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ON SOME INEQUALITIES FOR SOLUTIONS OF EQUATIONS DESCRIBING THE MOTION OF A VISCOUS COMPRESSIBLE HEAT-CONDUCTING CAPILLARY FLUID BOUNDED BY A FREE SURFACE

Abstract. We derive inequalities for a local solution of a free boundary problem for a viscous compressible heat-conducting capillary fluid. The inequalities are crucial in proving the global existence of solutions belonging to certain anisotropic Sobolev–Slobodetskiĭ space and close to an equilibrium state.

1. Introduction. The aim of the paper is to obtain some inequalities for a local solution of equations of motion of a viscous compressible heat-conducting capillary fluid bounded by a free surface. The motion of such a fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ (which depends on time $t \in \mathbb{R}^1_+$) is described by the following system with the boundary and initial conditions (see [3], [4]):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(u,p) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \varkappa\Delta\theta \\
&\qquad - \frac{\mu}{2}\sum_{i,j=1}^3 (v_{ix_j} + v_{jx_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,
\end{aligned} \tag{1.1}
$$

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(1.1) [cont.]

$$v\cdot\overline{n}=-\varphi_t/|\nabla\varphi| \qquad \text{on } \widetilde{S}^T,$$

$$\partial\theta/\partial n=\overline{\theta} \qquad \text{on } \widetilde{S}^T,$$

$$\varrho|_{t=0}=\varrho_0,\quad v|_{t=0}=v_0,\quad \theta|_{t=0}=\theta_0 \qquad \text{in } \Omega,$$

where $\widetilde{\Omega}^T\equiv\bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, $\Omega_0=\Omega$ is an initial domain, $\widetilde{S}^T\equiv\bigcup_{t\in(0,T)}S_t\times\{t\}$, $S_t=\partial\Omega_t$, $\varphi(x,t)=0$ describes S_t, $\overline{n}$ is the unit outward vector normal to the boundary, i.e. $\overline{n}=\nabla\varphi/|\nabla\varphi|$. Moreover, $v=v(x,t)$ is the velocity of the fluid, $\varrho=\varrho(x,t)$ the density, $\theta=\theta(x,t)$ the temperature, $r=r(x,t)$ the heat sources per unit mass, $\overline{\theta}=\overline{\theta}(x,t)$ the heat flow per unit surface, $p=p(\varrho,\theta)$ the pressure, $c_v=c_v(\varrho,\theta)$ the specific heat at constant volume, μ and ν the viscosity coefficients, $\varkappa$ the coefficient of heat conductivity, σ the coefficient of surface tension, and p_0 the external (constant) pressure.

From the thermodynamic considerations we have

$$\nu>\tfrac{1}{3}\mu>0,\qquad \varkappa>0,\qquad c_v>0,\qquad \sigma>0.$$

Further, $\mathbb{T}=\mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\mathbb{T}(v,p)=\{T_{ij}\}_{i,j=1,2,3}=\{D_{ij}(v)-p\delta_{ij}\}_{i,j=1,2,3},$$

where

$$\mathbb{D}(v)=\{D_{ij}(v)\}_{i,j=1,2,3}=\{\mu S_{ij}(v)+(\nu-\mu)\delta_{ij}\operatorname{div}v\}_{i,j=1,2,3}$$

and $\mathbb{S}(v)=\{v_{ix_j}+v_{jx_i}\}_{i,j=1,2,3}$ is the velocity deformation tensor.

Finally, we denote by H the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\overline{n}=\Delta_{S_t}(t)x,\qquad x=(x_1,x_2,x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be determined by $x=x(s_1,s_2,t)$, $(s_1,s_2)\in\mathbb{U}\subset\mathbb{R}^2$. Then we have

$$\begin{aligned}\Delta_{S_t}(t)&=g^{-1/2}\left(\frac{\partial}{\partial s_\gamma}g^{-1/2}\widehat{g}_{\gamma\delta}\frac{\partial}{\partial s_\delta}\right)\\&=g^{-1/2}\left(\frac{\partial}{\partial s_\gamma}g^{1/2}g^{\gamma\delta}\frac{\partial}{\partial s_\delta}\right)\quad(\gamma,\delta=1,2),\end{aligned}$$

where the convention summation over repeated indices is assumed, $g=\det\{g_{\gamma\delta}\}_{\gamma,\delta=1,2}$, $g_{\gamma\delta}=\frac{\partial x}{\partial s_\gamma}\cdot\frac{\partial x}{\partial s_\delta}$, $\{g^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\gamma\delta}\}$ and $\{\widehat{g}_{\gamma\delta}\}$ is the matrix of algebraic complements of $\{g_{\gamma\delta}\}$.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t=\{x\in\mathbb{R}^3: x=x(\xi,t),\ \xi\in\Omega\}$, where $x=x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{dx}{dt}=v(x,t),\qquad x|_{t=0}=\xi\in\Omega,\qquad \xi=(\xi_1,\xi_2,\xi_3). \tag{1.2}$$

Integrating (1.2) we obtain

$$x = \xi + \int_0^t u(\xi, t')\, dt' \equiv X_u(\xi, t),$$

where $u(\xi, t) = v(X_u(\xi, t), t)$ and $x = X_u(\xi, t)$ describes the relation between the Eulerian x and Lagrangian ξ coordinates. Moreover, by $(1.1)_5$, $S_t = \{x : x = x(\xi, t),\ \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic conditions $(1.1)_5$ the total mass is conserved, i.e.

$$\int_{\Omega_t} \varrho(x, t)\, dx = \int_{\Omega} \varrho_0(\xi)\, d\xi = M.$$

Now, assume that $p_\varrho > 0$, $p_\theta > 0$ for $\varrho, \theta \in \mathbb{R}^1_+$ and consider the equation

$$p\bigg(\frac{M}{\frac{4}{3}\pi R_e^3}, \theta_e\bigg) = p_0 + \frac{2\sigma}{R_e}. \tag{1.3}$$

We assume that there exist $R_e > 0$ and $\theta_e > 0$ satisfying (1.3). Then we have the following definition.

Definition 1.1. Let $r = \overline{\theta} = 0$. By an *equilibrium state* we mean a solution $(v, \theta, \varrho, \Omega_t)$ of problem (1.1) such that $v = 0$, $\theta = \theta_e$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\varrho_e = M/(\frac{4}{3}\pi R_e^3)$, Ω_e is a ball of radius R_e, and $R_e > 0$ and $\theta_e > 0$ satisfy equation (1.3).

Next, we introduce

$$\varrho_\sigma = \varrho - \varrho_e, \qquad \theta_\sigma = \theta - \theta_e.$$

Then problem (1.1) takes the form

$$\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v, p_\sigma) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho_{\sigma t} + v\cdot\nabla\varrho_\sigma + \varrho\operatorname{div} v = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_{\sigma t} + v\cdot\nabla\theta_\sigma) - \varkappa\Delta\theta_\sigma + \theta p_\theta \operatorname{div} v &&\\
&\qquad = \frac{\mu}{2}\sum_{i,j=1}^3 (v_{ix_j} + v_{jx_i})^2 + (\nu - \mu)(\operatorname{div} v)^2 + \varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}(v, p_\sigma)\overline{n} - \sigma(H + H_e)\overline{n} = 0 && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\varphi_t/|\nabla\varphi| && \text{on } \widetilde{S}^T,\\
&\partial\theta_\sigma/\partial n = \overline{\theta} && \text{on } \widetilde{S}^T\\
&\varrho_\sigma|_{t=0} = \varrho_{\sigma 0} = \varrho_0 - \varrho_e, \quad \theta_\sigma|_{t=0} = \theta_{\sigma 0} = \theta_0 - \theta_e, &&\\
&v|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned} \tag{1.4}$$

where

$$p_\sigma = p - \sigma H_e - p_0, \qquad H_e = 2/R_e.$$

On the other hand we can write

$$p_\sigma(\varrho,\theta) = p_1\varrho_\sigma + p_2\theta_\sigma, \tag{1.5}$$

where

$$p_1(\varrho,\theta) = \int_0^1 p_\varrho(\varrho_e + s(\varrho-\varrho_e),\theta)\,ds,$$
$$p_2(\theta) = \int_0^1 p_\theta(\varrho_e,\theta_e + s(\theta-\theta_e))\,ds.$$

Problem (1.4) written in Lagrangian coordinates has the following form:

$$\begin{aligned}
&\eta u_t - \operatorname{div}_u \mathbb{T}_u(u,p_\sigma) = 0 && \text{in } \Omega^T = \Omega\times(0,T),\\
&\eta_{\sigma t} + \eta\operatorname{div}_u u = 0 && \text{in } \Omega^T,\\
&\eta c_v\vartheta_{\sigma t} + \vartheta p_\vartheta \operatorname{div}_u u - \varkappa\nabla_u^2\vartheta_\sigma \\
&\quad = \frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\cdot\nabla_\xi u_j + \xi_{x_j}\cdot\nabla_\xi u_i)^2 \\
&\qquad - (\nu-\mu)(\operatorname{div}_u u)^2 + \eta k && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,p_\sigma)\overline{n}_u - \sigma(H+H_e)\overline{n}_u = 0 && \text{on } S^T,\\
&\overline{n}_u\cdot\nabla_u\vartheta_\sigma = \overline{\vartheta} && \text{on } S^T,\\
&\eta_\sigma|_{t=0} = \varrho_{\sigma 0}, \quad \vartheta_\sigma|_{t=0} = \theta_{\sigma 0}, \quad u|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t), \vartheta(\xi,t) = \theta(X_u(\xi,t),t)$, $\eta_\sigma = \eta - \varrho_e$, $\vartheta_\sigma = \vartheta - \theta_e$, $k(\xi,t) = r(X_u(\xi,t),t)$, $\overline{\vartheta}(\xi,t) = \overline{\theta}(X_u(\xi,t),t)$, $\overline{n}_u(\xi,t) = \overline{n}(X_u(\xi,t),t)$, $\nabla_u = \xi_{ix}\partial_{\xi i} = \{\xi_{ix_j}\partial_{\xi_i}\}_{j=1,2,3}$, $\mathbb{T}_u(u,p) = -pI + \mathbb{D}_u(u)$, $I = \{\delta_{ij}\}_{i,j=1,2,3}$,

$$\begin{aligned}
\mathbb{D}_u(u) &= \{D_{uij}(u)\}_{i,j=1,2,3}\\
&= \{\mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu-\mu)\delta_{ij}\operatorname{div}_u u\}_{i,j=1,2,3},
\end{aligned}$$

$\operatorname{div}_u u = \nabla_u\cdot u = \partial_{x_i}\xi_k\partial_{\xi_k}u_i$, $\operatorname{div}_u\mathbb{T}_u(u,p) = \{\partial_{x_j}\xi_k\partial_{\xi_k}T_{uij}(u,p)\}_{i=1,2,3}$ and $\partial_{x_i}\xi_k$ are elements of the matrix ξ_x which is inverse to the matrix $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$.

