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NOTE ON THE ANOVA OF A COMPLETELY CONFOUNDED FACTORIAL EXPERIMENT

Abstract. The purpose of this paper is to present a modern approach to the analysis of variance (ANOVA) of disconnected resolvable group divisible partially balanced incomplete block (GDPBIB) designs with factorial structure and with some interaction effects completely confounded. A characterization of a factorial experiment with completely confounded interaction is given. The treatment effect estimators and some relations between the matrix $\mathbf{F}$ of the reduced normal equations and the information matrix $\mathbf{A}$ are given. Moreover the ANOVA of the sum of squares for adjusted treatment effects and the matrix $\mathbf{F}$ with its eigenvalues and orthonormal eigenvectors for the case of a completely confounded factorial experiment are presented. A special form of a generalized inverse (g-inverse) of $\mathbf{F}$ is introduced (Theorems 3.2.1–3.2.4). The corresponding numerical example has been worked out by Oktaba (1956) and Oktaba, Rejmak and Warteresiewicz (1956) by applying Galois fields and congruences.

1. Introduction. The literature on confounded factorial experiments is very rich. The classic papers are: Yates (1933, 1937) and Fisher (1942). Many statisticians have been interested in the problem of confounding: Bose and Kishen (1940), Finney (1945), Nair (1938), Pearce (1963, 1970, 1976, 1983), C.R. Rao (1951), Cochran and Cox (1950), Kempthorne (1952), Scheffé (1959), Oktaba (1956, 1970, 1971), Caliński and Kageyama (2000, 2003).

In this paper the modern approach to the ANOVA of GD-PIB designs with factorial structure and with some interaction effects completely confounded is presented. The results are given in the theorems. Some of them have special proofs.

The intra-block ANOVA of disconnected incomplete block designs can be based on the method M (Oktaba, 1971), or on the graphical method "O" of constructing independent contrasts (Oktaba, 1958). The corresponding numerical example has been worked out by applying Galois fields and congruences to a real field factorial experiment of the type $4 \times 4 \times 4$ with completely confounded interaction component (Oktaba, 1956; Oktaba et al. 1956).

$\mathbf{1}_k$ represents the $k \times 1$ column vector of 1's, $\mathbf{I}_k$ the $k \times k$ identity matrix, $\mathbf{J}_k = \mathbf{1}_k \mathbf{1}'_k$ the $k \times k$ matrix of 1's, $\mathbf{A}$ an $n \times m$ matrix. trace($\mathbf{A}$) stands for the trace of the square matrix $\mathbf{A}$; rank($\mathbf{A}$) is the number of non-zero eigenvalues of $\mathbf{A}$, and $|\mathbf{A}|$ denotes its determinant. $\mathbf{A}^-$ is a g-inverse of the matrix $\mathbf{A}$ iff $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$. The Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$.

2. The intrablock ANOVA of a block design $D(t, b, k, r)$, connected or disconnected. A block design is denoted as $D(t, b, k, r)$ if there are t treatments arranged in b blocks, each of size k, and every treatment is replicated r times.

A resolvable block design has $t = fk$ treatments (combinations of levels of factors) arranged in $b = fr$ blocks (f blocks forming one replication) (Street and Street, 1987, Chapter 8).

The intra-block linear model of $n = tr = kb$ observations obtained from the design $D(t, b, k, r)$ can be written as

$$(2.1) \quad y_{ijw} = \mu + \tau_i + \beta_j + e_{ijw}, \quad i = 1, \ldots, t; \; j = 1, \ldots, b; \; w = 1, \ldots, n_{ij},$$

where y_{ijw} is the wth observation from the ith treatment in the jth block, μ is the general mean, τ_i is the ith treatment effect, β_j is the jth block effect and e_{ijw}'s are residuals distributed identically and independently with

$$\mathrm{E}(e_{ijw}) = 0, \quad \mathrm{Var}(e_{ijk}) = \sigma^2, \quad \mathrm{Cov}(e_{ijw}, e_{i'j'w'}) = 0$$

for all $ijw \neq i'j'w'$.

The model (2.1) can be expressed as (Pearce, Caliński and Marshall, 1974)

$$(2.2) \qquad \underset{n1}{\mathbf{y}} = \mu\mathbf{1}_n + \underset{nt}{\boldsymbol{\Delta}'}\underset{t1}{\boldsymbol{\tau}} + \underset{nb}{\mathbf{D}'}\underset{b1}{\boldsymbol{\beta}} + \underset{n1}{\mathbf{e}} = \mathbf{X}\boldsymbol{\theta} + \mathbf{e},$$

where $\mathbf{e} = (e_1, \ldots, e_n)'$ is an n-component residual vector, having $\mathrm{E}(\mathbf{e}) = \mathbf{0}$ and $\mathrm{Cov}(\mathbf{e}) = \sigma^2\mathbf{I}_n$, $\mathbf{y} = (y_1, \ldots, y_n)'$ is an n-component vector of observations, $\mathbf{X} = (\mathbf{1}_n, \underset{nt}{\boldsymbol{\Delta}'}, \underset{nb}{\mathbf{D}'})$ is the $n \times (1 + t + b)$ design matrix partitioned into the n-component vector $\mathbf{1}_n$, an $n \times t$ design matrix for treatments $\boldsymbol{\Delta}'$ and an $n \times b$ design matrix for blocks $\mathbf{D}'$, and where $\boldsymbol{\theta} = (\mu, \boldsymbol{\tau}', \boldsymbol{\beta}')'$ is a $(1 + b + t)$-component vector with $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_t)'$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_b)'$. The elements of the $\boldsymbol{\Delta}$ and $\mathbf{D}$ matrices are 0 or 1. Let $\mathbf{y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\theta}, \sigma^2\mathbf{I}_n)$ (i.e., has an

n-variable normal distribution). We also have $\mathbf{T} = \boldsymbol{\Delta}\mathbf{y} = (T_1, \ldots, T_t)'$, the vector of treatment totals, $\mathbf{B} = \mathbf{D}\mathbf{y} = (B_1, \ldots, B_b)'$, the vector of block totals, and $\mathbf{1}_t'\mathbf{T} = \mathbf{1}_b'\mathbf{B} = \mathbf{1}_n'\mathbf{y} = G$, the grand total.

