n-(2l+2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k)!! k!!} \quad \text{for} \quad k = 0, 2, 4, \ldots, n.$$

Indeed, it is easy to see that this is true for $k = n$. Suppose that it is true for $k + 2$, that is,

$$C_{k+2}^n = -\frac{\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k-2)!! (k+2)!!}.$$ 

Then

$$C_k^n = \frac{1}{n-k} \left( -d_{k+1}^n + (k+2) \frac{-\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k-2)!! (k+2)!!} \right)$$

$$= \frac{d_{k+1}^n}{n-k} - \frac{-\sum_{l=(k+2)/2}^{n/2-1} (n-(2l+2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k)!! k!!}$$

$$= \frac{\sum_{l=k/2}^{n/2-1} (n-(2l+2))!! (2l)!! d_{2l+1}^n + \lambda_n}{(n-k)!! k!!},$$

which completes the proof of the theorem.
The method works as follows. From the first terms of the Poincaré series (2), i.e. \( C^2_0 = C^2_2 = 1/2 \) and \( C^2_1 = 0 \) one calculates \( d^3_k \) for \( k = 0, 1, 2, 3 \) and hence \( C^3_k \) for \( k = 0, 1, 2, 3, 4 \). The next step is calculating \( d^4_k \) for \( k = 0, 1, 2, 3, 4 \) and finally we obtain \( V_4 \) and \( C^4_k \) for \( k = 0, 1, 2, 3, 4 \). The process continues in an analogous way. The method has been implemented in Mathematica 2.2.

A particular case: quadratic systems. We apply the above expressions to quadratic systems. In this case all \( a^*_k \) and \( b^*_k \) are zero except \( a^2_0, a^2_1, a^2_2 \) and \( b^2_0, b^2_1, b^2_2 \). Therefore in the expression

\[
d^n_k = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (la^n_{k-l+1} + (m + 1 - l)b^n_{k-l}) C^{m+1}_l,
\]

we have \( n - m = 2 \), i.e. \( m = n - 2 \), and the expression takes the form

\[
d^n_k = \sum_{l=0}^{n-1} (la^2_{k-l+1} + (n - l - 1)b^2_{k-l}) C^{n-1}_l,
\]

where we can omit the upper index of \( a^2_0, a^2_1, a^2_2 \) and \( b^2_0, b^2_1, b^2_2 \) because it is always 2. Taking into account that the subscripts \( k - l + 1 \) and \( k - l \) must be 0, 1, 2, we have respectively \( l = k + 1, l = k, l = k - 1 \) and \( l = k, l = k - 1, l = k - 2 \) with \( 0 \leq l \leq n - 1 \). Then

\[
d^n_k = (k + 1)a_0 C^{n-1}_{k+1} + (ka_1 + (n - 1 - k)b_0) C^{n-1}_k + (k - 1)a_2 + (n - k)b_1) C^{n-1}_{k-1} + (n + 1 - k)b_2 C^{n-1}_{k-2},
\]

and \( C^{n-1}_l = 0 \) if the restriction \( 0 \leq l \leq n - 1 \) is not satisfied. Then the Poincaré–Lyapunov constants for quadratic systems are

\[
V_n = \frac{\sum_{l=0}^{n/2} (n - (2l + 1))!! (2l - 1)!! d^n_{2l}}{\sum_{l=0}^{n/2} (n - (2l + 1))!! (2l - 1)!! (n/2)_l}, \quad n = 4, 6, 8, \ldots,
\]

where

\[
d^n_{2l} = (2l + 1)a_0 C^{n-1}_{2l+1} + (2la_1 + (n - 2l)b_0) C^{n-1}_{2l} + ((2l - 1)a_2 + (n - 2l)b_1) C^{n-1}_{2l-1} + (n + 1 - 2l)b_2 C^{n-1}_{2l-2}.
\]

Application to more general systems is based on finding the expression \( d^n_k \) and it is easy to see that the contributions to \( d^n_k \) of each homogeneous term of the system are independent.