In this paper we derive estimates for problem (1.1) (see Theorems 3.2 and 3.4) which are essential in the proof of the global-in-time existence of solutions to (1.1) such that $(u,\vartheta_\sigma,\eta_\sigma)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)\cap C(0,T;W_2^{1+\alpha}(\Omega))$, $\alpha\in(3/4,1)$ (see definitions in Section 2) and close to the equilibrium state. Problem (1.1) was already examined in [8], where the global existence of more regular solutions was proved. Moreover, the free boundary problem for a viscous barotropic compressible capillary fluid has been considered in [6], [7] and [10].

2. Notation. By $W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)$, $k\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1)$, we denote the Sobolev–Slobodetskiĭ space with the norm

$$\|u\|_{W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)}=\sum_{|\beta|+2i\le k}\|\partial_x^\beta\partial_t^i u\|^2_{L_2(\Omega^T)}+\sum_{|\beta|=k}\int_0^T\int_\Omega\int_\Omega\frac{|\partial_x^\beta u(x,t)-\partial_{x'}^\beta u(x',t)|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\,dt+\int_\Omega\int_0^T\int_0^T\frac{|\partial_t^{[k/2]}u(x,t)-\partial_{t'}^{[k/2]}u(x,t')|^2}{|t-t'|^{1+\alpha+k-2[k/2]}}\,dx\,dt\,dt',$$

where $\partial_x^\beta=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}$, $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi-index, $|\beta|=\beta_1+\beta_2+\beta_3$. Similarly we can define the norms in $W_2^{k+\alpha}(\Omega)$ and $W_2^{k+\alpha,k/\alpha+\alpha/2}(S^T)$.

Moreover, we shall use the notation:

$$\|u\|_{W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)}=\|u\|_{k+\alpha,\Omega^T};$$

$$\|u\|_{W_2^{k+\alpha}(Q)}=\|u\|_{k+\alpha,Q},\quad Q\in\{\Omega,S,S^1\}\ (S^1\text{ is the unit sphere});$$

$$\|u\|_{L_p(Q)}=|u|_{p,Q},\quad p\in[1,\infty],\ Q\in\{\Omega,S\};$$

$$\|u\|_{L_2(Q)}=\|u\|_{0,Q},\quad Q\in\{\Omega,S,\Omega^T,S^T\};$$

$$\|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T}=\Big[\|u\|^2_{\alpha+2,\Omega^T}+T^{-\alpha}\Big(\|u_t\|^2_{0,\Omega^T}+\sum_{|\beta|=2}\|\partial_x^\beta u\|^2_{0,\Omega^T}\Big)+\sup_{t\le T}\|u(\cdot,t)\|^2_{\alpha+1,\Omega}\Big]^{1/2};$$

$$\|u\|^{(\alpha,\alpha/2)}_{Q^T}=(\|u\|^2_{\alpha,Q^T}+T^{-\alpha}\|u\|^2_{0,Q^T})^{1/2},\quad Q\in\{\Omega,S\};$$

$$[u]_{\alpha,\Omega^T,x}=\bigg(\int_0^T dt\int_\Omega\int_\Omega\frac{|u(x,t)-u(x',t)|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\bigg)^{1/2};$$

$$[u]_{\alpha,\Omega^T,t}=\bigg(\int_\Omega dx\int_0^T\int_0^T\frac{|u(x,t)-u(x,t')|^2}{|t-t'|^{1+2\alpha}}\,dt\,dt'\bigg)^{1/2}.$$

Next, we define the isotropic Besov spaces by introducing the norm (see [1], Sect. 18)

$$\|u\|_{B_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{i=1}^n\bigg(\int_0^{h_0}dh\int_{\mathbb{R}^n}dx\,\frac{|\Delta_i^m(h)\partial_{x_i}^k u|^p}{h^{1+(l-k)p}}\bigg)^{1/p},$$

where $p \in [1, \infty]$,

$$\Delta_i^m(h)f(x) = \sum_{j=0}^{m} (-1)^{m-j} c_{jm} f(x + jhe_i),$$

$c_{jm} = \binom{m}{j} = m!/(j!(m-j)!)$, $x \in \mathbb{R}^n$, e_i is the unit vector of the ith coordinate axis, $i = 1, \dots, n$ and $m > l - k$, $m, k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{R}_+$, $l \notin \mathbb{Z}$.

It is proved in [2] (see also [1], Th. 18.2) that the Besov space norms are equivalent for all m, k satisfying $m > l - k$.

Now, we define the Sobolev–Slobodetskiĭ spaces by intoducing the norm

$$\|u\|_{W_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^{n} \bigg(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx \frac{|\Delta_i(h)\partial_{x_i}^{[l]} u|^p}{h^{1+p(l-[l])}} \bigg)^{1/p},$$

where $\Delta_i(h) = \Delta_i^l(h)$, $l \notin \mathbb{Z}$, $[l]$ is the integer part of l.

By the Golovkin theorem [2] the norms of the spaces $B_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ are equivalent.

Now, we recall the following imbedding for Besov spaces (see [1], Sect. 18):

$$\partial_x^\sigma B_p^l(\mathbb{R}^n) \subset B_q^\varrho(\mathbb{R}^n) \quad \text{for } \frac{n}{p} - \frac{n}{q} + |\sigma| + \varrho \le l.$$

Moreover, for

$$\varkappa = \frac{1}{l}\bigg(\frac{n}{p} - \frac{n}{q} + |\sigma| + \varrho\bigg) < 1$$

we have the interpolation inequality

$$\|\partial_x^\sigma u\|_{B_q^\sigma(\mathbb{R}^n)} \le \varepsilon^{1-\varkappa} \|u\|_{B_p^l(\mathbb{R}^n)} + c\varepsilon^{-\varkappa} \|u\|_{L_p(\mathbb{R}^n)}.$$

In the above notation $B_p^l(\mathbb{R}^n)$ with $l \in \mathbb{Z}_+$ is the Sobolev space.

All the above remarks can be applied to spaces of functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$ (which has the cone property), and by using a partition of unity we can also define spaces of traces on the boundary of Ω and formulate the corresponding trace theorems.

Next, we define

$$\|u\|_{L_{p_1,p_2}(\Omega^T)} = \bigg(\int_0^T dt \bigg(\int_\Omega |u(x,t)|^{p_1} dx \bigg)^{p_2/p_1} \bigg)^{1/p_2}$$

and

$$\|u\|_{\overline{L}_{p_1,p_2}(\Omega^T)} = \bigg(\int_\Omega dx \bigg(\int_0^T |u(x,t)|^{p_1} dt \bigg)^{p_2/p_1} \bigg)^{1/p_2},$$

where $p_i \in [1, \infty]$, $i = 1, 2$.

We have the following imbeddings (see [1], Sect. 18):

$$\partial_x^{\alpha_1}\partial_t^{\alpha_2}W_2^{l,l/2}(\Omega^T)$$
$$\subset\begin{cases} L_{p_1,p_2}(\Omega^T) & \text{if } n/2-n/p_1+2/2-2/p_2+|\alpha_1|+2\alpha_2\le l,\\ \overline{L}_{p_1,p_2}(\Omega^T) & \text{if } n/2-n/p_2+2/2-2/p_1+|\alpha_1|+2\alpha_2\le l,\\ L_p(0,T;B_q^\sigma(\Omega)) & \text{if } n/2-n/q+2/2-2/p+|\alpha_1|+2\alpha_2+\sigma\le l,\\ L_p(\Omega;B_q^\sigma(0,T)) & \text{if } n/2-n/p+2/2-2/q+|\alpha_1|+2\alpha_2+2\sigma\le l,\end{cases}$$

where Ω has the cone property.

Moreover, the corresponding interpolation inequalities hold.

3. Inequalities for global existence. In [9] the following local existence theorem is proved.

THEOREM 3.1. *Let* $S\in W_2^{5/2+\alpha}$, $\varrho_0\in W_2^{1+\alpha}(\Omega)$, $v_0\in W_2^{1+\alpha}(\Omega)$, $\theta_0\in W_2^{1+\alpha}(\Omega)$, $\alpha\in[3/4,1)$, $\varrho_0\ge\varrho_*>0$, $c_v\in C^2(\mathbb{R}^2)$, $c_v>0$, $p\in C^3(\mathbb{R}^2)$, *assume that* r *and* $\overline{\theta}$ *have continuous derivatives of order one and two*, r, r_{x_k} *and* $\overline{\theta},\overline{\theta}_{x_k}$ *satisfy the Hölder condition with exponent* $\overline{\alpha}\ge 1/2$ *and suppose the following compatibility conditions are satisfied*:

$$\Pi_0\mathbb{D}(v_0)\overline{n}_0=0 \qquad \text{on } S,$$
$$\overline{n}_0\cdot\mathbb{D}(v_0)\overline{n}_0=\overline{n}_0\cdot(p(\varrho_0\theta_0)-p_0)\overline{n}_0+\sigma\overline{n}_0\cdot\Delta_S(0)\xi \qquad \text{on } S,$$
$$\overline{n}_0\cdot\nabla_\xi\theta_0=\overline{\theta}(\xi,0) \qquad \text{on } S.$$

Then there exists $T>0$ *such that there exists a unique solution of problem* (1.1) *such that* $(u,\vartheta,\eta)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times C(0,T;W_2^{1+\alpha}(\Omega))\cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$.

