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REGULARITY PROPERTIES OF THE ATTRACTOR TO THE NAVIER–STOKES EQUATIONS

Abstract. Existence of a global attractor for the Navier–Stokes equations describing the motion of an incompressible viscous fluid in a cylindrical pipe has been shown already. In this paper we prove the higher regularity of the attractor.

1. Introduction. We consider viscous incompressible fluid motions in a finite cylinder with large inflow and outflow and under boundary slip conditions. The following initial-boundary value problem is examined:

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\bar n = 0 && \text{on } S_1^T = S_1\times(0,T),\\
&\nu\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + \gamma v\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1^T,\\
&v\cdot\bar n = d && \text{on } S_2^T = S_2\times(0,T),\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_2^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,\\
&\int_\Omega p\,dx = 0,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$, $S=S_1\cup S_2=\partial\Omega$, $v=v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$ is the velocity vector of the fluid motion, $p=p(x,t)\in\mathbb{R}^1$ the pressure, $f=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ the external force field, $\bar n$ the unit outward vector normal to the boundary S, and $\bar\tau_\alpha$, $\alpha=1,2$, are tangent

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vectors to S. $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI,$$

where ν is the constant viscosity coefficient, I the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally $\gamma>0$ is the slip coefficient.

By $\Omega\subset\mathbb{R}^3$ we denote a cylindrical type domain parallel to the x_3 axis with arbitrary cross section. We assume that S_1 is the part of the boundary which is parallel to the x_3-axis and S_2 is perpendicular to x_3. Hence

$$\begin{aligned}S_1&=\{x\in\mathbb{R}^3:\varphi(x_1,x_2)=c_0,\ -a<x_3<a\},\\ S_2(-a)&=\{x\in\mathbb{R}^3:\varphi(x_1,x_2)<c_0,\ x_3=-a\},\\ S_2(a)&=\{x\in\mathbb{R}^3:\varphi(x_1,x_2)<c_0,\ x_3=a\},\end{aligned}$$

where a, c_0 are given positive numbers and $\varphi(x_1,x_2)=c_0$ describes a sufficiently smooth closed curve in the plane $x_3=\text{const}$.

To describe the inflow and outflow we define

$$d_1=-v\cdot\bar n|_{S_2(-a)},\qquad d_2=v\cdot\bar n|_{S_2(a)},\tag{1.2}$$

so $d_i\ge0$, $i=1,2$, and by $(1.1)_{2,3}$ and (1.2) we have the compatibility condition

$$\Phi\equiv\int\limits_{S_2(-a)}d_1\,dS_2=\int\limits_{S_2(a)}d_2\,dS_2,\tag{1.3}$$

where Φ is the flux.

Let us introduce an extension $\alpha=\alpha(x,t)\in\mathbb{R}$ such that

$$\alpha|_{S_2(-a)}=d_1,\qquad \alpha|_{S_2(a)}=d_2.\tag{1.4}$$

Then equations $(1.1)_{2,3,6}$ and (1.3) imply the compatibility condition

$$\int\limits_{\Omega}\alpha_{,x_3}\,dx=-\int\limits_{S_2(-a)}\alpha|_{x_3=-a}\,dS_2+\int\limits_{S_2(a)}\alpha|_{x_3=a}\,dS_2=0.$$

In [14, 15, 16] the long time existence of solutions is proved in non-axially cylindrical domains. In [17] the existence is proved in Besov spaces and in [2, 9] the proof of existence is simplified so as to use Sobolev spaces only. In [8] the global existence of solution is proved by prolongation of long time solutions from [9]. In [17] the inflow-outflow problem is considered, and in [3] global existence by prolongation of long time solutions from [2] is proved.

To formulate the main result of [3] we need the notation

$$\begin{aligned}\Gamma^2(t) &= |\alpha|^2_{2,S_1} + |\alpha_{,t}|^2_{6/5,\Omega} + |\alpha_{,x_3t}|^2_{6/5,\Omega}\\ &\quad + (1+\|\alpha\|^2_{1,3,\Omega})|\nabla\alpha|^2_{2,\Omega} + |f|^2_{6/5,\Omega},\end{aligned}$$

$$\Gamma_1^2(kT,t) = \|\alpha\|^2_{L_\infty(kT,t;L_2(\Omega))} + \|\alpha_{,x_3}\|^2_{L_\infty(kT,t;L_2(\Omega))} + \int_{kT}^t \|\alpha(t')\|^2_{1,2,\Omega}\,dt',$$

