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TWO-POINT PRIORS AND MINIMAX ESTIMATION OF A BOUNDED PARAMETER UNDER CONVEX LOSS

Abstract. The problem of minimax estimation of a parameter θ when θ is restricted to a finite interval $[\theta_0, \theta_0 + m]$ is studied. The case of a convex loss function is considered. Sufficient conditions for existence of a minimax estimator which is a Bayes estimator with respect to a prior concentrated in two points θ_0 and $\theta_0 + m$ are obtained. An example is presented.

1. Introduction. The problem of minimax estimation of a bounded real parameter θ has been considered in many particular models. The square loss function and normal mean were considered by Casella and Strawderman [3], the case of the Linex loss function and normal mean was treated by Bischoff, Fieger and Wulfert [2], the Linex loss and Poisson model by Wan, Zou and Lee [10], the scale invariant squared loss and Poisson model by Johnstone and MacGibbon [8]. The binomial model was considered in Marchand and MacGibbon [9].

There are also many papers which consider general distributions, namely: DasGupta [4] (square loss function and multiparameter families), Eichenauer-Hermann and Fieger [6] (scale parameter family), Eichenauer-Hermann and Ickstadt [7] (L_p-loss and location parameter family), Bischoff [1] (L_p-loss and scale parameter family), van Eeden and Zidek [5] (bounded scale parameter and scale-invariant squared error loss). In several of the above mentioned papers the minimax estimator obtained is a Bayes estimator with respect to a prior concentrated in two points.

We show that suitable two-point priors on the endpoints of a sufficiently small interval parameter space are least favourable and that the corresponding Bayes estimators are minimax for many convex losses under general conditions on density functions. The Linex loss, square loss, scale invariant

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squared loss are special cases, and many location parameter families and scale parameter families satisfy our conditions for densities. Our result is a generalization of some results obtained for particular loss functions (see [8], [9]) and it is a generalization of DasGupta's result [4] to the case of an asymmetric loss and one-parameter families. We use the convexity technique. A similar method has been used to find minimax estimators for various special distributions by many authors. See references [2], [5] and [10] for examples.

Let $\mathcal{X}$ denote a sample space and X the observed random variable. Let $\{P_\theta : \theta \in \Theta = [\theta_0, \theta_0 + m]\}$, where θ_0 and $\theta_0 + m$ are fixed real numbers, $m > 0$, be a family of probability measures of X with densities $f(\cdot, \theta)$ with respect to a fixed measure μ on the space $\mathcal{X}$. Let $L(\theta, a)$ be a loss if an estimate a is chosen when in fact θ is the true value of the parameter. Let $\delta : \mathcal{X} \to \Theta$ be an estimator, and

$$R(\theta, \delta) = \int_{\mathcal{X}} L(\theta, \delta(x)) f(x, \theta)\, \mu(dx)$$

the risk function.

We will use the following general result (see van Eeden and Zidek [5]).

THEOREM 1. *Suppose*

(1) δ_B *is a Bayes rule with respect to a prior distribution* Π *such that* $\Pi\{\theta_0\} + \Pi\{\theta_0 + m\} = 1$;
(2) $R(\theta_0, \delta_B) = R(\theta_0 + m, \delta_B)$;
(3) *the function* $\theta \mapsto R(\theta, \delta_B)$ *is convex on* Θ.

Then δ_B *is a minimax rule and* Π *is a least favourable distribution. If* δ_B *is the unique Bayes rule, then* δ_B *is the unique minimax rule.* ∎

We now list our assumptions about the loss function:

L1. $L(\theta, a) = h(c(\theta)(\theta - a))$.
L2. The function $h : \mathbb{R} \to [0, \infty)$ is of class C^3.
L3. $h(0) = 0$.
L4. $h'(x) < 0$ iff $x < 0$; $h'(x) > 0$ iff $x > 0$; $h''(x) > 0$ for all x.
L5. The function $c : [\theta_0, \infty) \to (0, \infty)$ is of class C^2.

