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ON EXISTENCE OF SOLUTIONS FOR THE NONSTATIONARY STOKES SYSTEM WITH BOUNDARY SLIP CONDITIONS

Abstract. Existence of solutions for equations of the nonstationary Stokes system in a bounded domain $\Omega \subset \mathbb{R}^3$ is proved in a class such that velocity belongs to $W_p^{2,1}(\Omega \times (0,T))$, and pressure belongs to $W_p^{1,0}(\Omega \times (0,T))$ for $p > 3$. The proof is divided into three steps. First, the existence of solutions with vanishing initial data is proved in a half-space by applying the Marcinkiewicz multiplier theorem. Next, we prove the existence of weak solutions in a bounded domain and then we regularize them. Finally, the problem with nonvanishing initial data is considered.

1. Introduction. In a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary S we consider the Stokes equation with an initial condition and boundary slip conditions. Our problem is described by the following system:

$$
\begin{aligned}
&\partial_t v - \nu \Delta v + \nabla q = F,\\
&\operatorname{div} v = G,\\
&\overline{n} \cdot \mathbb{D}(v) \cdot \overline{\tau}_\alpha + \gamma v \cdot \overline{\tau}_\alpha|_{S^T} = b_\alpha \quad (\alpha = 1,2),\\
&v \cdot \overline{n}|_{S^T} = b_3,\\
&v|_{t=0} = v_0,
\end{aligned}
\tag{1.1}
$$

where $\mathbb{D}(v) = \{\nu(\partial_i v_j + \partial_j v_i)\}$ is the stress tensor, $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ the velocity vector, $q(x,t)$ the pressure, ν the constant viscosity coefficient, γ a positive constant (slip coefficient), $\overline{n}$ the external normal vector to S, and $\overline{\tau}_\alpha$ $(\alpha = 1,2)$ tangent vectors to S.

The aim of this paper is to prove the existence of regular solutions of (1.1). The main result of this paper is formulated in

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Theorem 1. *Let* $F \in L_p(\Omega^T)$, $G \in W_p^{1,0}(\Omega^T)$ *and* $S \in W_p^{2-1/p}$. *Assume that there exist* $A, B \in L_p(\Omega^T)$ *such that* $\partial G/\partial t - \operatorname{div} F = \operatorname{div} B + A$, *where* $\operatorname{diam}\operatorname{supp} A < 2\lambda$, $v_0 \in W_p^{2-2/p}(\Omega)$, $b_3 \in W_p^{2-1/p,1-1/2p}(S^T)$, $b_\alpha \in W_p^{1-1/p,1/2-1/2p}(S^T)$ *for* $\alpha = 1,2$. *Then there exists a unique solution of problem* (1.1) *such that* $v \in W_p^{2,1}(\Omega^T)$, $q \in W_p^{1,0}(\Omega^T)$ *for* $p > 3$ *and*

$$\begin{aligned} (1.2)\qquad \|v\|_{W_p^{2,1}(\Omega^T)} + \|q\|_{W_p^{1,0}(\Omega^T)} &\le C(T)[\|F\|_{L_p(\Omega^T)} + \|G\|_{W_p^{1,0}(\Omega^T)} \\ &\quad + \|B\|_{L_p(\Omega^T)} + \lambda \cdot \|A\|_{L_p(\Omega^T)} + \|v_0\|_{W_p^{2-2/p}(\Omega)} \\ &\quad + \|b'\|_{W_p^{1-1/p,1/2-1/2p}(S^T)} + \|b_3\|_{W_p^{2-1/p,1-1/2p}(S^T)}], \end{aligned}$$

where $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$, $b' = (b_1, b_2, 0)^T$ *and* $C(T)$ *is an increasing positive function of* T.

To prove Theorem 1 we apply the methods from [2]. First we consider problem (1.1) in a half-space with vanishing initial data. By applying the Fourier transform with respect to tangential directions and the Laplace transform with respect to time we prove existence and find an appropriate estimate employing the Marcinkiewicz theorem on multipliers. Next exploiting the regularizer technique the existence of solutions to problem (1.1) with vanishing initial data is proved in a bounded domain. Finally, by extending the initial data we prove Theorem 1.

2. Notation. In our considerations we use the anisotropic Sobolev–Slobodetskiĭ spaces $W_p^{k,l}(\Omega^T)$, where $k, l \in \mathbb{R}_+$, $p \ge 1$, $Q^T = Q \times (0,T)$ and Q is either Ω or S, with the norm

$$\|v\|^p_{W_p^{k,l}(Q^T)} = \|v\|^p_{W_p^{k,0}(Q^T)} + \|v\|^p_{W_p^{0,l}(Q^T)},$$

where

$$\|v\|^p_{W_p^{k,0}(Q^T)} = \int_0^T \|v\|^p_{W_p^k(Q)}\,dt, \qquad \|v\|^p_{W_p^{0,l}(Q^T)} = \int_\Omega \|v\|^p_{W_p^l((0,T))}\,dx$$

and

$$\begin{aligned} \|v\|^p_{W_p^k(Q)} = &\sum_{|\alpha|\le[k]} \|D_x^\alpha v\|^p_{L_p(Q)} \\ &+ \sum_{|\alpha|=[k]} \int_Q\int_Q \frac{|D_x^\alpha v(x,t) - D_{x'}^\alpha v(x',t)|^p}{|x-x'|^{s+p(k-[k])}}\,dx\,dx', \end{aligned}$$

$$\|v\|_{W_p^l(0,T)} = \sum_{i\le[l]} \|\partial_t^i v\|^p_{L_p(0,T)} + \sum_{|i|=[l]} \int_0^T\int_0^T \frac{|\partial_t^i v(x,t) - \partial_{t'}^i v(x,t')|^p}{|t-t'|^{1+p(l-[l])}}\,dt\,dt',$$

where $s \equiv \dim\Omega$, $[m]$ is the integral part of m, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_s}^{\alpha_s}$ where

$\alpha = (\alpha_1, \ldots, \alpha_s)$ is a multiindex. We will use the Fourier–Laplace transform: $\mathcal{F}_{t,x}[\phi](\xi, \xi_0) \equiv \widehat{\phi}(\xi_0, \xi)$, where

$$\widehat{\phi}(\xi_0, \xi) = c \int_0^\infty e^{-st} \int_{\mathbb{R}^3} e^{-i\xi x} \phi(x, t)\, dx\, dt$$

and the inverse transform

$$\mathcal{F}^{-1}_{t,x}[\widehat{\phi}](x, t) = \frac{1}{(2\pi)^{(3+1)/2}} \int_0^\infty e^{st} \int_{\mathbb{R}^3} e^{i\xi x} \widehat{\phi}(\xi_0, \xi)\, d\xi\, d\xi_0,$$

where $s = i\xi_0$, $\xi = (\xi_1, \ldots, \xi_s)$, $x = (x_1, \ldots, x_s)$ and

$$x \cdot \xi = \sum_{j=1}^{d} x_j \xi_j.$$

In the case when $Q^T = \mathbb{R}^d \times \mathbb{R}$ we can apply the Fourier transform and define the Bessel-potential spaces given by the norm

$$\begin{aligned} \|\phi\|_{H^{m,n}_p(\mathbb{R}^{d+1})} = \|\phi\|_{L_p(\mathbb{R}^{d+1})} &+ \|\mathcal{F}^{-1}_{t,x}[|\xi|^m \widehat{\phi}(\xi, \xi_0)\|_{L_p(\mathbb{R}^{d+1})} \\ &+ \|\mathcal{F}^{-1}_{t,x}[|\xi_0|^n \widehat{\phi}(\xi, \xi_0)]\|_{L_p(\mathbb{R}^{d+1}))}, \end{aligned}$$

where $\widehat{\phi}(\xi_0, \xi)$ is the Fourier transform of $\phi(x, t)$.

In the proof we will use the following results.

THEOREM 2.1 (Marcinkiewicz theorem [2]). *Suppose that a function $\psi : \mathbb{R}^m \to \mathbb{C}$ is smooth enough and there exists a constant $M > 0$ such that for every $x \in \mathbb{R}^m$ we have*

$$|x_{j_1} \cdots x_{j_k}| \left| \frac{\partial^k \psi}{\partial x_{j_1} \cdots \partial x_{j_k}} \right| \le M, \qquad 0 \le k \le m,\ 1 \le j_1 < \cdots < j_k \le m.$$

Then the operator

$$Pg(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} dy\, e^{ixy} \psi(y) \int_{\mathbb{R}^m} e^{-iyz} g(z)\, dz$$

is bounded in $L_p(\mathbb{R}^m)$ and

$$\|Pg\|_{L_p(\mathbb{R}^m)} \le A_{p,m} M \|g\|_{L_p(\mathbb{R}^m)}.$$

PROPOSITION 2.2 (see [2]). *Let $\phi \in W^{m,n}_p(\Omega^T)$. If*

$$K = \sum_{i=1}^{3} \left(\alpha_i + \frac{1}{p} - \frac{1}{q} \right) \frac{1}{m} + \left(\beta + \frac{1}{p} - \frac{1}{q} \right) \frac{1}{n} < 1$$

then

$$\|D^\beta_t D^\alpha_x \phi\|_{L_q(\Omega^T)} \le \varepsilon \|\phi\|_{W^{m,n}_p(\Omega^T)} + c(\varepsilon) \|\phi\|_{L_2(\Omega^T)},$$

where $q \ge p \ge 2$, $\varepsilon \in (0, 1)$ and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

PROPOSITION 2.3 (see [3]). *Let* $p \geq 2$, $\phi \in W_p^{2,1}(\Omega^T)$ *and* $\phi|_{t=0} = 0$. *Then*

$$\|\phi\|_{W_p^{1,1/2}(\Omega^T)} \leq c(T^{1/p} + T^{1/2})\|\phi\|_{W_p^{2,1}(\Omega^T)}.$$

3. Problem (1.1) in the half-space with vanishing initial data. To solve problem (1.1) we apply the regularizer technique so it must be considered locally. We have two kinds of subdomains: interior and boundary. Since considerations near the boundary are more complicated we restrict ourselves to such subdomains. Therefore we consider problem (1.1) in the half-space $x_3 > 0$ with $v|_{t=0} = 0$ and $F = 0$. Then (1.1) takes the form

$$\begin{aligned}
&\partial_t v - \nu\Delta v + \nabla q = 0,\\
&\operatorname{div} v = 0,\\
&\nu(\partial_1 v_3 + \partial_3 v_1) + \gamma v_1|_{x_3=0} = b_1,\\
&\nu(\partial_2 v_3 + \partial_3 v_2) + \gamma v_2|_{x_3=0} = b_2,\\
&v_3|_{x_3=0} = b_3,\\
&v|_{t=0} = 0.
\end{aligned}\tag{3.1}$$

To solve (3.1) we will use the Fourier transform

$$\widehat{v}(\xi_0, \xi', x_3) = \mathcal{F}_{t,x'}[v](\xi_0, \xi', x_3) = \int_0^\infty e^{-st} \int_{\mathbb{R}^2} e^{-ix'\xi'} v(x,t)\, dx'\, dt,$$

$$\widehat{q}(\xi_0, \xi', x_3) = \mathcal{F}_{t,x'}[q](\xi_0, \xi', x_3) = \int_0^\infty e^{-s_0 t} \int_{\mathbb{R}^2} e^{-ix'\xi'} q(x,t)\, dx'\, dt,$$

where $s = i\xi_0$, $\xi' = (\xi_1, \xi_2)$, $x' = (x_1, x_2)$.