$$\begin{aligned}l_1^2(kT,t) &= c\exp c(|d_1|^6_{3,6,S_2\times(kT,t)} + |\nabla\alpha|^2_{3,2,\Omega\times(kT,t)})\\ &\quad\cdot\Big(\int_{kT}^t \Gamma^2(t')dt' + \Gamma_1^2(kT,t) + |v(kT)|^2_{2,\Omega}\Big),\end{aligned}$$

$$\begin{aligned}G(kT,t) &= l_1(kT,t) + \|d_1\|_{3/2,2,S_2\times(kT,t)} + |f|_{2,\Omega\times(kT,t)}\\ &\quad + |F_3|_{10/7,\Omega\times(kT,t)} + |d_1|_{\infty,\Omega\times(kT,t)}, \quad \text{where} \quad F_3 = (\operatorname{rot} v)_3,\end{aligned}$$

$$G'(kT,t) = |g|_{2,\Omega\times(kT,t)} + l_1(kT,t) + \|d_{1,x'}\|_{3/2,2,S_2\times(kT,t)},$$

where $g = f_{,x_3}$,

$$\begin{aligned}\eta(kT,t) &= \|d_{1,x'}\|_{L_\infty(kT,t;H^1(S_2))} + \|d_{1,t}\|_{L_2(kT,t;H^1(S_2))}\\ &\quad + \|f_3\|_{L_2(kT,t;L_{4/3}(\Omega))} + \|g\|_{L_2(kT,t;L_{6/5}(\Omega))} + \frac{1}{T}\,l_1(kT,t),\end{aligned}$$

where $t \in (kT,(k+1)T)$.

THEOREM 1.1 (global existence). *Assume that* $t \in (kT,(k+1)T)$, $k \in \mathbb{N}\cup\{0\} \equiv \mathbb{N}_0$,

$$\int_{kT}^t \Gamma^2(t')dt' \le \int_0^T \Gamma^2(t')dt',$$

$$\Gamma_1(kT,t) \le \Gamma_1(0,T), \qquad G(kT,t) \le G(0,T), \qquad G'(kT,t) \le G'(0,T),$$

$$l_1(kT,t) \le l_1(0,T), \qquad \eta(kT,t) \le \eta(0,T).$$

Assume that $\eta(kT,(k+1)T)$ *is so small that there exists a positive constant* A *such that*

$$\varphi(A,G(kT,t))\eta(kT,t) + G'(kT,t) < A, \qquad t \in (kT,(k+1)T], \tag{1.5}$$

where φ *is some positive increasing function. Then there exists a solution to* (1.1) *such that*

$$\|v_{,x_3}\|_{W_2^{2,1}((kT,(k+1)T)\times\Omega)} \le A, \qquad \|v\|_{W_2^{2,1}((kT,(k+1)T)\times\Omega)} \le c(A^2+1), \tag{1.6}$$

where $A > 0$ *is a constant chosen for a given* T *and independent of* $k \in \mathbb{N}$.

In [4] we proved the existence of the global attractor for problem (1.1). The attractor is bounded in $H^1(\Omega)$. In this paper we show the H^2 regularity for this attractor.

Now we formulate the main result.

THEOREM 1.2. *There exists a global attractor $\mathcal{A}$ in V for the semiprocess $\{S(t)\}_{t\ge0}$ defined by* (3.1). *The attractor is bounded in $H^2(\Omega)$ and compact and connected in V. It attracts bounded sets in V, where V is defined by* (2.2).

2. Notation and auxiliary results. To simplify the presentation we introduce the following notation:

$$
\begin{aligned}
&|u|_{p,Q} = \|u\|_{L_p(Q)}, && Q\in\{\Omega^T,S^T,\Omega,S\},\ p\in[1,\infty],\\
&\|u\|_{s,Q} = \|u\|_{H^s(Q)}, && Q\in\{\Omega,S\},\ s\in\mathbb{R}_+\cup\{0\},\\
&\|u\|_{s,Q^T} = \|u\|_{W_2^{s,s/2}(Q^T)}, && Q\in\{\Omega,S\},\ s\in\mathbb{R}_+\cup\{0\},\\
&|u|_{p,q,Q^T} = \|u\|_{L_q(0,T;L_p(Q))}, && Q\in\{\Omega,S\},\ p,q\in[1,\infty],\\
&\|u\|_{s,q,Q^T} = \|u\|_{W_q^{s,s/2}(Q^T)}, && Q\in\{\Omega,S\},\ s\in\mathbb{R}_+\cup\{0\},\ q\in[1,\infty],\\
&\|u\|_{s,q,Q} = \|u\|_{W_q^{s}(Q)}, && Q\in\{\Omega,S\},\ s\in\mathbb{R}_+\cup\{0\},\ q\in[1,\infty].
\end{aligned}
$$