In order to derive global estimates we assume the following condition: Ω_t is diffeomorphic to a ball, so S_t can be described by

$$|x|=r=R(\omega,t),\qquad \omega\in S^1,$$

where S_1 is the unit sphere and we consider the motion near the equilibrium state (see Definition 1.1).

First we obtain an energy type inequality.

THEOREM 3.2. *Assume that* (v,ϱ,θ) *is the local solution to problem* (1.1). *Assume that* $\varrho^*=|\varrho|_{\infty,\Omega^T}$, $\theta^*=|\theta|_{\infty,\Omega^T}$, $\varrho_*=\min_{\overline{\Omega}^T}\varrho$, $\theta_*=\min_{\overline{\Omega}^T}\theta$. *Assume that* $\alpha\in[3/4,1)$, p_ϱ, p_θ, c_v *are positive. Then*

$$\tag{3.1} \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v^2+\frac{p_1}{\varrho}\varrho_\sigma^2+\frac{\varrho c_v p_2}{\theta p_\theta}\theta_\sigma^2\right)dx + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}g^{\gamma\delta}\int_0^t v_{s_\gamma}\,dt'\cdot\int_0^t v_{s_\delta}\,dt'\,ds+c_0(\|v\|_{1,\Omega_t}^2+\|\theta_{\sigma x}\|_{0,\Omega_t}^2)$$

$$\le \varepsilon(\|\varrho_\sigma\|^2_{0,\Omega_t} + \|\theta_\sigma\|^2_{0,\Omega_t})$$
$$+ \varepsilon_1\Big(\|v\|^2_{2,\Omega_t} + \Big\|\int_0^t v\,dt'\Big\|^2_{0,S_t} + \|H(\cdot,0) + 2/R_e\|^2_{0,S^1}\Big)$$
$$+ a_1(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*)(\|v\|^2_{0,\Omega_t} + \|r\|^2_{0,\Omega_t} + \|\overline{\theta}\|^2_{0,S_t})$$
$$+ a_2(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*)[(\|\theta_\sigma\|^2_{1+\alpha,\Omega_t} + \|\varrho_\sigma\|^2_{1+\alpha,\Omega_t})(\|\theta_{\sigma t}\|^2_{0,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t})$$
$$+ (\|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1,\Omega_t})\|\theta_\sigma\|^2_{1+\alpha,\Omega_t} + \|v\|^2_{1,\Omega_t}(\|\varrho_\sigma\|^2_{1+\alpha,\Omega_t} + \|\theta_\sigma\|^2_{1+\alpha,\Omega_t})^2],$$

where ε *and* ε_1 *are sufficiently small constants, and* a_1 *and* a_2 *are positive continuous functions of their arguments.*

Proof. Multiplying $(1.4)_1$ by v, integrating over Ω_t, using the equation of continuity, integrating by parts and using the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho v^2\,dx + \int_{\Omega_t} |\mathbb{D}(v)|^2\,dx - \int_{\Omega_t} (p_1\varrho_\sigma + p_2\theta_\sigma)\operatorname{div} v\,dx - \sigma\int_{S_t} (\Delta_{S_t}x + H_e\overline{n})\cdot v\,ds = 0. \tag{3.2}$$

Muliplying $(1.4)_2$ by $\frac{p_1}{\varrho}\varrho_\sigma$ and $(1.4)_3$ by $\frac{p_2}{\theta p_\theta}\theta_\sigma$, integrating the results over Ω_t and adding to (3.2) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho v^2\,dx &+ \int_{\Omega_t} (\varrho_{\sigma t} + v\cdot\nabla\varrho_\sigma)\frac{p_1}{\varrho}\varrho_\sigma\,dx + \int_{\Omega_t} \frac{\varrho c_v}{\theta p_\theta}p_2\theta_\sigma(\theta_{\sigma t} + v\cdot\nabla\theta_\sigma)\,dx \\ &+ \int_{\Omega_t} |\mathbb{D}(v)|^2\,dx - \int_{\Omega_t} \frac{\varkappa}{\theta p_\theta}p_2\theta_\sigma\Delta\theta_\sigma\,dx - \sigma\int_{S_t}(\Delta_{S_t}x + H_e\overline{n})\cdot v\,ds \\ &= \int_{\Omega_t} \frac{p_2}{\theta p_\theta}\theta_\sigma\Big[\frac{\mu}{2}\sum_{i,j=1}^3 (v_{ix_j} + v_{jx_i})^2 + (\nu-\mu)(\operatorname{div} v)^2\Big]dx + \int_{\Omega_t} \frac{\varrho p_2}{\theta p_\theta}\theta_\sigma r\,dx.\end{aligned} \tag{3.3}$$

By using the equation of continuity $(1.1)_2$ the second term on the l.h.s. of (3.3) takes the form

$$\frac{d}{dt}\int_{\Omega_t} \frac{1}{2}\frac{p_1}{\varrho}\varrho_\sigma^2\,dx + I_1, \tag{3.4}$$

where

$$I_1 = -\frac{1}{2}\int_{\Omega_t} \varrho_\sigma^2\Big[\varrho\partial_t\Big(\frac{p_1}{\varrho^2}\Big) + \varrho v\cdot\nabla\Big(\frac{p_1}{\varrho^2}\Big)\Big]dx.$$

Hence

$$
\begin{aligned}
|I_1| &\le a_3(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*\Big) \int\limits_{\Omega_t} |\varrho_\sigma|^2[|\varrho_{\sigma t}| + |\theta_{\sigma t}| + |v|(|\varrho_{\sigma x}| + |\theta_{\sigma x}|)]\,dx \\
&\le \varepsilon\|\varrho_\sigma\|^2_{0,\Omega_t} + a_4(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*)[\|\varrho_\sigma\|^2_{1+\alpha,\Omega_t}(\|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\theta_{\sigma t}\|^2_{0,\Omega_t}) \\
&\quad + \|\varrho_\sigma\|^2_{1+\alpha,\Omega_t}\|v\|^2_{1,\Omega_t}(\|\varrho_\sigma\|^2_{1+\alpha,\Omega_t} + \|\theta_\sigma\|^2_{1+\alpha,\Omega_t})],
\end{aligned}
$$

where a_3, a_4 etc. denote positive continuous functions. Similarly, the third term on the l.h.s. of (3.3) takes the form

$$
\frac{d}{dt}\int\limits_{\Omega_t} \frac{1}{2}\frac{\varrho c_v p_2}{\theta p_\theta}\theta_\sigma^2 + I_2, \tag{3.5}
$$

where

$$
I_2 = -\frac{1}{2}\int\limits_{\Omega_t} \theta_\sigma^2\bigg[\varrho\partial_t\bigg(\frac{c_v p_2}{\theta p_\theta}\bigg) + \varrho v\cdot\nabla\bigg(\frac{c_v p_2}{\theta p_\theta}\bigg)\bigg]\,dx,
$$

so

$$
\begin{aligned}
|I_2| &\le a_5(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*) \int\limits_{\Omega_t} \theta_\sigma^2[|\varrho_{\sigma t}| + |\theta_{\sigma t}| + |v|(|\varrho_{\sigma x}| + |\theta_{\sigma x}|)]\,dx \\
&\le \varepsilon\|\theta_\sigma\|^2_{0,\Omega_t} + a_6(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*)[\|\theta_\sigma\|^2_{1+\alpha,\Omega_t}(\|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\theta_{\sigma t}\|^2_{0,\Omega_t}) \\
&\quad + \|\theta_\sigma\|^2_{1+\alpha,\Omega_t}\|v\|^2_{1,\Omega_t}(\|\varrho_\sigma\|^2_{1+\alpha,\Omega_t} + \|\theta_\sigma\|^2_{1+\alpha,\Omega_t})].
\end{aligned}
$$

By the boundary condition $(1.4)_6$ the fifth term on the l.h.s. of (3.3) is equal to

$$
-\int\limits_{S_t} \frac{\varkappa p_2}{\theta p_\theta}\theta_\sigma\overline{\theta}\,ds + \int\limits_{\Omega_t} \frac{\varkappa p_2}{\theta p_\theta}|\nabla\theta_\sigma|^2\,dx + \int\limits_{\Omega_t} \nabla\bigg(\frac{\varkappa p_2}{\theta p_\theta}\bigg)\cdot\theta_\sigma\nabla\theta_\sigma\,dx. \tag{3.6}
$$

Denoting the last expression in (3.6) by I_3 we obtain

$$
\begin{aligned}
|I_3| &\le a_5(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*) \int\limits_{\Omega_t} |\theta_\sigma||\theta_{\sigma x}|(|\varrho_{\sigma x}| + |\theta_{\sigma x}|)\,dx \\
&\le \varepsilon\|\theta_{\sigma x}\|^2_{0,\Omega_t} + a_7(\varrho^*, 1/\varrho_*, \theta^*, 1/\theta_*, \varepsilon) \\
&\quad \cdot (\|\theta_\sigma\|^2_{1+\alpha,\Omega_t}\|\varrho_\sigma\|^2_{1,\Omega_t} + \|\theta_\sigma\|^2_{1+\alpha,\Omega_t}\|\theta_\sigma\|^2_{1,\Omega_t}).
\end{aligned} \tag{3.7}
$$

In view of the considerations from Lemma 4.1 of [10] the boundary term on the l.h.s. of (3.3) takes the form

$$
\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t} g^{\gamma\delta}\int\limits_0^t v_{s_\gamma}\,dt'\cdot\int\limits_0^t v_{s_\delta}\,dt'\,ds + I_4, \tag{3.8}
$$

where

(3.9) $$|I_4| \le \varepsilon_1\Big(\Big\|\int_0^t v\,dt'\Big\|_{0,S_t}^2 + \|v\|_{1,\Omega_t}^2 + \|H(\cdot,0)+2/R_e\|_{0,S^1}^2\Big) + a_8\Big\|\int_0^t v\,dt'\Big\|_{2,S_t}^2 + \|v\|_{2,\Omega_t}^2 + a_9\|v\|_{0,\Omega_t}^2,$$

where $\varepsilon_1 \in (0,1)$.