3. Applications. When we apply our method to particular cases of system (1) we can determine the Poincaré–Lyapunov constants more explicitly. The system of Proposition 1 was studied in [7] and [13] with \( a_4 = b_4 = 0 \) and in [23] with \( b_2 = a_2 \). Here we present the following result.
Consider the system

\[
\begin{aligned}
\dot{x} &= -y + a_2 x^2 + a_3 x^3 + a_4 x^4, \\
\dot{y} &= x + b_2 y^2 + b_3 y^3 + b_4 y^4,
\end{aligned}
\]

where \(a_i\) and \(b_i\) are real numbers. Then the origin is a center if and only if one of the following conditions holds: \(a_2 - b_2 = a_3 + b_3 = a_4 - b_4 = 0\), \(a_2 + b_2 = a_3 + b_3 = a_4 + b_4 = 0\), \(a_2 = a_3 = a_4 = b_3 = 0\) and \(b_2 = b_3 = b_4 = a_3 = 0\).

**Proof.** (a) **Sufficiency.** Every group of conditions gives the necessary symmetries to show that system (8) is reversible and then the origin is a center (the symmetry principle, see [18], p. 135).

(b) **Necessity.** The first Poincaré–Lyapunov constant is \(V_4 = a_3 + b_3\) so taking \(b_3 = -a_3\) we see that the second Poincaré–Lyapunov constant takes the form

\[V_6 = 5a_2^2 a_3 - 6a_2^3 b_2 - 22a_4 b_2 - 5a_3 b_2^2 + 6a_2 b_3^2 + 22a_2 b_4.\]

If \(a_2\) is different from zero we can express \(b_4\) in terms of the other parameters. In this case the third Poincaré–Lyapunov constant is

\[V_8 = \frac{1}{a_2} (b_2 + a_2)(b_2 - a_2)(-235a_2^3 a_3 - 1254a_3 a_4 + 84a_4^2 b_2 - 285a_3^2 b_2 + 1100a_2 a_4 b_2 - 357a_2 a_3 b_2^2 - 216a_2 b_3^2),\]

and the vanishing of the second factor, that is, \(b_2 = a_2\) gives the first condition of Proposition 1. Next, \(b_2 = -a_2\) corresponds to the second condition. From the last factor of \(V_8\), we can isolate \(a_4\) if \(57a_3 - 50a_2 b_2\) is different from zero, and the vanishing of the next Poincaré–Lyapunov constants implies \(a_2 = b_2 = 0\). If \(57a_3 - 50a_2 b_2\) is zero, that is, \(a_3 = 50a_2 b_2/57\), the last factor of \(V_8\) takes the form \(a_2 b_2(a_2^2 + b_2^2)\), which implies \(b_2 = 0\), and we obtain the fourth condition of Proposition 1.

If \(a_2\) is zero then the second Poincaré–Lyapunov constant is \(V_6 = b_2(22a_4 + 5a_3 b_2)\). Let \(22a_4 + 5a_3 b_2\) be zero with \(b_2 \neq 0\), that is, \(a_4 = -5a_3 b_2/22\); in this case \(V_8 = a_3 b_2(235b_2^2 + 1254b_4)\). The case \(a_3 = 0\) corresponds to the third condition of Proposition 1. In the case \(b_4 = -235b_2^2/1254\) the next Poincaré–Lyapunov constants imply \(a_3 = b_2 = 0\). Finally, if \(b_2 = 0\) the Poincaré–Lyapunov constant \(V_8\) is zero and \(V_{10} = a_3(a_4 - b_4)(a_4 + b_4)\). The vanishing of the factors \(a_4 - b_4\) and \(a_4 + b_4\) corresponds to particular cases of the first and second conditions respectively. When \(a_3 = 0\) with \((a_4 - b_4)(a_4 + b_4) \neq 0\) we have \(V_{12} = 0\) and \(V_{14} = a_4 b_4(a_4 - b_4)(a_4 + b_4)\). The cases \(a_4 = 0\) and \(b_4 = 0\) correspond to particular cases of the third and fourth conditions respectively.

Consider the system \(\dot{x} = -y + x f(x, y)\), \(\dot{y} = x + y f(x, y)\) with \(f(x, y) = \sum_{i=1}^{3} f_i(x, y)\) where \(f_i(x, y)\) are homogeneous polynomials of degree \(i\). Any
center at the origin of this type of systems is necessarily isochronous (all the closed orbits around the center have the same period), since in polar coordinates \((r, \varphi)\) the angle \(\varphi\) satisfies the equation \(\dot{\varphi} = 1\). This type of isochronous centers are called uniformly isochronous centers (see [9]). If \(f_2 = f_3 = 0\) the origin is automatically a center because the system has 
\[ R(x, y) = (1 - a_2x + a_1y)^{-3} \]
as integrating factor. These classes of systems have been studied in [8] with \(f_3 = 0\). Here we present the center conditions for \(f_2 = 0\) and \(f_3 \neq 0\).