Our assumptions about the density function $f(x, \theta)$ are:

F1. $f(x, \theta_0) + f(x, \theta_0 + m) \neq 0$ for all $x \in \mathcal{X}$.
F2. $P_\theta\{x : f(x, \theta_0) f(x, \theta_0 + m) > 0\} > 0$ for $\theta = \theta_0$ and $\theta = \theta_0 + m$.
F3. The derivatives $\frac{\partial}{\partial\theta} f(x, \theta)$ and $\frac{\partial^2}{\partial\theta^2} f(x, \theta)$ exist for every $\theta \in [\theta_0, \theta_0 + m]$ and almost all $x \in \mathcal{X}$.
F4. There exist μ-integrable functions $g_i(x)$, $i = 0, 1, 2$, such that

$$|f(x, \theta)| \le g_0(x), \quad \left|\frac{\partial}{\partial\theta} f(x, \theta)\right| \le g_1(x), \quad \left|\frac{\partial^2}{\partial\theta^2} f(x, \theta)\right| \le g_2(x)$$

for all $\theta \in [\theta_0, \theta_0 + m]$ and almost all $x \in \mathcal{X}$.

Let

$$\begin{aligned} A &= \{x : f(x,\theta_0)f(x,\theta_0+m) > 0\},\\ A_0 &= \{x : f(x,\theta_0) = 0 \wedge f(x,\theta_0+m) > 0\},\\ A_1 &= \{x : f(x,\theta_0) > 0 \wedge f(x,\theta_0+m) = 0\}. \end{aligned}$$

Let c_0 and c_m denote $c(\theta_0)$ and $c(\theta_0+m)$, respectively. We will suppress θ wherever possible and write c instead of $c(\theta)$, c' instead of $dc(\theta)/d\theta$, c'' instead of $d^2c(\theta)/d\theta^2$.

Let Π_β, $\beta \in [0,1]$, be a prior such that

$$\Pi_\beta(\theta_0) = \beta, \qquad \Pi_\beta(\theta_0+m) = 1-\beta.$$

Let $\delta^{\beta,m}$ be the Bayes estimator if the prior is Π_β. Note that $\delta^{\beta,m}(x) = \theta_0+m$ if $f(x,\theta_0) = 0$, and $\delta^{\beta,m}(x) = \theta_0$ if $f(x,\theta_0+m) = 0$, when $\beta \in (0,1)$. If $\beta = 0$ and $f(x,\theta_0+m) = 0$ then we put $\delta^{0,m}(x) = \theta_0$, and if $\beta = 1$ and $f(x,\theta_0) = 0$ then we put $\delta^{1,m}(x) = \theta_0+m$. The existence of $\delta^{\beta,m}$ for x satisfying $f(x,\theta_0)f(x,\theta_0+m) \neq 0$ follows from the properties of the function h. The estimator $\delta^{\beta,m}$ is a solution of the equation

$$\varrho_{\beta,m}(\delta) = 0,$$

where

$$\varrho_{\beta,m}(\delta) = -c_0h'(c_0(\theta_0-\delta))\beta f(x,\theta_0) - c_mh'(c_m(\theta_0+m-\delta))(1-\beta)f(x,\theta_0+m)$$

is an increasing function of δ and

$$\begin{aligned} \varrho_{\beta,m}(\theta_0) &= -c_mh'(c_mm)(1-\beta)f(x,\theta_0+m) < 0,\\ \varrho_{\beta,m}(\theta_0+m) &= -c_0h'(-c_0m)\beta f(x,\theta_0) > 0. \end{aligned}$$

From now on we suppress x wherever possible and write δ instead of $\delta(x)$.