Assuming $b' \in W_p^{1-1/p,1/2-1/2p}(x_3 = 0)$ and $b_3 \in W_p^{2-1/p,1-1/2p}(x_3 = 0)$ and $b|_{t=0} = 0$ we can extend b by zero for $t < 0$, where $b' = (b_1, b_2, 0)^T$ and $b = (b_1, b_2, b_3)^T$. Without loss of generality we can put $\nu, \gamma = 1$ and consider a reduced system of the form

$$\begin{aligned}
&L(\partial_x, \partial_t)(v, q) = 0,\\
&\operatorname{div} v = 0,\\
&A(v)|_{x_3=0} = b,\\
&v|_{t=0} = 0,
\end{aligned}\tag{3.2}$$

where L, A are differential operators such that

$$L(v, p) = \partial_t v - \Delta v + \nabla q,$$

$$A(v) = \begin{pmatrix} \{\partial_i v_j + \partial_j v_i\}|_{x_3=0} = b_i \ (i = 1, 2) \\ v_3|_{x_3} = b_3 \end{pmatrix}.$$

Because A and L are linear operators we can consider three differential problems and the solution of (3.2) will be the sum of solutions of those systems.

First we assume that $b_2 = b_3 \equiv 0$ and $b_1 \neq 0$, second that $b_2 \neq 0$ and $b_1 = b_3 \equiv 0$, and third that $b_3 \neq 0$ and $b_1 = b_2 \equiv 0$.

First we consider the following system:

$$
\begin{aligned}
&\partial_t v - \Delta v + \nabla q = 0,\\
&\operatorname{div} v = 0,\\
&\partial_1 v_3 + \partial_3 v_2|_{x_3=0} = b_1,\\
&\partial_2 v_3 + \partial_3 v_2|_{x_3=0} = 0,\\
&v_3|_{x_3=0} = 0,\\
&v|_{t=0} = 0.
\end{aligned}
\tag{3.3}
$$

For the Fourier transforms system (3.3) takes the form

$$
\begin{aligned}
&\left(-\frac{d^2}{dx_3^2} + r^2\right)\widehat{v}_i + i\xi_i\widehat{q} = 0 \qquad (\text{for } i = 1,2),\\
&\left(-\frac{d^2}{dx_3^2} + r^2\right)\widehat{v}_3 + \frac{d\widehat{q}}{dx_3} = 0,\\
&i\xi_1\widehat{v}_1 + i\xi_2\widehat{v}_2 + \frac{d\widehat{v}_3}{dx_3} = 0,
\end{aligned}
\tag{3.4}
$$

with boundary conditions

$$
\begin{aligned}
&i\xi_1\widehat{v}_3 + \frac{d\widehat{v}_1}{dx_3}\bigg|_{x_3=0} = \widehat{b}_1,\\
&i\xi_2\widehat{v}_3 + \frac{d\widehat{v}_2}{dx_3}\bigg|_{x_3=0} = 0,\\
&\widehat{v}_3|_{x_3=0} = 0,\\
&\widehat{q} \to 0, \quad \widehat{v} \to 0 \qquad (x_3 \to \infty),
\end{aligned}
\tag{3.5}
$$

where $r^2 = s + |\xi'|^2$, $\arg r \in (-\pi/4, \pi/4)$. Solving $(3.4)_{1,2,3}$ with $(3.5)_4$ we get (see also [3])

$$
\begin{aligned}
&\widehat{v} = \Phi(\xi' s)e^{-rx_3} + \phi(x,\xi')(i\xi_1, i\xi_2, -|\xi'|)e^{-|\xi'|x_3},\\
&\widehat{q} = -s\phi(\xi', s)e^{-|\xi'|x_3},
\end{aligned}
\tag{3.6}
$$

where

$$
\Phi = \left(\Phi_1, \Phi_1, \frac{1}{T}\left(i\xi_1\Phi_1 + i\xi_2\Phi_2\right)\right)
$$

and φ, Φ_1, Φ_2 are functions which must be calculated from the boundary conditions.

From (3.6) and boundary conditions $(3.5)_{1,2,3}$ we get a solution of (3.4) and (3.5):

$$\begin{aligned}
\widehat{v}_1 &= \frac{1}{r}\widehat{b}_1 e^{-rx_3} + \frac{\xi_1^2}{s}\biggl(\frac{1}{|\xi'|} - \frac{1}{r}\biggr)\widehat{b}_1 e^{-rx_3} + \frac{\xi_1^2}{s}\,\frac{|\xi'|-r}{|\xi'|}\,\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r}\,\widehat{b}_1,\\
\widehat{v}_2 &= \biggl(\frac{1}{|\xi'|} - \frac{1}{r}\biggr)\frac{\xi_1\xi_2}{s}\,\widehat{b}_1 e^{-rx_3} + \frac{\xi_1\xi_2}{|\xi'|}\,\frac{1}{|\xi'|-r}\cdot(e^{-|\xi'|x_3} - e^{-rx_3})\widehat{b}_1,\\
\widehat{v}_3 &= \frac{i\xi_1\widehat{b}_1}{r^2}\,e^{-rx_3} + \biggl(\frac{1}{|\xi'|^2} - \frac{1}{r^2}\biggr)\frac{i\xi_1}{s}\,|\xi'|^2\widehat{b}_1 e^{-rx_3}\\
&\quad + \frac{i\xi_1|\xi'|^2}{s}\cdot\frac{|\xi'|-r}{|\xi'|^2}\,\widehat{b}_1\,\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r},\\
\widehat{p} &= \frac{i\xi_1}{|\xi'|}\,\widehat{b}_1 e^{-|\xi'|x_3}.
\end{aligned}$$

Now we get

Lemma 3.1. *Let $b_1 \in W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)$. Then there exists a solution of problem* (3.3) *such that $v(t,x) \in W_{p,\mathrm{loc}}^{2,1}(\mathcal{D}^4)$, $q(t,x) \in W_{p,\mathrm{loc}}^{1,0}(\mathcal{D}^4)$ and*

$$\begin{aligned}
&\|D_x^2 v\|_{L_p(\mathcal{D}^4)} + \|D_t v\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)},\\
&\|\nabla q\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}\\
&\|v\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)},
\end{aligned}$$

where $\mathcal{D}^4 = \mathbb{R}^2_{x'} \times [0,\infty)_{x_3} \times \mathbb{R}_t$.

First we consider the pressure; its Fourier transform is $\widehat{q} = (i\xi_1/|\xi'|) \cdot \widehat{b}_1 e^{-rx_3}$. After the inverse Fourier transform we obtain

$$\begin{aligned}
q(x,t) = \mathcal{F}^{-1}_{t,x'}[\widehat{q}](x,t) &= c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\,\widehat{q}(\xi_0,\xi',x_3)\,d\xi_0\,d\xi'\\
&= c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\cdot\frac{i\xi_1}{|\xi'|}\,\widehat{b}_1 e^{-|\xi'|x_3}\,d\xi_0\,d\xi'
\end{aligned}$$

To estimate the tangential derivatives of the pressure

$$D_{x_1}q(x,t) = c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{i\xi'x'}\,\frac{i\xi_1}{|\xi'|}\cdot\xi_1\widehat{b}_1 e^{-|\xi'|x_3}\,d\xi_0\,d\xi'$$

we use $|i\xi_1/|\xi'|\,| \le 1$, so similarly to [3], [2] we have

$$\|D_{x'}q\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

Now we consider the x_3 derivative,

$$
\begin{aligned}
D_{x_3} q &= D_{x_3}\Big[c\int\limits_0^\infty e^{st}\int\limits_{\mathbb{R}^2} e^{ix'\xi'}\widehat{q}(\xi_0,\xi',x_3)\,d\xi_0\,d\xi'\Big] \\
&= c\int\limits_0^\infty e^{st}\int\limits_{\mathbb{R}^2} e^{ix'\xi'}(D_{x_3}\widehat{q})d\xi_0 d\xi' = c\int\limits_0^\infty e^{st}\int\limits_{\mathbb{R}^2} D_{x_3}\mathcal{F}^{-1}_{t,x'}(\widehat{q})e^{ix'\xi'}\,d\xi_0\,d\xi' \\
&= c\int\limits_0^\infty e^{st}\int\limits_{\mathbb{R}^2} \xi_1 b_1\cdot e^{-|\xi'|x_3}\cdot e^{ix'\xi'}d\xi_0\,d\xi' = \mathcal{F}^{-1}_{t,x}(i\xi_1\widehat{b}_1 e^{-|\xi'|x_3}).
\end{aligned}
$$

This implies that

$$\|D_{x_3}q\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

Hence

$$\|D_x q\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

Now let us consider $\widehat{v}_1,\widehat{v}_2,\widehat{v}_3$. We put

$$\widehat{v}_1 = \widehat{v}_{1A}+\widehat{v}_{1B}+\widehat{v}_{1C},\quad \widehat{v}_2 = \widehat{v}_{2A}+\widehat{v}_{2B},\quad \widehat{v}_3 = \widehat{v}_{3A}+\widehat{v}_{3B}+\widehat{v}_{3C},$$

where

$$
\begin{aligned}
\widehat{v}_{1A} &= -\frac{1}{r}\,\widehat{b}_1 e^{-rx_3},\\
\widehat{v}_{1B} &= \frac{\xi_1^2}{s}\Big(\frac{1}{|\xi'|}-\frac{1}{r}\Big)\widehat{b}_1 e^{-rx_3},\\
\widehat{v}_{1C} &= \frac{\xi_1^2}{s}\,\frac{|\xi'|-r}{|\xi'|}\Big(\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\Big)b_1,\\
\widehat{v}_{2A} &= \Big(\frac{1}{|\xi'|}-\frac{1}{r}\Big)\frac{\xi_1\xi_2}{s}\,\widehat{b}_1\cdot e^{-rx_3},\\
\widehat{v}_{2B} &= \frac{\xi_1\xi_2}{s}\,\frac{|\xi'|-r}{|\xi'|}\,\widehat{b}_1\Big(\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\Big),\\
\widehat{v}_{3A} &= \frac{i\xi_1}{r^2}\,\widehat{b}_1 e^{-rx_3},\\
\widehat{v}_{3B} &= \Big(\frac{1}{|\xi'|^2}-\frac{1}{r^2}\Big)\frac{i\xi_1}{s}\,|\xi'|^2\widehat{b}_1 e^{-rx_3},\\
\widehat{v}_{3C} &= \frac{i\xi_1|\xi'|}{s}\,\frac{|\xi'|-r}{|\xi'|^2}\cdot\widehat{b}_1\Big(\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\Big).
\end{aligned}
$$

Since $\mathcal{F}^{-1}_{t,x}$ is linear we have

$$\mathcal{F}^{-1}_{t,x}(\widehat{v}_1)\equiv v_1 = v_{1A}+v_{1B}+v_{1C},\qquad v_2 = v_{2A}+v_{2B}$$

and $v_3 = v_{3A} + v_{3B} + v_{3C}$. We need only examine v_{1A}, v_{2B}, v_{3B}. We consider

$$D^2_{x_1} v_{1A} = \mathcal{F}^{-1}_{t,x}[-\xi_1^2 \widehat{v}_{1A}] = c \int_0^\infty e^{st} \int_{\mathbb{R}^2} e^{ix'\xi'} \cdot \frac{\xi_1}{r} (\xi_1 \cdot \widehat{b}_1) e^{-rx_3} \, d\xi_0 \, d\xi'.$$

