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DEFAULTABLE BONDS WITH AN INFINITE NUMBER OF LÉVY FACTORS

Abstract. A market with defaultable bonds where the bond dynamics is in a Heath–Jarrow–Morton setting and the forward rates are driven by an infinite number of Lévy factors is considered. The setting includes rating migrations driven by a Markov chain. All basic types of recovery are investigated. We formulate necessary and sufficient conditions (generalized HJM conditions) under which the market is arbitrage-free. Connections with consistency conditions are discussed.

Introduction. The paper is concerned with the market containing a risk-free bond and defaultable bonds issued by companies. A defaultable bond will default with a certain probability before or at maturity time T. The probabilities of defaults depend on economic conditions of the firms and are reflected by *rating classes* assigned by rating agencies like Standard&Poor's or Moody's. If a default does not occur an owner of the bond receives, as in the case of default-free bond, one currency unit. In the case of default the owner obtains a part of the promised payoff. This part depends on the credit rating of the issuer of the bond and on the adopted recovery scheme. To model defaultable bonds we use the intensity based models which are the basic way of modeling (see e.g. Bielecki and Rutkowski [1], Lando [24]). In contrast to most papers on the subject, which use Brownian motion for modeling (see e.g. Duffie and Singleton [12], Bielecki and Rutkowski [1], [3]), we apply the theory of Lévy processes which admit discontinuous trajectories and contain many standard processes like Brownian motion, Poisson processes, and generalized hyperbolic Lévy motion.

It is well known that using Lévy processes to modeling has many advantages (see e.g. Eberlein and Özkan [14], Eberlein and Kluge [13], Cont

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and Tankov [9], Özkan and Schmidt [25]) such as better calibration procedure for real-world and also risk-neutral data. Eberlein and Raible [15] and Eberlein and Özkan [14] used finite-dimensional Lévy processes with exponential moments in some neighborhood of zero to model the term structure of defaultable forward rates. They generalize the approach of Bielecki and Rutkowski [1] to defaultable bonds with rating migration. This approach is in the spirit of the Heath, Jarrow and Morton (hereafter HJM) methodology [17]. They assume that the real-world defaultable forward rates dynamics as well as recovery schemes are exogenously specified and they establish existence of an arbitrage-free model that supports these objects. More precisely, they show that if the intensity matrix process satisfies the so called "consistency condition" then one can construct a rating migration process and price processes of defaultable bonds with credit migration that are, under an appropriate measure, local martingales after discounting. The consistency conditions are interpreted as conditions on the intensity matrix of the rating migration process. We should stress that these conditions do not determine the intensity matrix process uniquely, so actually there can be infinitely many transition matrix processes satisfying those systems of equations. Neither Bielecki and Rutkowski nor Eberlein and Özkan attempt to generalize the HJM condition to a condition on the drift term which guarantees that the HJM postulate is satisfied, i.e. that the discounted bond prices are local martingales (see Definition 1.1).

In this paper we do this in the case of defaultable bonds. We cover all situations of practical importance. The same question in the infinite-dimensional case was considered by Schmidt [27] with Brownian motion as a noise and by Özkan and Schmidt [25] with Lévy noise and recovery of market value. [27] gives necessary and sufficient conditions for discounted prices of defaultable bonds to be martingales in the case of rating based recovery of market value and recovery of treasury value. Özkan and Schmidt's [25] approach is based on Musiela parameterization and requires more stringent conditions on the model than ours, since the Itô formula for processes with values in Hilbert spaces is used. As we notice in Remark 3.3 their result is not true without some additional assumptions.

In this paper we give the generalized HJM conditions in the case of defaultable bonds and typical recovery schemes. We consider fractional recovery of market value, fractional recovery of treasury value and fractional recovery of par value. The multiple default case introduced by Schönbucher [29], which allows one to consider company reorganization, is discussed as well. From the very beginning we assume that the Lévy processes may be infinite-dimensional. The importance of treating models with an infinite number of factors was stressed in recent papers of Carmona and Tehranchi [7], Ekeland and Taflin [16], Cont [8] and Schmidt [27].

In Section 1 we recall basic facts on forward rates driven by Lévy processes and the HJM-type condition for non-defaultable bonds provided that the market is arbitrage-free. Next, in Section 2, we describe credit risk models with and without rating migration. The rating classes vary according to a conditional continuous time Markov chain and the default time is equal to the time of entering the worst rating class. In the main part (Section 3) we give HJM-type conditions for defaultable bonds with credit migration. These conditions depend on the form of recovery and the rating migration process. From a structural point of view, all equations follow a similar pattern, where one has the classical HJM drift condition plus an additional term, depending on the particular recovery rate. All is proved under a natural assumption on the default risk-adjusted short-term interest rate (Hypothesis (H1)). More precisely, under Hypothesis (H1) we prove that in the general case the HJM postulate is equivalent to the generalized HJM condition. It is worth mentioning that in a model in which all processes are continuous, we do not need to assume (H1). Namely, (H1) plus the generalized HJM condition is then equivalent to the HJM postulate. We also formulate generalized HJM conditions in terms of the derivative of the logarithm of the moment generating function of the Lévy noise (Section 4). These forms are much more useful in applications (see e.g. [18]). In Section 5 we formulate, following [2], consistency conditions involving the recovery structure, default intensities and bond prices. We prove that these conditions are equivalent to the HJM type conditions derived in the previous sections. Hence, under (H1), we can extend and generalize the results of [1] and [14] to the case of infinite-dimensional Lévy processes. The proofs of our results are given in the last section of the paper. The present paper is a version of the preprint [20].

Summing up, the main contributions of the paper are necessary and sufficient conditions (generalized HJM conditions) on the coefficients in the definition of the forward rates ensuring that the discounted prices of defaultable bonds are martingales. These conditions are given for all typical recovery schemes and with infinite-dimensional Lévy noise as the source of uncertainty in the dynamics of defaultable forward rates, which is the most general Lévy noise one can use. Our assumptions on the Lévy processes are weaker than having exponential moments in some neighborhood of zero, as in [14], [13] and [25].

1. Preliminaries. We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We take Lévy processes with values in some abstract separable Hilbert space U as the source of uncertainty in our model. Let Z be a *Lévy process*, i.e. a càdlàg process with independent and stationary increments and values in U with inner product denoted by $\langle\cdot,\cdot\rangle_U$. Let $\mathcal{F}_t^0 = \sigma(Z(s); s \le t)$ be the σ-field generated by $Z(t), t \ge 0$, and $\mathcal{F}_t$ be the completion of $\mathcal{F}_t^0$ by all sets of $\mathbf{P}$ probability zero. It is known that this filtration is right continuous, so it satisfies the "usual conditions". We can associate with $Z(t)$ a measure of its jumps, denoted by μ, i.e. for any $A \in \mathcal{B}(U)$ such that $\overline{A} \subset U \setminus \{0\}$ we have

$$\mu([0,t], A) = \sum_{0<s\le t} \mathbb{1}_A(\Delta Z(s)).$$

The measure ν defined by

$$\nu(A) = \mathbf{E}(\mu([0,1], A))$$

is called the *Lévy measure* of the process Z. Stationarity of increments implies that $\mathbf{E}(\mu([0,t], A)) = t\nu(A)$. The Lévy–Khinchin formula shows that the characteristic function of the Lévy process has the form

$$\mathbf{E}e^{i\langle\lambda, Z(t)\rangle_U} = e^{t\psi(\lambda)},$$

with

$$\psi(\lambda) = i\langle a, \lambda\rangle_U - \frac{1}{2}\langle Q\lambda, \lambda\rangle_U + \int_U (e^{i\langle\lambda,x\rangle_U} - 1 - i\langle\lambda, x\rangle_U \mathbb{1}_{\{|x|_U\le 1\}}(x))\, \nu(dx),$$

where $a \in U$, Q is a symmetric nonnegative nuclear operator on U, ν is a measure on U with $\nu(\{0\}) = 0$ and

$$\int_U (|x|_U^2 \wedge 1)\, \nu(dx) < \infty.$$

Let b be the Laplace transform of ν restricted to the complement of the ball $\{y : |y|_U \le 1\}$,

$$b(u) = \int_{|y|_U>1} e^{-\langle u,y\rangle_U} \nu(dy), \tag{1.1}$$

and set

$$B = \{u \in U : b(u) < \infty\}.$$

Z has the well-known *Lévy–Itô decomposition*:

$$Z(t) = at + W(t) + Z_0(t),$$

where

$$Z_0(t) = \int_0^t \int_{|y|_U\le 1} y\,(\mu(ds, dy) - ds\,\nu(dy)) + \int_0^t \int_{|y|_U>1} y\,\mu(ds, dy),$$

and W is a Wiener process with values in U and covariance operator Q.

Let $r(t)$, $t \ge 0$, be the short rate process and

$$B_t = e^{\int_0^t r(\sigma)\, d\sigma}.$$

Let $B(t,\theta)$, $0 \le t \le \theta \le T^*$, be the market price at time t of a risk-free bond paying 1 at maturity time θ; T^* is a finite horizon of the model. The *forward* *rate curve* is a function $f(t,\theta)$ defined for $t \le \theta$ and such that

$$B(t,\theta) = e^{-\int_t^\theta f(t,s)\,ds}. \tag{1.2}$$

It is convenient to assume that once a bond has matured its cash equivalent goes to the bank account. Thus $B(t,\theta)$, the market price at time t of a bond paying 1 at maturity time θ, is also defined for $t \ge \theta$ by the formula

$$B(t,\theta) = e^{\int_\theta^t r(\sigma)\,d\sigma}. \tag{1.3}$$

We postulate here the following dynamics for forward rates:

$$df(t,\theta) = \alpha(t,\theta)dt + \langle \sigma(t,\theta), dZ(t)\rangle_U, \tag{1.4}$$

where for each θ the processes $\alpha(t,\theta)$, $\sigma(t,\theta)$, $t \le \theta$, are assumed to be predictable with respect to the given filtration $(\mathcal{F}_t)$ and such that the integrals in (1.4) are well defined. Sometimes we use another form of SDE for forward rates,

$$df(t) = \tilde{\alpha}(t)dt + \tilde{\sigma}(t)dZ(t), \tag{1.5}$$

where $\tilde{\alpha}(t)$ is a function on $[0,T^*]$ given by $\tilde{\alpha}(t)(\theta) = \alpha(t,\theta)$ and $\tilde{\sigma}(t)$ is a linear operator from U into $L^2[0,T^*]$ defined by

$$(\tilde{\sigma}(t)u)(\theta) = \langle \sigma(t,\theta), u\rangle_U.$$

For $t > \theta$ we put

$$\alpha(t,\theta) = \sigma(t,\theta) = 0. \tag{1.6}$$

So we will assume that for given T^*, the integrals in the definition of f exist in the sense of the Hilbert space $H = L^2[0,T^*]$ with scalar product $(\cdot,\cdot)$. We will regard the coefficients α and σ as, respectively, H- and $L(U,H)$-valued predictable processes.