2. Main result

THEOREM 2. *There exists $M_0 > 0$ such that for every $m \in (0, M_0)$ there exists $\beta^* \in [0,1]$ such that the Bayes estimator $\delta^{\beta^*,m}$ for a prior Π_{β^*} is the minimax estimator of θ under the loss function $L(\theta,a) = h(c(\theta)(\theta-a))$. The value β^* satisfies the equation*

$$R(\theta_0, \delta^{\beta^*,m}) = R(\theta_0+m, \delta^{\beta^*,m}).$$

The two-point prior Π_{β^*} is least favourable.

We have divided the proof into a sequence of lemmas.

LEMMA 1. *For all $x \in \mathcal{X}$ the estimator $\delta^{\beta,m}$ satisfies:*

(i) *if $f(x,\theta_0)f(x,\theta_0+m) \neq 0$ then $\delta^{0,m}(x) = \theta_0+m$ and $\delta^{1,m} = \theta_0$;*
(ii) *$\delta^{\beta,m}$ is a differentiable function of $\beta \in (0,1)$;*
(iii) *$\delta^{\beta,m}$ is a strictly decreasing function of β for every $m > 0$ and x such that $f(x,\theta_0)f(x,\theta_0+m) \neq 0$.*

Proof. If $\beta = 0$ then $\Pi_\beta(\theta_0 + m) = 1$, if $\beta = 1$ then $\Pi_\beta(\theta_0) = 1$, which proves (i). The estimator $\delta^{\beta,m}$ satisfies

$$c_0 h'(c_0(\theta_0 - \delta^{\beta,m}))\beta f(x,\theta_0) + c_m h'(c_m(\theta_0 + m - \delta^{\beta,m}))(1-\beta) f(x,\theta_0+m) = 0.$$

Differentiating in β we obtain

$$\frac{\partial}{\partial\beta}\delta^{\beta,m}(x) = \frac{c_0 h'(c_0(\theta_0 - \delta^{\beta,m}))f(x,\theta_0) - c_m h'(c_m(\theta_0 + m - \delta^{\beta,m}))f(x,\theta_0+m)}{c_0^2 h''(c_0(\theta_0 - \delta^{\beta,m}))\beta f(x,\theta_0) + c_m^2 h''(c_m(\theta_0 + m - \delta^{\beta,m}))(1-\beta) f(x,\theta_0+m)}.$$

The denominator is greater than 0 for all $\beta \in (0,1)$, x and $m > 0$. If $f(x,\theta_0)f(x,\theta_0+m) \neq 0$ then for $\beta \in (0,1)$,

$$\delta^{\beta,m} \in (\theta_0, \theta_0 + m), \qquad h'(\theta_0 - \delta^{\beta,m}) < 0, \qquad h'(\theta_0 + m - \delta^{\beta,m}) > 0,$$

which proves (iii). ∎

LEMMA 2. *The risk function $R(\theta, \delta^{\beta,m})$ is a continuous function of β and*

$$R(\theta_0, \delta^{0,m}) = h(-c_0 m)P_{\theta_0}(A), \qquad R(\theta_0 + m, \delta^{0,m}) = 0$$
$$R(\theta_0, \delta^{1,m}) = 0, \qquad R(\theta_0 + m, \delta^{1,m}) = h(c_m m)P_{\theta_0+m}(A).$$

Proof. We have

$$R(\theta, \delta^{\beta,m}) = \int_A h(c(\theta - \delta^{\beta,m}))f(x,\theta)\,\mu(dx) + h(c(\theta - \theta_0 - m))P_\theta(A_0) + h(c(\theta - \theta_0))P_\theta(A_1)$$

and $\delta^{\beta,m}$ is a continuous function of β and

$$\forall x \quad \theta_0 \le \delta^{\beta,m}(x) \le \theta_0 + m.$$

Thus from the Lebesgue dominated convergence theorem we obtain the continuity of R.