From $\xi_1/r \le \xi_1/|\xi_1| \le 1$ and from the Marcinkiewicz theorem we need only estimate

$$\mathcal{F}^{-1}_{t,x}(\xi_1 e^{-rx_3}) \equiv \partial_{x_1}\theta_1 = \frac{x_3 \cdot x_1}{2t^{7/2}} \, e^{-|x|^2/4t},$$

where $\theta_1 \equiv \mathcal{F}_{t,x}(e^{-rx_3}) = -(x_3/t^{5/2}) e^{-|x|^2/4t}$ (see [3]). We put as in [3]

$$I_1(t,x) = \int_{\mathbb{R}^2} dy' \int_{\mathbb{R}} dt' \, \partial_{t'} \partial_{y_1} \theta_1(t', y', x_3)[b_1(x'-y', t-t') - b_1(x', t-t')]. \tag{3.7}$$

Now we get

$$\begin{aligned} &\|I_1\|_{L_p(\mathbb{R}^2_{x'} \times \mathbb{R}_t)} \quad (3.8)\\ &\le \int_{\mathbb{R}^2} dy' \Big(\int_{\mathbb{R}^2_{x'} \times \mathbb{R}_t} |b_1(x'-y', t-t') - b_1(x', t-t')|^p \, dx' \, dt' \Big)^{1/p} \cdot \int_{\mathbb{R}} dt \, \partial_{y_1}\theta_1 \\ &\le \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dy' \, |\partial_{y_1}\theta_1| \psi(y'), \end{aligned}$$

where

$$\psi(y') \equiv \|b_1(x'-y', t-t') - b_1(x', t-t')\|_{L_p(\mathbb{R}^3)}.$$

In view of the Hölder inequality, we obtain

$$\begin{aligned} \|I_1\|_{L_p(\mathbb{R}^3)} \le c x_3 \Big(\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^2} dy' \, \frac{1}{t^{(7/2-m)p}} \, e^{-x_3^2/4t} \cdot e^{-|y'|^2/4t} \cdot \psi^p(y') \Big)^{1/p} \\ \times \Big(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{1}{t^{mq}} \, x_1^q e^{-|x|^2/4t} \Big)^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$ so $q = p/(r-1)$. Let

$$\begin{aligned} S_1 &\equiv \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{x_1^q}{t^{mq}} \cdot e^{-|x|^2/4t} \\ &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{x_1^q}{t^{mq}} \, e^{-x_1^2/4t} \cdot e^{-x_2^2/4t} \cdot e^{-x_3^2/4t}. \end{aligned}$$

Setting $x_2 = w\sqrt{t}$ so $dx_2 = t^{1/2} dw$ we have

$$
\begin{aligned}
(3.9)\qquad S_1 &= \int\limits_{\mathbb{R}_t} dt \int\limits_{\mathbb{R}^2} dx_1\, t^{1/2}\, dw\, \frac{1}{t^{mq}}\, x_1^q e^{-x_1^2/4t} e^{-w^2/4} e^{-x_3^2/4t} \\
&= \int\limits_{\mathbb{R}_t} dt\, t^{1/2}\, \frac{1}{t^{mq}} \int\limits_{\mathbb{R}} dx_1\, x_1^q e^{-x_1^2/4t} \cdot e^{-x^3/4t} \Big(\int\limits_{\mathbb{R}} e^{-w^2/4}\, dw \Big) \\
&\le c \int\limits_{\mathbb{R}} dt \int\limits_{\mathbb{R}} dx_1\, t^{1/2}\, \frac{x_1^q}{t^{mq}}\, e^{-x_1^2/4t} \cdot e^{-x_3^2/4t}.
\end{aligned}
$$

Taking $x_1 = wt^{1/2}$ we have $dx_1 = dwt^{1/2}$ so

$$
\begin{aligned}
(3.10)\qquad & \int\limits_0^\infty dt \int\limits_{\mathbb{R}} dx_1\, t^{1/2}\, \frac{x_1^q}{t^{mq}}\, e^{-x_1^2/4t} e^{-x_3^2/4t} \\
&= c \int\limits_0^\infty dt\, \frac{t^{1/2}}{t^{mq}} \int\limits_{\mathbb{R}} t^{1/2}\, dw\, w^q t^{q/2} e^{w^2/4} e^{-x_3^2/4t} \\
&= c \int\limits_0^\infty dt\, \frac{t^{q/2+1}}{t^{mq}}\, e^{-x_3^2/4t} \int\limits_{\mathbb{R}} dw\, w^q e^{-w^2/4} = c \int\limits_0^\infty \frac{dt\, t^{q/2+1}}{t^{mq}}\, e^{-x_3^2/4t}.
\end{aligned}
$$

Taking $t = |x_3|^2/w$, we get

$$
\int\limits_0^\infty dw\, w^{-2} |x_3|^2\, \frac{\frac{|x_3|^{q+2}}{w^{q/2+1}}}{\frac{|x_3|^{2mq}}{w^{mq}}}\, e^{-w/4} = x_3^{q-2mq+4} \int\limits_0^\infty dw\, \frac{e^{-w/4}}{w^{q/2+3-mq}}.
$$

To have the integral bounded we assume

$$
q/2 + 3 - mq < 1 \quad \text{so} \quad m > 7/2 - 2/p.
$$

For $m > 7/2 - 2/p$ we obtain the estimate

$$
\Big(\int\limits_{\mathbb{R}_t \times \mathbb{R}^2_{x'}} dt\, dx'\, \frac{x_1^q}{t^{mq}}\, e^{-|x|^2/4t} \Big)^{1/q} \le c x_3^{(4+q-2mq)/q},
$$

where

$$
[(4 + q - 2mq)/q] \cdot p = 5p - 2mp - 4,
$$

so

$$
\begin{aligned}
&\|I_2\|^p_{L_p(\mathcal{D}^4)} \\
&\qquad \le c \int\limits_0^\infty dx_3 \cdot x^p \cdot x_3^{5p-2mp-4} \int\limits_0^\infty dt \int\limits_{\mathbb{R}^2} dy'\, \frac{e^{-x_3^2/4t}}{t^{(7/2-m)p}} \cdot e^{-|y'|^2/4t} |y'|^\alpha\, \frac{\psi(y')}{|y'|^\alpha}.
\end{aligned}
$$

Now we consider the integral

$$
(3.11)\qquad S_2 \equiv \int\limits_0^\infty dx_3\, x_3^{6p-2mp-4} \int\limits_0^\infty dt\, \frac{1}{t^{(7/2-m)p}}\, e^{-x_3^2/4t} \cdot e^{-|y'|^2/4t} \cdot |y'|^\alpha.
$$

Taking $x_3 = wt^{1/2}$, we have

$$S_2 = c\int_0^\infty dt\,\frac{t^{3p-mp-3/2}}{t^{(7/2-m)p}}\,e^{-|y'|^2/4t}|y'|^\alpha.$$

For $t = |y|^2/w$ we get

$$S_2 = c\int_0^\infty dw\,\frac{|y'|^{6p-2mp-3+2-7p+2mp+\alpha}}{w^{3p-mp-3/2+2-7/2p+mp}}\cdot e^{-w/4}.$$

To have S_2 bounded and independent of y' we need

$$6p-2mp-3+2-7p+2mp+\alpha = 0$$

so

$$\alpha = p+1 \quad\text{and}\quad 3p-\frac{3}{2}+2-\frac{7}{2}p < 1 \text{ so } p > -1. \tag{3.12}$$

Thus from (3.11) and (3.12) we get

$$\begin{aligned}&\|I_1\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)},\\ &\|D^2_{x_1}v_1\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.\end{aligned}\tag{3.13}$$

Let us consider

$$D^2_{x_2}v_{1A} = \mathcal{F}^{-1}_{t,x}[-\xi_2^2\widehat{v}_{1A}] = c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\,\frac{\xi_2}{r}\,(\xi_2\widehat{b}_1)e^{-rx_3}\,d\xi_0\,d\xi'.$$

Since $\xi_2/r \le \xi_2/\xi_2 = 1$ it is enough to examine the inverse Fourier transform of $\xi_2\cdot e^{-rx_3}$. As above we get

$$\|D^2_{x_2}v_{1A}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.\tag{3.14}$$

Now we consider v_{2B}:

$$\begin{aligned}v_{2B} &= c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{x'\xi'}\left(\frac{\xi_1\xi_2}{s}\,\frac{(|\xi'|-r)(|\xi'|+r)}{|\xi'|(|\xi'|+r)}\widehat{b}_1\left(\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\right)\right)d\xi_0\,d\xi'\\ &= \int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\,\frac{\xi_2\cdot\xi_1}{|\xi'|(|\xi'|+r)}\,\widehat{b}_1\left(\xi_1^2\,\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\right)d\xi_0\,d\xi'.\end{aligned}$$

Because we can find $M > 0$ such that

$$\left|\frac{\xi_2\xi_1}{|\xi'|(|\xi'|+r)}\right| \le M,$$

by the Marcinkiewicz theorem and by [3] we only need to consider $|D^2_{x'}\theta_3|$, where

$$\theta_3 \equiv \mathcal{F}^{-1}_{t,x'}\left[\frac{e^{-rx_3}-e^{-|\xi'|x_3}}{r-|\xi'|}\right].$$

Then from [3] we get

$$\begin{aligned} \|D^2_{x_1} v_{2B}\|_{L_p(\mathcal{D}^4)} &\le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}, \\ \|D^2_{x_2} v_{1B}\|_{L_p(\mathcal{D}^4)} &\le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}. \end{aligned} \tag{3.15}$$

Considering

$$v_{3B} = c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{i\xi' x'}\,\frac{i\xi_1}{s}\,\frac{r^2-|\xi'|^2}{r^2|\xi'|^2}\cdot|\xi'|^2\widehat{b}_1 e^{-rx_3},$$

we get

$$D^2_{x_j} v_{3B} = c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\,\frac{i\xi_1\xi_j^2}{r^2}\,\widehat{b}_1 e^{-rx_3}\,d\xi_0\,d\xi' \qquad (j=1,2). \tag{3.16}$$

Since there exists $M>0$ such that $\xi_1\cdot\xi_j/r^2\le M$ we obtain

$$\|D^2_{x'} v_{3B}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} \quad \text{for } x'=(x_1,x_2). \tag{3.17}$$

Now using (3.13)–(3.17) we have

$$\|D^2_{x'} v\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}. \tag{3.18}$$

We have to estimate $D^2_{x_3}v$ and $D_t v$ in the L_p norm. It is enough to show estimates for v_{1B} and v_{1C}. We get

$$\begin{aligned} &D^2_{x_j} v_{1B} \\ &= \mathcal{F}^{-1}_{t,x}(-\xi_j^2\widehat{v}_{1B}) \\ &= c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\cdot\frac{\xi_1^2}{s}\left(\frac{r-|\xi'|}{|\xi'|r}\right)\widehat{b}_1\xi_j^2 e^{-rx_3}\,d\xi_0\,d\xi' \\ &= c\int_0^\infty e^{st}\int_{\mathbb{R}^2} e^{ix'\xi'}\cdot\frac{\xi_1^2\cdot\xi_j}{r|\xi'|(r+|\xi'|)}\,(\xi_j e^{-rx_3})\,d\xi_0\,d\xi' \qquad (j=1,2). \end{aligned} \tag{3.19}$$