It follows from (1.4) that for $t \le \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)\,ds + \int_0^t \langle \sigma(s,\theta), dZ(s)\rangle_U,$$

and for $t \ge \theta$, according to (1.6),

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)\,ds + \int_0^\theta \langle \sigma(s,\theta), dZ(s)\rangle_U.$$

Thus the process $f(t,\theta)$ for $t \ge \theta$ is constant for each $\theta > 0$, say equal to $f(\theta,\theta)$, and it can be identified with the short rate process

$$r(\theta) = f(\theta,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)\,ds + \int_0^\theta \langle \sigma(s,\theta), dZ(s)\rangle_U.$$

Definition 1.1. We say that *the HJM postulate holds* if the discounted bond prices

$$\hat{B}(t,\theta) = \frac{B(t,\theta)}{B_t}, \quad t \le \theta,$$

are local martingales for each $\theta \in [0, T^*]$.

Since for $t > u$ we have $f(t,u) = f(u,u)$, it follows that

$$B_t = e^{\int_0^t f(u,u)\,du} = e^{\int_0^t f(t,u)\,du},$$

and thus the discounted bond prices can be written as

$$\hat{B}(t,\theta) = e^{-\int_t^\theta f(t,u)\,du} e^{-\int_0^t f(t,u)\,du} = e^{-\int_0^\theta f(t,u)\,du},$$

and hence the HJM postulate is that the processes $\hat{B}(\cdot,\theta)$, $\theta \in [0,T^*]$, given by

$$\hat{B}(t,\theta) = e^{-\int_0^\theta f(t,u)\,du} = e^{-(f(t),\mathbb{1}_{[0,\theta]})},$$

are local martingales. We will assume that the processes Z, α and σ satisfy the following conditions:

(A1a) The processes α and σ are predictable and with probability one have bounded trajectories (the bound may depend on ω).

(A1b) For arbitrary $r > 0$ the function b given by (1.1) is bounded on $\{u : |u| \le r,\, b(u) < \infty\}$.

(A2) For all $\theta \le T^*$, $\mathbf{P}$-almost surely,

$$\int_t^\theta \sigma(t,v)\,dv \in B \tag{1.7}$$

for almost all $t \in [0,\theta]$.

Our goal is to find a condition which ensures that the HJM postulate holds (in fact this condition will, under some assumptions, be equivalent to the HJM postulate). By analogy to the classical case we call this condition the *generalized HJM condition* or the *HJM-type condition.*

It is convenient to express the generalized HJM condition in terms of the *Laplace exponent* of the Lévy process Z, i.e. of the logarithm of the moment generating function of Z, that is, the functional $J : U \to \mathbb{R}$ given by

$$J(u) = -\langle u,a\rangle_U + \frac{1}{2}\langle Qu,u\rangle_U + \int_{\{|y|_U\le 1\}} (e^{-\langle u,y\rangle_U} - 1 + \langle u,y\rangle_U)\,\nu(dy) + \int_{\{|y|_U>1\}} (e^{-\langle u,y\rangle_U} - 1)\,\nu(dy).$$

The following theorem, under other assumptions, goes back to the paper [4] by Björk, Di Massi, Kabanov and Runggaldier (see also Eberlein and Özkan [14]). We present it following Jakubowski and Zabczyk [21].

Theorem A. *Assume* (A1) *and* (A2) *hold. The discounted bond prices are local martingales if and only if the following HJM-type condition holds:*

$$A(t,\theta) = J(\Sigma(t,\theta)) \tag{1.8}$$

for each $\theta \in [0, T^*]$ *and almost all* $t \le \theta$, *where*

$$A(t,\theta) := \int_t^\theta \alpha(t,v)\,dv, \quad \Sigma(t,\theta) := \int_t^\theta \sigma(t,v)\,dv.$$

Using integration by parts and the dynamics of the bond, we obtain

Theorem 1.2. *The processes of discounted price of the bond have the following dynamics:*

$$d\hat{B}(t,\theta) = \hat{B}(t-,\theta)\Big(\bar{a}(t,\theta)dt + \int_U [e^{-\langle \Sigma(t,\theta),y\rangle_U} - 1](\mu(dt,dy) - dt\nu(dy)) - \langle \Sigma(t,\theta), dW(t)\rangle_U\Big),$$

where

$$\bar{a}(t,\theta) = -A(t,\theta) + J(\Sigma(t,\theta)).$$

Corollary 1.3. *The process of discounted bond price can be written in the following integral form:*

$$\hat{B}(t,\theta) = \hat{B}(0,\theta)\exp\Big(-\int_0^t A(s,\theta)\,ds - \int_0^t \langle \Sigma(s,\theta), dZ(s)\rangle_U\Big),$$

and if the HJM-type condition (1.8) *holds, then*

$$\hat{B}(t,\theta) = \hat{B}(0,\theta)\exp\Big(-\int_0^t J(\Sigma(s,\theta))\,ds - \int_0^t \langle \Sigma(s,\theta), dZ(s)\rangle_U\Big).$$

In what follows we assume that condition (1.8) is fulfilled.

2. Description of credit risk models. In the default-free world, by a bond maturing at time θ with face value 1 we mean a financial instrument whose payoff is 1 at time θ. In a defaultable case we have several variants describing the amount and timing of so called *recovery rate* which is paid to bondholders if default has occurred before the bond's maturity. If we denote by τ the default time, then, generally speaking, the payoff of the defaultable bond is as follows:

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}} \cdot \text{recovery payment}.$$

If δ is a recovery rate process, then the recovery payment can take different forms (see e.g. [2] and references there):

- $\delta(t)D(\tau-,\theta)B_\theta/B_\tau$, *fractional recovery of market value*: at time of default bondholders receive a fraction of the pre-default market value of the defaultable bond (i.e. of $D(\tau-,\theta)$):

$$D(\theta,\theta)=\mathbb{1}_{\{\tau>\theta\}}+\mathbb{1}_{\{\tau\le\theta\}}\cdot\delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau},$$

  where $\delta(t)$ is an $\mathbb{F}$-predictable process with values in $[0,1]$.
- δ, *fractional recovery of treasury value*: a fixed fraction δ of the bond's face value is paid to bondholders at the bond's maturity date θ:

$$D^\delta(\theta,\theta)=\mathbb{1}_{\{\tau>\theta\}}+\mathbb{1}_{\{\tau\le\theta\}}\cdot\delta.$$

- $\delta B_\theta/B_\tau$, *fractional recovery of par value*: a fixed fraction δ of the bond's face value is paid to bondholders at default time τ:

$$D^\Delta(\theta,\theta)=\mathbb{1}_{\{\tau>\theta\}}+\mathbb{1}_{\{\tau\le\theta\}}\cdot\delta\frac{B_\theta}{B_\tau}.$$

Our objective is to derive the HJM drift condition for models with different kinds of recovery and with migration of credit ratings.

2.1. Models with rating migration. We give a short description of a model with rating migration; for details see Bielecki and Rutkowski [2]. We assume that the credit rating migration process C^1, which is a càdlàg process, is modeled by a conditional Markov chain relative to $\mathbb{F}$ with values in the set of rating classes $\mathcal{K}=\{1,\dots,K\}$, where state $i=1$ represents the highest rank, $i=K-1$ the lowest rank, and $i=K$ the default event. With state i, $i\le K-1$, there is associated the pre-default term structure g_i. We assume that the instantaneous defaultable forward rates have dynamics $g_i(t,\theta)$ given by

$$dg_i(t,\theta)=\alpha_i(t,\theta)dt+\langle\sigma_i(t,\theta),dZ_i(t)\rangle_U,\qquad i\in\{1,\dots,K-1\},$$

where $Z_i(t)$ are Lévy processes with values in U. By the Lévy–Itô decomposition, each $Z_i(t)$ has the form

$$Z_i(t)=a_it+W_i(t)+\int_0^t\int_{|y|_U\le1}y\,(\mu_i(ds,dy)-ds\,\nu_i(dy))+\int_0^t\int_{|y|_U>1}y\,\mu_i(ds,dy),$$

where $a_i\in U$ and μ_i is the jump measure of Z_i. Let

$$D_i(t,\theta)=e^{-\int_t^\theta g_i(t,u)\,du}$$

and denote the discounted values of D_i by $\hat{D}_i(t,\theta)=D_i(t,\theta)/B_t$. Applying the Itô lemma as in the default-free case we have (below, J_i corresponds to Z_i in the same way as J corresponds to Z)

THEOREM 2.1. *The dynamics of the process* $\hat{D}_i(t,\theta)$ *is given by*

$$d\hat{D}_i(t,\theta) = \hat{D}_i(t-,\theta)\Big((g_i(t,t) - f(t,t) + \bar{a}_i(t,\theta))\,dt + \int_U [e^{-\langle \Sigma_i(t,\theta),y\rangle_U} - 1]\,(\mu_i(dt,dy) - dt\,\nu_i(dy)) - \langle \Sigma_i(t,\theta), dW_i(t)\rangle_U\Big),$$

where $\bar{a}_i(t,\theta)$ *satisfies*

$$\bar{a}_i(t,\theta) = -A_i(t,\theta) + J_i(\Sigma_i(t,\theta)), \tag{2.1}$$

and we denote

$$A_i(t,\theta) = \int_t^\theta \alpha_i(t,v)\,dv, \qquad \Sigma_i(t,\theta) = \int_t^\theta \sigma_i(t,v)\,dv.$$

To preserve the interpretation of rating classes, i.e. the fact that higher rated bonds are more expensive than lower rated ones, it is reasonable to assume that

$$g_{K-1}(t,\theta) > g_{K-2}(t,\theta) > \cdots > g_1(t,\theta) > f(t,\theta)$$

for all $t \in [0,\theta]$ and all $\theta \in [0,T^*]$. This condition implies that inter-rating spreads are positive.