For $\beta = 0$ we obtain

$$R(\theta, \delta^{0,m}) = h(c(\theta - \theta_0 - m))(P_\theta(A) + P_\theta(A_0)) + h(c(\theta - \theta_0))P_\theta(A_1).$$

For $\beta = 1$ we obtain

$$R(\theta, \delta^{1,m}) = h(c(\theta - \theta_0))(P_\theta(A) + P_\theta(A_1)) + h(c(\theta - \theta_0 - m))P_\theta(A_0). \ ∎$$

LEMMA 3. *For every $m > 0$ there exists a unique $\beta^*(m) \in (0,1)$ such that*

$$R(\theta_0, \delta_m) = R(\theta_0 + m, \delta_m), \quad \textit{where} \quad \delta_m = \delta^{\beta^*(m),m}.$$

Proof. Let $F_m(\beta) = R(\theta_0, \delta^{\beta,m}) - R(\theta_0 + m, \delta^{\beta,m})$. Then F_m is a continuous function of β and $F_m(0) = h(-c_0 m)P_{\theta_0}(A) > 0$ and $F_m(1) = -h(c_m m)P_{\theta_0+m}(A) < 0$.

Both $R(\theta_0, \delta^{\beta,m})$ and $-R(\theta_0 + m, \delta^{\beta,m})$ are decreasing functions of β. To show this for $R(\theta_0, \delta^{\beta,m})$ (the proof for $R(\theta_0 + m, \delta^{\beta,m})$ is similar) take $\beta_2 > \beta_1$ and $\beta_1, \beta_2 \in (0,1)$. From Lemma 1 and property **L4** for x satisfying $f(x,\theta_0)f(x,\theta_0+m) \neq 0$ we have

$$h(c_0(\theta_0 - \delta^{\beta_2,m})) < h(c_0(\theta_0 - \delta^{\beta_1,m})).$$

Now assumption **F2** gives

$$R(\theta_0, \delta^{\beta_2,m}) < R(\theta_0, \delta^{\beta_1,m}).$$

Hence F_m is decreasing as a sum of decreasing functions, and thus $\beta^*(m)$ is unique. ∎

Let $\delta_m(x) = \delta^{\beta^*(m),m}(x)$ for $m > 0$, and $\delta_0(x) = \theta_0$.

LEMMA 4. *The function δ_m is continuous in m for $m \geq 0$.*

Proof (van Eeden and Zidek [5]). It is enough to show that $\beta^* : [0,\infty] \to [0,1]$ is continuous (see Lemma 1). Lemma 3 implies that β^* is a unique solution of the equation

$$F_m(\beta) = R(\theta_0, \delta^{\beta,m}) - R(\theta_0 + m, \delta^{\beta,m}) = 0.$$

The function F for fixed β is continuous in m, because $\delta^{\beta,m}(x)$ is continuous in m and $\delta^{\beta,m}(x)$ is uniformly bounded as a function of x in a neighbourhood of m_0 for every $m_0 > 0$.

Take $\varepsilon > 0$. Let β_1 and β_2 be numbers such that

$$0 \leq \beta_1 < \beta^*(m) < \beta_2 \leq 1, \qquad |\beta_1 - \beta_2| < \varepsilon.$$

Then

$$F_m(\beta_1) > F_m(\beta^*(m)) = 0 > F_m(\beta_2)$$

and there exists $B > 0$ such that

$$\forall 0 < b < B \qquad |F_{m+b}(\beta_1) - F_m(\beta_1)| < \tfrac{1}{2} F_m(\beta_1),$$
$$\forall 0 < b < B, \qquad |F_{m+b}(\beta_2) - F_m(\beta_2)| < \tfrac{1}{2} |F_m(\beta_2)|.$$

Hence

$$F_{m+b}(\beta_2) < 0 < F_{m+b}(\beta_1),$$

thus $\beta_1 < \beta^*(m+b) < \beta_2$ and

$$|\beta^*(m) - \beta^*(m+b)| < \varepsilon. \ ∎$$

LEMMA 5. *For every $m \geq 0$ and $\theta \geq \theta_0$ the second derivative $\frac{\partial^2}{\partial \theta^2} R(\theta, \delta_m)$ exists and is a continuous function of m uniformly in $\theta \in [\theta_0, \theta_0 + M]$ for each $M > 0$.*