Taking into account (3.3)–(3.9) we obtain estimate (3.1). ∎

Next, we obtain estimates for $\sup_{t_1\le t\le T}\|u\|_{2+\alpha,\Omega}^2$, $\sup_{t_1\le t\le T}\|\vartheta_\sigma\|_{2+\alpha,\Omega}^2$ and $\sup_{t_1\le t\le T}\|\eta_\sigma\|_{1+\alpha,\Omega}^2$, where $(u,\vartheta_\sigma,\eta_\sigma)$ is the local solution of problem (1.6) and $t_1>0$. To do this we use the argument from [5] (see Theorem 6).

Let $\zeta_\lambda \in C^\infty$ be a function such that $\zeta_\lambda(t)=1$ for $t\ge t_0+\lambda$ ($t_0>0$, $\lambda>0$, $t_0+\lambda<T$), $\zeta_\lambda(t)=0$ for $t\le t_0+\lambda/2$, $0\le\zeta_\lambda(t)\le 1$, $|\dot\zeta_\lambda(t)|\le C/\lambda$, where $\dot\zeta_\lambda = d\zeta/dt$. Let $w_\lambda = w\zeta_\lambda$, where $w\in\{u,\vartheta_\sigma,\eta_\sigma,k,\overline{\vartheta}\}$. Then $(u_\lambda,\vartheta_{\sigma\lambda},\eta_{\sigma\lambda})$ satisfies the problem

$$\begin{aligned}
&\eta u_{\lambda t} - \mu\nabla_u^2 u_\lambda - \nu\nabla_u\nabla_u\cdot u_\lambda = -p_\eta\nabla_u\eta_{\sigma\lambda} - p_\vartheta\nabla_u\vartheta_{\sigma\lambda} + \eta u\dot\zeta_\lambda && \text{in } \Omega^T,\\
&\Pi_0\Pi_u\mathbb{D}_u(u_\lambda)\overline{n}_u = 0 && \text{on } S^T,\\
&\overline{n}_0\cdot\mathbb{D}_u(u_\lambda)\overline{n}_u - \sigma\overline{n}_0\cdot\Delta_u(t')\int_0^t u_\lambda(t')\,dt' \\
&\quad = \int_0^t\Big[\dot\zeta_\lambda\overline{n}_0\cdot\mathbb{T}_u(u,p_\sigma)\overline{n}_u - \sigma\overline{n}_0\cdot\zeta_\lambda\dot\Delta_u(t')\Big(\xi+\int_0^t u(t'')\,dt''\Big)\\
&\qquad - \zeta_\lambda\partial_{t'}\Big(\frac{2}{R_e}\overline{n}_0\cdot\overline{n}_u\Big)\Big]\,dt' + (p_1\eta_{\sigma\lambda}+p_2\vartheta_{\sigma\lambda})\overline{n}_0\cdot\overline{n}_u\\
&\quad \equiv \int_0^t B(t')\,dt' + (p_1\eta_{\sigma\lambda}+p_2\vartheta_{\sigma\lambda})\overline{n}_0\cdot\overline{n}_u && \text{on } S^T,\\
&u_\lambda|_{t=0} = 0 && \text{in } \Omega,\\
&\eta c_v\vartheta_{\sigma\lambda t} + \vartheta p_\vartheta\operatorname{div}_u u_\lambda - \varkappa\nabla_u^2\vartheta_{\sigma\lambda}\\
&\quad = \frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\cdot\nabla_\xi u_{j\lambda} + \xi_{x_j}\cdot\nabla_\xi u_{i\lambda})\\
&\qquad \cdot(\xi_{x_i}\cdot\nabla_\xi u_j + \xi_{x_j}\cdot\nabla_\xi u_i)\\
&\qquad - (\nu-\mu)\operatorname{div}_u u_\lambda\operatorname{div}_u u + \eta k_\lambda + \eta c_v\vartheta_\sigma\dot\zeta_\lambda && \text{in } \Omega^T,\\
&\overline{n}_u\cdot\nabla_u\vartheta_{\sigma\lambda} = \overline{\vartheta}_\lambda && \text{on } S^T,\\
&\vartheta_{\sigma\lambda}|_{t=0} = 0 && \text{in } \Omega,\\
&\eta_{\sigma\lambda t} + \eta\nabla_u\cdot u_\lambda = \eta_\sigma\dot\zeta_\lambda && \text{in } \Omega^T,\\
&\eta_{\sigma\lambda}|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}$$

where $\Pi_0 g = g - \overline{n}_0(\overline{n}_0 \cdot g)$, $\Pi_u g = g - \overline{n}_u(\overline{n}_u \cdot g)$, and $\overline{n}_0$ is the unit outward vector normal to S.

Next, we introduce the differences

$$w^{(s)}(\xi,t) = w_\lambda(\xi,t) - w'_\lambda(\xi,t),$$

where $w \in \{u, \eta_\sigma, \vartheta_\sigma, k, \overline{\vartheta}\}$, $w'_\lambda(\xi,t) = w_\lambda(\xi, t-s)$, and

$$w_*^{(s)}(\xi,t) = w(\xi,t) - w'(\xi,t),$$

where $w \in \{u, \eta, \vartheta\}$, $w'(\xi,t) = w(\xi, t-s)$.

Then we obtain the following problem:

$$\begin{aligned}
&\eta u_t^{(s)} - \mu\nabla_u^2 u^{(s)} - \nu\nabla_u\nabla_u \cdot u^{(s)} \\
&\quad = -p_\eta \nabla_u \eta_\sigma^{(s)} - p_\vartheta \nabla_u \vartheta_\sigma^{(s)} \\
&\qquad - \eta_*^{(s)} u'_{\lambda t} + \mu(\nabla_u^2 - \nabla_{u'}^2)u'_\lambda \\
&\qquad + \nu(\nabla_u\nabla_u - \nabla_{u'}\nabla_{u'}) \cdot u'_\lambda - p'_\eta(\nabla_u - \nabla_{u'})\eta'_{\sigma\lambda} \\
&\qquad - p'_\vartheta(\nabla_u - \nabla_{u'})\vartheta'_{\sigma\lambda} - (p_\eta - p'_\eta)\nabla_{u'}\eta'_{\sigma\lambda} \\
&\qquad - (p_\vartheta - p'_\vartheta)\nabla_{u'}\vartheta'_{\sigma\lambda} + \eta_*^{(s)} u\dot{\zeta}_\lambda \\
&\qquad + \eta' u(\dot{\zeta}_\lambda - \dot{\zeta}'_\lambda) + \eta' u_*^{(s)}\dot{\zeta}'_\lambda \equiv F && \text{in } \Omega^T, \\
&\Pi_0 \Pi_u \mathbb{D}_u(u^{(s)})\overline{n}_u \\
&\quad = -\Pi_0(\Pi_u \mathbb{D}_u(u'_\lambda)\overline{n}_u - \Pi_{u'}\mathbb{D}_{u'}(u'_\lambda)\overline{n}_{u'}) \equiv \Pi_0 G && \text{on } S^T, \\
&\overline{n}_0 \cdot \mathbb{D}_u(u^{(s)})\overline{n}_u - \sigma\overline{n}_0 \cdot \Delta_u(t)\int_0^t u^{(s)}(t')\,dt' \\
&\quad = p_1(\eta,\vartheta)\eta_\sigma^{(s)}\overline{n}_0 \cdot \overline{n}_u + p_2(\eta,\vartheta)\vartheta_\sigma^{(s)}\overline{n}_0 \cdot \overline{n}_u \\
&\qquad - \overline{n}_0(\mathbb{D}_u(u'_\lambda)\overline{n}_u - \mathbb{D}_{u'}(u'_\lambda)\overline{n}_{u'}) \\
&\qquad + \sigma\overline{n}_0 \cdot \int_0^t (\Delta_u(t') - \Delta_{u'}(t'))u'_\lambda\,dt' \\
&\qquad + \int_0^t (B(t') - B'(t'))\,dt' + p_1(\eta,\vartheta)\eta'_{\sigma\lambda}\overline{n}_0 \cdot (\overline{n}_u - \overline{n}_{u'}) \\
&\qquad + p_2(\eta,\vartheta)\vartheta'_{\sigma\lambda}\overline{n}_0 \cdot (\overline{n}_u - \overline{n}_{u'}) \\
&\qquad + [p_1(\eta,\vartheta) - p_1(\eta',\vartheta')]\eta'_{\sigma\lambda}\overline{n}_0 \cdot \overline{n}_{u'} \\
&\qquad + [p_2(\eta,\vartheta) - p_2(\eta',\vartheta')]\vartheta'_{\sigma\lambda}\overline{n}_0 \cdot \overline{n}_{u'} \\
&\quad \equiv H_1 + \int_0^t H_2(t')\,dt' && \text{on } S^T, \\
&u^{(s)}|_{t=t_0+\lambda/2} = 0 && \text{in } \Omega,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\eta c_v\vartheta^{(s)}_{\sigma t}-\varkappa\nabla^2_u\vartheta^{(s)}_\sigma=\varkappa(\nabla^2_u-\nabla^2_{u'})\vartheta'_{\sigma\lambda}-\eta^{(s)}_*c_v\vartheta'_{\sigma\lambda t}-\eta'c_{v\eta}\eta^{(s)}_*\vartheta'_{\sigma\lambda t}\\
&\qquad-\eta'c_{v\vartheta}\vartheta^{(s)}_*\vartheta'_{\sigma\lambda t}+\eta^{(s)}_*c_v\vartheta_\sigma\dot\zeta_\lambda+\eta'c_{v\eta}\eta^{(s)}_*\vartheta_\sigma\dot\zeta_\lambda+\eta'c_{v\vartheta}\vartheta^{(s)}_*\vartheta_\sigma\dot\zeta_\lambda\\
&\qquad+\eta'c'_v(\dot\zeta_\lambda-\dot\zeta'_\lambda)\vartheta_\sigma+\eta'c'_v\vartheta^{(s)}_*\dot\zeta'_\lambda\\
&\qquad-\vartheta p_\vartheta\operatorname{div}_u u^{(s)}-\vartheta p_\vartheta(\nabla_u-\nabla_{u'})\cdot u'_\lambda\\
&\qquad-\vartheta^{(s)}_*p_\vartheta\operatorname{div}_{u'}u'_\lambda-\vartheta'(p_\vartheta-p'_\vartheta)\operatorname{div}_{u'}u'_\lambda\\
&\qquad+\frac{\mu}{2}\sum_{i,j=1}^{3}[(\xi_{x_i}\cdot\nabla_\xi u_{j\lambda}+\xi_{x_j}\cdot\nabla_\xi u_{i\lambda})(\xi_{x_i}\cdot\nabla_\xi u_j+\xi_{x_j}\cdot\nabla_\xi u_i)\\
&\qquad-(\xi'_{x_i}\cdot\nabla_\xi u'_{j\lambda}+\xi'_{x_j}\cdot\nabla_\xi u'_{i\lambda})(\xi'_{x_i}\cdot\nabla_\xi u'_j+\xi'_{x_j}\cdot\nabla_\xi u'_i)] \qquad\qquad (3.10)\ \text{[cont.]}\\
&\qquad-(\nu-\mu)(\operatorname{div}_u u_\lambda\operatorname{div}_u u-\operatorname{div}_{u'}u'_\lambda\operatorname{div}_{u'}u')+\eta^{(s)}_*k_\lambda+\eta'k^{(s)}\equiv I\quad\text{in }\Omega^T,\\
&\overline{n}_u\cdot\nabla_u\vartheta^{(s)}_\sigma=-\overline{n}_u\cdot(\nabla_u-\nabla_{u'})\vartheta'_{\sigma\lambda}-(\overline{n}_u-\overline{n}_{u'})\cdot\nabla_{u'}\vartheta'_{\sigma\lambda}+\overline{\vartheta}^{(s)}\equiv J\quad\text{on }S^T,\\
&\vartheta^{(s)}_\sigma|_{t=t_0+\lambda/2}=0\quad\text{in }\Omega,\\
&\eta^{(s)}_{\sigma t}+\eta\nabla_u\cdot u^{(s)}\\
&\qquad=\eta(\nabla_u-\nabla_{u'})\cdot u'_\lambda+\eta^{(s)}_*\nabla_{u'}\cdot u'_\lambda+\eta^{(s)}_*\dot\zeta_\lambda+\eta_\sigma(\dot\zeta_\lambda-\dot\zeta'_\lambda)\quad\text{in }\Omega^T,\\
&\eta^{(s)}_{\sigma t}|_{t=t_0+\lambda/2}=0\quad\text{in }\Omega.
\end{aligned}$$