Given two filtrations $\mathbb{F}$ and $\mathbb{G}$, the $\mathbb{F}$-conditional infinitesimal generator of the process C^1 describing the credit rating migration at time t given the σ-field $\mathcal{F}_t$ has the form

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \cdots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

where the off-diagonal processes $\lambda_{i,j}(t)$, $i \neq j$, are nonnegative processes adapted to $\mathbb{F} \subseteq \mathbb{G}$, and the diagonal elements are negative and determined by off-diagonals,

$$\lambda_{i,i}(t) = -\sum_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}(t).$$

For our purposes we specify $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ where $\mathbb{F} = \mathbb{F}^{\hat{Z}} \vee \mathbb{F}^{\Lambda}$, $\hat{Z} = (Z, Z_1, \ldots, Z_K)$ and $\mathbb{H} = \mathbb{F}^{C^1}$, i.e. $\mathcal{F}_t = \mathcal{F}_t^{\hat{Z}} \vee \mathcal{F}_t^{\Lambda}$, $\mathcal{G}_t = \mathcal{F}_t^{\hat{Z}} \vee \mathcal{F}_t^{\Lambda} \vee \mathcal{F}_t^{C^1}$. A detailed construction of C^1 in this case can be found in Bielecki and Rutkowski [2], [3] or Lando [23].

To describe the credit risk we also need, besides the credit migration process C^1 defined above, the process C^2 of the previous ratings. If we denote by $\tau_1, \tau_2, \ldots$ the consecutive jump times of the credit migration process C^1, then for $t \in [\tau_k, \tau_{k+1})$,

$$C^1(t) := C^1(\tau_k), \qquad C^2(t) := C^1(\tau_{k-1}).$$

We denote by $C(t)$ the two-dimensional credit rating process defined by

$$C(t) = (C^1(t), C^2(t)).$$

Therefore the pre-default term structure depending on $C^1(t)$ is given by

$$g(t,u) = g_{C^1(t)}(t,u) = \mathbb{1}_{\{C^1(t)=1\}} g_1(t,u) + \cdots + \mathbb{1}_{\{C^1(t)=K-1\}} g_{K-1}(t,u).$$

We sum up to $K-1$, since the last Kth rating corresponds to the default event

$$\tau = \inf\{t > 0 : C^1(t) = K\}.$$

It is obvious that each recovery rate depends on the credit rating before default, i.e.

$$\delta(t) = \delta_{C^2(t)}(t) = \mathbb{1}_{\{C^2(t)=1\}} \delta_1(t) + \cdots + \mathbb{1}_{\{C^2(t)=K-1\}} \delta_{K-1}(t),$$

where δ_i is a recovery rate process connected with the ith rating and such that $\delta_i(t) \in [0,1)$.

We make a standard assumption on the relationship between short term spread, recovery and the intensity of migration into the default state for defaultable bonds (see e.g. Jarrow et al. [22], Duffie and Singleton [11]).

Hypothesis (H1).

$$g_i(t,t) - f(t,t) = \lambda_{i,K}(t)(1 - \delta_i(t)), \quad i = 1, \ldots, K-1,\ t < T^*. \tag{2.2}$$

Hypothesis (H1) postulates that the intensity of migration from rating i into the default state K is equal to the short term credit spread for rating i divided by one minus recovery from rating i. Of course, this does not mean that the forward rates f, g are strongly linked. It only means that we cannot arbitrarily specify the intensities of the migration into the default state K if we have specified f, g_i and the recovery δ. Of course, (2.2) implies

$$g_{C^1(t)}(t,t) = f(t,t) + (1 - \delta_{C^1(t)}(t))\lambda_{C^1(t),K}(t), \quad t < T^*. \tag{2.3}$$

Hypothesis (H1) is natural, which can be seen from the following facts.

Remark 2.2. If the price of a defaultable bond with fractional recovery of market value is given in a traditional way (see Duffie and Singleton [12]), then it is given by the intensity proces λ and the risk-free short term rate r in the following way:

$$\mathbb{1}_{\{\tau>t\}} \hat{D}(t,\theta) = \mathbb{1}_{\{\tau>t\}} \mathbf{E}(e^{-\int_t^\theta [r(u) + (1-\delta(u))\lambda(u)]du} \mid \mathcal{F}_t).$$

Then, for bounded λ and r, we have

$$\begin{aligned} g_1(t,t) :=& -\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}\ln \mathbf{E}(e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t) \\ &= -\lim_{\theta\downarrow t}\frac{\frac{\partial}{\partial\theta}\mathbf{E}(e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)}{\mathbf{E}(e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)} \\ &= -\lim_{\theta\downarrow t}\frac{\mathbf{E}(\frac{\partial}{\partial\theta}e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)}{\mathbf{E}(e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)} \\ &= \lim_{\theta\downarrow t}\frac{\mathbf{E}([r(\theta)+(1-\delta(\theta))\lambda(\theta)]e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)}{\mathbf{E}(e^{-\int_t^\theta [r(u)+(1-\delta(u))\lambda(u)]\,du} \mid \mathcal{F}_t)} \\ &= r(t)+(1-\delta(t))\lambda(t), \end{aligned}$$

so (2.2) holds.

For models with ratings we can make a similar observation. We illustrate this in the next propositions where we assume that the price of a bond has a natural form (2.4) (see e.g. Jakubowski and Niewęgłowski [19]) and we demonstrate that this form of price implies (2.2).

Proposition 2.3. *Let a market of defaultable bonds with fractional recovery of par value be such that the price D of a bond maturing at $\theta > 0$ is, on the set $\{C_t = i\}$, $i \neq K$, equal to*

$$\begin{aligned} (2.4)\qquad & D(t,\theta)\mathbb{1}_{\{C_t=i\}} \\ &= \mathbb{1}_{\{C_t=i\}}\sum_{j=1}^{K-1}\mathbf{E}\Big(e^{-\int_t^\theta r(v)\,dv}p_{i,j}(t,\theta)+\delta_j\int_t^\theta e^{-\int_t^u r(v)\,dv}p_{i,j}(t,u)\lambda_{j,K}(u)\,du\,\Big|\,\mathcal{F}_t\Big) \end{aligned}$$

for $t<\theta$, where δ_j is the recovery rate for rating j and $p(t,u)$ is the solution to the (random) conditional Kolmogorov forward equation

$$dp(t,u) = p(t,u)\Lambda(u)du, \qquad p(t,t) = \mathbb{I},$$

with the intensity matrix process Λ. Assume that r and Λ are bounded processes. Then

$$g_i(t,t) = r(t) + (1-\delta_i)\lambda_{i,K}(t)$$

for $i<K$ and $t<\theta$.

As we announced in the Introduction, the proof of this proposition, as well as other proofs, are given in the last section of the paper.

It is worth noticing that the same conclusions can be drawn for other kinds of recovery. Next, we assume that the credit migration process and bond price processes have no common jumps.

Hypothesis (H2). For the consecutive jump times $(\tau_k)_{k\geq 0}$ of the credit migration process and for all $\theta \in [0, T^*]$ we have

$$\mathbf{P}(\Delta B(\tau_k,\theta) \neq 0) = 0, \quad \mathbf{P}(\Delta D_i(\tau_k,\theta) \neq 0) = 0, \quad \forall i = 1,\dots,K-1.$$

Remark 2.4. For the credit migration process $(C^1(t))_{t\in[0,T^*]}$ constructed in Bielecki and Rutkowski [2], [3], Hypothesis (H2) is satisfied. This follows from Proposition 2.7 below.

We also impose the following natural assumption (see [2] and Blanchet-Scalliet and Jeanblanc [5]):

Hypothesis (H3). For given filtrations $\mathbb{F}$ and $\mathbb{G}$ with $\mathbb{F} \subseteq \mathbb{G}$, every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

In the rest of the paper we assume (H1)–(H3) for all semimartingales under consideration.

2.2. Models without rating migration. We recall the classical description of such models. The default time τ is a $\mathbb{G}$-stopping time, and $\mathbb{G} = \mathbb{F}\vee\mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ are filtrations generated by observing the market and observing the default time, i.e. $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$, respectively. Let $(H(t))_{t\geq 0}$ be the *default indicator process*, i.e.

$$H(t) = \mathbb{1}_{\{\tau\leq t\}}. \tag{2.5}$$

We assume that τ admits an $\mathbb{F}$-*martingale intensity* $(\lambda_t)_{t\geq 0}$ which is an $\mathbb{F}$-adapted process such that M_t given by the formula

$$M_t = H(t) - \int_0^{t\wedge\tau} \lambda_u\,du = H(t) - \int_0^t (1 - H(u))\lambda_u\,du \tag{2.6}$$

follows a $\mathbb{G}$-martingale (see Bielecki and Rutkowski [1]).

Since we allow for enlarging the filtration, we need some additional assumptions under which an $\mathbb{F}$-Lévy process is a $\mathbb{G}$-Lévy process. So we assume Hypothesis (H3) holds for filtrations $\mathbb{F}$ and $\mathbb{G}$. In Bielecki and Rutkowski [1] Hypothesis (H3) is also called Condition (M.1) or *Martingale Invariance Property of* $\mathbb{F}$ *with respect to* $\mathbb{G}$. In our case, if τ is an $\mathbb{F}$-stopping time, then $\mathbb{G} = \mathbb{F}$ by definition, so Hypothesis (H3) holds. If τ is not an $\mathbb{F}$-stopping time, then Hypothesis (H3) is equivalent to conditional independence of the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ given $\mathcal{F}_t$ for any $t \in \mathbb{R}_+$ (see Lemma 6.1.1 in Bielecki and Rutkowski [1]).