Proof. We have

$$\frac{\partial}{\partial \theta} R(\theta, \delta_m) = \int_{\mathcal{X}} [c'(\theta - \delta_m) + c] h'(c(\theta - \delta_m)) f(x,\theta)\, \mu(dx) + \int_{\mathcal{X}} h(c(\theta - \delta_m)) \frac{\partial}{\partial \theta} f(x,\theta)\, \mu(dx),$$

$$\frac{\partial^2}{\partial\theta^2}R(\theta,\delta_m)=\int_{\mathcal{X}} h''(c(\theta-\delta_m))[c'(\theta-\delta_m)+c]^2 f(x,\theta)\,\mu(dx)$$
$$+\int_{\mathcal{X}}[c''(\theta-\delta_m)+2c']h'(c(\theta-\delta_m))f(x,\theta)\,\mu(dx)$$
$$+2\int_{\mathcal{X}} h'(c(\theta-\delta_m))[c'(\theta-\delta_m)+c]\frac{\partial}{\partial\theta}f(x,\theta)\,\mu(dx)$$
$$+\int_{\mathcal{X}} h(c(\theta-\delta_m))\frac{\partial^2}{\partial\theta^2}f(x,\theta)\,\mu(dx).$$

The existence of the derivatives follows from the Lebesgue theorem and assumptions **F3, F4** and **L2, L5**. The continuity of $\frac{\partial^2}{\partial\theta^2}R(\theta,\delta_m)$ follows from assumptions **F3** and **F4** and the continuity of δ_m.

It remains to prove that the continuity is uniform with respect to θ. We need to show that

$$\forall\varepsilon,m,M>0\ \exists\eta>0\qquad H(m_1,m,\theta)=\left|\frac{\partial^2}{\partial\theta^2}R(\theta,\delta_{m_1})-\frac{\partial^2}{\partial\theta^2}R(\theta,\delta_m)\right|<\varepsilon$$

if $|m-m_1|<\eta$ and $\theta\in[\theta_0,\theta_0+M]$. We have

$$H(m_1,m,\theta)$$
$$\le(\theta c'+c)^2\int_{\mathcal{X}}|h''(c(\theta-\delta_{m_1}))-h''(c(\theta-\delta_m))|g_0(x)\,\mu(dx)$$
$$+2|c'(\theta c'+c)|\int_{\mathcal{X}}|h''(c(\theta-\delta_{m_1}))\delta_{m_1}-h''(c(\theta-\delta_m))\delta_m|g_0(x)\,\mu(dx)$$
$$+c'^2\int_{\mathcal{X}}|h''(c(\theta-\delta_{m_1}))\delta_{m_1}^2-h''(c(\theta-\delta_m))\delta_m^2|g_0(x)\,\mu(dx)$$
$$+2|c'|\int_{\mathcal{X}}|h'(c(\theta-\delta_{m_1}))\delta_{m_1}-h'(c(\theta-\delta_m))\delta_m|g_1(x)\,\mu(dx)$$
$$+2|\theta c'+c|\int_{\mathcal{X}}|h'(c(\theta-\delta_{m_1}))-h'(c(\theta-\delta_m))|g_1(x)\,\mu(dx)$$
$$+\int_{\mathcal{X}}|h(c(\theta-\delta_{m_1}))-h(c(\theta-\delta_m))|g_2(x)\,\mu(dx)$$
$$+|c''|\int_{\mathcal{X}}|h'(c(\theta-\delta_{m_1}))\delta_{m_1}-h'(c(\theta-\delta_m))\delta_m|g_0(x)\,\mu(dx)$$
$$+|\theta c''+2c'|\int_{\mathcal{X}}|h'(c(\theta-\delta_{m_1}))-h'(c(\theta-\delta_m))|g_0(x)\,\mu(dx),$$