First, we prove

LEMMA 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied and let* $\alpha\in(3/4,1)$. *Then*

$$(\|u^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2+(\|\vartheta^{(s)}_\sigma\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2\le c_1(K)\overline{K}s^{1+\overline{\omega}_1},\tag{3.11}$$

where $0<s<t_0$, $\overline{\omega}_1>0$ *is a constant*, $\overline{K}=\|u\|^2_{2+\alpha,\Omega^T}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^T}$,

$$\begin{aligned}K=\overline{K}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^T}\\
+\sup_{0\le t\le T}\|u\|^2_{1+\alpha,\Omega}+\sup_{0\le t\le T}\|\vartheta_\sigma\|^2_{1+\alpha,\Omega}+\sup_{0\le t\le T}\|\eta_\sigma\|^2_{1+\alpha,\Omega},\end{aligned}$$

$c_1(K)$ *is a positive nondecreasing continuous function of* K, $Q_\lambda=\Omega\times(t_0+\lambda,T)$, *and* $\lambda\in(0,1)$.

Proof. By Theorem 1.2 of [7] and Lemma 3.2 of [9] we have

$$\begin{aligned}\|u^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda}&+\|\vartheta^{(s)}_\sigma\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda}\\
&\le c(\|F\|^{(\alpha,\alpha/2)}_{Q_{\lambda/2}}+\|I\|^{(\alpha,\alpha/2)}_{Q_{\lambda/2}}+\|G\|_{\alpha+1/2,G_{\lambda/2}}\\
&\quad+\|H_1\|_{\alpha+1/2,G_{\lambda/2}}+\|J\|_{\alpha+1/2,G_{\lambda/2}}+\|H_2\|^{(\alpha-1/2,\alpha/2-1/4)}_{G_{\lambda/2}}).\end{aligned}\tag{3.12}$$

We have to estimate the terms on the right-hand side of (3.12).

First, we estimate $K_1 = p_\vartheta \nabla_u \vartheta_\sigma^{(s)}$. We have

$$\begin{aligned}
[K_1]^2_{\alpha,Q_{\lambda/2},\xi} \le c \int\limits_{t_0+\lambda/2}^T \int\limits_\Omega \int\limits_\Omega \frac{|\eta_\sigma(\xi)-\eta_\sigma(\xi')|^2 |\vartheta^{(s)}_{\sigma\xi}(\xi)|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt \\
+ c \int\limits_{t_0+\lambda/2}^T \int\limits_\Omega \int\limits_\Omega \frac{|\vartheta_\sigma(\xi)-\vartheta_\sigma(\xi')|^2 |\vartheta^{(s)}_{\sigma\xi}(\xi)|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt \\
+ c \int\limits_{t_0+\lambda/2}^T \int\limits_\Omega \int\limits_\Omega \frac{|\int_0^t (u_\xi - u_{\xi'})\, dt'|^2 |\vartheta^{(s)}_{\sigma\xi}(\xi)|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt \\
+ c \int\limits_{t_0+\lambda/2}^T \int\limits_\Omega \int\limits_\Omega \frac{|\vartheta^{(s)}_{\sigma\xi}(\xi)-\vartheta^{(s)}_{\sigma\xi'}(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt \equiv \sum_{i=1}^4 J_i.
\end{aligned}$$

First, we get

$$\begin{aligned}
(3.13)\qquad J_4 &= c \int\limits_{t_0+\lambda/2}^T \|\vartheta^{(s)}_{\sigma\xi}\|^2_{\alpha,\Omega}\, dt \\
&\le \varepsilon \int\limits_{t_0+\lambda/2}^T \|\vartheta^{(s)}_\sigma\|^2_{2+\alpha,\Omega}\, dt + c(\varepsilon)\|\vartheta^{(s)}_\sigma\|^2_{0,Q_{\lambda/2}} \\
&\le \varepsilon \|\vartheta^{(s)}_\sigma\|^2_{2+\alpha,Q_{\lambda/2}} + c(\varepsilon)s^{1+\alpha} \sup_{0<s<t_0} \int\limits_{t_0+\lambda/2}^T \frac{|\vartheta^{(s)}_\sigma|^2_{2,\Omega}}{s^{1+\alpha}}\, dt \\
&\le \varepsilon \|\vartheta^{(s)}_\sigma\|^2_{2+\alpha,Q_{\lambda/2}} + c(\varepsilon)\overline{K} s^{1+\alpha}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
J_1 &\le c \sup_{t_0+\lambda/2\le t\le T} \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|\eta_\sigma(\xi)-\eta_\sigma(\xi')|^4}{|\xi-\xi'|^{3+4(1/4+\alpha)}}\, d\xi\, d\xi' \bigg)^{1/2} \\
&\quad \cdot \int\limits_{t_0+\lambda/2}^T \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|\vartheta^{(s)}_{\sigma\xi}(\xi)|^4}{|\xi-\xi'|^2}\, d\xi\, d\xi' \bigg)^{1/2} dt \\
&\le c \sup_{t_0+\lambda/2\le t\le T} \|\eta_\sigma\|^2_{1+\alpha,\Omega} \int\limits_{t_0+\lambda/2}^T \|\vartheta^{(s)}_{\sigma\xi}\|^2_{\alpha,\Omega}\, dt \\
&\le \varepsilon \|\vartheta^{(s)}_\sigma\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K} s^{1+\alpha},
\end{aligned}$$

where we have used the imbeddings $W_2^{1+\alpha}(\Omega) \subset W_4^{1/4+\alpha}(\Omega)$, $W_2^\alpha(\Omega) \subset L_4(\Omega)$ for $\alpha \ge 3/4$ and we have estimated $\int_{t_0+\lambda/2}^T \|\vartheta^{(s)}_{\sigma\xi}\|^2_{\alpha,\Omega}\, dt$ in the same way as in (3.13).

Similarly, we estimate J_2 and J_3.

Summarizing the above considerations we get

$$[K_1]^2_{\alpha,Q_{\lambda/2},\xi} \le \varepsilon\|\vartheta_\sigma^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}, \tag{3.14}$$

where $c(K)$ is a positive nondecreasing continuous function.

Next, we calculate

$$\begin{aligned}
[K_1]^2_{\alpha/2,Q_{\lambda/2},t} &\le c\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T \frac{|\eta_\sigma(t)-\eta_\sigma(t')|^2|\vartheta_{\sigma\xi}^{(s)}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\quad + c\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T \frac{|\vartheta_\sigma(t)-\vartheta_\sigma(t')|^2|\vartheta_{\sigma\xi}^{(s)}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\quad + c\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T \frac{|\int_{t'}^t u_\xi\,d\tau|^2|\vartheta_{\sigma\xi}^{(s)}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\quad + c\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T \frac{|\vartheta_{\sigma\xi}^{(s)}(t)-\vartheta_{\sigma\xi}^{(s)}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&= \sum_{i=5}^8 J_i.
\end{aligned}$$

We estimate J_8 in the same way as J_4.