Moreover, if τ is not an $\mathbb{F}$-stopping time, the assumption that τ has intensity can be given in an alternative form, through the assumption that the process $F_t := \mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t)$ is increasing and absolutely continuous with respect to Lebesgue measure. This means that there exists a nonnegative $\mathbb{F}$-adapted process f_t such that

$$F_t := \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \int_0^t f_u \, du.$$

If we assume that $F_t < 1$, $t \ge 0$, then we can find an $\mathbb{F}$-adapted process $(\lambda_t)_{t\ge0}$ such that

$$1 - F_t = \mathbf{P}(\tau > t \,|\, \mathcal{F}_t) = e^{-\int_0^t \lambda_u \, du}. \tag{2.7}$$

This process $(\lambda_t)_{t\ge0}$ is given by the formula

$$\lambda_t := \frac{f_t}{1 - F_t}, \tag{2.8}$$

and one can easily check that $(\lambda_t)_{t\ge0}$ is the $\mathbb{F}$-martingale intensity of τ. Moreover,

$$\mathbf{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbf{E}(e^{-\int_t^T \lambda_u \, du} \,|\, \mathcal{F}_t).$$

If

$$\mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_\infty) \quad \forall t \in \mathbb{R}_+, \tag{2.9}$$

then F_t is increasing. Bielecki and Rutkowski [1] show that this condition (called Condition (F.1a)) is equivalent to Hypothesis (H3) (see Lemma 6.1.2 in [1]).

Example 2.5. Assume that τ is a random time with density $f > 0$ and probability distribution F independent of the σ-field $\mathbb{F}$. Then the $\mathbb{F}$-intensity of τ is a deterministic function given by

$$\lambda_t = \frac{f_t}{1 - F_t}.$$

Indeed, independence implies $\mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \le t) = F_t$. Moreover M given by (2.6) is a $\mathbb{G}$-martingale, so λ_t is an $\mathbb{F}$-martingale intensity.

Example 2.6 (Canonical construction of default time, see Section 6.5 in [1]). If the probability space is sufficiently rich to support a random variable U uniformly distributed on $[0, 1]$ and independent of $\mathbb{F}$, then for a given $\mathbb{F}$ adapted nonnegative process $(\lambda_t)_{t\ge0}$ satisfying $\int_0^\infty \lambda_u \, du = \infty$ we can construct a default time τ with intensity $(\lambda_t)_{t\ge0}$ by the formula

$$\tau := \inf\{t \ge 0 : e^{-\int_0^t \lambda_u du} \le U\}.$$

One can easily show that $(\lambda_t)_{t\ge0}$ is the $\mathbb{F}$-intensity of τ (formula (2.7) holds), and hence also the $\mathbb{F}$-martingale intensity. Under this construction (2.9) holds, which implies Hypothesis (H3).

We also have

Proposition 2.7. *Let (X_t) be an $\mathbb{F}$-semimartingale, and τ a random time given by the canonical construction with (λ_t) a strictly positive $\mathbb{F}$-intensity of τ. Then $\mathbf{P}(\Delta X_\tau \ne 0) = 0$.*

We do not want to assume that τ is given through the canonical construction, so we assume (H3) throughout the rest of the paper. But we emphasize that if τ is given through the canonical construction, then Hypothesis (H3) is redundant.

REMARK 2.8. This model is a special case of the model with rating migration. Indeed, taking $K=2$, $C(t)=1+H(t)$ and the intensity λ given by (2.8), we obtain the previous model (for details see [2, p. 396]). The conditional generator of C is of the form

$$\Lambda(t)=\begin{pmatrix}-\lambda_t & \lambda_t\\ 0 & 0\end{pmatrix}.$$

3. The generalized HJM conditions for credit risk. We consider three types of recovery payment described in the previous section and fractional recovery with multiple defaults. Since we investigate them separately, we use the same notation D for price processes with different recovery payments (so D has different meanings in different subsections). Let us recall that all results are obtained under assumptions (A1), (A2) for all Lévy processes considered, and (H1)–(H3).

3.1. Models with rating migration

3.1.1. *Fractional recovery of market value with rating migration.* Let us focus on defaultable bonds with fractional recovery of market value $D(t,\theta)$. This kind of bond pays one unit of cash if default has not occurred before maturity θ, i.e., if the default time satisfies $\tau>\theta$, and if the bond defaults before θ we have recovery payment at the default time which is a fraction $\delta(t)$ of its market value just before the default time, so the recovery payment is equal to $\delta(\tau)D(\tau-,\theta)$. Therefore, in the case of rating migration, the price process of the defaultable bond with credit migration and fractional recovery of market value should satisfy

$$D(\theta,\theta)=\mathbb{1}_{\{\tau>\theta\}}+\mathbb{1}_{\{\tau\le\theta\}}\delta_{C^2(\tau)}(\tau)D_{C^2(\tau)}(\tau-,\theta)\frac{B_\theta}{B_\tau},$$

where $\tau=\inf\{t>0: C^1(t)=K\}$. Hence we postulate that for $t\le\theta$,

$$\begin{aligned}D(t,\theta)&=\mathbb{1}_{\{C^1(t)\neq K\}}D_{C^1(t)}(t,\theta)+\mathbb{1}_{\{C^1(t)=K\}}\delta_{C^2(\tau)}(\tau)D_{C^2(\tau)}(\tau-,\theta)\frac{B_t}{B_\tau}\\&=\sum_{i=1}^{K-1}\mathbb{1}_{\{C^1(t)\neq K\}}\mathbb{1}_{\{C^1(t)=i\}}D_i(t,\theta)\\&\quad+\sum_{i=1}^{K-1}\mathbb{1}_{\{C^1(t)=K\}}\mathbb{1}_{\{C^2(t)=i\}}\delta_i(\tau)D_i(\tau-,\theta)\frac{B_t}{B_\tau}.\end{aligned}$$

For $i \neq K$ we define an auxiliary process $H_{i,K}$ by the formula

$$H_{i,K}(t) = \sum_{0<u\le t} H^i(u-)H^K(u), \qquad \forall t \in \mathbb{R}_+.$$

This process counts the number of jumps of the migration process $C^1(t)$ from state i to state K up to time t. Using the processes H_i and $H_{i,K}$ we can write D in the form

$$D(t,\theta) = \sum_{i=1}^{K-1} \left(H_i(t)D_i(t,\theta) + H_{i,K}(t)\delta_i(\tau)D_i(\tau-,\theta)\frac{B_t}{B_\tau} \right). \tag{3.1}$$

THEOREM 3.1. *The processes of discounted prices of a defaultable bond with credit migration and fractional recovery of market value are local martingales if and only if the following condition holds: for all $\theta \in [0,T^*]$ and for almost all $t \le \theta$, on the set $\{\tau > t\}$,*

$$\begin{aligned} A_{C^1(t)}(t,\theta) = J_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)) \\ + \sum_{i=1,\, i\neq C^1(t)}^{K-1} \left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),i}(t). \end{aligned} \tag{3.2}$$

It is worth pointing out that from the proof of Theorem 3.1 we immediately obtain

THEOREM 3.2. *If the processes $D_i, \bar{a}_i, \Lambda$ have continuous trajectories then the HJM postulate is equivalent to the following two conditions:* (2.3) *and the HJM-type condition* (3.2).

So in this case the HJM postulate implies the HJM-type condition (3.2) without assuming Hypothesis (H1). Theorem 3.2 has appeared for the first time in [27], but in terms of the derivative of the Laplace exponent (see items (i) and (ii) of Theorem 4.2). The same conclusion is true for other types of recovery with analogous proofs, but we do not formulate these facts as separate statements.

REMARK 3.3. Theorem 3.2 is not true in the case of Lévy noise: see an example in the last section. Therefore, Theorem 4.2 in [25], which was proved under the stronger assumption than ours (since in the proof the Itô formula for processes with values in Hilbert spaces is used), is not true without some additional assumption.

3.1.2. *Fractional recovery of treasury value with rating migration.* The holder of a defaultable bond with fractional recovery of treasury value receives 1 if there is no default by θ, and if default has occurred before maturity θ, then a fixed amount $\delta \in [0,1]$ is paid to the bondholder at maturity. Therefore, since paying δ at maturity θ is equivalent to paying $\delta B(\tau,\theta)$ at the default time τ, in the case of fractional recovery of treasury value with rating migration we have

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}}\delta_{C^2(\tau)},$$

hence

$$\begin{aligned} D(t,\theta) &= \mathbb{1}_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + \mathbb{1}_{\{C^1(t)=K\}}\delta_{C^2(t)}B(t,\theta) \\ &= \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=i\}} D_i(t,\theta) + \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=K\}}\mathbb{1}_{\{C^2(t)=i\}}\delta_i B(t,\theta), \end{aligned}$$

or equivalently,

$$D(t,\theta) = \sum_{i=1}^{K-1} (H_i(t)D_i(t,\theta) + H_{i,K}(t)\delta_i B(t,\theta)). \tag{3.3}$$

THEOREM 3.4. *The processes of discounted prices of a defaultable bond with fractional recovery of treasury value are local martingales if and only if the following condition holds: for all $\theta \in [0,T^*]$ and for almost all $t \le \theta$, on the set $\{\tau > t\}$,*

$$\begin{aligned} A_{C^1(t)}(t,\theta) = {} & J_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)) \\ & + \delta_{C^1(t)}\left[\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) \\ & + \sum_{j=1,\, j\neq C^1(t)}^{K-1} \left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t). \end{aligned} \tag{3.4}$$

3.1.3. *Fractional recovery of par value with rating migration.* In the case of fractional recovery of par value the holder of a defaultable bond receives one unit of cash if there is no default prior to maturity, and if the bond has defaulted, a fixed fraction δ of the par value is paid at the default time. Therefore the payoff at maturity has the form

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}}\delta_{C^2(t)}\frac{B_\theta}{B_\tau},$$

hence

$$\begin{aligned} D(t,\theta) &= \mathbb{1}_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + \mathbb{1}_{\{C^1(t)=K\}}\delta_{C^2(t)}\frac{B_t}{B_\tau} \\ &= \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)\neq K\}}\mathbb{1}_{\{C^1(t)=i\}} D_i(t,\theta) + \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=K\}}\mathbb{1}_{\{C^2(t)=i\}}\delta_i \frac{B_t}{B_\tau}, \end{aligned}$$

or equivalently,

$$D(t,\theta) = \sum_{i=1}^{K-1} \left(H_i(t)D_i(t,\theta) + H_{i,K}(t)\delta_i \frac{B_t}{B_\tau}\right).$$