The $t \times b$ incidence matrix $\underset{tb}{\mathbf{N}} = \boldsymbol{\Delta}\mathbf{D}'$ has elements n_{ij} ($i = 1, \ldots, t;\ j = 1, \ldots, b$), where n_{ij} is the number of 1's in the jth block receiving the ith treatment.

Let

$$(2.3) \qquad \underset{nn}{\boldsymbol{\phi}} = \mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}, \quad \mathrm{rank}(\underset{tn}{\boldsymbol{\Delta}}\ \underset{nn}{\boldsymbol{\phi}}) = h < t.$$

The normal equations under the constraints

$$(2.4) \qquad \mathbf{1}_b'\boldsymbol{\beta} = 0 \quad \text{and} \quad \mathbf{1}_t'\boldsymbol{\tau} = 0$$

are

$$(2.5) \qquad \begin{cases} G = n\widehat{\mu}, \quad \underset{b1}{\mathbf{B}} = k\mathbf{1}_b\widehat{\mu} + k\widehat{\boldsymbol{\beta}} + \mathbf{N}'\widehat{\boldsymbol{\tau}}, \\[2mm] \underset{t1}{\mathbf{T}} = r\mathbf{1}_t\widehat{\mu} + \mathbf{N}\widehat{\boldsymbol{\beta}} + r\widehat{\boldsymbol{\tau}}. \end{cases}$$

The reduced normal equations for studying the treatment effects $\widehat{\boldsymbol{\tau}}$ are

$$(2.6) \qquad r\underset{tt}{\mathbf{F}}\,\widehat{\boldsymbol{\tau}} = \underset{t1}{\mathbf{Q}},$$

where

$$(2.7) \qquad \mathbf{F} = \frac{1}{r}\mathbf{A} = \mathbf{I}_t - \mathbf{M},$$

$$(2.8) \qquad \mathbf{A} = r\mathbf{F} = \boldsymbol{\Delta}\boldsymbol{\phi}\boldsymbol{\Delta}'$$

is the information matrix, $\mathbf{NN}'$ is the so-called concordance (John, 1980) or concurrence matrix (Pearce, 1963), $\mathbf{M} = \frac{1}{kr}\mathbf{NN}'$ is Jones' (1959) $\mathbf{M}$ matrix of order $t \times t$, and

$$(2.9) \qquad \underset{t1}{\mathbf{Q}} = \mathbf{T} - \frac{1}{k}\mathbf{NB} = \underset{tn}{\boldsymbol{\Delta}}\ \underset{nn}{\boldsymbol{\phi}}\ \underset{n1}{\mathbf{y}} = (Q_1, \ldots, Q_t)'$$

is the vector of adjusted treatment totals. The matrix $\boldsymbol{\phi}$ defined in (2.3) has the properties

$$(2.10) \qquad \boldsymbol{\phi}' = \boldsymbol{\phi}, \quad \boldsymbol{\phi}\boldsymbol{\phi} = \boldsymbol{\phi}, \quad \boldsymbol{\phi}\mathbf{1}_n = 0, \quad \boldsymbol{\phi}\mathbf{D}' = 0.$$

The residual sum of squares is given by

$$(2.11) \qquad SS_e = \mathbf{y}'\boldsymbol{\psi}\mathbf{y},$$

where $\boldsymbol{\psi} = \boldsymbol{\phi} - \boldsymbol{\phi}\boldsymbol{\Delta}'\mathbf{A}^-\boldsymbol{\Delta}\boldsymbol{\phi}$ (see Caliński and Kageyama, 2000, p. 65).

The term

$$(2.12) \qquad SS_{\mathrm{tr\,adj}} = \frac{1}{r}\mathbf{Q}'\mathbf{F}^-\mathbf{Q}$$

represents, in the intra-block ANOVA, the treatment sum of squares adjusted for blocks.

DEFINITION 2.1. An incomplete block design $D(t, b, k, r)$ is *disconnected* if rank$(\mathbf{F}) = t - f = h$, where $f > 1$.

If $f = 1$ the design is *connected*. Then all the elementary contrasts are estimable in the intra-block analysis.

The orthogonal and non-orthogonal experimental designs are discussed in (Oktaba, 2002, pp. 87–89).

The condition $\mathbf{N} = \frac{1}{n}\mathbf{r}\mathbf{k}'$ for orthogonality between blocks and treatments of a block design is given by Caliński, Ceranka and Mejza (1979, pp. 178–189) who proposed a general definition of an orthogonal block design. Caliński and Kageyama (2000, Sections 2.2, 2.3 and 3.6; 2003, Section 7.2) also discuss the orthogonal designs.

3. Factorial disconnected incomplete block design with some of interaction/main effects completely confounded with blocks

3.1. *Some methods of analysis.* (i) The general method of statistical interpretation of the data obtained in incomplete block designs with effects of interaction (interaction components) completely confounded with blocks has been described by many authors. It is based on congruences and Galois fields. The theory was applied to a block design of the type $4 \times 4 \times 4$ (3 factors at four levels each) with the effects of three interaction components completely confounded with blocks. The factorial experiment was laid down at the Experimental Agricultural Station, Felin, Lublin University of Agriculture (Oktaba, 1956; Oktaba *et al.*, 1956).