Next, we have

$$\begin{aligned}
J_5+J_7 &\le c\int_0^T|u_\xi|_{\infty,\Omega}\,dt\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T\frac{|\vartheta_\sigma^{(s)}(t')|^2}{|t-t'|^\alpha}\,d\xi\,dt\,dt' \\
&\le c\|u\|^2_{2+\alpha,\Omega^T}\int_{t_0+\lambda/2}^T|\vartheta_{\sigma\xi}^{(s)}|^2_{2,\Omega}\,dt \\
&\le \varepsilon\|\vartheta_\sigma^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
J_6 &\le c\int_\Omega\int_{t_0+\lambda/2}^T\int_{t_0+\lambda/2}^T\frac{|\int_{t'}^t\vartheta_{\sigma\tau}\,d\tau|^2|\vartheta_{\sigma\xi}^{(s)}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\le c\int_{t_0+\lambda/2}^T|\vartheta_{\sigma\tau}|^2_{4,\Omega}\,dt\int_{t_0+\lambda/2}^T|\vartheta_{\sigma\xi}^{(s)}|^2_{4,\Omega}\,dt \le c\|\vartheta_\sigma\|^2_{\alpha,Q_{\lambda/2}}\int_{t_0+\lambda/2}^T|\vartheta_{\sigma\xi}^{(s)}|^2_{4,\Omega}\,dt \\
&\le \varepsilon\|\vartheta_\sigma^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}.
\end{aligned}$$

Summarizing the above considerations we get

$$[K_1]^2_{\alpha/2,Q_{\lambda/2},t} \le \varepsilon\|\vartheta_\sigma^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}. \tag{3.15}$$

Estimates (3.14) and (3.15) yield

$$(\|p_\vartheta\nabla_u\vartheta_\sigma^{(s)}\|^{(\alpha,\alpha/2)}_{Q_{\lambda/2}})^2 \le \varepsilon\|\vartheta_\sigma^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}. \tag{3.16}$$

The next term we consider is $K_2 = \eta' c_{v\vartheta}\vartheta_*^{(s)}\vartheta'_{\sigma\lambda t}$.
First, we have

$$\begin{aligned}
[K_2]^2_{\alpha,Q_{\lambda/2},\xi} \le{}& c\int_{t_0+\lambda/2}^T\int_\Omega\int_\Omega \frac{|\eta_\sigma(\xi)-\eta_\sigma(\xi')|^2|\vartheta_*^{(s)}|^2|\vartheta'_{\sigma\lambda t}|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&+ c\int_{t_0+\lambda/2}^T\int_\Omega\int_\Omega \frac{|\vartheta_\sigma(\xi)-\vartheta_\sigma(\xi')|^2|\vartheta_*^{(s)}|^2|\vartheta'_{\sigma\lambda t}|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&+ c\int_{t_0+\lambda/2}^T\int_\Omega\int_\Omega \frac{|\vartheta_*^{(s)}(\xi)-\vartheta_*^{(s)}(\xi')|^2|\vartheta'_{\sigma\lambda t}|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&+ c\int_{t_0+\lambda/2}^T\int_\Omega\int_\Omega \frac{|\vartheta_*^{(s)}(\xi)|^2|\vartheta'_{\sigma\lambda t}(\xi)-\vartheta'_{\sigma\lambda t}(\xi')|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
\equiv{}& \sum_{i=9}^{12} J_i.
\end{aligned}$$

We estimate

$$\begin{aligned}
J_9 \le{}& c\sup_{t_0+\lambda/2\le T\le T}|\vartheta_*^{(s)}|^2_{\infty,\Omega}\sup_{t_0+\lambda/2\le t\le T}\bigg(\int_\Omega\int_\Omega\frac{|\eta_\sigma(\xi)-\eta_\sigma(\xi')|^4}{|\xi-\xi'|^{3+4(1/4+\alpha)}}\bigg)^{1/2} \\
&\cdot\int_{t_0+\lambda/2}\bigg(\int_\Omega\int_\Omega\frac{|\vartheta'_{\sigma\lambda t}|^4}{|\xi-\xi'|^2}\,d\xi\,d\xi'\bigg)^{1/2}dt.
\end{aligned}$$

Using the imbeddings $W_2^{1+\alpha}(\Omega)\subset W_4^{1/4+\alpha}(\Omega)$ and $W_2^\alpha(\Omega)\subset L_4(\Omega)$ (which hold for $\alpha\ge 3/4$) and the interpolation inequality

$$\sup_{t_0+\lambda/2\le t\le T}|\vartheta_*^{(s)}|^2_{\infty,\Omega} \le \varepsilon_1^{1-\varkappa}\|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c\varepsilon_1^{-\varkappa}\|\vartheta_*^{(s)}\|^2_{0,Q_{\lambda/2}}$$

(which holds for $\varkappa = 5/(4+2\alpha)$ and $\alpha > 1/2$) we get

$$\begin{aligned}
J_9 \le{}& c\bigg[\varepsilon_1^{1-\varkappa}\|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c\varepsilon_1^{-\varkappa}s^{1+\alpha}\sup_{0<s<t_0}\int_{t_0+\lambda/2}^T\frac{|\vartheta_*^{(s)}|^2_{2,\Omega}}{s^{1+\alpha}}\,dt\bigg] \\
&\cdot\sup_{t_0+\lambda/2\le t\le T}\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\vartheta_{\sigma t}\|^2_{\alpha,Q_{\lambda/2}} \\
\le{}& \varepsilon\|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha},
\end{aligned}$$

where we have taken $\varepsilon_1 = (\varepsilon/(cK^2))^{1/(1-\varkappa)}$ and $c(K)$ is a positive nondecreasing continuous function of K.

In the same way we estimate J_{10}.

Next, we have

$$J_{11} \le c \sup_{t_0+\lambda/2\le t\le T} \bigg(\int_\Omega\int_\Omega \frac{|\vartheta_*^{(s)}(\xi) - \vartheta_*^{(s)}(\xi')|^4}{|\xi-\xi'|^{3+4(1/4+\alpha-\delta)}} \bigg)^{1/2} \cdot \int_{t_0+\lambda/2}^T \bigg(\int_\Omega\int_\Omega \frac{|\vartheta_{\sigma\lambda t}|^4}{|\xi-\xi'|^{2+2\delta}} \bigg)^{1/2} dt,$$

where $\delta > 0$ is a sufficiently small constant such that $2+2\delta < 3$. Using the interpolation inequality

$$\|\vartheta_*^{(s)}\|_{L\infty(t_0+\lambda/2,T;W_4^{1/4+\alpha-\delta}(\Omega))} \le \varepsilon_1^{1-\varkappa} \|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c\varepsilon_1^{-\varkappa}\|\vartheta_*^{(s)}\|^2_{0,Q_{\lambda/2}}$$

with $\varkappa = (2+\alpha-\delta)/(2+\alpha)$ and the imbedding $W_2^\alpha(\Omega) \subset L_4(\Omega)$ (both holding for $\alpha \ge 3/4$) we obtain

$$J_{11} \le \varepsilon \|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}$$

where we have set $\varepsilon_1 = (\varepsilon/(cK))^{1/(1-\varkappa)}$.

Finally,

$$J_{12} \le c \sup_{t_0+\lambda/2\le t\le T} |\vartheta_*^{(s)}|^2_{\infty,\Omega} \|\vartheta_{\sigma t}\|^2_{\alpha,Q_{\lambda/2}} \le \varepsilon \|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}.$$

Taking into account the above considerations we get

$$[K_2]^2_{\alpha,Q_{\lambda/2},\xi} \le \varepsilon \|\vartheta_*^{(s)}\|^2_{2+\alpha,Q_{\lambda/2}} + c(K)\overline{K}s^{1+\alpha}, \tag{3.17}$$

where $c(K)$ is a positive nondecreasing continuous function of K.

Now, we consider

$$\begin{aligned}
[K_2]^2_{\alpha/2,Q_{\lambda/2},t} \le{}& c \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\eta_\sigma(t)-\eta_\sigma(t')|^2 |\vartheta_*^{(s)}|^2 |\vartheta'_{\sigma\lambda t}|^2}{|t-t'|^{1+\alpha}}\, d\xi\, dt\, dt' \\
&+ c \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\vartheta_\sigma(t)-\vartheta_\sigma(t')|^2 |\vartheta_*^{(s)}|^2 |\vartheta'_{\sigma\lambda t}|^2}{|t-t'|^{1+\alpha}}\, d\xi\, dt\, dt' \\
&+ c \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\vartheta_*^{(s)}(t)-\vartheta_*^{(s)}(t')|^2 |\vartheta'_{\sigma\lambda t}|^2}{|t-t'|^{1+\alpha}}\, d\xi\, dt\, dt' \\
&+ c \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\vartheta_*^{(s)}(t)|^2 |\vartheta'_{\sigma\lambda t}(t)-\vartheta'_{\sigma\lambda t}(t')|^2}{|t-t'|^{1+\alpha}}\, d\xi\, dt\, dt' \\
\equiv{}& \sum_{i=13}^{16} J_i.
\end{aligned}$$

First, we have

$$J_{13} \le c \sup_{t_0+\lambda/2\le t\le T} |\vartheta_*^{(s)}|_{\infty,\Omega}^2 \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t'}^t u_\xi\, d\tau|^2 |\vartheta'_{\sigma\lambda t}|^2}{|t-t'|^{1+\alpha}}\, d\xi\, dt\, dt'$$
$$\le c[\varepsilon_1^{1-\varkappa}\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c\varepsilon_1^{-\varkappa}\|\vartheta_*^{(s)}\|_{0,Q_{\lambda/2}}^2]$$
$$\cdot \int_{t_0+\lambda/2}^T |u_\xi|_{\infty,\Omega}^2\, dt \int_{t_0+\lambda/2}^T |\vartheta'_{\sigma\lambda t}|_{2,\Omega}\, dt$$
$$\le \varepsilon\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha},$$

where $\varkappa = 5/(4+2\alpha)$ and we have taken $\varepsilon_1 = (\varepsilon/(cK^2))^{1/(1-\varkappa)}$.

Next, we get

$$J_{14} \le c \sup_{t_0+\lambda/2\le t\le T} |\vartheta_*^{(s)}|_{\infty,\Omega}^2 \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t'}^t \vartheta_{\sigma\tau}\, d\tau|_{4,\Omega}^2 |\vartheta'_{\sigma\lambda t}|_{4,\Omega}^2}{|t-t'|^{1+\alpha}}\, dt\, dt'$$
$$\le c \sup_{t_0+\lambda/2\le t\le T} |\vartheta_*^{(s)}|_{\infty,\Omega}^2 \int_{t_0+\lambda/2}^T \|\vartheta_{\sigma t}\|_{\alpha,\Omega}^2\, dt \int_{t_0+\lambda/2}^T \|\vartheta'_{\sigma\lambda t}\|_{\alpha,\Omega}^2\, dt$$
$$\le \varepsilon\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha},$$

where we have used the same interpolation inequality as before and the imbedding $W_2^\alpha(\Omega) \subset L_4(\Omega)$, which holds for $\alpha \ge 3/4$.