Now the inequality

$$\left|\frac{\xi_1^2\xi_j}{r|\xi'|(r+|\xi'|)}\right| \le \left|\frac{|\xi'|^3}{r^2|\xi'|}\right| \le \left|\frac{|\xi'|^3}{|\xi'|^3}\right| = 1$$

implies

$$\|D^2_{x_j} v_{1B}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} \qquad (j=1,2).$$

We can see that

$$D^2_{x_j} v_{1C} = c\int_0^\infty e^{st}\int_{\mathbb{R}^2} \frac{\xi_1^2\cdot\widehat{b}_1}{|\xi'|(|\xi'|+r)}\,\xi_j^2\,\frac{e^{-|\xi'|x_3}-e^{-rx_3}}{|\xi'|-r}\,d\xi_0\,d\xi'.$$

There exists $M > 0$ such that

$$\left|\frac{|\xi|^2}{|\xi'|(|\xi'|+r)}\right| \le M$$

so we have (see [3])

$$\|D^2_{x_j} v_{1C}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

Since

$$v_{2A} = \mathcal{F}^{-1}_{t,x}\left[\frac{(r-|\xi'|)(r+|\xi'|)}{|\xi'|r(r+|\xi'|)}\,\frac{\xi_1\xi_2}{s}\,\widehat{b}_1 e^{-rx_3}\right],$$

it follows that

$$\|D^2_{x_j} v_{2A}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

In this same way we show

$$\|D^2_{x_j} v_{3A}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)},$$
$$\|D^2_{x_j} v_{3C}\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

From [3] we get

$$\|D_t v\|_{L_p(\mathcal{D}^4)} \le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

To estimate $D^2_{x_3} v$ we use

$$\frac{\partial^2 v}{\partial x_3^2} = \partial_t v - \frac{\partial^2 v}{\partial x_1^2} - \frac{\partial^2 v}{\partial x_2^2} + \nabla q.$$

Then we have

$$\begin{aligned}\|D^2_{x_3} v\|_{L_p(\mathcal{D}^4)} &\le c\|D_t v\|_{L_p(D^4)} + \|D^2_{x_1} v\|_{L_p(\mathcal{D}^4)} \\ &\quad + \|D^2_{x_2} v\|_{L_p(\mathcal{D}^4)} + \|D_x q\|_{L_p(\mathcal{D}^4)} \\ &\le c\|b_1\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.\end{aligned} \tag{3.20}$$

This concludes the proof of the lemma.

Now we have to consider two problems:

$$\begin{aligned}&\partial_t v - \Delta v + \nabla q = 0,\\ &\operatorname{div} v = 0,\\ &\partial_1 v_3 + \partial_3 v_1|_{x_3=0} = 0,\\ &\partial_2 v_3 + \partial_3 v_2|_{x_3=0} = b_2,\\ &v_3|_{x_3=0} = 0,\\ &v|_{t=0} = 0,\end{aligned}$$

and

$$(3.21)\qquad \begin{aligned} &\partial_t v - \Delta v + \nabla q = 0,\\ &\operatorname{div} v = 0,\\ &\partial_1 v_3 + \partial_3 v_1|_{x_3=0} = 0,\\ &\partial_2 v_3 + \partial_3 v_2|_{x_3=0} = 0,\\ &v_3|_{x_3=0} = b_3. \end{aligned}$$

From Lemma 3.1 with $b_2 \in W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)$ we get the estimate

$$(3.22)\qquad \|D_x^2 v\|_{L_p(\mathcal{D}^4)} + \|D_t v\|_{L_p(\mathcal{D}^4)} + \|\nabla q\|_{L_p(\mathcal{D}^4)} \le c\|b_2\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}.$$

Lemma 3.2. *Let $b_3 \in W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)$. Then there exists a solution of problem* (3.21) *such that $v \in W_p^{2,1}(\mathcal{D}^4)$, $p \in W_{p,\mathrm{loc}}^{1,0}(\mathcal{D}^4)$ and*

$$\begin{aligned} \|D_t v\|_{L_p(\mathcal{D}^4)} &\le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)},\\ \|D_x^2 v\|_{L_p(\mathcal{D}^4)} &\le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)},\\ \|D_x q\|_{L_p(\mathcal{D}^4)} &\le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}, \end{aligned}$$

where $\mathcal{D}^4 = \mathbb{R}^2_{x'} \times [0,\infty)_{x_3} \times [0,\infty)_t$, $\mathbb{R}^3_T = \mathbb{R}^2_{x'} \times [0,\infty)_t$.

Proof. First we will estimate

$$v_{1A} = \mathcal{F}^{-1}_{t,x}\left(\frac{i\xi_1}{r}\,\widehat{b}_3 e^{-rx_3}\right).$$

We have

$$D^2_{x_1} v_{1A} = \mathcal{F}^{-1}_{t,x}\left(\frac{\xi_1^3}{r}\,\widehat{b}_3 e^{-rx_3}\right).$$

From the Marcinkiewicz theorem it is enough to consider (see [3])

$$\mathcal{F}^{-1}_{t,x}(\xi_1^2 e^{-rx_3}\widehat{b}_3) = \frac{x_1 e^{-|x|^2/4t}}{2t^{7/2}} - \frac{x_1^2 x_3 e^{-|x|^2/4t}}{4t^{9/2}} \equiv A.$$

Since $\int_{\mathbb{R}^2} A\,dx' = 0$ we get

$$\int_{\mathbb{R}^2} A(y')\cdot b_3(x', t-t')\,dy' = 0,$$

so we can put

$$K(t,x) = \int_{\mathbb{R}^2} dy' \int_{\mathbb{R}} dt' A(y', x_3, t')(b_3(x'-y', t-t') - b_3(x', t-t'))$$

and consider the following integrals:

$$S_1 \equiv \int_{\mathbb{R}^2} dy' \int_{\mathbb{R}} dt' \left| \frac{y_1 e^{-x_3^2/4t} e^{-|y'|^2/4t}}{t^{7/2}} \right| \psi(y'),$$

$$S_2 \equiv \int_{\mathbb{R}^2} dy' \int_{\mathbb{R}} dt' \left| \frac{y_1^2 x_3 e^{-x_3^2/4t} e^{-|y'|^2/4t}}{t^{9/2}} \right| \psi(y'),$$

where $\psi(y') \equiv \|b_3(x'-y',t-t') - b_3(x',t-t')\|_{L_p(\mathbb{R}^3)}$. From the Minkowski and Hölder inequalities we get

$$\begin{aligned}(3.23)\quad \|K\|_{L_p(\mathbb{R}^2\times\mathbb{R}_t)} &\le cS_1 + cS_2 \\ &\le c\bigg(\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^2} dy' \, \frac{e^{-x_3^2/4t}}{t^{(7/2-m)p}} \, e^{-|y'|^2/4t} \psi^p(y')\bigg)^{1/p} \\ &\quad\cdot \bigg(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{x_1^q}{t^{mq}} \, e^{-|x|^2/4t}\bigg)^{1/q} \\ &\quad + cx_3 \bigg(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dy' \, \frac{e^{-x_3^2/4t} \cdot e^{-|y'|^2/4t}}{t^{(9/2-m)p}} \, \psi^p(y')\bigg)^{1/p} \\ &\quad\cdot \bigg(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{x_1^{2q}}{t^{mq}} \, e^{-|x|^2/4t}\bigg)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. From Lemma 3.1 we have

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \, \frac{x_1^q}{t^{mq}} \, e^{-|x|^2/4t} \le c x_3^{4+q-2mq}$$

(see (3.10)–(3.11)). This implies that

$$\|S_1\|^p_{L_p(\mathcal{D}^4)} \le c \int_0^\infty dx_3 x_3^{5p-2mp-4} \int_0^\infty dt \int_{\mathbb{R}^2} dy' \, \frac{e^{-x_3^2/4t}}{t^{(7/2-m)p}} \, e^{-|y'|^2/4t} |y'|^\alpha \, \frac{\psi^p(y')}{|y'|^\alpha}.$$

Let us consider the integral

$$\int_0^\infty dx_3 \, x_3^{5p-2mp-4} \int_0^\infty dt \, \frac{e^{-|y'|^2/4t}}{t^{(7/2-m)p}} \, |y'|^\alpha \equiv M.$$

Putting $x_3 = wt^{1/2}$ we have

$$M \le c \int_0^\infty dt \, t^{5/2p-mp-3/2} \, \frac{e^{-|y'|^2/4t}}{t^{(7/2-m)p}} \, |y'|^\alpha.$$

Now putting $t = |y'|^2/w$ we get

$$M \le c \int_0^\infty \frac{|y'|^{2+\alpha+5p-37p}}{w^{1/2+5/2p-7/2p}}\, e^{-w/4}\, dw. \tag{3.24}$$

To have the integral (3.24) independent of $|y'|$ and bounded we need

$$2 + \alpha + 5p - 3 - 7p = 0$$

and

$$\frac{1}{2} + \frac{5}{2}\, p - \frac{7}{2}\, p < 1$$

so $\alpha = 1 + 2p$ and $p > -1/2$. We can see that $2 + p(2 - 1/p) = 1 + 2p = \alpha$. This implies

$$\begin{aligned} \|S_1\|^p_{L_p(\mathcal{D}^4)} &\le c \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_{x'} \times \mathbb{R}_t} \frac{|b_3(x' - y', t - t') - b_3(x', t - t')|^p}{|y'|^\alpha}\, dx'\, dy'\, dt \\ &\le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}. \end{aligned} \tag{3.25}$$

We now estimate the second integral S_2. Let us consider the second term in (3.23). Putting $x_2 = w\sqrt{t}$ we obtain

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx'\, \frac{x_1^{2q} e^{-|x|^2/4t}}{t^{mq}} \le c \int_0^\infty dt \int_{\mathbb{R}} dx_1\, \frac{x_1^{2q}}{t^{mq}}\, t^{1/2} e^{-x_1^2/4t} e^{-x_3^2/4t}.$$

Putting $x_1 = wt^{1/2}$ in the following integral we get

$$\int_0^\infty dt \int_{\mathbb{R}} dx_1\, \frac{x_1^{2q}}{t^{mq}}\, t^{1/2} e^{-x_1^2/4t} e^{-x_3^2/4t} \le c \int_0^\infty dt\, \frac{t^{1+q}}{t^{mq}}\, e^{-x_3^2/4t}. \tag{3.26}$$

Inserting $t \equiv |x_3|^2/w$ into (3.26) we get

$$\int_0^\infty dt\, \frac{t^{1+q}}{t^{mq}}\, e^{-x_3^2/4t} \le x_3^{4+2q-2mq} \int_0^\infty \frac{e^{-w/4}}{w^{3+q-mq}}\, dw.$$

We have the estimate (see [3])

$$\left(\int_{\mathbb{R}^3} dt\, dx'\, \frac{x_1^{2q}}{t^{mq}}\, e^{-|x|^2/4t} \right)^{1/q} \le c x_3^{(4+2q-2mq)/q}.$$