(ii) The same experiment can be analysed by a method using the eigenvalues and the eigenvectors of the $\mathbf{F}$ matrix, and the theory of block designs with factorial structure (Nigam *et al.*, 1988, pp. 64–92; Tocher, 1952; Pearce *et al.*, 1974; Caliński, 1983). Estimations of the treatment effects and the sum of squares for adjusted treatments are expressed by formulas including the matrix $\mathbf{F}$. This sum of squares can be obtained by applying the graphical method "O" (Oktaba, 1971). The method M (Oktaba, 1971) is useful in the intra-block ANOVA of a disconnected block design.

3.2. *Characterization of a factorial experiment with interaction/main effects completely confounded.* Let us consider a block design as in Section 2. It is possible to have a one-to-one correspondence between treatments of a block design and the treatment combinations of a factorial experiment with r blocks involving the same set of treatment combinations. It is assumed that some interaction/main effects are completely confounded with blocks, so that $t = fk$ (cf. Sec. 2).

A factorial experiment with some confounded interaction/main effects is a simple design, so there is no problem with the ANOVA. The design is resolvable, because $b = rf$. The $t = fk$ treatment combinations satisfy

the following association scheme: the t treatment combinations are divided into f groups (arranged in different blocks) each of size k such that any two combinations of the same group are first associates (they occur together in $\lambda_1 = r$ blocks) and those of different groups are second associates (they occur together in $\lambda_2 = 0$ blocks).

The eigenvalues of $\mathbf{M}$ are:

$$(3.2.1) \quad \begin{cases} \mu_0 = 1, \\ \mu_1 = \dfrac{\theta_1}{rk} = 0 & \text{with multiplicity } \varrho_1 = t - f, \\ \mu_2 = \dfrac{\theta_2}{rk} = \dfrac{rk}{rk} = 1 & \text{with multiplicity } \varrho_2 = f - 1, \end{cases}$$

(Bose and Connor, 1952; Nigam *et al.*, 1988, pp. 33, 86–87; Jones, 1959, p. 176).

From (3.2.1) it follows that the losses of unconfounded effects are equal to zero. There are $h = (t-1) - (f-1) = t - f$ unconfounded (full efficiency) degrees of freedom which correspond to the eigenvalue 0 of the matrix $\mathbf{M}$ with multiplicity $\varrho_1 = h = t - f$.

Thus, we have only $h = t - f$ $(f > 1)$ linearly independent contrasts estimable in the intra-block analysis (full efficiency) and not $t - 1$ as in the case of a connected design.

THEOREM 3.2.1. *In the disconnected design $D(t, b, k, r)$ with a completely confounded component interaction (main effect) the treatment effects* $\widehat{\boldsymbol{\tau}}_{t1}$ *are*

$$(3.2.2) \quad \widehat{\boldsymbol{\tau}} = \frac{1}{r}\mathbf{F}^-\mathbf{Q} = \boldsymbol{\Omega}_p\mathbf{Q} = \mathbf{A}^-\mathbf{Q},$$

where

$$(3.2.3) \quad \begin{cases} \mathbf{F} = \mathbf{I}_f \otimes \mathbf{A}_1 = \mathbf{F}^-, \\ \operatorname{rank}(\mathbf{F}) = t - f = h, \\ \operatorname{rank}(\mathbf{A}_1) = k - 1, \end{cases}$$

$\mathbf{Q}$ *is as in (2.9) and* $\boldsymbol{\Omega}_p\mathbf{A}^- = \frac{1}{r}\mathbf{F}^-$, $\mathbf{A}_1 = \mathbf{I}_k - \frac{1}{k}\mathbf{J}_k$.

Proof. The incidence matrix $\mathbf{N}$ is

$$(3.2.4) \quad \underset{tb}{\mathbf{N}} = \underbrace{[\underset{tf}{\mathbf{L}}, \ldots, \underset{tf}{\mathbf{L}}]}_{r \text{ times}}, \quad \underset{tf}{\mathbf{L}} = \mathbf{I}_f \otimes \mathbf{1}_k.$$

Then

$$\mathbf{N}\mathbf{N}' = [\mathbf{L}, \ldots, \mathbf{L}]\begin{bmatrix} \mathbf{L}' \\ \vdots \\ \mathbf{L}' \end{bmatrix} = r\mathbf{L}\mathbf{L}' = r(\mathbf{I}_f \otimes \mathbf{J}_k) = \mathbf{I}_f \otimes r\mathbf{J}_k.$$

Thus we have

(3.2.5) $\qquad \mathbf{M} = \mathbf{M}' = \dfrac{1}{rk}\,\mathbf{N}\mathbf{N}' = \dfrac{1}{rk}\,(\mathbf{I}_f \otimes r\mathbf{J}_k) = \dfrac{1}{k}\,(\mathbf{I}_f \otimes \mathbf{J}_k)$

and

$$\operatorname{rank}\mathbf{M} = \operatorname{rank}(\mathbf{I}_f \otimes \mathbf{J}_k) = \operatorname{rank}\mathbf{I}_f \cdot \operatorname{rank}\mathbf{J}_k = f \cdot 1 = f.$$

Since $\mathbf{F} = \mathbf{I}_t - \mathbf{M} = \mathbf{I}_t - \frac{1}{rk}\mathbf{N}\mathbf{N}'$ and $t = fk$ we obtain

$$\mathbf{F} = \mathbf{I}_f \otimes \mathbf{A}_1, \quad \text{where} \quad \mathbf{A}_1 = \mathbf{I}_k - \dfrac{1}{k}\mathbf{J}_k.$$

Note that as $\mathbf{F1} = 0$, the matrix $\mathbf{F}$ is singular.