THEOREM 3.5. *The processes of discounted prices of a defaultable bond with fractional recovery of par value are local martingales if and only if the following condition holds: for all $\theta \in [0, T^*]$ and for almost all $t \le \theta$, on the set $\{\tau > t\}$,*

$$A_{C^1(t)}(t,\theta) = J_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)) + \delta_{C^1(t)}\left[\frac{1}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) + \sum_{j=1,\, j\neq C^1(t)}^{K-1}\left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t). \tag{3.5}$$

3.1.4. *Fractional recovery with multiple defaults and rating migration.* The HJM models with fractional recovery with multiple defaults were introduced by Schönbucher [29]. This model describes a situation where a company that has had to declare default is not liquidated but is restructured. After restructuring the firm may again default in the future. Schönbucher investigated defaultable bonds whose face value is reduced by a fraction L_{τ_i} at each default time τ_i, where L_s is an $\mathbb{F}$-predictable process taking values in $[0,1]$. Therefore, a holder of such a defaultable bond receives, at maturity θ,

$$D^m(\theta,\theta) = \prod_{\tau_i \le \theta} (1 - L_{\tau_i}).$$

If we introduce a process V_t by the formula

$$V_t = \prod_{\tau_i \le t} (1 - L_{\tau_i}),$$

then $D^m(\theta,\theta) = V_\theta$ and for $t \le \theta$, we give a natural definition

$$D^m(t,\theta) = V_t e^{-\int_t^\theta g_1(t,u)\,du} = V_t D_1(t,\theta). \tag{3.6}$$

Moreover, we assume that τ_i are jump times of a *Cox process* N_t (doubly stochastic Poisson process) with stochastic intensity process $(\gamma_t)_{t\ge 0}$. It can be shown that V_t solves the following SDE:

$$dV_t = -V_{t-} L_t dN_t, \tag{3.7}$$

and the process

$$M_t = N_t - \int_0^t \gamma_u\,du \tag{3.8}$$

is a $\mathbb{G}$-martingale (Lando [23]).

In this paper we add a rating migration process to the model. We assume that the default times are jumps of a Cox process with intensity $(\gamma_t)_{t\ge 0}$. Since the company is restructured after default, the rating migration process has no absorbing state and for the rating migration process C we take a càdlàg process, which is an $\mathbb{F}$-conditional Markov chain with values in the set $\{1, \ldots, K-1\}$ without absorbing state. Moreover, we assume that the process describing fractional losses does not depend on the credit migration process.

REMARK 3.6. Note that $1 - L_t$ can be interpreted as a recovery process and therefore we will denote it by $\delta(t)$. Thus $\delta(t) = 1 - L_t$.

Thus the bond price process should satisfy the following terminal condition:

$$D(\theta, \theta) = V_\theta = \prod_{\tau_i \le \theta} (1 - L_{\tau_i}) = \prod_{\tau_i \le \theta} \delta_{\tau_i},$$

and before maturity it should be given by the formula

$$D(t, \theta) = V_t D_{C^1(t)}(t, \theta) = V_t \sum_{i=1}^{K-1} H_i(t) D_i(t, \theta).$$

REMARK 3.7. In this case the filtration $\mathbb{G}$ is specified as $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^N \vee \mathbb{F}^C$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^N \vee \mathcal{F}_t^C$, and Hypothesis (H1), i.e. formula (2.2), takes the form

$$g_{C^1(t)}(t, t) = f(t, t) + (1 - \delta(t))\gamma_t \ .$$

THEOREM 3.8. *The discounted prices of a bond with fractional recovery with multiple defaults and rating migration are local martingales if and only if the following condition holds: for all $\theta \in [0, T^*]$ and for almost all $t \le \theta$, on the set $\{V_{t-} > 0\}$,*

$$A_{C^1(t)}(t, \theta) = J_{C^1(t)}(\Sigma_{C^1(t)}(t, \theta)) + \sum_{j=1,\, j \ne C^1(t)}^{K-1} \left[\frac{D_j(t-, \theta)}{D_{C^1(t)}(t-, \theta)} - 1 \right] \lambda_{C^1(t), j}(t). \tag{3.9}$$

3.2. Models without rating migration. As we know, taking $K = 2$ in the model with rating migration we obtain results for models of defaultable bonds with one credit rate, so for models without rating migration. To give a clear picture of markets with defaultable bonds, and for the sake of completeness, we formulate the HJM drift conditions for these models:

THEOREM 3.9. *The discounted prices of a defaultable bond are local martingales if and only if the following condition holds for all $\theta \in [0, T^*]$ and for almost all $t \le \theta$, on the set $\{\tau > t\}$:*

(1) *for fractional recovery of market value,*

$$A_1(t, \theta) = J_1(\Sigma_1(t, \theta)); \tag{3.10}$$

(2) *for fractional recovery of treasury value,*

$$A_1(t,\theta) = J_1(\Sigma_1(t,\theta)) + \delta\left(\frac{B(t-,\theta)}{D_1(t-,\theta)} - 1\right)\lambda_t; \tag{3.11}$$

(3) *for fractional recovery of par value,*

$$A_1(t,\theta) = J_1(\Sigma_1(t,\theta)) + \delta\left(\frac{1}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{3.12}$$

THEOREM 3.10. *The discounted prices of a defaultable bond with multiple defaults and fractional recovery are local martingales if and only if the following condition holds: for all $\theta \in [0,T^*]$ and for almost all $t \le \theta$, on the set $\{V_{t-} > 0\}$,*

$$A_1(t,\theta) = J_1\left(\Sigma_1(t,\theta)\right). \tag{3.13}$$

4. The generalized HJM condition in terms of the derivative of the Laplace exponent. If the derivative of the Laplace exponent exists, then the generalized HJM conditions have simpler forms. To obtain these forms we use some facts on such derivatives, including

LEMMA 4.1. *Let G be a functional defined on an open subset B_1 of U, of the form*

$$G(x) = \int_U (e^{-\langle x,y\rangle_U} - 1 + \mathbb{1}_{|y|_U \le 1}(y)\langle x,y\rangle_U)\,\nu(dy),$$

where ν is a Lévy measure which has exponential moments

$$\int_{\{|y|_U>1\}} e^{\langle c,y\rangle_U}\,\nu(dy) < \infty \tag{4.1}$$

for all $c \in B_1$. Then G is differentiable at each $x \in B_1$ and

$$DG(x) = -\int_U (e^{-\langle x,y\rangle_U} - \mathbb{1}_{|y|_U \le 1}(y))y\,\nu(dy).$$

The proof is straightforward. We use the existence of exponential moments of the form (4.1) for all $c \in B_1$.

Hence, for models with ratings, after straightforward calculations we obtain the HJM conditions in terms of the derivatives of the Laplace exponents J_i, $i = 1,\dots,k-1$.

THEOREM 4.2. *Assume that for $i = 1,\dots,K-1$,*

$$\int_{\{|y|\ge 1\}} e^{-\langle u,y\rangle_U}\,\nu_i(dy) < \infty \tag{4.2}$$

for all u from some neighborhood of the set in which $\Sigma_i(t,\theta)$ takes values. Then

(i) *Condition* (3.2) *for fractional recovery of market value and condition* (3.9) *for fractional recovery with multiple defaults have the form*

$$\alpha_{C^1(t)}(t,\theta) = \langle DJ_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)), \sigma_{C^1(t)}(t,\theta)\rangle_U + \sum_{i=1,\, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))\,du}.$$

(ii) *Condition* (3.4) *for fractional recovery of treasury value has the form*

$$\begin{aligned}\alpha_{C^1(t)}(t,\theta) &= \langle DJ_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)), \sigma_{C^1(t)}(t,\theta)\rangle_U \\ &+ \sum_{i=1,\, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))\,du} \\ &+ \delta_{C^1(t)}\lambda_{C^1(t),K}(g_{C^1(t)}(t-,\theta) - f(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-f(t-,u))\,du}.\end{aligned}$$

(iii) *Condition* (3.5) *for fractional recovery of par value has the form*

$$\begin{aligned}\alpha_{C^1(t)}(t,\theta) &= \langle DJ_{C^1(t)}(\Sigma_{C^1(t)}(t,\theta)), \sigma_{C^1(t)}(t,\theta)\rangle_U \\ &+ \sum_{i=1,\, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))\,du} \\ &+ \delta_{C^1(t)}\lambda_{C^1(t),K}\, g_{C^1(t)}(t-,\theta)e^{\int_t^\theta g_{C^1(t)}(t-,u)\,du}.\end{aligned}$$

For infinite-dimensional Brownian motion items (i) and (ii) of Theorem 4.2 were proved in [27]. As a simple consequence of Theorem 4.2 we obtain

Corollary 4.3. *Under the assumption of Theorem* 4.2 *on* J_1*, the conditions for models without ratings have a simpler form, namely:*

(i) *Condition* (3.10) *for fractional recovery of market value has the form*

$$\alpha_1(t,\theta) = \langle DJ_1(\Sigma_1(t,\theta)), \sigma_1(t,\theta)\rangle_U.$$

(ii) *Condition* (3.11) *for fractional recovery of treasury value has the form*

$$\begin{aligned}\alpha_1(t,\theta) &= \langle DJ_1(\Sigma_1(t,\theta)), \sigma_1(t,\theta)\rangle_U \\ &\quad + \delta\lambda_t(g_1(t-,\theta) - f(t-,\theta))e^{\int_t^\theta (g_1(t-,u)-f(t-,u))\,du}.\end{aligned}$$

(iii) *Condition* (3.12) *for fractional recovery of par value has the form*

$$\alpha_1(t,\theta) = \langle DJ_1(\Sigma_1(t,\theta)), \sigma_1(t,\theta)\rangle_U + \delta\lambda_t g_1(t-,\theta)e^{\int_t^\theta g_1(t-,u)\,du}.$$

5. Comparison of consistency conditions and generalized HJM conditions. In this section we investigate the relationships between consistency conditions formulated by Bielecki and Rutkowski and the generalized

HJM conditions introduced in the previous section. The papers [1] and [14] provide an exogenously specified term structure of defaultable forward rates corresponding to a given finite collection of credit ratings and then the authors look for an arbitrage-free model that supports these objects. They are interested in whether there exists a rating migration process such that defaultable bond price processes have prespecified defaultable forward rates. They require this system of prices to be consistent in the sense that the discounted defaultable price processes are local martingales under an appropriately chosen equivalent probability measure. They provide conditions for the intensity matrix processes which guarantee this kind of "consistency", which means that the HJM postulate is satisfied. Hence the "consistency condition" should be related in some way to the HJM drift condition derived in the previous section.