and

$$|h''(c(\theta-\delta_{m_1}))-h''(c(\theta-\delta_m))|\le\sup_{z\in[-S,S]}|h'''(z)|\,|\delta_{m_1}-\delta_m|C,$$
$$|h'(c(\theta-\delta_{m_1}))-h'(c(\theta-\delta_m))|\le\sup_{z\in[-S,S]}|h''(z)|\,|\delta_{m_1}-\delta_m|C,$$
$$|h(c(\theta-\delta_{m_1}))-h(c(\theta-\delta_m))|\le\sup_{z\in[-S,S]}|h'(z)|\,|\delta_{m_1}-\delta_m|C,$$

where $S = \sup_{\theta\in[\theta_0,\theta_0+M]} c(\theta)M$, $C = \sup_{\theta\in[\theta_0,\theta_0+M]} c(\theta)$, h satisfies **L2**, c satisfies **L5**, $\theta \in [\theta_0, \theta_0 + M]$, δ_m is continuous and bounded for $\theta \in [\theta_0, \theta_0 + M]$ and g_0, g_1, g_2 are integrable. ∎

LEMMA 6. *There exists M_0 such that for every $m \in (0, M_0)$,*

$$\frac{\partial^2}{\partial\theta^2} R(\theta, \delta_m) > 0 \quad \text{for every } \theta \in [\theta_0, \theta_0 + m].$$

Proof. We have

$$\frac{\partial^2}{\partial\theta^2} R(\theta, \delta_m)\bigg|_{m=0} = h''(0)c^2(\theta_0) > 0,$$

and thus the assertion follows from Lemma 5. ∎

Now using Lemmas 3 and 6 and Theorem 1 we obtain the assertion of Theorem 2.

3. Example. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the uniform distribution $U(0, \theta)$, where $\theta \in [a, b]$ is unknown, a, b are known and $b > a > 0$. Let $m = b - a$. We estimate θ under the LINEX loss function

$$L(\theta, d) = \exp(c(\theta - d)) - c(\theta - d) - 1,$$

where $c > 0$ is fixed. Set $X = (X_1, \ldots, X_n)$. If $X = x$ then the Bayes estimate $\delta^{\beta,m}(x)$ of θ for a prior that puts mass β and $1 - \beta$ at a and b respectively, is given by

$$\delta^{\beta,m}(x) = \begin{cases} \dfrac{1}{c} \ln \dfrac{e^{ca}\beta b^n + e^{cb}(1-\beta)a^n}{\beta b^n + (1-\beta)a^n} & \text{if } 0 < x_{n:n} < a, \\ b & \text{if } x_{n:n} \in [a, b), \end{cases}$$

where $X_{n:n} = \max(X_1, \ldots, X_n)$.

The risk of the estimator $\delta^{\beta,m}$ is equal to

$$R(\theta, \delta^{\beta,m}) = \left(e^{c\theta}\frac{B}{A} - c\theta - \ln\frac{B}{A} - 1\right)\frac{a^n}{\theta^n} + (e^{c\theta-cb} - c\theta + cb - 1)\left(1 - \frac{a^n}{\theta^n}\right),$$

where

$$B = \beta b^n + (1-\beta)a^n, \tag{3.1}$$

$$A = e^{ca}\beta b^n + e^{cb}(1-\beta)a^n. \tag{3.2}$$

We would like to find the value of M such that $R(\cdot, \delta^{\beta,m})$ is a convex function of $\theta \in [a, a + m]$ if $m < M$ and $\beta \in (0, 1)$. We have

$$\frac{\partial}{\partial\theta} R(\theta, \delta^{\beta,m}) = \frac{-na^n}{\theta^{n+1}}\left(e^{c\theta}\frac{B}{A} - \ln\frac{B}{A} - e^{c\theta-cb} - cb\right) + \frac{ca^n}{\theta^n}e^{c\theta}\frac{B}{A} + \frac{\theta^n - a^n}{\theta^n}ce^{c\theta-cb} - c,$$

$$\frac{\partial^2}{\partial\theta^2}R(\theta,\delta^{\beta,m}) = \frac{a^n e^{c\theta}}{\theta^{n+2}}\left(\frac{B}{A} - e^{-cb}\right)(n(n+1) - 2cn\theta + c^2\theta^2) + c^2 e^{c\theta-cb} - \frac{n(n+1)a^n}{\theta^{n+2}}\left(\ln\frac{B}{A} + cb\right).$$