By Lemma 4.3 (Appendix) $\mathbf{A}_1 = \mathbf{A}_1^2 = \mathbf{A}_1^-$, so

$$\mathbf{F}^- = (\mathbf{I}_f \otimes \mathbf{A}_1)^- = \mathbf{I}_f^- \otimes \mathbf{A}_1^- = \mathbf{I}_f \otimes \mathbf{A}_1 = \mathbf{F}$$

and by (2.8),

$$\mathbf{A}^- = (r\mathbf{F})^- = \dfrac{1}{r}\,\mathbf{F}^- = \dfrac{1}{r}\,(\mathbf{I}_f \otimes \mathbf{A}_1) = \boldsymbol{\Omega}_p,$$

where

$$\boldsymbol{\Omega}_p = \dfrac{1}{r}\,\mathbf{F} = \dfrac{1}{r}\,\mathbf{F}^-,$$

$$\operatorname{rank}(\mathbf{A}^-) = \operatorname{rank}(\mathbf{F}^-) = \operatorname{rank}(\mathbf{F}) = \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{I}_f) \cdot \operatorname{rank}(\mathbf{A}_1)$$

$$= f \cdot \operatorname{rank}\left(\mathbf{I}_k - \dfrac{1}{k}\,\mathbf{J}_k\right) = f(k-1) = fk - f = t - f = h \quad \text{(say)},$$

$$\operatorname{rank}(\mathbf{A}_1) = \operatorname{rank}\left(\mathbf{I}_k - \dfrac{1}{k}\,\mathbf{J}_k\right) = k - 1 \quad \text{[cf. (4.3), Lemma 4.2].} \ \blacksquare$$

Evidently the vector $\boldsymbol{\tau}$ of treatment parameters is not estimable in the intra-block ANOVA. Of course, some linear functions of $\boldsymbol{\tau}$, say $v'\boldsymbol{\tau}$, are estimable (see, e.g., Theorems 4.1 and 4.2 in Caliński, 1983). Note that $\widehat{\boldsymbol{\tau}}$ is simply a solution, nonunique, of the equation (2.6).

THEOREM 3.2.2. *The sum of squares for adjusted treatments $SS_{\mathrm{tr\,adj}}$ in the intra-block ANOVA of a disconnected design $D(t, b, k, r)$ with a component interaction/main effect completely confounded can be expressed in the form*

(3.2.6) $\qquad SS_{\mathrm{tr\,adj}} = \mathbf{Q}'\boldsymbol{\Omega}_p\mathbf{Q} = \dfrac{1}{r}\,\mathbf{Q}'\mathbf{Q} = \dfrac{1}{r}\displaystyle\sum_{i=1}^{t} T_i^2 - \dfrac{1}{rk}\sum_{j=1}^{f} H_j^2$

with $\operatorname{rank}(\mathbf{F}) = t - f = h$ degrees of freedom, where

$$
(3.2.7) \quad
\begin{cases}
\underset{tt}{\boldsymbol{\Omega}_p} = \dfrac{1}{r} \underset{tt}{\mathbf{F}} = \dfrac{1}{r} \left(\mathbf{I}_f \otimes \underset{kk}{\mathbf{A}_1} \right), \\[2ex]
\underset{t1}{\mathbf{Q}} = [\underset{1k}{\mathbf{Q}'_I}, \ldots, \underset{1k}{\mathbf{Q}'_f}] = \underset{t1}{\mathbf{T}} - \dfrac{1}{k} \underset{tb}{\mathbf{N}} \underset{b1}{\mathbf{B}} = \underset{t1}{\mathbf{T}} - \underset{t1}{\mathbf{H}}, \\[2ex]
\underset{1k}{\mathbf{Q}'_I} = [\mathbf{Q}_1, \ldots, \mathbf{Q}_k], \\[2ex]
\mathbf{Q}'_{II} = [\mathbf{Q}_{k+1}, \ldots, \mathbf{Q}_{2k}], \ \ldots, \ \mathbf{Q}'_f = [\mathbf{Q}_{k(f-1)+1}, \ldots, \mathbf{Q}_{fk}], \\[2ex]
H_I = \displaystyle\sum_{i=1}^{k} T_i, \quad H_{II} = \sum_{i=k+1}^{2k} T_i, \ \ldots, \ H_f = \sum_{i=k(f-1)+1}^{fk} T_i, \\[3ex]
\mathbf{H} = \dfrac{1}{k} \underset{tb}{\mathbf{N}} \underset{b1}{\mathbf{B}} = \dfrac{1}{k} \begin{bmatrix} H_I \cdot \mathbf{1}_k \\ \ldots \\ H_f \cdot \mathbf{1}_k \end{bmatrix}, \quad t = fk, \\[3ex]
\underset{t1}{\mathbf{T}} = [\underset{1k}{T'_I}, \ldots, T'_f]' \\[2ex]
\quad = [T_1, \ldots, T_k \vdots T_{k+1}, \ldots, T_{2k} \vdots \ldots \vdots T_{k(f-1)+1}, \ldots, T_t]'.
\end{cases}
$$