Now we investigate this relation. First, note that the consistency conditions in Bielecki and Rutkowski [1] and Eberlein and Özkan [14] are given under a real-world probability measure, and our generalized HJM conditions are related to a risk-neutral world. So we formulate consistency conditions assuming that we are in a risk-neutral world. We start with the case of fractional recovery of market value with rating migration. We say that the *consistency condition* (cf. [2], [14]) holds if the equalities

$$\begin{aligned}(5.1)\qquad \sum_{i=1,\, i\neq C^1(t)}^{K-1} & [(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),i}(t)] \\ & + (\delta_{C^1(t)}(t)D_{C^1(t)}(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),K}(t) \\ & + (g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta))D_{C^1(t)}(t-,\theta) = 0\end{aligned}$$

are satisfied on the set $\{C^1(t) \neq K\}$ for all $\theta \in [0, T^*]$ and all $t \leq \theta$. We recall that $\bar{a}_i(t,\theta)$ is defined by (2.1).

The following theorem states that the consistency condition and HJM type condition are equivalent under Hypothesis (H1).

THEOREM 5.1. *Assume that Hypothesis* (H1) *holds. For defaultable bonds with credit migration and fractional recovery of market value the consistency condition* (5.1) *holds if and only if the HJM type condition* (3.2) *holds.*

This theorem allows us to generalize, under (H1), the results of [1] and [14] to the case of infinite-dimensional Lévy processes.

COROLLARY 5.2. *Assume that Hypothesis* (H1) *holds. If the consistency condition* (5.1) *holds, then the market is arbitrage-free.*

Moreover, we also have an inverse implication:

Corollary 5.3. *Assume that Hypothesis* (H1) *holds. If the HJM postulate is satisfied, then the consistency condition* (5.1) *holds.*

In the case of other kinds of recovery we have a similar situation although consistency conditions have a slightly different form. For fractional recovery of treasury value with rating migration the consistency condition is of the form

$$\begin{aligned}(5.2)\qquad \sum_{i=1,\, i\neq C^1(t)}^{K-1} &[(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),i}(t)] \\ &+ (\delta_{C^1(t)} B(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),K}(t) \\ &+ (g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta))D_{C^1(t)}(t-,\theta) = 0\end{aligned}$$

on the set $\{C^1(t) \neq K\}$ for all $\theta \in [0, T^*]$ and all $t \leq \theta$.

In the case of fractional recovery of par value with rating migration the consistency condition has the form

$$\begin{aligned}(5.3)\qquad \sum_{i=1,\, i\neq C^1(t)}^{K-1} &[(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),i}(t)] \\ &+ (\delta_{C^1(t)}(t) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),K}(t) \\ &+ (g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta))D_{C^1(t)}(t-,\theta) = 0.\end{aligned}$$

In the case of fractional recovery with multiple defaults with rating migration the consistency condition has the form

$$\begin{aligned}(5.4)\qquad \sum_{i=1,\, i\neq C^1(t)}^{K-1} &[(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),i}(t)] \\ &+ (\delta_t D_{C^1(t)}(t-,\theta) - D_{C^1(t)}(t-,\theta))\lambda_t \\ &+ (g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta))D_{C^1(t)}(t-,\theta) = 0.\end{aligned}$$

Arguing as in Theorem 5.1 we obtain

Theorem 5.4. *Assume that Hypothesis* (H1) *holds. Then*

(i) *The HJM-type condition* (3.4) *for defaultable bonds with credit migration and fractional recovery of treasury value is equivalent to the consistency condition* (5.2).
(ii) *The HJM-type condition* (3.5) *for defaultable bonds with credit migration and fractional recovery of par value is equivalent to the consistency condition* (5.3).
(iii) *The HJM-type condition* (3.9) *for defaultable bonds with credit migration and multiple defaults with fractional recovery is equivalent to the consistency condition* (5.4).

It is worth noticing that we can formulate and prove results analogous to those in Corollaries 5.2 and 5.3 for all kinds of recovery. We have just shown that if Hypothesis (H1) holds then the HJM-type conditions are equivalent to equalities known as consistency conditions. We should stress, however, that Bielecki and Rutkowski [1] and also Eberlein and Özkan [14] treated these conditions as conditions on the intensity matrix process. They show that if we specify a real-world dynamics of defaultable forward rates, and then construct a migration process with intensity matrix satisfying the "consistency condition", then we obtain an arbitrage-free model of defaultable bonds. Note that if we specify the transition intensity matrix then we cannot specify the volatilities arbitrarily. More precisely, this means that if the transition intensity matrix is specified and we are in an arbitrage-free framework then we calculate prices (conditional prices, i.e. on the sets $\{C_t = i\}$) and then extract from them the defaultable forward rates, to finally get the volatilities. In our framework Hypothesis (H1) gives the intensities of migration to the default state, and by specifying the volatilities and then choosing $\lambda_{i,j}$ for $j \neq K$ in such a way that the generalized HJM condition holds we have specified a risk-neutral dynamics of defaultable forward rates.

6. Proofs. In what follows we use the following facts from Bielecki and Rutkowski [2] (see also [26], [6]). If $H_i(t) = \mathbb{1}_{\{i\}}(C^1(t))$, then

$$M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)\,du$$

is a $\mathbb{G}$-martingale. Recall that

$$H_{i,j}(t) = \sum_{0<u\le t} H^i(u-)H^j(u), \qquad \forall t \in \mathbb{R}_+.$$

For arbitrary $i, j \in \mathcal{K}, i \neq j$, the processes

$$M_{i,j}(t) = H_{i,j}(t) - \int_0^t \lambda_{i,j}(u) H_i(u)\,du = H_{i,j}(t) - \int_0^t \lambda_{C^1(u),j}(u) H_i(u)\,du$$

and

$$M_K(t) = H_K(t) - \int_0^t \sum_{i=1}^{K-1} \lambda_{i,K} H_i(u)\,du = H_K(t) - \int_0^t \lambda_{C^1(u),K}(1 - H_K(u))\,du$$

are $\mathbb{G}$-martingales.

Using these facts and assumption (H2) we obtain very useful representations of $d(H_i(t)D_i(t,\theta)/B_t)$:

$$
\begin{aligned}
(6.1)\qquad d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) &= d(H_i(t))\frac{D_i(t-,\theta)}{B_t} + H_i(t-)d\Big(\frac{D_i(t,\theta)}{B_t}\Big)\\
&\quad + \underbrace{d\Big[H_i(\cdot), \frac{D_i(\cdot,\theta)}{B.}\Big]^c_t}_{=0} + \underbrace{\Delta H_i(t)\Delta\frac{D_i(t,\theta)}{B_t}}_{=0}\\
&= d(H_i(t))\frac{D_i(t-,\theta)}{B_t} + H_i(t-)d\Big(\frac{D_i(t,\theta)}{B_t}\Big).
\end{aligned}
$$

Since the process $M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)\,du$ is a $\mathbb{G}$-martingale, using (6.1) we obtain

$$
\begin{aligned}
(6.2)\qquad & d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)\\
&= \frac{D_i(t-,\theta)}{B_t}\Big(dM_i(t) + \lambda_{C^1(t),i}(t)dt\\
&\quad + H_i(t-)(g_i(t,t) - f(t,t) + \bar{a}_i(t,\theta))dt\\
&\quad + H_i(t-)\int_U [e^{\langle \Sigma_i(t,\theta),y\rangle_U} - 1]\,(\mu_i(dt,dy) - dt\,\nu_i(dy))\\
&\quad - H_i(t-)\langle \Sigma_i(t,\theta), dW_i(t)\rangle_U\Big).
\end{aligned}
$$

Proof of Proposition 2.3. Let $D_i(t,\theta)$ be the price process on the set $\{C_t = i\}$, i.e.

$$
\begin{aligned}
D_i(t,\theta) := D(t,\theta)\mathbb{1}_{\{C_t=i\}} = \sum_{j=1}^{K-1} \mathbf{E}\Big(& e^{-\int_t^\theta r_v\,dv} p_{i,j}(t,\theta)\\
&+ \delta_j \int_t^\theta e^{-\int_t^u r_v\,dv} p_{i,j}(t,u)\lambda_{j,K}(u)\,du \,\Big|\, \mathcal{F}_t\Big).
\end{aligned}
$$

Obviously,

$$
(6.3)\qquad g_i(t,t) := -\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}\ln D_i(t,\theta) = -\lim_{\theta\downarrow t}\frac{\frac{\partial}{\partial\theta}D_i(t,\theta)}{D_i(t,\theta)}.
$$

First note that, by definition of D_i,

$$
\lim_{\theta\downarrow t} D_i(t,\theta) = 1
$$

for $i \neq K$. Let

$$
\begin{aligned}
A_{i,j}(t,\theta) &:= e^{-\int_t^\theta r_v\,dv} p_{i,j}(t,\theta),\\
B_{i,j}(t,\theta) &:= \delta_j \int_t^\theta e^{-\int_t^u r_v\,dv} p_{i,j}(t,u)\lambda_{j,K}(u)\,du.
\end{aligned}
$$