Now consider the algebraic equality

$$[(4 + 2q - 2mq)/q]p = 6p - 2mp - 4,$$

which implies that

$$\|S_2\|^p_{L_p(\mathcal{D}^4)} \le c \int_0^\infty dx_3\, x_3^{7p-2mp-4} \int_0^\infty dt \int_{\mathbb{R}^2} dy'\, \frac{e^{-x_3^2/4t} e^{-|y'|^2/4t}}{t^{(9/2-m)p}}\, \frac{\psi(y')^p}{|y'|^\alpha}\, |y'|^\alpha.$$

We first consider the integral

$$\int_0^\infty dx_3\, x_3^{7p-2mp-4} \int_0^\infty dt\, \frac{e^{-x_3^2/4t} e^{-|y'|^2/4t}}{t^{(9/2-m)p}}\, |y'|^\alpha.$$

Putting $x_3 = wt^{1/2}$ we get the integral

$$c\int_0^\infty dt\, \frac{t^{7/2p-mp-3/2}}{t^{(9/2-m)p}}\, e^{-|y'|^2/4t}|y'|^\alpha.$$

Now putting $t = |y'|^2/w$ we have

$$c\int_0^\infty \frac{|y'|^{7p-2mp-3+2+\alpha-9p+2mp}}{w^{2+7/2p-mp-3/2-9/2p+mp}}\, e^{-w/4}\, dw. \tag{3.27}$$

To have the integral (3.27) independent of $|y'|$ and bounded we need

$$7p - 1 + \alpha - 9p = 0$$

and

$$2 + \frac{7}{2}p - \frac{3}{2} - \frac{9}{2}p < 1.$$

This implies that

$$\alpha = 1 + 2p \quad \text{and} \quad p > -1/2.$$

From (2.1) we can see that $2 + p(2 - 1/p) = 1 + 2p = \alpha$. And this implies that

$$\|S_2\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}. \tag{3.28}$$

From (3.25), (3.28) and (3.23) we get

$$\begin{aligned}\|K\|^p_{L_p(\mathcal{D}^4)} &\le c_1\|S_1\|^p_{L_p(\mathcal{D}^4)} + c_2\|b_3\|^p_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} \\ &\Rightarrow \|D^2_{x_1} v_{1A}\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}.\end{aligned}$$

Now we write the explicit form of $\widehat{v}_1, \widehat{v}_2, \widehat{v}_3$:

$$\begin{aligned}\widehat{v}_1 &= \frac{i\xi_1\widehat{b}_3}{r}\, e^{-rx_3} + \left(\frac{1}{r} - \frac{1}{\xi'|}\right)\frac{i\xi(|\xi'|^2 + r^2)}{s}\,\widehat{b}_3 e^{-rx_3} \\ &\quad - \frac{(|\xi'| - r)}{|\xi'|}\,\frac{i\xi_1(|\xi'|^2 + r^2)}{s}\,\widehat{b}_3\,\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'| - r} \\ \widehat{v}_2 &= \frac{i\xi_2\widehat{b}_3}{r}\, e^{-rx_3} + \left(\frac{1}{r} - \frac{1}{|\xi'|}\right)\frac{i\xi_2(|\xi'|^2 + r^2)}{s} \\ &\quad \cdot \widehat{b}_3\,\frac{-(|\xi'| - r)}{|\xi'|}\,\frac{i\xi_2(|\xi'|^2 + r^2)}{s}\, b_3\,\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'| - r}\end{aligned}$$

$$\widehat{v}_3 = \frac{-|\xi'|^2}{r^2}\, b_3 e^{-rx_3} + \left(\frac{1}{|\xi'|^2} - \frac{1}{r^2}\right) \frac{|\xi'|^2(|\xi'|^2+r^2)}{s}\, \widehat{b}_3 e^{-rx_3}$$
$$+ \frac{(|\xi'|-r)(|\xi'|^2+r^2)}{s}\, \widehat{b}_3\, \frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r}$$
$$\widehat{q} = \frac{|\xi'|^2+r^2}{|\xi'|}\, \widehat{b}_3 e^{-|\xi'|x_3}.$$

As in the proof of Lemma 3.1, we can put

$$\widehat{v}_{1A} + \widehat{v}_{1B} + \widehat{v}_{1C} = \widehat{v}_1, \quad \widehat{v}_2 = \widehat{v}_{2A} + \widehat{v}_{2B} + \widehat{v}_{2C}, \quad \widehat{v}_3 = \widehat{v}_{3A} + \widehat{v}_{3B} + \widehat{v}_{3C}.$$

Repeating the considerations for v_{1A} we have

$$\|D^2_{x_2} v_{1A}\|_{L_p(\mathcal{D}^4)} + \|D^2_{x'} v_{2A}\|_{L_p(\mathcal{D}^4)} + \|D^2_{x'} v_{2B}\|_{L_p(\mathcal{D}^4)}$$
$$+ \|D^2_{x'} v_{3A}\|_{L_p(\mathcal{D}^4)} + \|D^2_{x} v_{3B}\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}.$$

Now we have to consider $\widehat{v}_{\alpha A}$, $\widehat{v}_{\alpha B}$ and $\widehat{v}_{\alpha C}$ with the term

$$\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r}$$

for $\alpha = 1,2,3$. Let us note the algebraic equality

$$\frac{(|\xi'|-r)(|\xi'|^2+r^2)}{s} = \frac{(|\xi'|^2-r^2)(|\xi'|^2+r^2)}{(|\xi'|+r)\cdot s}.$$

Since $r^2 = |\xi'|^2 + s$ we have $|\xi'|^2 - r^2 = -s$ and this implies that

$$\frac{(|\xi'|-r)(|\xi'|^2+r^2)}{s} = -\frac{|\xi'|^2+r^2}{|\xi'|+r}.$$

It is enough to consider $\widehat{v}_{3C}$ of the following form:

$$\widehat{v}_{3C} = \frac{(|\xi'|-r)(|\xi'|^2+r^2)}{s}\, \widehat{b}_3 \left(\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r}\right).$$

This implies that

$$D^2_{xj} v_{3C} = \mathcal{F}^{-1}_{t,x'}\left[\frac{|\xi'|^2+r^2}{|\xi'|+r} \cdot \xi_j^2 \cdot \widehat{b}_3 \left(\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi' - r}\right)\right].$$

From the Marcinkiewicz theorem it is enough to consider

$$\mathcal{F}^{-1}_{t,x'}\left[\xi_1^3 \widehat{b}_3 \left(\frac{e^{-|\xi'|x_3} - e^{-rx_3}}{|\xi'|-r}\right)\right]. \tag{3.29}$$

Denoting (3.29) by A we get

$$|A| \le c\left|\frac{x_j}{t^{1/2}(x^2+t)^3}\right|, \qquad \int\limits_{\mathbb{R}^2} A\, dx' = 0.$$

Now we put

$$\theta_3 \equiv \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} dy' \, |A(t', y', x_3)| [b(x' - y', t - t') - b(x', t - t')].$$

We have

$$\|\theta_3\|_{L_p(\mathbb{R}^3)} \le \int_0^\infty dt \int_{\mathbb{R}^2} \frac{dy'}{t^{1/2}(y_1^2 + x_3^2 + t)^3} \, \frac{\psi(y')}{y_j},$$

where

$$\psi(y') \equiv \|b(x' - y', t - t') - b(x', t - t')\|_{L_p(\mathbb{R}^3)}.$$

So we have the estimate

$$\|\theta_3\|_{L_p(\mathbb{R}^3)} \le \bigg(\int_{\mathbb{R}_t} dt \int_{\mathbb{R}^2} \frac{dy' \, y_j^{-p} \psi^p(y')}{t^{1/2}(y_1^2 + x_3^2 + t)^{3-mp}} \bigg)^{1/p}$$

$$\cdot \bigg(\int_0^\infty dt \int_{\mathbb{R}^2} \frac{dy'}{t^{1/2}(y^2 + x_3^2 + t)^{mq}} \bigg)^{1/q},$$

where $1/p + 1/q = 1$. From [3] we have the following inequality for the second integral:

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}} dy' \, \frac{1}{t^{1/2}(y^2 + x_3^2 + t)^{mq}} \le c \int_0^\infty \frac{dt}{t^{1/2}(x_3^2 + t)^{mq-1}}.$$

Hence choosing, as in [3], $m > \frac{3}{2} \frac{p-1}{p}$ we get

$$\bigg(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} \frac{dy'}{t^{1/2}(y^2 + x_3^2 + t)^{mq}} \bigg)^{1/q} \le c x_3^{(3-2mq)/q}.$$

Thus we have

$$\|\theta_3\|^p_{L_p(\mathcal{D}^4)} \le c \int_0^\infty dx_3 \, x_3^{(3-2mq)(p-1)} \int_{\mathbb{R}_+ \times \mathbb{R}^2} dt \, dy' \, \frac{|y'|^{-p} |y'|^{\alpha} \psi^p(y')}{t^{1/2}(x^2 + t)^{(2-m)p} |y'|^{\alpha}}.$$

We consider the integral of the form

$$\int_0^\infty dx_3 \, x_3^{(3-2mq)(p-1)} \int_0^\infty \frac{dt \, |y'|^{-p} |y'|^{\alpha}}{t^{1/2}(x^2 + t)^{(2-m)p}}.$$

Taking $w = t/x^2$ we get (for simplicity we write $x' = y'$)

$$\int_0^\infty dx_3 \int_0^\infty dw \, x^2 \, \frac{x_3^{(3-2mq)(p-1)} |x'|^{\alpha - p}}{|x| w^{1/2} (1 + w)^{(2-m)p} |x|^{2(2-m)p}} \equiv D,$$

where

$$D \le c \int_0^\infty dx_3 \, x_3^{(3-2mq)(p-1)} |x'|^{\alpha - p} |x|^{1-(4-2m)p}$$

for $(2-m)p > 1/2$. Putting $w = x_3/|x'|$ we obtain

$$D \le c \int_0^\infty dw \, \frac{|x'|^{1+\alpha-p+3p-3-2mp+1-4p+2mp} w^{3p-3-2mp}}{(1+w^2)^{((4-2m)p-1)/2}}$$

To have the integral independent of $|x'|$ we need

$$1+\alpha-p+3p-3+1-4p = 0 \ \Rightarrow\ \alpha = 2p+1.$$

We also need $3p-3-2mp-((4-2m)p-1) < -1$ so $p > -1$. This implies that

$$\|\theta_3\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)},$$

so

$$\|D^2_{x'} v_{3c}\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)},$$

which implies that

$$\|D^2_{x'} v\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}. \tag{3.30}$$

Similarly one can show that

$$\|D_x p\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}, \tag{3.31}$$

$$\|D_t v\|_{L_p(\mathcal{D}^4)} \le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}. \tag{3.32}$$

Now from $(3.48)_1$ and (3.30)–(3.32) we have

$$\frac{\partial^2 v}{\partial x_3^2} = \partial_t v - D^2_{x_1} v - D^2_{x_2} v + D_x q.$$

This implies that

$$\begin{aligned}\|D^2_{x_3} v\|_{L_p(\mathcal{D}^4)} &\le \|D_t v\|_{L_p(\mathcal{D}^4)} + \|D^2_{x'} v\|_{L_p(\mathcal{D}^4)} + \|D_x q\|_{L_p(\mathcal{D}^4)}\\ &\le c\|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}.\end{aligned}$$

Hence $v \in W^{2,1}_{p,\mathrm{loc}}(\mathcal{D}^4)$. Lemma 3.2 is proved.