By c we denote a generic constant which changes its magnitude from formula to formula. By $\bar c(\sigma)$, $\varphi(\sigma)$ we understand generic functions which are always positive and increasing. Finally, we do not distinguish scalar and vector-valued functions in notation.

We introduce the space

$$
V_2^k(\Omega^T) = \Big\{u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{H^k(\Omega)} + \Big(\int_0^T \|\nabla u(t)\|^2_{H^k(\Omega)}\,dt\Big)^{1/2} < \infty\Big\}, \quad k\in\mathbb{N}.
$$

Finally we introduce the quantities

$$
\begin{aligned}
&h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3},\\
&w = v_3, \quad \chi = v_{2,x_1} - v_{1,x_2}.
\end{aligned}
\tag{2.1}
$$

Let $\delta>0$ be fixed and

$$
\bar V = \{v\in C^\infty(\Omega) : \operatorname{div} v = 0,\ v\cdot\bar n|_{S_1} = 0,\ v\cdot\bar n|_{S_2} = d,\ |v_{,x_3}|_{2,\Omega} < \delta\},
$$

then

$$
\begin{aligned}
&H \equiv \text{ closure of } \bar V \text{ in } L_2\text{-norm},\\
&V \equiv \text{ closure of } \bar V \text{ in } H^1\text{-norm}.
\end{aligned}
\tag{2.2}
$$

LEMMA 2.1 (Korn inequality; see [14]). *Assume that $|\mathbb{D}(v)|^2_{2,\Omega} < \infty$, $v\cdot\bar n|_S = 0$, $\operatorname{div} v = 0$. If Ω is not axially symetric, then there exists a constant $c>0$ such that*

$$
\|v\|^2_{1,\Omega} \le c|\mathbb{D}(v)|^2_{2,\Omega}. \tag{2.3}
$$

First we need estimates and the uniform Gronwall inequality.

Lemma 2.2. *A solution* $v \in H^2(\Omega)$ *of the elliptic problem*

$$\begin{aligned}&\operatorname{div}\mathbb{D}(v) = f,\\ &v\cdot\bar n|_{S_1} = 0,\\ &v\cdot\bar n|_{S_2} = d,\\ &(\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + v\cdot\bar\tau_\alpha)_{|S_1} = 0, \quad \alpha = 1,2,\\ &\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha|_{S_2} = 0, \quad \alpha = 1,2,\end{aligned}$$

satisfies the estimate

$$\|v\|_{2,\Omega} \le c(|f|_{2,\Omega} + |v|_{2,\Omega} + \|d\|_{3/2,2,S_2} + \|v\cdot\tau_\alpha\|_{1/2,2,S_1}). \tag{2.4}$$

Lemma 2.3. *A solution* $(v,p) \in H^2(\Omega)\times H^1(\Omega)$ *of the elliptic problem*

$$\begin{aligned}&\operatorname{div}\mathbb{T}(v,p) = f,\\ &\operatorname{div} v = 0,\\ &v\cdot\bar n|_{S_1} = 0,\\ &v\cdot n|_{S_2} = d,\\ &(\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + v\cdot\bar\tau_\alpha)_{|S_1} = 0, \quad \alpha = 1,2,\\ &\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha|_{S_2} = 0, \quad \alpha = 1,2,\end{aligned}$$

satisfies the estimate

$$\|v\|_{2,\Omega} + |\nabla p|_{2,\Omega} \le c(|f|_{2,\Omega} + |v|_{2,\Omega} + \|d\|_{3/2,2,S_2} + \|v\cdot\tau_\alpha\|_{1/2,2,S_1}). \tag{2.5}$$

Lemmas 2.2 and 2.3 follow from the general theory on boundary value problems for Douglis–Nirenberg elliptic systems (see [10]).