Then

$$\frac{\partial}{\partial\theta}D_i(t,\theta) = \sum_{j=1}^{K-1}\mathbf{E}\Big(\frac{\partial}{\partial\theta}A_{i,j}(t,\theta) + \frac{\partial}{\partial\theta}B_{i,j}(t,\theta)\,\Big|\,\mathcal{F}_t\Big)$$

since r and Λ are bounded processes. Next we calculate the derivatives using the conditional Kolmogorov forward equation for $P(t,\theta)$, and letting $\theta\downarrow t$ we obtain

$$\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}A_{i,j}(t,\theta) = -r_t\varrho_{i,j} + \sum_{k=1}^{K}\varrho_{i,k}\lambda_{k,j}(t) = -r_t\varrho_{i,j} + \lambda_{i,j}(t),$$

$$\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}B_{i,j}(t,\theta) = \delta_j\varrho_{i,j}\lambda_{j,K}(t),$$

where $\varrho_{i,j}$ denotes the Kronecker delta. Therefore, by passing to the limit inside the conditional expectation we have

$$\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}D_i(t,\theta) = -r_t + \sum_{j=1}^{K-1}\lambda_{i,j}(t) + \delta_i\lambda_{i,K}(t) = -r_t - (1-\delta_i)\lambda_{i,K}(t),$$

since $\sum_{j=1}^{K-1}\lambda_{i,j}(t) = -\lambda_{i,K}(t)$. This and (6.3) complete the proof. ∎

Proof of Proposition 2.7. Let σ be a jump time of (X_t), and τ be a random time given by the canonical construction. Since $e^{-\int_0^\sigma \lambda_u\,du}$ is $\mathcal{F}_\sigma$-measurable, and U is uniformly distributed on $[0,1]$ and independent of $\mathcal{F}_\sigma$, we obtain

$$\begin{aligned}\mathbf{P}(\tau=\sigma) &= \mathbf{P}(e^{-\int_0^\tau \lambda_u\,du} = e^{-\int_0^\sigma \lambda_u\,du}) = \mathbf{P}(U = e^{-\int_0^\sigma \lambda_u\,du})\\ &= \mathbf{E}(\mathbf{E}(\mathbb{1}_{\{U=e^{-\int_0^\sigma\lambda_u\,du}\}}\,|\,\mathcal{F}_\sigma)) = \mathbf{E}(\mathbf{E}(\mathbb{1}_{\{U=x\}})|_{x=e^{-\int_0^\sigma\lambda_u\,du}}) = 0.\end{aligned}$$

Since a semimartingale is a càdlàg process, the set of jump times of X is countable, so

$$\mathbf{P}(\Delta X_\tau \neq 0) = \mathbf{P}\Big(\bigcup_{n\ge1}\{\tau=\sigma_n\}\Big) \le \sum_{n=0}^{\infty}\mathbf{P}(\tau=\sigma_n) = 0. \ ∎$$

Proof of Theorem 3.1. By (3.1),

$$d\Big(\frac{D(t,\theta)}{B_t}\Big) = \sum_{i=1}^{K-1}\Big(d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) + d\Big(H_{i,K}(t)\delta_i(\tau)\frac{D_i(\tau-,\theta)}{B_\tau}\Big)\Big). \tag{6.4}$$

The first term in this sum is given by (6.2), and the second has the form

$$d\Big(H_{i,K}(t)\delta_i(\tau)\frac{D_i(\tau-,\theta)}{B_\tau}\Big) = \delta_i(t)\frac{D_i(t-,\theta)}{B_t}d(H_{i,K}(t)).$$

Since the process $M_{i,K}(t) = H_{i,K}(t) - \int_0^t \lambda_{i,K}(u) H_i(u)\,du$ is a $\mathbb{G}$-martingale, we have

$$\delta_i(t)\frac{D_i(t-,\theta)}{B_t} d(H_{i,K}(t)) = \frac{D_i(t-,\theta)}{B_t}\delta_i(t) dM_{i,K}(t) + \frac{D_i(t-,\theta)}{B_t}\delta_i(t)\lambda_{i,K}(t) H_i(t) dt.$$

Combining these results we see that the drift term I of (6.4) is given by

$$\begin{aligned}
I(t,\theta) &= \int_0^t \sum_{i=1}^{K-1} H_i(s)\frac{D_i(s-,\theta)}{B_s}(g_i(s,s) - f(s,s) + \bar{a}_i(s,\theta) + \delta_i(s)\lambda_{i,K}(s))\,ds \\
&\quad + \int_0^t \sum_{i=1}^{K-1} \frac{D_i(s-,\theta)}{B_s}\lambda_{C^1(s),i}(s)\,ds \\
&= \int_0^t (1 - H_K(s))\frac{D_{C^1(s)}(s-,\theta)}{B_s}(g_{C^1(s)}(s,s) - f(s,s) + \bar{a}_{C^1(s)}(s,\theta) \\
&\quad + \delta_{C^1(s)}(s)\lambda_{i,K}(s))\,ds + \int_0^t \sum_{i=1}^{K-1} \frac{D_i(s-,\theta)}{B_s}\lambda_{C^1(s),i}(s)\,ds.
\end{aligned}$$

Since $D_{C^1(s)} > 0$ and

$$\begin{aligned}
&\sum_{i=1}^{K-1} \frac{D_i(s-,\theta)}{D_{C^1(s)}(s-,\theta)}\lambda_{C^1(s),i}(s) \\
&\qquad = \sum_{\substack{i=1 \\ i\neq C^1(s)}}^{K-1} \frac{D_i(s-,\theta)}{D_{C^1(s)}(s-,\theta)}\lambda_{C^1(s),i}(s) + \lambda_{C^1(s),C^1(s)}(s) \\
&\qquad = \sum_{\substack{i=1 \\ i\neq C^1(s)}}^{K-1} \left[\frac{D_i(s-,\theta)}{D_{C^1(s)}(s-,\theta)} - 1\right]\lambda_{C^1(s),i}(s) - \lambda_{C^1(s),K}(s),
\end{aligned}$$

we can split I into two parts: $I_1(s)$, independent of θ, and $I_2(s,\theta)$, depending on both s and θ, i.e.

$$I(t,\theta) = \int_0^t (1 - H_K(s))\frac{D_{C^1(s)}(s-,\theta)}{B_s}(I_1(s) + I_2(s,\theta))\,ds,$$

where

$$I_1(s) = g_{C^1(s)}(s,s) - f(s,s) - (1 - \delta_{C^1(s)}(s))\lambda_{C^1(s),K}(s)$$

and

$$I_2(s,\theta) = \bar{a}_{C^1(s)}(s,\theta) + \sum_{\substack{i=1 \\ i\neq C^1(s)}}^{K-1} \left[\frac{D_i(s-,\theta)}{D_{C^1(s)}(s-,\theta)} - 1\right]\lambda_{C^1(s),i}(s).$$

If (2.3) and (3.2) hold, then $I_1(t) = 0$ and $I_2(t,\theta) = 0$, which implies that the drift term $I(\cdot,\theta)$ vanishes for any θ, so the HJM postulate is satisfied.

Conversely, if the drift term vanishes for each $\theta \in [0,T^*]$, then on the set $\{C^1(t) \neq K\} = \{\tau > t\}$,

$$I_1(t) + I_2(t,\theta) = 0 \qquad \text{for almost all } t \in [0,\theta], \tag{6.5}$$

since $D_{C^1(s)}(s-,\theta)/B_s > 0$. From (2.3) we obtain $I_1(t) = 0$. Therefore $I_2(t,\theta) = 0$ for almost all $t \in [0,\theta]$, which is equivalent to (3.2). ∎

Proof of Theorem 3.2. Since I_1 and $I_2(\cdot,\theta)$ are right continuous, vanishing of the drift term I for each $\theta \in [0,T^*]$ implies that

$$I_1(t) + I_2(t,\theta) = 0, \qquad \forall t < \theta.$$

Since $I_2(\theta,\theta) = 0$ by definition, we obtain $I_1(\theta-) = 0$ for $\theta < T^*$. Hence we deduce that $I_1(t) = 0$ for $t < T^*$, which is equivalent to (2.3).

Proof of Remark 3.3. The reason why we could not omit the assumption (2.3) in Theorem 3.1, so Theorem 3.2 is not true without the continuity assumption, is that (6.5) does not imply $I_1(\theta) + I_2(\theta,\theta) = 0$ for $\theta < T^*$ a.s., which gives $I_1(\theta) = 0$ for $\theta < T^*$ a.s., i.e. (2.3).

An example that shows that this implication does not hold is obtained by taking as $\lambda_{i,K}$, $i = 1,\ldots,K-1$, some càdlàg processes such that

$$\mathbf{P}(\exists\theta \in [0,T^*] : |\lambda_{C^1(\theta),K}(\theta) - \lambda_{C^1(\theta),K}(\theta-)| > 0) > 0$$

and then defining

$$g_{C^1(t)}(t,t) := f(t,t) + (1-\delta_{C^1(t)})\lambda_{C^1(t),K}(t-).$$

We note that this choice of g_i gives

$$I_1(\theta) = (1-\delta_{C^1(\theta)})(\lambda_{C^1(\theta),K}(\theta-) - \lambda_{C^1(\theta),K}(\theta)),$$

so

$$\begin{aligned}&\mathbf{P}(\exists\theta \in [0,T^*] : |I_1(\theta)| > 0)\\ &\qquad = \mathbf{P}(\exists\theta \in [0,T^*] : |\lambda_{C^1(\theta),K}(\theta) - \lambda_{C^1(\theta),K}(\theta-)| > 0) > 0\end{aligned}$$

even though we have $I_1(s) = 0$ $ds \times d\mathbf{P}$ almost surely.