Proof. By (2.12) and (3.2.3) we get $SS_{\mathrm{tr\,adj}} = \frac{1}{r} \mathbf{Q}' \mathbf{F} \mathbf{Q}$. From

$$
\underset{k1}{\mathbf{Q}_j} = \underset{k1}{\mathbf{T}_j} - \dfrac{1}{k} H_j \cdot \mathbf{1}_k \quad (j = I, II, \ldots, f)
$$

and (3.2.7) we have

$$
SS_{\mathrm{tr\,adj}} = \underset{1t}{\mathbf{Q}'} \underset{tt}{\boldsymbol{\Omega}_p} \underset{t1}{\mathbf{Q}} = \dfrac{1}{r} [\underset{1k}{\mathbf{Q}'_I}, \ldots, \underset{1k}{\mathbf{Q}'_f}] (\mathbf{I}_f \otimes \underset{kk}{\mathbf{A}_1}) \begin{bmatrix} \mathbf{Q}_I \\ \vdots \\ \mathbf{Q}_f \end{bmatrix}
$$

$$
= \dfrac{1}{r} [\underset{1k}{\mathbf{Q}'_I}, \ldots, \underset{1k}{\mathbf{Q}'_f}] \begin{bmatrix} \mathbf{A}_1 & & 0 \\ & \ddots & \\ 0 & & \mathbf{A}_1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}_I \\ \vdots \\ \mathbf{Q}_f \end{bmatrix}
$$

$$
= \dfrac{1}{r} \sum_{i=1}^{f} \underset{1k}{\mathbf{Q}'_i} \underset{kk}{\mathbf{A}_1} \underset{k1}{\mathbf{Q}_i}.
$$

Since $\mathbf{Q}'_j \mathbf{J}_k = 0$ $(j = I, \ldots, f)$, we get

$$
\dfrac{1}{r} \mathbf{Q}'_I \mathbf{A}_1 \mathbf{Q}_I = \dfrac{1}{r} \mathbf{Q}'_I \left(\mathbf{I}_k - \dfrac{1}{k} \mathbf{J}_k \right) \mathbf{Q}_I = \dfrac{1}{r} \mathbf{Q}'_I \mathbf{Q}_I = \dfrac{1}{r} \sum_{i=1}^{k} \mathbf{Q}_i^2
$$

$$
= \dfrac{1}{r} \left(\mathbf{T}_I - \dfrac{1}{k} H_I \cdot \mathbf{1}_k \right)' \left(\mathbf{T}_I - \dfrac{1}{k} H_I \cdot \mathbf{1}_k \right) = \dfrac{1}{r} \sum_{i=1}^{k} T_i^2 - \dfrac{1}{rk} H_1^2.
$$

Analogously we obtain

$$\frac{1}{r}\,\mathbf{Q}'_{II}\mathbf{A}_1\mathbf{Q}_{II} = \frac{1}{r}\sum_{i=k+1}^{2k} T_i^2 - \frac{1}{rk}\,H_{II}^2,\ \ldots,$$

$$\frac{1}{r}\,\mathbf{Q}'_f\mathbf{A}_1\mathbf{Q}_f = \frac{1}{r}\sum_{i=(f-1)k+1}^{fk} T_i^2 - \frac{1}{rk}\,H_f^2.$$

Then

$$SS_{\mathrm{tr\,adj}} = \frac{1}{r}\sum_{j=1}^{t} T_i^2 - \frac{1}{rk}\sum_{j=1}^{f} H_j^2.\ \blacksquare$$

Note that the proof of Theorem 3.2.2 can be shortened if we use $\mathbf{F}^- = \mathbf{I}_t$. Moreover: if we take as $\mathbf{F}^-$ the matrix $\mathbf{F}$ or $\mathbf{I}_t$ we obtain the same, because $\mathbf{FQ} = \mathbf{Q}$, giving $\mathbf{Q}'\mathbf{FQ} = \mathbf{Q}'\mathbf{Q}$.

THEOREM 3.2.3. *Under the assumptions of Theorem* 3.2.1 *the following form of g-inverse* $\mathbf{F}^-$ *can be obtained*:

(3.2.8)
$$\mathop{\mathbf{F}^-}_{tt} = \begin{bmatrix} \mathop{\mathbf{C}^{-1}}_{t=f,t-f} & \mathop{\mathbf{0}}_{t-f,f} \\ \mathop{\mathbf{0}}_{f,t-f} & \mathop{\mathbf{0}}_{ff} \end{bmatrix},$$

where $t - f = h$ *and*

(3.2.9)
$$\begin{cases} \mathbf{C}^{-1} = \mathbf{I}_f \otimes (\mathbf{I}_{k-1} + \mathbf{J}_{k-1}) = \mathbf{I}_f \otimes (\mathbf{A}_1^*)^{-1}, \\ \mathbf{A}_1^* = \mathbf{I}_{k-1} - \dfrac{1}{k}\,\mathbf{J}_{k-1}. \end{cases}$$

Proof. By (3.2.3) the matrix $\mathbf{F}$ can be represented in the form

(3.2.10) $$\mathbf{F} = \mathbf{I}_f \otimes \mathbf{A}_1.$$

Let us note that the diagonal and off-diagonal elements of the matrix $\mathbf{A}_1$ are equal to $1 - 1/k$ and $-1/k$, respectively.

Let us omit in $\mathbf{A}_1$ the last row and the last column to get the matrix

(3.2.11) $$\mathbf{A}_1^* = \mathbf{I}_{k-1} - \frac{1}{k}\,\mathbf{J}_{k-1}.$$

We have

(3.2.12) $$\mathbf{C} = \mathbf{I}_f \otimes \mathbf{A}_1^*$$

and $\mathrm{rank}(\mathbf{C}) = \mathrm{rank}(\mathbf{I}_f)\,\mathrm{rank}(\mathbf{A}_1^*) = f \cdot (k - 1) = fk - f = t - f = h$. Since $\mathrm{rank}(\mathbf{F}) = h$ we can apply Lemma 4.1. Then

$$\mathbf{C}^{-1} = (\mathbf{I}_f \otimes \mathbf{A}_1^*)^{-1} = \mathbf{I}_f \otimes (\mathbf{A}_1^*)^{-1} = \mathbf{I}_f \otimes (\mathbf{I}_{k-1} + \mathbf{J}_{k-1})$$

since by Lemma 4.2 we have

$$(\mathbf{A}_1^*)^{-1} = \left(\mathbf{I}_{k-1} - \frac{1}{k}\,\mathbf{J}_{k-1}\right)^{-1} = \mathbf{I}_{k-1} + \mathbf{J}_{k-1}.\ \blacksquare$$