The next step in solving (1.1) is to consider the following problem in the whole space:

$$\begin{aligned} &v_{,t} - \nu \Delta v + \nabla q = f,\\ &\operatorname{div} v = 0,\\ &v|_{t=0} = 0.\end{aligned} \tag{3.33}$$

From [3] we have

LEMMA 3.3. *Let $f \in L_p(\mathbb{R}^4)$. Then there exists a unique solution of* (3.33) *such that $v \in W^{2,1}_{p,\mathrm{loc}}(\mathbb{R}^4)$, $q \in W^{1,0}_{p,\mathrm{loc}}(\mathbb{R}^4)$ and*

$$\|D_t v\|_{L_p(\mathcal{D}^4)} + \|D^2_x v\|_{L_p(\mathcal{D}^4)} + \|D_x p\|_{L_p(\mathcal{D}^4)} \le c\|f\|_{L_p(\mathbb{R}^4)}.$$

Moreover if $f \in L_p^{\mathrm{div}}(\mathbb{R}^4)$ *then* $q \equiv 0$, *where*

$$L_p^{\mathrm{div}}(\mathbb{R}^4) = \overline{\{f \in C_0^\infty(\mathbb{R}^4) : \operatorname{div} f = 0\}}^{\|\cdot\|_{L_p}}.$$

Using Lemmas 3.2 and 3.3 we deduce the main result of this section.

LEMMA 3.4. *Let* $p \geq 2$, $f \in L_p(\mathbb{R}^4)$, $G \in W_p^{1,0}(\mathbb{R}^4)$, $\partial_t G - \operatorname{div} f = \operatorname{div} B + A$ *for* $(A, B) \in L_p(\mathbb{R}^4)$ *and* $\operatorname{diam}\operatorname{supp} A < \lambda$. *Suppose that* $b = (b_1, b_2, 0)^T$, $b \in W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)$ *and* $b_3 \in W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)$. *Then there exists a unique solution of the system*

$$\begin{aligned} &\partial_t v - \nu \Delta v + \nabla q = f, \\ &\operatorname{div} v = G, \\ &\overline{e}_i \cdot \mathbb{D}(v) \cdot \overline{\tau}_i + \gamma v \cdot \overline{\tau}_i = b_i \quad (i = 1, 2), \\ &v \cdot \overline{e}_3 = b_3, \\ &v|_{t=0} = 0, \end{aligned} \tag{3.34}$$

such that $v \in W_p^{2,1}(\mathcal{D}^4)$, $q \in W_p^{1,0}(\mathcal{D}^4)$ *and*

$$\begin{aligned} &\|D_t v\|_{L_p(\mathcal{D}^4)} + \|D_x^2 v\|_{L_p(\mathcal{D}^4)} + \|v\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} + \|\nabla q\|_{L_p(\mathcal{D}^4)} \\ &\quad \leq c(T)[\|f\|_{L_p(\mathcal{D}^4)} + \|B\|_{L_p(\mathcal{D}^4)} + \lambda \cdot \|A\|_{L_p(\mathcal{D}^4)} \\ &\qquad + \|b\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} + \|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}], \end{aligned} \tag{3.35}$$

where $\mathcal{D}^4 = \mathbb{R}^2_{x'} \times [0, \infty)_{x_3} \times \mathbb{R}_t$.

Proof. Let us first consider the system

$$\begin{aligned} &\Delta \omega = G, \\ &\omega|_{x_3=0} = 0 \quad (\omega \to 0,\ x_3 \to \infty), \end{aligned} \tag{3.36}$$

where $G \in W_p^{1,0}(\mathcal{D}^4)$. From (3.36) we get (see [3])

$$\|\nabla \omega\|_{W_p^{2,0}(\mathcal{D}^4)} \leq c\|G\|_{W_p^{1,0}(\mathcal{D}^4)}.$$

We put $\nabla(\partial_t \omega) = \nabla(\partial_t \omega^{(1)}) + \nabla(\partial_t \omega^{(2)})$. Then $\Delta(\partial_t \omega^{(1)}) = \operatorname{div}(F + B)$, $\Delta(\partial_t \omega^{(2)}) = A$. From [3] we have

$$\begin{aligned} &\|\nabla(\partial_t \omega^{(1)})\|_{L_p(\mathcal{D}^4)} \leq c(\|f\|_{L_p(\mathcal{D}^4)} + \|B\|_{L_p(\mathcal{D}^4)}), \\ &\|\nabla(\partial_t \omega^{(2)})\|_{L_p(\mathcal{D}^4)} \leq c \cdot \lambda \|A\|_{L_p(\mathcal{D}^4)}, \end{aligned}$$

and this implies

$$\begin{aligned} &\|\nabla \omega\|_{W_p^{2,1}(\mathcal{D}^4)} \\ &\quad \leq c(\|G\|_{W_p^{1,0}(\mathcal{D}^4)} + \|f\|_{L_p(\mathcal{D}^4)} + \|B\|_{L_p(\mathcal{D}^4)} + \lambda\|A\|_{L_p(\mathcal{D}^4)}). \end{aligned} \tag{3.37}$$

Putting $v = u + \nabla\omega$ we have a perturbation of system (3.34):

$$
\begin{aligned}
&\partial_t u - \Delta u + \nabla q = f^*,\\
&\operatorname{div} u = 0,\\
&\overline{n}\cdot\mathbb{D}(u)\cdot\overline{\tau}_i + \gamma u\cdot\overline{\tau}_i|_S = b_i \quad (i=1,2),\\
&u\cdot\overline{n}|_S = b_3 - \nabla u|_S = b_3^*,\\
&u|_{t=0} = 0,
\end{aligned}
\tag{3.38}
$$

where

$$
\begin{aligned}
&f^* \equiv f - \nabla(\partial_t\omega) + \Delta\nabla\omega, \quad b_i^* \equiv b_i - \overline{n}\cdot\mathbb{D}(\omega)\cdot\overline{\tau}_i - \gamma\nabla\omega\cdot\overline{\tau}_i,\\
&b_3^* = b_3 - \nabla\omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|f^*\|_{L_p(\mathcal{D}^4)} \le \|f\|_{L_p(\mathcal{D}^4)} + c\|\nabla\omega\|_{W_p^{2,1}(\mathcal{D}^4)},\\
&\|b_i^*\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} \le \|b_i\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} + c\|\nabla\omega\|_{W_p^{2,1}(\mathcal{D}^4)}\\
&\hspace{25em}(i=1,2),\\
&\|b_3^*\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} \le \|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} + c\|\nabla\omega\|_{W_p^{2,1}(\mathcal{D}^4)}.
\end{aligned}
\tag{3.39}
$$

Now consider the problem

$$
\begin{aligned}
&\Delta q' = \operatorname{div} f^*,\\
&\frac{\partial q'}{\partial n}\bigg|_S = g.
\end{aligned}
\tag{3.40}
$$

Then $\|\nabla q'\|_{L_p(\mathcal{D}^4)} \le c\|f^*\|_{L_p(\mathcal{D}^4)}$ (see [3]) and $q' \in W_p^{1,0}(\mathcal{D}^4)$.

Now consider the problem

$$
\begin{aligned}
&\Delta q' = \operatorname{div} f^*,\\
&\frac{\partial q'}{\partial n}\bigg|_S = g, \quad \text{where} \quad g \equiv \nu\phi - b_{3,t} + f\cdot\overline{n}
\end{aligned}
$$

for $\phi \equiv \alpha_1\cdot v_n + \alpha_2\cdot v_{\tau_i,\tau_i} + \alpha_3\cdot v_{\tau_i} + \alpha_4\cdot v_{\tau_i,\tau_i} + \alpha_5 b_{i,\tau_i} + \alpha_6 b_{3,\tau_i\tau_i} + \gamma v + a_\beta b_{\beta,\tau_i} - a_\beta v\cdot n_{,\tau_i}$, where $v_n = (v_n\cdot\overline{n} + v_i\overline{\tau}_i)\cdot\overline{n}$, $v_{\tau_i} = v\cdot\overline{\tau}_i$ $(i=1,2)$. Then $\|\nabla q'\|_{L_p(\mathcal{D}^4)} \le c(\|f^*\|_{L_p(\mathcal{D}^4)} + \|b_i\|_{W_p^{1-1/p,1/2-1/p}(\mathbb{R}^3)} + \|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}$ (see [3]) and $q' \in W_p^{1,0}(\mathcal{D}^4)$. Now we put $q = q' + q''$, where p' is a solution of problem (3.40). This implies the following perturbation of system (3.38):

$$
\begin{aligned}
&\partial_t u - \Delta u + \nabla q'' = f^0,\\
&\operatorname{div} u = 0,\\
&\overline{n}\cdot\mathbb{D}(u)\cdot\overline{\tau}_i + \gamma u\cdot\overline{\tau}_i|_S = b_i \quad (i=1,2),\\
&u\cdot\overline{n}|_S = b_3,\\
&u|_{t=0} = 0,
\end{aligned}
$$

where $f^0 \equiv f^* - \nabla q'$ and div $f^0 = 0$. From Lemmas 3.2 and 3.3 (the case with div $f = 0$) we get

$$\begin{aligned}\|D_t u\|_{L_p(\mathcal{D}^4)} + \|D_x^2 u\|_{L_p(\mathcal{D}^4)} + \|\nabla q''\|_{L_p(\mathcal{D}^4)} \\ \le c(T)[\|f^0\|_{L_p(\mathcal{D}^4)} + \|b^*\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} + \|b_3^*\|_{W_p^{2-1/2,1-1/2p}(\mathbb{R}^3)}].\end{aligned}$$

Now from $v = u + \nabla\omega$, $q = q' + q''$, $b_i = b_i^* + \overline{n} \cdot \mathbb{D}(\nabla\omega) \cdot \overline{\tau}_i + \gamma\nabla\omega \cdot \overline{\tau}_i$ $(i = 1, 2)$, $b_3 = b_3^* + \nabla\omega$ we obtain

$$\begin{aligned}\|D_t v\|_{L_p(\mathcal{D}^4)} + \|D_x^2 v\|_{L_p(\mathcal{D}^4)} + \|\nabla q\|_{L_p(\mathcal{D}^4)} + \|v\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)} \\ \le c(T)[\|f\|_{L_p(\mathcal{D}^4)} + \|B\|_{L_p(\mathcal{D}^4)} + \lambda\|A\|_{L_p(\mathcal{D}^4)} \\ + \|b\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)} + \|b_3\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^3)}].\end{aligned}$$

This concludes the proof of the lemma.