Proof of Theorem 3.4. By (3.3), the discounted value of a defaultable bond with fractional recovery of treasury value equals

$$d\left(\frac{D(t,\theta)}{B_t}\right) = \sum_{i=1}^{K-1}\left(d\left(H_i(t)\frac{D_i(t,\theta)}{B_t}\right) + d\left(H_{i,K}(t)\delta_i\frac{B(t,\theta)}{B_t}\right)\right). \tag{6.6}$$

By integration by parts we have

$$d\Big(H_{i,K}(t)\delta_i\frac{B(t,\theta)}{B_t}\Big) = \delta_i\frac{B(t-,\theta)}{B_t}\Big(dM_{i,K}(t) + (\lambda_{i,K}(t)H_i(t) + \bar{a}(t,\theta)H_{i,K}(t-))\,dt + H_{i,K}(t-)\int_U [e^{-\langle \Sigma(t,\theta),y\rangle_U} - 1](\mu(dt,dy) - dt\,\nu(dy)) - H_{i,K}(t-)\langle \Sigma(t,\theta), dW(t)\rangle_U\Big).$$

Together with (6.2) this implies that the drift term I in (6.6) is given by

$$I = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{i=1}^{K-1} H_i(t-)\frac{D_i(t-,\theta)}{B_t}\Big((g_i(t,t)-f(t,t)+\bar{a}_i(t,\theta))+\delta_i\frac{B(t-,\theta)}{D_i(t-,\theta)}\lambda_{i,K}(t)\Big)\,dt,$$

$$I_2 = \sum_{j=1}^{K-1} \frac{D_j(t-,\theta)}{B_t}\lambda_{C^1(t),j}(t)dt,$$

$$I_3 = \frac{B(t-,\theta)}{B_t}\bar{a}(t,\theta)\sum_{i=1}^{K-1}\delta_i H_{i,K}(t-)dt = H_K(t)\Big(\frac{B(t-,\theta)}{B_t}\bar{a}(t,\theta)\sum_{i=1}^{K-1}\delta_i \mathbb{1}_{\{C^2(t)=i\}}dt\Big).$$

Now $I_3 = 0$, because the HJM-type condition for default-free bonds holds (condition (1.8)). Moreover

$$(6.7)\qquad I_2 = \sum_{j=1}^{K-1}\frac{D_j(t-,\theta)}{B_t}\lambda_{C^1(t),j}(t)dt = \sum_{j=1}^{K-1}\frac{D_j(t-,\theta)}{B_t}\sum_{i=1}^{K-1}H_i(t)\lambda_{i,j}(t)dt = \sum_{i=1}^{K-1}H_i(t)\frac{D_i(t-,\theta)}{B_t}\Big(\sum_{\substack{j=1\\ j\neq i}}^{K-1}\frac{D_j(t-,\theta)}{D_i(t-,\theta)}\lambda_{i,j}(t) + \lambda_{i,i}(t)\Big)dt = \sum_{i=1}^{K-1}H_i(t)\frac{D_i(t-,\theta)}{B_t}\Big(\sum_{\substack{j=1\\ j\neq i}}^{K-1}\Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1\Big]\lambda_{i,j}(t) - \lambda_{i,K}(t)\Big)dt.$$

Since $H_i(t) = 1$ on the set $\{C^1(t) = i\}$ and zero on its complement we can write

$$I_1 + I_2 = (1 - H_K(t)) \frac{D_{C^1(t)}(t-,\theta)}{B_t} \Big(g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta) - (1 - \delta_{C^1(t)})\lambda_{C^1(t),K}(t) + \delta_{C^1(t)} \Big[\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),K}(t) + \sum_{\substack{j=1 \\ j \neq C^1(t)}}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),j}(t) \Big) dt,$$

where we have also used the fact that we sum only up to $K-1$. We conclude the argument as in Theorem 3.1. ∎

Proof of Theorem 3.5. We have

$$d\Big(\frac{D(t,\theta)}{B_t}\Big) = \sum_{i=1}^{K-1} \Big(d\Big(H_i(t) \frac{D_i(t,\theta)}{B_t} \Big) + d\Big(H_{i,K}(t) \frac{\delta_i}{B_\tau} \Big) \Big). \tag{6.8}$$

The first part was calculated before (see (6.2)). The second part can be written using the martingale $M_{i,K}$ as

$$\frac{\delta_i}{B_t} dH_{i,K}(t) = \frac{\delta_i}{B_t} dM_{i,K}(t) + \frac{\delta_i}{B_t} H_i(t)\lambda_{i,K}(t)dt.$$

Hence by (6.7) the drift term I of (6.8) is given by

$$\begin{aligned} I &= \sum_{i=1}^{K-1} \Big(\frac{D_i(t-,\theta)}{B_t} (\lambda_{C^1(t),i}(t) + H_i(t-)(g_i(t,t) - f(t,t) + \bar{a}_i(t,\theta)))dt \\ &\quad + \frac{\delta_i}{B_t} H_i(t)\lambda_{i,K}(t)dt \Big) \\ &= \sum_{i=1}^{K-1} H_i(t-) \frac{D_i(t-,\theta)}{B_t} \Big(g_i(t,t) - f(t,t) + \delta_i \Big[\frac{1}{D_i(t-,\theta)} - 1 \Big] \lambda_{i,K}(t) \\ &\quad + \sum_{j=1,\, j\neq i}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1 \Big] \lambda_{i,j}(t) - (1-\delta_i)\lambda_{i,K}(t) + \bar{a}_i(t,\theta) \Big) dt \\ &= (1 - H_K(t)) \frac{D_{C^1(t)}(t-,\theta)}{B_t} \Big(g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta) \\ &\quad - (1 - \delta_{C^1(t)})\lambda_{C^1(t),K}(t) + \delta_{C^1(t)} \Big[\frac{1}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),K}(t) \\ &\quad + \sum_{\substack{j=1 \\ j \neq C^1(t)}}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),j}(t) \Big) dt. \end{aligned}$$

Arguing as before we complete the proof. ∎

Proof of Theorem 3.8. By the Itô lemma,

$$(6.9)\quad d\Big(\frac{D(t,\theta)}{B_t}\Big) = V_{t-}\sum_{i=1}^{K-1} d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) + \Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)dV_t + \Delta\Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)\Delta V_t = I_1 + I_2 + I_3.$$

By (H2) we have $I_3 = 0$.

By (3.7), (3.8) and the fact that $D_i(\cdot,\theta)$ is a càdlàg process, we obtain

$$\begin{aligned}-I_2 &= \Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)V_{t-}L_t dM_t + \Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)V_{t-}L_t\gamma_t dt\\ &= \Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)V_{t-}L_t dM_t + \Big(\sum_{i=1}^{K-1} H_i(t)\frac{D_i(t-,\theta)}{B_t}\Big)V_{t-}L_t\gamma_t dt.\end{aligned}$$

Hence, taking into account (6.9), (6.2), we see that the drift term of $d(\frac{D(t,\theta)}{B_t})$ is given by

$$\sum_{j=1}^{K-1} V_{t-}\frac{D_j(t-,\theta)}{B_t}\lambda_{C^1(t),j}(t)dt + \sum_{i=1}^{K-1} H_i(t)V_{t-}\frac{D_i(t-,\theta)}{B_t}\Big(g_i(t,t) - f(t,t) + \bar{a}_i(t,\theta) - L_t\gamma_t\Big)dt.$$

Since

$$\begin{aligned}\sum_{j=1}^{K-1} V_{t-}\frac{D_j(t-,\theta)}{B_t}\lambda_{C^1(t),j}(t) &= \sum_{i=1}^{K-1} H_i(t)V_{t-}\frac{D_i(t-,\theta)}{B_t}\sum_{j=1}^{K-1}\frac{D_j(t-,\theta)}{D_i(t-,\theta)}\lambda_{i,j}(t)\\ &= \sum_{i=1}^{K-1} H_i(t)V_{t-}\frac{D_i(t-,\theta)}{B_t}\Big(\sum_{j=1,\, j\neq i}^{K-1}\frac{D_j(t-,\theta)}{D_i(t-,\theta)}\lambda_{i,j}(t) + \lambda_{i,i}(t)\Big)\\ &= \sum_{i=1}^{K-1} H_i(t)V_{t-}\frac{D_i(t-,\theta)}{B_t}\sum_{j=1,\, j\neq i}^{K-1}\Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1\Big]\lambda_{i,j}(t),\end{aligned}$$

the drift term is given by

$$\sum_{i=1}^{K-1} H_i(t)V_{t-}\frac{D_i(t-,\theta)}{B_t}\Big(g_i(t,t) - f(t,t) + \bar{a}_i(t,\theta) - L_t\gamma_t + \sum_{j=1,\, j\neq i}^{K-1}\Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1\Big]\lambda_{i,j}(t)\Big)dt.$$

Arguing as in the previous sections, we obtain the theorem. ∎

Proof of Theorem 4.2. Theorem 4.2 follows from Lemma 4.1 by using the following facts on derivatives (the details are left to the reader):

(i) for fractional recovery of treasury value,

$$\frac{\partial}{\partial\theta}\left(\frac{B(t-,\theta)}{D_1(t-,\theta)}-1\right)=(g_1(t-,\theta)-f(t-,\theta))e^{\int_t^\theta (g_1(t-,u)-f(t-,u))\,du},$$

(ii) for fractional recovery of par value,

$$\frac{\partial}{\partial\theta}\left(\frac{1}{D_1(t-,\theta)}-1\right)=g_1(t-,\theta)e^{\int_t^\theta g_1(t-,u)\,du},$$

(iii) for fractional recovery of market value with rating migration,

$$\frac{\partial}{\partial\theta}\left(\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right)=(g_{C^1(t)}(t-,\theta)-g_i(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))\,du},$$

(iv) for fractional recovery of treasury value with rating migration,

$$\frac{\partial}{\partial\theta}\left(\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right)=(g_{C^1(t)}(t-,\theta)-f(t-,\theta))e^{\int_t^\theta (g_{C^1(t)}(t-,u)-f(t-,u))\,du},$$

(v) for fractional recovery of par value with rating migration,

$$\frac{\partial}{\partial\theta}\left(\frac{1}{D_{C^1(t)}(t-,\theta)}-1\right)=g_{C^1(t)}(t-,\theta)e^{\int_t^\theta g_{C^1(t)}(t-,u)\,du}. \blacksquare$$

Proof of Theorem 5.1. Under Hypothesis (H1) we can write (5.1) in the form

$$\begin{aligned}\sum_{i=1,\,i\neq C^1(t)}^{K-1} &[(D_i(t-,\theta)-D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),i}(t)]\\ &-(1-\delta_{C^1(t)}(t))D_{C^1(t)}(t-,\theta)\lambda_{C^1(t),K}(t)\\ &+\big((1-\delta_{C^1(t)}(t))\lambda_{C^1(t),K}(t)+\bar{a}_{C^1(t)}(t,\theta)\big)\,D_{C^1(t)}(t-,\theta)=0.\end{aligned}$$

By definition of $\bar{a}_i(t,\theta)$ we see that this condition is equivalent to

$$\begin{aligned}\sum_{i=1,\,i\neq C^1(t)}^{K-1}\left(\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right)\lambda_{C^1(t),i}(t)&\\ -A_{C^1(t)}(t,\theta)+J_{C^1(t)}(\Sigma_i(t,\theta))=0,&\end{aligned}\tag{6.10}$$

which is exactly (3.2).

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