REMARK 3.2.1. By using (3.2.8) we get the following formula for $SS_{\text{tr adj}}$:

$$(3.2.13) \quad SS_{\text{tr adj}} = \mathbf{Q}'\boldsymbol{\Omega}_p\mathbf{Q} = \frac{1}{r}\mathop{\mathbf{Q}'}_{1b}\begin{bmatrix} \mathbf{I}_f \otimes (\mathbf{I}_{k-1} + \mathbf{J}_{k-1}) & \mathop{\mathbf{0}}_{t-f,f} \\ \mathop{\mathbf{0}}_{f,t-f} & \mathop{\mathbf{0}}_{ff} \end{bmatrix} \mathbf{Q}$$

$$= \frac{1}{r}\mathop{\mathbf{Q}'_1}_{1,t-f}\left[\mathbf{I}_f \otimes (\mathbf{I}_{k-1} + \mathbf{J}_{k-1})\right]\mathop{\mathbf{Q}_1}_{t-f,1},$$

where

$$\mathop{\mathbf{Q}}_{t1} = \begin{bmatrix} \mathop{\mathbf{Q}_1}_{t-f,1} \\ \mathop{\mathbf{Q}_2}_{f,1} \end{bmatrix},$$

$\mathbf{Q}_1$ corresponds to treatments except those with numbers $k, 2k, \ldots, fk$ in $\mathbf{Q}$ and

$$\boldsymbol{\Omega}_p = \frac{1}{r}\mathbf{F}^- = \frac{1}{r}\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} \mathbf{I}_f \otimes (\mathbf{I}_{k-1} + \mathbf{J}_{k-1}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

THEOREM 3.2.2. *Under the assumptions of Theorem 3.2.1 the treatment sum of squares adjusted for blocks* $SS_{\text{tr adj}}$ *can be represented in the following special form:*

$$(3.2.14) \qquad SS_{\text{tr adj}} = \frac{1}{r}\mathbf{Q}'\mathbf{F}^-\mathbf{Q} = \frac{1}{r}\mathop{\mathbf{Q}'}_{1t}\sum_{i=1}^{h}\mathop{\mathbf{p}_i}_{t1}\mathop{\mathbf{p}'_i}_{1t}\mathop{\mathbf{Q}}_{t1},$$

where $\mathop{\mathbf{p}_1}_{t1}, \ldots, \mathop{\mathbf{p}_h}_{t1}$ *are h orthonormal eigenvectors of the matrix* $\mathbf{F}$ *corresponding to the eigenvalue 1, with multiplicity h, of the matrix* $\mathop{\mathbf{F}}_{tt}$, *i.e., t-component vectors satisfying the conditions*

$$(3.2.15) \quad \mathbf{F}\mathbf{p}_i = \mathbf{p}_i, \quad \mathbf{p}'_i\mathbf{p}_{i'} = \delta_{ii'} \quad (\delta_{ij} \text{ is the Kronecker delta})$$
$$\textit{for } i, i' = 1, 2, \ldots, h, \textit{ where } h = t - f.$$

Proof. Since $\mathbf{F}^- = \mathbf{F}$ by using Theorem 3.2.2, $SS_{\text{tr adj}} = \mathbf{Q}'\boldsymbol{\Omega}_p\boldsymbol{\Omega} = \frac{1}{r}\mathbf{Q}'\mathbf{F}^-\mathbf{Q} = \frac{1}{r}\mathbf{Q}'\mathbf{F}\mathbf{Q}$. The spectral decomposition of the matrix $\mathbf{F}$ in the case of the design with completely confounded interaction because of eigenvalues $\varepsilon_1 = \cdots = \varepsilon_h = 1$ is given by $\mathbf{F} = \mathbf{F}' = \sum_{i=1}^{h}\varepsilon_i\mathbf{p}_i\mathbf{p}'_i = \sum_{i=1}^{h}\mathbf{p}_i\mathbf{p}'_i$, where $\mathbf{F}\mathbf{p}_i = \varepsilon_i\mathbf{p}_i$, $\mathbf{p}'_i\mathbf{p}_i = 1$, $\mathbf{p}'_i\mathbf{p}_j = 0$ $(i \neq j)$.

Now we prove (3.2.15). Since $\mathop{\mathbf{F}}_{tt}\mathop{\mathbf{p}_i}_{t1} = \varepsilon_i\mathop{\mathbf{p}_i}_{t1}$, with eigenvalues $\varepsilon_i = 1$, $i = 1, \ldots, h$, we have

$$\mathbf{F}\mathbf{p}_i = \mathbf{p}_i, \quad \mathop{(\mathbf{I}_t - \mathbf{F})}_{tt}\mathop{\mathbf{p}_i}_{t1} = \mathop{\mathbf{0}}_{t1}$$

so that

$$(3.2.16) \qquad \mathop{\mathbf{M}}_{tt}\mathop{\mathbf{p}_i}_{t1} = \mathop{\mathbf{0}}_{t1}.$$