For every $\theta \geq a$ and $b > a > 0$ the following inequalities hold:

$$c^2\theta^2 - 2cn\theta + n^2 + n = (c\theta - n)^2 + n \geq n, \qquad \frac{B}{A} \geq e^{-cb}.$$

To prove convexity of $R(\cdot,\delta^{\beta,m})$ if $\theta \geq a$ it suffices to show that

$$na^n e^{ca}\left(\frac{B}{A} - e^{-cb}\right) - n(n+1)a^n\left(\ln\frac{B}{A} + cb\right) + c^2a^{n+2}e^{ca-cb} > 0.$$

Substituting (3.1) and (3.2) and dividing both sides by a^n we obtain

$$g(\beta,m) > 0,$$

where

$$g(\beta,m) = \frac{n[\beta b^n + (1-\beta)a^n]}{\beta b^n + e^{cm}(1-\beta)a^n} - ne^{-cm} + c^2a^2e^{-cm} - n(n+1)\ln\frac{[\beta b^n + (1-\beta)a^n]e^{cm}}{\beta b^n + e^{cm}(1-\beta)a^n}$$

and $m = b - a > 0$. We have

$$\begin{aligned}\frac{\partial}{\partial\beta}g(\beta,m) &= \frac{nb^na^n(e^{cm}-1)}{[\beta b^n + e^{cm}(1-\beta)a^n]^2} - \frac{n(n+1)b^na^n(e^{cm}-1)}{[\beta b^n + (1-\beta)a^n][\beta b^n + e^{cm}(1-\beta)a^n]} \\ &= \frac{-nb^na^n(e^{cm}-1)[n\beta b^n + (1-\beta)a^n((n+1)e^{cm}-1)]}{[\beta b^n + (1-\beta)a^n][\beta b^n + e^{cm}(1-\beta)a^n]^2} < 0,\end{aligned}$$

hence g is a decreasing function of $\beta \in (0,1)$. Therefore $g(\beta,m) > 0$ for $\beta \in (0,1)$ iff

$$g(1,m) = n - n(n+1)cm + (c^2a^2 - n)e^{-cm} > 0.$$

The function $g(1,m)$ is a decreasing function of $m > 0$ and $g(1,0) > 0$ and $\lim_{m\to\infty} g(1,m) = -\infty$.

Hence, for a given $a > 0$, if $m \in (0,M)$, where M is a solution of the equation

$$n - n(n+1)cM + (c^2a^2 - n)e^{-cM} = 0,$$

then $R(\theta,\delta^{\beta,m})$ is a convex function of $\theta \in [a,a+m]$ for every $\beta \in (0,1)$. It follows that the Bayes estimator $\delta^{\beta,m}$ for β satisfying $R(a,\delta^{\beta,m}) = R(a+m,\delta^{\beta,m})$ is a minimax estimator.

Table 1 presents the values of M for some n, a and c.

Table 1. Values of M for some n, a and c

	$n=1$			$n=5$			$n=10$		
	c			c			c		
a	0.2	1	3	0.2	1	3	0.2	1	3
0.5	0.049	0.190	0.262	0.002	0.010	0.027	0.0005	0.003	0.007
1	0.192	0.500	0.481	0.008	0.038	0.090	0.002	0.010	0.028
2	0.660	1.033	0.761	0.032	0.138	0.229	0.008	0.038	0.090
5	2.500	2.048	1.197	0.192	0.551	0.540	0.049	0.202	0.297
10	5.168	2.990	1.559	0.688	1.160	0.845	0.192	0.559	0.549

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