Now, we have

$$J_{15} \le c \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t'}^t \vartheta_{*\tau}^{(s)}\, d\tau|_{4,\Omega}^2 |\vartheta'_{\sigma\lambda t}|_{4,\Omega}^2}{|t-t'|^{1+\alpha}}\, dt\, dt'$$
$$\le c[\varepsilon_1^{1-\varkappa}\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c\varepsilon_1^{-\varkappa}\|\vartheta_*^{(s)}\|_{0,Q_{\lambda/2}}^2]\|\vartheta'_{\sigma\lambda t}\|_{\alpha,Q_{\lambda/2}}^2$$
$$\le \varepsilon\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha},$$

where $\varkappa = 3/(4\alpha)$, $\alpha > 3/4$ and $\varepsilon_1 = (\varepsilon/(cK))^{1/(1-\varkappa)}$.

We estimate J_{16} similarly to J_{12}.

Summarizing the above estimates we get

$$[K_2]_{\alpha/2,Q_{\lambda/2},t}^2 \le \varepsilon\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha}. \tag{3.18}$$

Inequalities (3.17) and (3.18) yield

$$(\|\eta' c_{v\vartheta}\vartheta_*^{(s)}\vartheta'_{\sigma\lambda t}\|_{Q_{\lambda/2}}^{(\alpha,\alpha/2)})^2 \le \varepsilon\|\vartheta_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha}. \tag{3.19}$$

Notice now that

$$
\begin{aligned}
&\operatorname{div}_u u_\lambda \operatorname{div}_u u - \operatorname{div}_{u'} u'_\lambda \operatorname{div}_{u'} u' \\
&\quad = \operatorname{div}_u u^{(s)} \operatorname{div}_u u + (\operatorname{div}_u - \operatorname{div}_{u'}) u'_\lambda \operatorname{div}_u u + \operatorname{div}_{u'} u'_\lambda (\operatorname{div}_u - \operatorname{div}_{u'}) u \\
&\qquad + \operatorname{div}_{u'} u'_\lambda \operatorname{div}_{u'} u_*^{(s)} \\
&\quad \equiv \sum_{i=3}^{6} K_i.
\end{aligned}
$$

Consider for example K_4. We have

$$
\begin{aligned}
&[K_4]^2_{\alpha,Q_{\lambda/2},\xi} \\
&\quad \le c \int\limits_{t_0+\lambda/2}^{T} \int\limits_{\Omega}\int\limits_{\Omega} \frac{|\int_{t-s}^t (u_\xi - u_{\xi'})\,d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&\qquad + c \int\limits_{t_0+\lambda/2}^{T} \int\limits_{\Omega}\int\limits_{\Omega} \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi} - u'_{\lambda\xi'}|^2 |u_\xi|^2}{|\xi-\xi'|^{3+2+\alpha}}\,dt\,d\xi\,d\xi' \\
&\qquad + c \int\limits_{t_0+\lambda/2}^{T} \int\limits_{\Omega}\int\limits_{\Omega} \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi}|^2 |\int_0^t (u_\xi - u_{\xi'})\,d\tau|^2 |u_\xi|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&\qquad + c \int\limits_{t_0+\lambda/2} \int\limits_{\Omega}\int\limits_{\Omega} \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi - u_{\xi'}|^2}{|\xi-\xi'|^{3+2\alpha}}\,dt\,d\xi\,d\xi' \\
&\quad \equiv \sum_{i=17}^{20} J_i.
\end{aligned}
$$

First, we estimate

$$
\begin{aligned}
J_{17} \le c \int\limits_{t_0+\lambda/2}^{T} |u'_{\lambda\xi}|^2_{\infty,\Omega} \bigg(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|\int_{t-s}^t (u_\xi - u_{\xi'})\,d\tau|^4}{|\xi-\xi'|^{3+4(1/4+\alpha-\delta)}}\,d\xi\,d\xi' \bigg)^{1/2} \\
\cdot \bigg(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u_\xi|^4}{|\xi-\xi'|^{2+2\delta}}\,d\xi\,d\xi' \bigg)^{1/2} dt,
\end{aligned}
$$

where $\delta > 0$ is so small that $2+2\delta < 3$. Using the interpolation inequality

$$
\Big\| \int\limits_{t-s}^{t} u_\xi\,d\tau \Big\|^2_{W_4^{1/4+\alpha-\delta}(\Omega)} \le \varepsilon^{1-\varkappa} \Big\| \int\limits_{t-s}^{t} u\,d\tau \Big\|^2_{2+\alpha,\Omega} + c\varepsilon^{-\varkappa} \Big\| \int\limits_{t-s}^{t} u\,d\tau \Big\|^2_{0,\Omega}
$$

with $\varkappa = (2+\alpha-\delta)/(2+\alpha)$ and the imbedding $W_2^\alpha(\Omega) \subset L_4(\Omega)$ (which both hold for $\alpha \ge 3/4$) and taking $\varepsilon = s$ we obtain

$$\begin{aligned} J_{17} &\le c \sup_{t_0+\lambda/2 \le t \le T} \|u\|^2_{1+\alpha,\Omega} \|u\|^2_{2+\alpha,Q_{\lambda/2}} \\ &\quad \cdot \Big(s^{2-\varkappa} \|u\|^2_{2+\alpha,Q_{\lambda/2}} + c s^{-\varkappa} s^2 \sup_{t_0+\lambda/2 \le t \le T} \|u\|^2_{0,\Omega}\Big) \\ &\le c(K)\overline{K} s^{1+\omega_1}, \end{aligned}$$

where $\omega_1 > 0$.

Next, we have

$$\begin{aligned} J_{18} \le c \int\limits_{t_0+\lambda/2}^{T} |u_\xi|^2_{\infty,\Omega} \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|u_\xi - u_{\xi'}|^4}{|\xi - \xi'|^{3+4(1/4+\alpha)}} \, d\xi \, d\xi' \bigg)^{1/2} \\ \cdot \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|\int_{t-s}^t u_\xi \, d\tau|^4}{|\xi - \xi'|^2} \, d\xi \, d\xi' \bigg)^{1/2} dt. \end{aligned}$$

Using the imbedding $\partial^\sigma_\xi W^{2+\alpha}_2(\Omega) \subset W^{1/4+\alpha}_4(\Omega)$ with $|\sigma| = 1$ and the interpolation inequality

$$\bigg| \int\limits_{t-s}^{t} u_\xi \, d\tau \bigg|^2_{4,\Omega} \le \varepsilon^{1-\varkappa} \bigg\| \int\limits_{t-s}^{t} u \, d\tau \bigg\|^2_{2+\alpha,\Omega} + c\varepsilon^{-\varkappa} \bigg\| \int\limits_{t-s}^{t} u \, d\tau \bigg\|^2_{0,\Omega}$$

(where $\varkappa = 7/(8+4\alpha)$, $\varepsilon = s$) we obtain as before

$$J_{18} \le c(K)\overline{K} s^{1+\omega_2},$$

where $\omega_2 > 0$. In the same way we estimate J_{20}.

Finally, we get

$$\begin{aligned} J_{19} &\le c \int\limits_{t_0+\lambda/2}^{T} \bigg| \int\limits_{t-s}^{t} u_\xi \, d\tau \bigg|^2_{\infty,\Omega} |u'_{\lambda\xi}|^2_{\infty,\Omega} \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|\int_0^t (u_\xi - u_{\xi'}) \, d\tau|^4}{|\xi - \xi'|^{3+4(1/4+\alpha)}} \, d\xi \, d\xi' \bigg)^{1/2} \\ &\quad \cdot \bigg(\int\limits_\Omega \int\limits_\Omega \frac{|u_\xi|^4}{|\xi - \xi'|^2} \, d\xi \, d\xi' \bigg)^{1/2} dt \\ &\le c(K)\overline{K} s^{1+\omega_3}, \qquad \omega_3 > 0. \end{aligned}$$

Taking into account the above considerations we obtain

$$[K_4]^2_{\alpha,Q_{\lambda/2},\xi} \le c(K)\overline{K} s^{1+\omega_4}, \tag{3.20}$$

where $\omega_4 > 0$.

Now, consider

$$\begin{aligned} &[K_4]^2_{\alpha/2,Q_{\lambda/2},t} \\ &\quad \le c \int\limits_\Omega \int\limits_{t_0+\lambda/2}^{T} \int\limits_{t_0+\lambda/2}^{T} \frac{|\int_{t-s}^t u_\xi \, d\tau - \int_{t-s-r}^t u_\xi \, d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi|^2}{|t - t'|^{1+\alpha}} \, d\xi \, dt \, dt' \end{aligned}$$

$$+ c\int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi}(t)-u'_{\lambda\xi}(t')|^2 |u_\xi|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'$$

$$+ c\int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi|^2 |\int_{t'}^t u_\xi\,d\tau|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'$$

$$+ c\int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t-s}^t u_\xi\,d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi(t)-u_\xi(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'$$

$$\equiv \sum_{i=21}^{24} J_i.$$

We estimate

$$\begin{aligned}
J_{21} &\le \int_\Omega \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{|\int_{t'}^t (u_\xi(\tau)-u_\xi(\tau-s))\,d\tau|^2 |u'_{\lambda\xi}|^2 |u_\xi|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\le c\int_\Omega \int_{t_0+\lambda/2}^T \int_0^T |u_\xi(\tau)-u_\xi(\tau-s)|^2\,d\tau\,|u'_{\lambda\xi}|^2 |u_\xi|^2\,d\xi\,dt \\
&\le c\int_{t_0+\lambda/2}^T |u_\xi^{(s)}|_{4,\Omega}^2\,dt \int_0^T |u_\xi|_{8,\Omega}^4\,dt \\
&\le c\|u\|_{2+\alpha,\Omega^T}^4 \Big(\varepsilon_1^{1-\varkappa}\|u_*^{(s)}\|_{2+\alpha,\Omega^T}^2 + c\varepsilon_1^{-\varkappa}s^{1+\alpha}\sup_{0<s<t_0}\int_{t_0+\lambda/2}^T \frac{|u_*^{(s)}|_{2,\Omega}^2}{s^{1+\alpha}}\,dt\Big) \\
&\le \varepsilon\|u_*^{(s)}\|_{2+\alpha,Q_{\lambda/2}}^2 + c(K)\overline{K}s^{1+\alpha},
\end{aligned}$$

where $\varkappa = 7/(8+4\alpha)$ and $\varepsilon_1 = (\varepsilon/(cK^2))^{1/(1-\varkappa)}$.