Since $\mathbf{M}$ is as in (3.2.5) we get

$$\frac{1}{rk}\left(\mathbf{I}_f \otimes r\mathbf{J}_k\right)\mathop{\mathbf{p}_i}_{t1} = \mathop{\mathbf{0}}_{t1}, \quad \text{i.e.} \quad \left(\mathbf{I}_f \otimes \mathbf{J}_k\right)\mathop{\mathbf{p}_i}_{t1} = \mathop{\mathbf{0}}_{t1},$$

and

$$
\underbrace{\begin{bmatrix} \mathbf{J}_k & & 0 \\ & \ddots & \\ 0 & & \mathbf{J}_k \end{bmatrix}}_{f \text{ columns}}
\begin{bmatrix} \mathbf{p}_{Ii} \\ {\scriptstyle k1} \\ \mathbf{p}_{IIi} \\ {\scriptstyle k1} \\ \cdots \\ \mathbf{p}_{fi} \\ {\scriptstyle k1} \end{bmatrix}
= \underset{t1}{\mathbf{0}}, \quad \underset{t1}{\mathbf{p}_i} = (\underbrace{\underset{k1}{\mathbf{p}_{Ii}}, \ldots, \underset{k1}{\mathbf{p}_{fi}}}_{f \text{ times}})' = (\underset{1k}{\mathbf{p}'_{Ii}}, \ldots, \underset{1k}{\mathbf{p}'_{fi}})',
$$

with

$$
(3.2.17) \quad \underset{k1}{\mathbf{p}_{Ii}} = \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ki} \end{bmatrix}, \quad \underset{k1}{\mathbf{p}_{IIi}} = \begin{bmatrix} p_{(k+1),i} \\ \vdots \\ p_{2k,i} \end{bmatrix}, \ldots, \underset{k1}{\mathbf{p}_{fi}} = \begin{bmatrix} p_{k(f-1)+1,i} \\ \vdots \\ p_{t,i} \end{bmatrix}
$$

for $i = 1, \ldots, h$ and $t = fk$. Hence we obtain (3.2.16). ∎

To solve equation (3.2.16) it is enough to find h orthonormal eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_h$ of the matrix $\mathbf{F}$.

REMARK 3.2.2. Formula (3.2.14) expresses the partition of the $SS_{\text{tr adj}}$ in ANOVA into h sums of squares with one degree of freedom each.

The treatment contrasts $\sum_{i=1}^{k} l_{ijm} Q_{ijm}$, $\sum_{i=1}^{k} l_{ijm} = 0$ $(j = 1, \ldots, k-1;$ $m = 1, \ldots, f)$, $t = fk$, have one degree of freedom each as in the following formula:

$$
(3.2.18) \qquad SS_{\text{tr adj}} = \sum_{m=1}^{f} \sum_{j=1}^{k-1} \frac{(\sum_{i=1}^{k} l_{ijm} Q_{ijm})^2}{r \sum_{i=1}^{k} l_{ijm}^2}
$$

with $h = f(k-1) = t - f$ degrees of freedom, where Q_{ijm} are treatment adjusted totals; the indices i, j, m in l_{ijm} denote: i—number of coefficients in the contrast, j—number of contrasts in the block, and m—number of blocks.

In intra-block ANOVA we have $\sum_{j=1}^{k} l_{jm} Q_{jm} = \sum_{j=1}^{k} l_{jm}(T_{jm} - C_m) = \sum_{j=1}^{k} l_{jm} T_{jm}$, since $Q_{jm} = T_{jm} - C_m$, where $C_m = \text{const}$ within blocks for $m = 1, \ldots, f$.

It may be instructive to write

$$
SS_{\text{tr adj}} = \frac{1}{r} \sum_{i=1}^{h} (\mathbf{p}'_i \mathbf{Q})^2 = r \sum_{i=1}^{h} (\widehat{\mathbf{p}'_i \boldsymbol{\tau}})^2,
$$

where $\widehat{\mathbf{p}'_i \boldsymbol{\tau}}$ is the best linear unbiased estimator (BLUE) of the contrast $\mathbf{p}'_i \boldsymbol{\tau}$.

The analysis of variance for testing treatment effects may be given in the following table:

Table 1. ANOVA of the completely confounded component of three-factor interaction of the factorial experiment

Source	D.F.	SS	M.S.
Block (ignoring treatments)	$rf - 1$	SS_{bl}	—
Treatments	$t - f$ $= \text{rank}\,\mathbf{F}$ $= h$	$SS_{\text{tr adj}}$ $= \frac{1}{r}\,\underset{1t}{\mathbf{Q}'}\,\underset{}{\mathbf{F}}\,\underset{t1}{\mathbf{Q}}$ $= \frac{1}{r}\,\underset{1t}{\mathbf{Q}'}\,\sum_{i=1}^{h}\,\underset{t1}{\mathbf{p}_i}\,\underset{1t}{\mathbf{p}_i'}\,\underset{t1}{\mathbf{Q}}$	$\frac{SS_{\text{tr adj}}}{h}$
Residual	$f(k-1)(r-1) = \nu_e$	$SS_e = \mathbf{y}'\psi\mathbf{y}$	$\frac{SS_e}{\nu_e}$
Total	$n - 1$	$SS_y = \sum_{i=1}^{n} y_i^2 - C_y$	—

Here $SS_{\text{bl}} = \frac{1}{k}\sum_{j=1}^{b} B_j^2 - C_y$, $C_y = \frac{1}{n}(\sum_{i=1}^{n} y_i)^2$. $SS_{\text{tr adj}}$ can be written as $\frac{1}{r}\sum_{i=1}^{h}(\mathbf{p}_i'\mathbf{Q})^2$.