4. Problem in a bounded domain. In this section we prove the existence of solutions of the following problem in the bounded domain Ω^T:

$$\begin{aligned} &\partial_t v - \Delta v + \nabla q = f, \\ &\operatorname{div} v = 0, \\ &\overline{n} \cdot \mathbb{D}(v) \cdot \overline{\tau}_i + \gamma v \cdot \overline{\tau}_i|_S = b_i, \\ &v \cdot \overline{n}|_S = b_3, \\ &v|_{t=0} = 0, \end{aligned} \tag{4.1}$$

where similarly to [3] we restrict our considerations to the case

$$f \in L_p^{\mathrm{div}}(\Omega^T) = \overline{\{f \in C^\infty(\Omega^T) : \operatorname{div} f = 0\}}^{\|\cdot\|_{L_p}}.$$

LEMMA 4.1. *If* $f \in L_2^{\mathrm{div}}(\Omega^T) \cap L_p^{\mathrm{div}}(\Omega^T)$ *then there exists a unique solution of* (4.1) *such that* $v \in W_2^{2,1}(\Omega^T) \cap W_p^{2,1}(\Omega^T)$ *and* $q \in W_p^{1,0}(\Omega^T)$, *where* (v, p) *is a weak solution of* (4.1), *and*

$$\begin{aligned}\|v\|_{W_2^{2,1}(\Omega^T)} + \|v\|_{W_p^{2,1}(\Omega^T)} + \|q\|_{W_p^{1,0}(\Omega^T)} \\ \le c(T)(\|f\|_{L_2(\Omega)} + \|f\|_{L_p(\Omega^T)} + \|B\|_{L_p(\Omega^T)} + \lambda\|A\|_{L_p(\Omega^T)}).\end{aligned} \tag{4.2}$$

Proof. We introduce a partition of unity. Let us define two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that $\omega^{(k)} \subset \Omega^{(k)} \subset \Omega$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega$ with $k \in \mathfrak{M} \cup \mathfrak{N}$, where $\Omega^{(k)} \cap S = \emptyset$ if $k \in \mathfrak{M}$ and $\omega^{(k)} \cap S \neq \emptyset$ if $k \in \mathfrak{N}$. We assume that $\sup \operatorname{diam} \Omega^{(k)} \le 2\lambda$ for some λ small enough. Let $\xi^{(k)}$ be a smooth function such that $0 \le \xi^{(k)} \le 1$ and

$$\xi^{(k)}(x) = \begin{cases} 1 & \text{for } x \in \omega^{(k)}, \\ 0 & \text{for } x \in \Omega \setminus \Omega^{(k)}, \end{cases}$$

and

$$|D_x^\nu \xi^{(k)}(x)| \le c/\lambda^{|\nu|}, \qquad 1 \le \sum_k (\xi^{(k)})^2 \le N_0.$$

By $\eta^{(k)}$ we denote a "center" (a point inside) of $\omega^{(k)}$ for $k \in \mathfrak{M}$ and a center of $\omega^{(k)} \cap S$ for $k \in \mathfrak{N}$. Let us consider a local coordinate system $y = (y_1, y_2, y_3)$ with center at $\eta^{(k)}$. If $k \in \mathfrak{N}$ then the part $\widetilde{S}^{(k)} = S \cap \Omega^{(k)}$ of the boundary is described by $y_3 = F(y_1, y_2)$. We choose the coordinates such that $F(0) = 0$ and $\nabla F(0) = 0$. From $S \in W_p^{2-1/p}$ we see that $F \in W_p^{2-1/p}$. Extending F to $\overline{F}$ in such a way that $\overline{F}(y_1, y_2, 0) = F(y_1, y_2)$ and $\overline{F} \in W_p^2$ we have $\overline{F} \in C^{1+\alpha}$ using the embedding theorem

$$C^{1+\alpha}(\Omega) \hookleftarrow W_p^2(\Omega) \quad \text{for } 3/p + 1 + \alpha < 2, \text{ i.e. } \alpha < 1 - 3/p.$$

Now we can transform $\Omega^{(k)}$ into a half-space by the transformation $Z = \Phi_k(y) = (\mathrm{Id} - \overline{F})(y)$. Let $y = Y_k(x)$ be a transformation to the local coordinates y which consists of translations and rotations. Let us introduce the variables $V = \xi v$ and $Q = \xi q$ for $\xi \equiv \xi^{(l)}$ and $l \in \mathfrak{M}$. Then the equations (4.1) take the form

$$\begin{aligned} &\partial_t V - \Delta V + \nabla Q = \xi \cdot f - 2\nabla v \cdot \nabla \xi - \Delta \xi \cdot v + q \cdot \nabla \xi \equiv F', \\ &\operatorname{div} V = v \cdot \nabla \xi \equiv G', \\ &V|_{t=0} = 0. \end{aligned} \tag{4.3}$$

Now we obtain a condition on the new functions F', G', A', B', where A' and B' are to be defined. We have

$$\partial_t G' - \operatorname{div} F' = \operatorname{div} B' + A', \tag{4.4}$$

where

$$\begin{aligned} B' &= \xi B + 2\nabla\xi \cdot \nabla v + v\Delta\xi - 2q = q \cdot \nabla\xi + \nabla\xi \times \operatorname{rot} v, \\ A' &= \xi \cdot A - B\nabla\xi + q\Delta\xi. \end{aligned} \tag{4.5}$$

By Lemma 3.4 we see that solutions of (4.1) satisfy

$$\begin{aligned} &\|V\|_{W_k^{2,1}(\mathcal{D}^4)} + \|Q\|_{W_k^{1,0}(\mathcal{D}^4)} \\ &\quad \le c(T)(\|F'\|_{L_k(\mathcal{D}^4)} + \|G'\|_{W_k^{1,0}(\mathcal{D}^4)} + \lambda\|A'\|_{L_k(\mathcal{D}^4)} + \|B'\|_{L_k(\mathcal{D}^4)}), \end{aligned} \tag{4.6}$$

where $k > 2$. By Proposition 2.3 we have (see [3])

$$\begin{aligned} &\|F'\|_{L_k(\Omega^T)} \\ &\quad \le c\bigg(\|f\|_{L_k(\Omega^T)} + \frac{T^{1/k} + T^{1/2}}{\lambda}\|v\|_{W_k^{2,1}(\Omega^T)} + \frac{1}{\lambda}\|q\|_{L_k(\Omega^T)}\bigg). \end{aligned} \tag{4.7}$$

Now consider the following problem:

$$\Delta q = 0, \qquad \left.\frac{\partial q}{\partial n}\right|_{S^T} = g, \tag{4.8}$$

where $g = \nu\phi - b_{3,t}$ (see (3.40)). This implies that $\|q\|_{W_p^{2,0}} \le c\|v\|_{W_p^{1,0}(\mathcal{D}^4)}$. Then

$$\|\nabla q\|_{L_k(\Omega^T)}\frac{1}{\lambda} \le \frac{c}{\lambda}(T^{1/p} + T^{1/2})\|v\|_{W_p^{2,1}(\mathcal{D}^4)}.$$

Hence by Proposition 2.3 we obtain

$$\|V\|_{W_k^{2,1}(\mathcal{D}^4)} + \|Q\|_{W_k^{1,0}(\mathcal{D}^4)} \le c(T)\bigg(\|f\|_{L_k(\mathcal{D}^4)} + \frac{T^{1/k} + T^{1/2}}{\lambda}\|v\|_{W_k^{2,1}(\mathcal{D}^4)} + \lambda\|A\|_{L_k(\mathcal{D}^T)} + \|B\|_{L_k(\mathcal{D}^T)}\bigg). \tag{4.9}$$

We would like to have a similar estimate when $k \in \mathbb{N}$. We write system (4.1) in Z-coordinates (see [3]):

$$\begin{aligned} &\partial_t V - \Delta_Z V + \nabla_Z Q = F' + L_1(\partial_Z - \nabla\overline{F}\cdot\partial_Z)(V,Q) \\ &\qquad\qquad\qquad - L_1(\partial_Z)(V,P) \equiv F'', \\ &\operatorname{div}_Z V = G' + \nabla\overline{F}\cdot\partial_Z V \equiv G'', \\ &\overline{e}_{z_3}\cdot\mathbb{D}(V)\cdot\overline{\tau}_i + \gamma V\cdot\xi\overline{e}_{z_3} = \Phi(\overline{n}(\partial_i\eta\cdot v_j + \partial\eta_j\cdot v_i)) \\ &\qquad\qquad\qquad + \nabla\overline{F}\cdot\mathbb{D}(V) + \gamma \equiv b_i'', \\ &V\cdot\overline{e}_{z_3}|_{S^T} = b_3'', \qquad V|_{t=0} = 0. \end{aligned} \tag{4.9a}$$

To apply Lemma 3.4 we need new A'' and B'' which satisfy

$$\partial_t G'' - \operatorname{div} F'' = \operatorname{div} B'' + A''. \tag{4.11}$$

We consider, as in [3], the following system:

$$\begin{aligned} &\Delta b = V_{j,t}\cdot\overline{F}_{i,ji}, \\ &b|_{Z_3=0} = 0, \\ &b \to 0 \quad \text{if } Z_3 \to \infty. \end{aligned} \tag{4.12}$$

By Proposition 2.2 (see [3])

$$\|\nabla b\|_{L_k(0,T;L_{k'+\varepsilon}(\mathbb{R}^3_+))} \le c\lambda^{3\varepsilon/(k'+\varepsilon)k}\|\partial_t V\|_{L_k(\mathcal{D}^4)}. \tag{4.13}$$

We put

$$\begin{aligned} &B'' \equiv B' - [L_1(\partial_Z - \nabla\overline{F}\cdot\partial_Z)(V,Q) - L_1(\partial_Z)(V,Q)] + \overline{F}_{i,j}V_j - \nabla b, \\ &A'' = A'. \end{aligned}$$

From [3] we get

$$\|F''\|_{L_k} \le \|F'\|_{L_k} + c(\delta + c(\delta)T^a + \lambda^\alpha)\|V\|_{W_k^{2,1}} + c\lambda^\alpha\|Q\|_{W_k^{1,0}}.$$

From Lemma 3.4 we have (see [3])

$$\begin{aligned}(4.14)\qquad &\|V\|_{W^{2,1}_k}+\|Q\|_{W^{1,0}_k}\\ &\le c(\|F'\|_{L_k}+\|G'\|_{W^{1,0}_k}+\|B'\|_{L_k}+\|A'\|_{L_k})\\ &\quad+c\lambda^\alpha\|Q\|_{W^{1,0}_k}+c\bigg(\|\overline{F}_{i,j}V_j\|_{L_k}+\|\nabla b\|_{L_k}\\ &\quad+\|\nabla\overline{F}\partial_Z V\|_{W^{1,0}_k}+\|L_1(\partial_Z-\nabla\overline{F}\cdot\partial_Z)(V,Q)\\ &\quad-L_1(\partial_Z)(V,Q)\|_{L_k(\mathcal{D}^4)}+\frac{T^{1/k}+T^{1/2}}{\lambda}\|v\|_{W^{2,1}_k}\bigg).\end{aligned}$$

Now we need to estimate

$$\begin{aligned}&\|L_1(\partial_Z-\nabla\overline{F}\cdot\partial_Z)(V,Q)-L_1(\partial_Z)(V,Q)\|_{L_k(\mathcal{D}^4)}\\ &\qquad\le c\|\nabla^2\overline{F}\nabla V\|_{L_k(\mathcal{D}^4)}+c\|\nabla\overline{F}\nabla^2V\|_{L_k(\mathcal{D}^4)}c\|\nabla\overline{F}|Q|\,\|_{L_k(\mathcal{D}^4)}.\end{aligned}$$

We estimate only the first term of the r.h.s.:

$$\|\,|\nabla^2\overline{F}|\,|\nabla V|\,\|_{L_k(\mathbb{R}^3_T)}\le\|\nabla^2\overline{F}\|_{L_r(\mathbb{R}^3_+)}\|\nabla V\|_{L_l(\mathbb{R}^3_+)},$$

where $l=kr/(r-k)$, and we have

$$\|\nabla V\|_{L_k(0,T;L_l(\mathbb{R}^3_+))}\le(\delta+c(\delta)T^a)\|V\|_{W^{2,1}_r(\mathcal{D}^4)},$$

where $\delta,a>0$ and $c(\delta)$ tends to infinity as $\delta\to0$ (see [3, p. 96]). Taking $c\beta<1/2$, where

$$\beta=\delta+c(\delta)T^a+\lambda^\alpha+\frac{T^{1/2}+T^{1/k}}{\lambda}+\lambda^{\varepsilon/k}$$

is a small parameter coming from the right hand sides of (4.13) and (4.14), and $f\in L^{\mathrm{div}}_p(\Omega^T)$, and using similar estimates to [3] we get

$$\begin{aligned}(4.15)\qquad &\|v\|_{W^{2,1}_p(\Omega^T)}+\|q\|_{W^{1,0}_p(\Omega^T)}\\ &\qquad\le c(\|f\|_{L_p(\Omega^T)}+\|B\|_{L_p(\Omega^T)}+\lambda\|A\|_{L_p(\Omega^T)}).\end{aligned}$$

From (4.14) and (4.15) we get the assertion of the lemma.