Next, we have

$$\begin{aligned}
J_{22} &\le c\int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \Big|\int_{t-s}^t u_\xi\,d\tau\Big|_{\infty,\Omega}^2 |u_\xi|_{4,\Omega}^2 \frac{|u'_{\lambda\xi}(t)-u'_{\lambda\xi}(t')|_{4,\Omega}^2}{|t-t'|^{1+\alpha}}\,dt\,dt' \\
&\le c\Big(\varepsilon_1^{1-\varkappa}s\int_{t_0+\lambda/2}^T \|u\|_{2+\alpha,\Omega}^2\,dt + c\varepsilon_1^{-\varkappa}s^2\sup_{t_0+\lambda/2\le t\le T}\|u\|_{0,\Omega}^2\Big) \\
&\quad\cdot \sup_{t_0+\lambda/2\le t\le T}|u_\xi|_{4,\Omega}^2 \int_{t_0+\lambda/2}^T \int_{t_0+\lambda/2}^T \frac{\|u(t)-u(t')\|_{2,\Omega}^2}{|t-t'|^{1+\alpha}}\,dt\,dt' \\
&\le c(K)\overline{K}s^{1+\omega_5},
\end{aligned}$$

where $\varkappa = 5/(4+2\alpha)$, $\omega_5 > 0$ and we have taken $\varepsilon_1 = s$.

We estimate J_{24} in the same way.

Finally,

$$J_{23} \le c \int_{t_0+\lambda/2}^{T} \Big| \int_{t-s}^{t} u_\xi \, d\tau \Big|^2_{\infty,\Omega} |u'_{\lambda\xi}|^2_{4,\Omega} |u_\xi|^2_{4,\Omega} \int_{t'}^{t} |u_\xi|^2_{\infty,\Omega} \, d\tau \, dt$$
$$\le c\overline{K} \int_{t_0+\lambda/2}^{T} \Big(\varepsilon_1^{1-\varkappa} \Big\| \int_{t-s}^{t} u \, d\tau \Big\|^2_{2+\alpha,\Omega} + c\varepsilon_1^{-\varkappa} \Big\| \int_{t-s}^{t} u \, d\tau \Big\|^2_{0,\Omega} \Big) |u_\xi|^4_{4,\Omega} \, dt$$
$$\le c(K)\overline{K} s^{1+\omega_6},$$

where $\varkappa = 5/(4+2\alpha)$, $\omega_6 > 0$ and we have taken $\varepsilon_1 = s$.

By the above calculations we get

$$[K_4]^2_{\alpha/2, Q_{\lambda/2}, t} \le \varepsilon \|u_*^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}} + c(K)\overline{K} s^{1+\omega_7}, \tag{3.21}$$

where $\omega_7 > 0$.

Hence, (3.20) and (3.21) yield

$$(\|(\mathrm{div}_u - \mathrm{div}_{u'}) u'_\lambda \, \mathrm{div}_u \, u\|^{(\alpha,\alpha/2)}_{Q_{\lambda/2}})^2 \le \varepsilon \|u_*^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}} + c(K)\overline{K} s^{1+\omega_7}, \tag{3.22}$$

where $\omega_7 > 0$.

The other terms on the right-hand side of (3.12) are estimated exactly in the same way and we obtain for them estimates similar to (3.16), (3.19) and (3.22).

This yields the estimate

$$\begin{aligned} (\|u^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2 + (\|\vartheta_\sigma^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2 \\ \le \varepsilon (\|u^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}} + \|u_*^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}} + \|\vartheta_\sigma^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}} \\ + \|\vartheta_*^{(s)}\|^2_{2+\alpha, Q_{\lambda/2}}) + c(K)\overline{K} s^{1+\varpi_1}, \end{aligned} \tag{3.23}$$

where $\varpi_1 > 0$ is a constant. Since $u^{(s)} = u_*^{(s)}$ and $\vartheta_\sigma^{(s)} = \vartheta_*^{(s)}$ on Q_λ inequality (3.23) yields

$$Y(\lambda) \le 2\varepsilon Y(\lambda/2) + c(K)\overline{K} s^{1+\varpi_1},$$

where

$$Y(\lambda) = (\|u^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2 + (\|u_*^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2$$
$$+ (\|\vartheta_\sigma^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2 + (\|\vartheta_*^{(s)}\|^{(\alpha+2,\alpha/2+1)}_{Q_\lambda})^2.$$

Therefore, after iteration we get

$$Y(\lambda) \le \frac{1}{1-2\varepsilon} c(K)\overline{K} s^{1+\varpi_1},$$

where we assume $\varepsilon < 1/2$.

Hence estimate (3.11) holds. ∎

Lemma 3.3 implies

THEOREM 3.4. *Let* $(u,\vartheta,\eta)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)\cap C(0,T;W_2^{1+\alpha}(\Omega))$ $(\alpha\in(3/4,1))$ *be the local solution of problem* (1.1). *Then for any* $0<t_0<T$ *and* $\lambda>0$ *we have* $u\in C(t_0+\alpha,T;W_2^{2+\alpha}(\Omega))$ *and*

$$\sup_{t_0+\lambda\le t\le T}\|u\|_{2+\alpha,\Omega}^2\le \overline{c}_1(K)\overline{K}, \tag{3.25}$$

$$\sup_{t_0+\lambda\le t\le T}\|\vartheta_\sigma\|_{2+\alpha,\Omega}^2\le \overline{c}_2(K)\overline{K}, \tag{3.26}$$

where $\overline{K}=\|u\|_{2+\alpha,\Omega^T}^2+\|\vartheta_\sigma\|_{2+\alpha,\Omega^T}^2$,

$$\begin{aligned}K=\overline{K}&+\sup_{0\le t\le T}\|u\|_{1+\alpha,\Omega}^2\\&+\sup_{0\le t\le T}\|\vartheta_\sigma\|_{1+\alpha,\Omega}^2+\|\eta_\sigma\|_{1+\alpha,\Omega^T}^2+\sup_{0\le t\le T}\|\eta_\sigma\|_{1+\alpha,\Omega}^2,\end{aligned}$$

and $\overline{c}_i(K)$ $(i=1,2)$ *are positive nondecreasing continuous functions of* K.

Proof. First, we have

$$\begin{aligned}&\|\,\|u(\cdot)\|_{2+\alpha,\Omega}\|_{B_{2,\infty}^{1/2+\overline{w}_1/2}(t_0+\lambda,T)} \qquad (3.27)\\
&\le\sup_{0\le s\le t_0}\int_{t_0+\lambda}^T\frac{|\,\|u(t)\|_{2+\alpha,\Omega}-\|u(t-s)\|_{2+\alpha,\Omega}|^2}{s^{1+\overline{w}_1}}\,dt+\int_{t_0+\lambda}^T\|u(t)\|_{2+\alpha,\Omega}^2\,dt\\
&\le\sup_{0\le s\le t_0}\int_{t_0+\lambda}^T\frac{\|u^{(s)}\|_{2+\alpha,\Omega}^2}{s^{1+\overline{w}_1}}\,dt+\int_{t_0+\lambda}^T\|u(t)\|_{2+\alpha,\Omega}^2\,dt.\end{aligned}$$

Now, the imbedding $B_{2,\infty}^{1/2+\overline{w}_1/2}(t_0+\lambda,T)\subset B_{\infty,\infty}^{\overline{w}_1/2}(t_0+\lambda,T)$ (which means that $\|u\|_{2+\alpha,\Omega}$ is continuous on $[t_0+\lambda,T]$) and inequalities (3.28) and (3.11) give (3.25).

Estimate (3.26) can be obtained in the same way as (3.25).

This completes the proof of the theorem. ∎

References

[1] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Theorems*, Nauka, Moscow, 1975 (in Russian); English transl.: Scripta Ser. in Math., Halsted Press, 1979.

[2] K. K. Golovkin, *On equivalent norms for fractional spaces*, Trudy Mat. Inst. Steklov. 66 (1962), 364–383 (in Russian).

[3] L. Landau and E. Lifschitz, *Hydrodynamics*, Nauka, Moscow, 1986 (in Russian).

[4] J. Serrin, *Mathematical principles of classical fluid mechanics*, in: Handbuch der Physik, Bd. VIII/1, Springer, Berlin, 1959.

[5] V. A. Solonnikov, *On an unsteady flow of a finite mass of a liquid bounded by a free surface*, Zap. Nauchn. Sem. LOMI 152 (1986), 137–157 (in Russian); English transl.: J. Soviet Math. 40 (1988), 672–686.

[6] V. A. Solonnikov and A. Tani, *Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid*, preprint, Paderborn University.

[7] —, —, *Free boundary problem for a viscous compressible flow with a surface tension*, in: *Constantin Carathéodory*: *An International Tribute*, T. M. Rassias (ed.), World Sci., 1991, 1270–1303.

[8] E. Zadrzyńska, *On nonstationary motion of a fixed mass of a general viscous compressible heat conducting capillary fluid bounded by a free boundary*, Appl. Math. (Warsaw) 25 (1999), 489–511.

[9] E. Zadrzyńska and W. M. Zajączkowski, *Local existence of solutions of a free boundary problem for equations of compressible viscous heat conducting capillary fluids*, J. Appl. Anal., to appear.

[10] W. M. Zajączkowski, *On nonstationary motion of a compressible barotropic viscous capillary fluid bounded by a free surface*, SIAM J. Math. Anal. 25 (1994), 1–84.

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