Note that $SS_e = \mathbf{y}'\phi\mathbf{y} - SS_{\text{tr adj}} = \mathbf{y}'\psi\mathbf{y}$, where $\psi = \phi - \phi\Delta'\mathbf{A}^-\Delta\phi$ (see, e.g., Caliński and Kageyama, 2000, p. 65). The test function for the hypothesis $\tau_1 = \tau_2 = \cdots = \tau_t$ is $F = \frac{SS_{\text{tr adj}}}{h} : \frac{SS_e}{\nu_e}$ with h and ν_e degrees of freedom. In the case of $D(t, b, k, r) = D(4^3, 12, 16, 3)$ and $f = 4$ we have $h = 4^3 - 4 = 60$ and $\nu_e = f(k-1)(r-1) = 4(16-1)(3-1) = 120$ degrees of freedom. The total number of observations is $n = rt = 3 \cdot 4^3 = 192$ (number of plots in the field experiment, Oktaba, 1956; Oktaba *et al.*, 1956). Further remarks are given in Sections 3.3 and 3.4.

Table 1 is general for any case of an orthogonal disconnected block design, where $f - 1$ contrasts are completely confounded with blocks (not necessarily contrasts concerning interaction effects).

3.3. *The graphical method "O" of constructing sets of orthonormal eigenvectors of the matrix* $\mathbf{F}$ *in the disconnected fixed design* $D(t, b, k, r)$ *with completely confounded interaction/main effect.* The method of constructing sets of mutually orthogonal contrasts which we called the graphical method "O" was given by Oktaba (1958). It is based upon the theorem of number theory concerning the systematic expansion of natural numbers to any natural numerical base (Sierpiński, 1925). This is the method of linking t points to the base (natural number) $a \geq 2$. The graphical configuration obtained from performing conjunctions is uniquely determined by the expansion of natural numbers. This method can be used in ANOVA and specially in ANOVA of a disconnected fixed design $D(t, b, k, r)$ with completely confounded interaction. This is stated in Theorem 3.3.1. It also presents constructing an orthonormal matrix $\mathbf{F}$ which can be found directly from the configuration of conjunctions in the graph.

THEOREM 3.3.1. *Under the assumptions of Theorem 3.2.1 the sum* $\sum_{i=1}^{h} \mathbf{p}_i\mathbf{p}_i'$, *which is as in formula* (3.2.14) *can be found by the graphical method "O".*

Proof. From Theorem 3.2.4 by (3.2.18), (3.2.10) and (3.2.11) we can use the graphical method "O" with respect to each of the f blocks of size k, where $t = fk$, $h = t - f = \text{rank}(\mathbf{F}^-)$. For each of the f blocks we get $k-1$ independent contrasts, so we obtain $h = f(k-1) = t - f$ orthonormal eigenvectors corresponding to h eigenvalues $\varepsilon = 1$ with multiplicity h. These eigenvectors fulfill conditions (3.2.15), as it should be. The same graph can be constructed for each of the f incomplete blocks. ∎

REMARK 3.3.1. It is evident that the graphical method can be replaced by any other method. But this one seems to be very simple and natural. The choice of method depends on the problem in which the experimenter is interested.

4. Appendix. Some matrix results. The main results required for this paper are given in this Appendix. They are stated without proof as they are straightforward and can be found in most books on statistics (e.g. Rao, 1973).

LEMMA 4.1. *If* $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{C}) = r$ *and* $\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{bmatrix}$, *then*

$$(4.1) \qquad \mathbf{A}^- = \begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^-)$ *and* $\mathbf{A}^-\mathbf{A}\mathbf{A}^- = \mathbf{A}^-$.

LEMMA 4.2 (Rao, 1973, p. 67). *We have*

$$(4.2) \qquad |\mathbf{Z}_{a,b}| = \begin{vmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \multicolumn{4}{c}{\dotfill} \\ b & b & \dots & a \end{vmatrix} = (a-b)^{k-1}[a + (k-1)b],$$

where $\mathbf{Z}_{a,b}$ *is a* $k \times k$ *matrix. Hence:*

$$(4.3) \quad \text{If} \quad a = 1 - \frac{1}{k}, \ b = -\frac{1}{k}, \quad \text{then} \quad |\mathbf{Z}_{a,b}| = 0.$$

$$(4.4) \quad \text{If} \quad b = a - 1, \quad \text{then} \quad |\mathbf{Z}_{a,a-1}| = \begin{vmatrix} a & a-1 & \dots & a-1 \\ a-1 & a & \dots & a-1 \\ \multicolumn{4}{c}{\dotfill} \\ a-1 & a-1 & \dots & a \end{vmatrix}$$

$$= k(a-1) + 1 = |\mathbf{I}_k + (a-1)\mathbf{J}_k|.$$

LEMMA 4.3. *For* $k = 2, 3, 4, \ldots$ *the g-inverse* $\mathbf{A}_1^-$ *of the matrix*

$$(4.5) \qquad \mathbf{A}_1 = \mathbf{I}_k - \frac{1}{k}\mathbf{J}_k,$$

is of the form

$$(4.6) \qquad \begin{cases} \mathbf{A}_1^- = \left(\mathbf{I}_k - \frac{1}{k}\mathbf{J}_k\right)^- = \begin{bmatrix} 1 - \frac{1}{k} & -\frac{1}{k} & \cdots & -\frac{1}{k} \\ -\frac{1}{k} & 1 - \frac{1}{k} & \cdots & -\frac{1}{k} \\ -\frac{1}{k} & -\frac{1}{k} & \cdots & 1 - \frac{1}{k} \end{bmatrix} = \mathbf{A}_1 = \mathbf{A}_1^2, \\[2em] \mathrm{trace}(\mathbf{A}_1) = \mathrm{rank}(\mathbf{A}_1). \end{cases}$$

LEMMA 4.4.

$$(4.7) \qquad \left(\mathbf{I}_{k-1} - \frac{1}{k}\mathbf{J}_{k-1}\right)^{-1} = \mathbf{I}_{k-1} + \mathbf{J}_{k-1},$$

where $\mathbf{I}_k$ *is the identity matrix, and* $\mathbf{J}_{k-1}$ *the matrix with all elements equal to* 1.

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