5. Proof of Theorem 1. Our aim is to show existence of solution of problem (1.1). Let us first consider an extension $\overline{v}\in W^{2,1}_p(\Omega^T)$ of v such that $\overline{v}|_{t=0}=v_0$, $v_0\in W^{2-2/p}_p(\Omega)$ and $\|\overline{v}\|_{W^{2,1}_p(\Omega^T)}\le c\|v_0\|_{W^{2-2/p}_p(\Omega)}$. Assume that the solution of (1.1) has the form

$$v=v'+\overline{v}\quad\text{and}\quad q=q(x,t).$$

Then problem (1.1) reduces to

$$(5.1)\qquad \begin{aligned} &\partial_t v' - \nu\Delta v' + \nabla q = F',\\ &\operatorname{div} v' = G',\\ &\overline{n}\cdot\mathbb{D}(v')\cdot\overline{\tau}_i + \gamma v'\cdot\overline{\tau}_i|_{S^T} = b_i' \qquad (i=1,2),\\ &v'\cdot\overline{n}|_{S^T} = b_3,\\ &v'|_{t=0} = 0, \end{aligned}$$

where

$$\begin{aligned} &F' = -\partial_t\overline{v} + \nu\Delta\overline{v} + f, \quad G' = G - \operatorname{div}\overline{v}, \quad b_3' = b_3 - \overline{v}\cdot\overline{n},\\ &b_i = b_i - \overline{n}\cdot\mathbb{D}(\overline{v})\cdot\tau_i - \gamma\overline{v}\cdot\overline{\tau}_i. \end{aligned}$$

Now we have

$$\begin{aligned} \partial_t G' - \operatorname{div} F' &= \partial_t G - \partial_t\operatorname{div}\overline{v} + \partial_t\operatorname{div}\overline{v} - \nu\nabla\Delta\overline{v} - \operatorname{div} f\\ &= \operatorname{div} B + A - \nu\nabla\Delta\overline{v} \quad \text{so} \quad \partial_t G' - \operatorname{div} F' = \operatorname{div} B' + A, \end{aligned}$$

where $B' = B - \nu\Delta\overline{v}$ and we get the following estimates:

$$\begin{aligned} &\|F'\|_{L_p} \le \|f\|_{L_p} + c\|v_0\|_{W_p^{2-2/p}},\\ &\|G'\|_{W_p^{1,0}} \le \|G\|_{W_p^{1,0}} + c\|v_0\|_{W_p^{2-2/p}},\\ &\|b_\alpha'\|_{W_p^{1-1/p,1/2-1/2p}} \le \|b_\alpha\|_{W_p^{1-1/p,1/2-1/2p}} + c\|v_0\|_{W_p^{2-2/p}} \qquad (\alpha=1,2),\\ &\|b_3'\|_{W_p^{2-1/p,1-1/2p}} \le \|b_3\|_{W_p^{2-1/p,1-1/2p}} + c\|v_0\|_{W_p^{2-2/p}}. \end{aligned}$$

We need to consider the following system (see [1]):

$$(5.2)\qquad \begin{aligned} &\Delta\psi = G',\\ &\psi|_{S^T} = 0, \end{aligned} \qquad \text{where} \quad \omega = \nabla\psi.$$

From (5.2) we have $\omega \in W_p^{2,0}(\Omega^T)$ and

$$\begin{aligned} \Delta\partial_t\psi = \partial_t G' &= \operatorname{div}(B' + F') + A = \operatorname{div}(B - \nu\Delta\overline{v} + f - \partial_t\overline{v} + \nu\Delta\overline{v}) + A\\ &= \operatorname{div}(B + f - \partial_t\overline{v}) + A = \operatorname{div}(B + f) + \widetilde{A}, \end{aligned}$$

where $\widetilde{A} = A = \nabla\partial_t\overline{v}$. Now putting $\Delta\partial_t\psi^{(1)} \equiv \operatorname{div}(B+f)$, $\Delta\partial_t\psi^{(2)} = \widetilde{A}$ and using similar considerations we have

$$\|\nabla\partial_t\psi\|_{L_p} \le c[\|f\|_{L_p} + \|B'\|_{L_p} + \lambda\|A\|_{L_p}].$$

This implies that $\omega = \nabla\psi \in W_p^{2,1}(\Omega^T)$ and of course

$$\|\omega\|_{W_p^{2,1}(\Omega^T)} \le c(\|G\|_{W_p^{1,0}(\Omega^T)} + \|f\|_{L_p(\Omega^T)} + \|B'\|_{L_p(\Omega^T)} + \lambda\|A\|_{L_p(\Omega^T)}),$$

and from $G'|_{t=0} = 0$ we have $\omega|_{t=0} = 0$. Taking $v' \equiv v'' + \nabla\psi$ in problem

(5.1) we obtain

$$
\begin{aligned}
&\partial_t v'' - \nu \Delta v'' + \nabla q = F'',\\
&\operatorname{div} v'' \equiv 0,\\
&\overline{n} \cdot \mathbb{D}(v'') \cdot \overline{\tau}_i + \gamma v'' \cdot \overline{\tau}_i|_{S^T} = b_i'' \quad (i = 1, 2),\\
&v'' \cdot \overline{n}|_{S^T} = b_3'',\\
&v''|_{t=0} = 0,
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
F'' &= F' - \nabla \partial_t \psi + \nu \nabla \Delta \psi,\\
b_i'' &= b_i' - \overline{n} \cdot \mathbb{D}(\nabla \psi) \cdot \overline{\tau}_i - \gamma \nabla \psi \cdot \overline{\tau}_i \quad (i = 1, 2), \qquad b_3'' = b_3' - \nabla \psi \cdot \overline{n}.
\end{aligned}
$$

Now we have the estimates

$$
\begin{aligned}
&\|F''\|_{L_p} \le \|F'\|_{L_p} + c\|\omega\|_{W_p^{2,1}},\\
&\|b_i''\|_{W_p^{1-1/p,1/2-1/2p}} \le \|b_i'\|_{W_p^{1-1/p,1/2-1/2p}} + c\|\omega\|_{W_p^{2,1}} \qquad (i = 1, 2),\\
&\|b_3''\|_{W_p^{2-1/p,1-1/2p}} \le \|b_3'\|_{W_p^{2-1/p,1-1/2p}} + c\|\omega\|_{W_p^{2,1}}.
\end{aligned}
\tag{5.4}
$$

Next we have to reduce system (5.3) to problem (4.1). Consider the following problem:

$$
\begin{aligned}
\Delta q &= \operatorname{div} F'',\\
\frac{\partial q}{\partial n}\bigg|_{S^T} &= g,
\end{aligned}
$$

where $g = F''\overline{n} - b_{3,t}'' + \nu \phi''$, $\phi'' = \nu \Delta v'' \cdot \overline{n}$.

For this elliptic problem we have the following estimate:

$$
\|q\|_{W_p^2(\Omega)} \le c(\|F''\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(S)}),
$$

where

$$
q(x,t) = \int_\Omega G_1(x-y) \operatorname{div} F''_{(y)}\, dy + \int_S G_2(x-y) g(y,t)\, dy.
$$

Then

$$
\nabla q(x,t) = \int_\Omega \nabla G_1 \operatorname{div} F''(y,t)\, dy + \int_S \nabla G_2 g(y,t)\, dy,
$$

$$
\|\nabla q\|_{L_p(\Omega^T)} \le \bigg\| \int_\Omega \nabla^2 G_1 F''(y,t)\, dy \bigg\|_{L_p(\Omega^T)} + \bigg\| \int_S \nabla G_2 g(y,t)\, dy \bigg\|_{L_p(S^T)}.
$$

Now we can consider only one term

$$
\int_S \nabla G_2 \cdot b''_{,\tau\tau} = -\int_S (\nabla G_2)_{,\tau} b_{,\tau}.
$$

This implies that

$$\Big\| \int_S \nabla G_2 b''_{,\tau\tau} \Big\|_{L_p(S^T)} \le \Big\| \int_S \nabla^2 G_2 b''_{,\tau} \Big\|_{L_p(S^T)} \le c\|b''_{,\tau}\|_{L_p(S^T)} \le c\|b''\|_{W_p^{1-1/p,1/2-1/p}(S^T)}.$$

Then

$$\|\nabla q\|_{L_p(\Omega^T)} \le c\|F''\|_{L_p(\Omega^T)} + c\|b''_i\|_{W_p^{1-1/p,1/2-1/2p}(S^T)} + c\|b''_3\|_{W_p^{2-1/p,1-1/2p}(S^T)}).$$

Now putting $q \equiv q' + \widetilde{q}$, where $\widetilde{q}$ is a solution of the system

$$\Delta\widetilde{q} = \operatorname{div} F'', \quad \Delta q' = \operatorname{div} F',$$
$$\frac{\partial\widetilde{q}}{\partial n}\Big|_{S^T} = h, \quad \frac{\partial q'}{\partial n} = g', \quad \text{where } h = g - g',$$

we get

$$\begin{aligned}
&\partial_t v'' - \nu\Delta v'' + \nabla q' = F'' - \nabla\widetilde{q} = F''', \\
&\operatorname{div} v'' = 0, \\
&\overline{n} \cdot \mathbb{D}(v'') \cdot \tau_i + \gamma v'' \cdot \tau_i|_{S^T} = b''_i \quad (i = 1, 2), \\
&v'' \cdot \overline{n}|_{S^T} = b''_3, \\
&v''|_{t=0} = 0,
\end{aligned}$$

where

$$\operatorname{div} F''' = \operatorname{div} F'' - \Delta\widetilde{q} = 0.$$

Now using Lemma 4.1 we obtain a solution of system (5.5) which gives a solution of system (1.1) given by

$$v = \overline{v} + \nabla\psi + v'' + \phi \quad \text{and} \quad q = q' + \widetilde{q}.$$

Now it is enough to continue the solution to the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$, $\ldots, T \le T_0$.

This proves Theorem 1 and concludes our considerations.

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