**Proposition 2.** Consider the system

\[
\begin{align*}
\dot{x} &= -y + x(a_1 x + a_2 y + a_6 x^2 + a_7 x^2 y + a_8 x y^2 + a_9 y^3), \\
\dot{y} &= x + y(a_1 x + a_2 y + a_6 x^3 + a_7 x^2 y + a_8 x y^2 + a_9 y^3),
\end{align*}
\]

where \(a_i\) are real numbers. Then the origin is a center if and only if

\[
\begin{align*}
a_1(a_7 + 3a_9) - a_2(a_8 + 3a_6) &= 0 \quad \text{and} \\
(3a_1a_2^2 - 3a_3^2)a_6 + (3a_1a_2 - a_1^3)a_7 - 2a_1^2a_2a_8 &= 0.
\end{align*}
\]

**Proof.** (a) Sufficiency. Suppose that the two conditions of Proposition 2 hold. If \(a_1 = a_2 = 0\) then the system has an integrating factor of the form

\[
R(x, y) = (1 - (a_7 + 2a_9)x^3 + 3a_6 x^2 y - 3a_9 x y^2 + (2a_6 + a_8)y^3)^{-5/3},
\]

which is defined at the origin and therefore the origin is a center. If \(a_1 = 0\) with \(a_2 \neq 0\) the first condition of Proposition 2 reads \(a_2(3a_6 + a_8) = 0\), which implies \(a_8 = -3a_6\). In this case the second condition is \(a_2^3a_6 = 0\) and therefore \(a_6 = 0\). System (9) with \(a_1 = a_6 = a_8 = 0\) is invariant under the change of variables \((x, y, t) \mapsto (x, -y, -t)\) and so the origin is a center. If \(a_2 = 0\) with \(a_1 \neq 0\) the first condition of Proposition 2 reads \(a_1(3a_9 + a_7) = 0\), which implies \(a_7 = -3a_9\). In this case the second condition is \(a_1^3a_9 = 0\) and therefore \(a_9 = 0\). System (9) with \(a_2 = a_7 = a_9 = 0\) is invariant under the change of variables \((x, y, t) \mapsto (-x, y, -t)\) and so the origin is also a center. Finally if \(a_1a_2 \neq 0\) from the first condition of Proposition 2 we can isolate \(a_9\), and from the second condition of Proposition 2 we obtain \(a_8\) in terms of the other parameters. In this case we make a rotation with \(\tan \alpha = -a_2/a_1\); in the new variables \((X, Y)\), system (9) is invariant under the change of variables \((X, Y, t) \mapsto (-X, Y, -t)\) and therefore has a center at the origin.

(b) Necessity. The first Poincaré–Lyapunov constant \(V_4\) is zero. The second and third Poincaré–Lyapunov constants are the two conditions of Proposition 2.

To find the maximum number of small-amplitude limit cycles which can bifurcate from the origin, the method is to find a fine focus of maximum order. From our calculations it is easy to see that if \(a_2 = b_2 = a_3 = b_3 = 0\) then \(V_4 = V_6 = V_8 = V_{10} = V_{12} = 0\) and \(V_{14} = a_4b_4(a_4 - b_4)(a_4 + b_4)\), which is different from zero if \(a_4b_4 \neq 0\) and \(a_4 \neq b_4\), and therefore...
we obtain a fine focus of order six for system (8). In the same way if $a_1 = 0$ and $a_8 = -3a_6$ then $V_4 = V_6 = 0$ and $V_8 = a_2^3a_6$, which is different from zero if $a_2$ and $a_6$ are different from zero, and therefore we obtain a fine focus of order three for system (9). Therefore we obtain the following result

**Proposition 3.** The maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least six for system (8) and three for system (9).

The Poincaré–Lyapunov constants of systems (8) and (9) are available from the e-mail address gine@eup.udl.es.

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