Lemma 2.4 (uniform Gronwall inequality; see [12, Ch. 3, Lemma 1.1]). *Let* $g,h,y : [t_0,\infty) \to (0,\infty)$ *be continuous functions. Assume that for some* $r > 0$ *and all* $t > t_0$ *we have*

$$y'(t) \le g(t)y(t) + h(t),$$

$$\int_t^{t+r} g(s)ds \le a_1, \qquad \int_t^{t+r} h(s)ds \le a_2, \qquad \int_t^{t+r} y(s)ds \le a_3.$$

Then y *satisfies the uniform estimate*

$$y(t+r) \le (a_3/r + a_2)e^{a_1} \quad \text{for } t > t_0. \tag{2.6}$$

We formulate the main results from [4].

Theorem 2.1. *There exists a unique global attractor* $\mathcal{A}_\Sigma$ *in* H *for the family of semiprocesses* $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma\in\Sigma$, *defined by* (3.1). *The attractor is bounded in* V *and compact and connected in* H. *It attracts bounded sets in* H.

Theorem 2.2. *Let* f *denote the external field force in the nonstationary problem and* f_∞ *the external field force in the stationary problem. Assume* *that* $f_\infty \in L_{6/5}(\Omega)$. *If*

$$\|f(t) - f_\infty\|_{L_{6/5}(\Omega)} \xrightarrow[t\to\infty]{} 0 \quad \textit{and} \quad \|d - d_\infty\|_{L_2(S_2)} \xrightarrow[t\to\infty]{} 0,$$

then the solution $v(t)$ *of problem* (1.1) *converges to the stationary solution* v_∞ *of problem* (1.1).

3. Existence of global attractor. In this section we prove the H^2 regularity of the global attractor to the problem (1.1). We start by recalling some facts and definitions from [1, Ch. 4].

Let us rewrite equation $(1.1)_1$ in the abstract form

$$v_{,t} = A(v,t) = A_{\sigma(t)}(v), \qquad t \in \mathbb{R}^+,$$

where the right-hand side depends explicitly on the time symbol $\sigma(t)$, which is the collection of all time-dependent coefficients of the equation (in the Navier–Stokes equations that will be the time-dependent external forces). By Φ we denote some metric or Banach space which contains values of $\sigma(t)$ for a.e. $t \in \mathbb{R}_+$. Moreover we assume that $\sigma(t)$, as a function of t, belongs to a topological space $\Xi := \{\xi(t), t \in \mathbb{R}_+ : \xi(t) \in \Psi \text{ for a.e. } t \in \mathbb{R}_+\}$.

Replacing the symbol $\sigma(t)$ by the shifted symbol $\sigma(t+h)$ should not change the attractor, hence we introduce a translation invariant subspace $\Sigma \subseteq \Xi$ called the symbol space. Translation invariance means that for all $\sigma(t) \in \Sigma$ the relation $T(h)\sigma(t) = \sigma(t+h) \in \Sigma$ holds, where $T(h) : \Xi \to \Xi$ is the shift operator. In our case, it will be convenient to set $\Sigma = \Sigma(\sigma_0) \equiv \{\sigma_0(t+h) : h \in \mathbb{R}^+\}$, where σ_0 is the time symbol of the initial equation and the closure is taken in the topology of Ξ.

Let $v(t)$ be a weak and global solution of problem (1.1) with initial data $v_0 = v(0)$. We define the family of semiprocesses $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$ acting on H, $U(t,\tau) : H \to H$, by the formula

$$v(t) = U_\sigma(t,\tau)v(\tau), \tag{3.1}$$

where $v(\tau)$ is the initial condition and $\Sigma \ni \sigma(t) = f(\cdot,t)$ is the external force.

By $\mathcal{B}(H)$ we denote the family of all bounded sets in H.

Definition 3.1 (see [1, Ch. 4, Definition 3.2]). A family of processes $\{u_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma \in \Sigma$, is said to be *uniformly bounded* if for any $B \in \mathcal{B}(H)$ we have

$$\bigcup_{\sigma\in\Sigma}\bigcup_{\tau\in\mathbb{R}^+}\bigcup_{t\ge\tau} U_\sigma(t,\tau)B \in \mathcal{B}(H).$$

Definition 3.2 (see [1, Ch. 4, Definition 3.3]). A set $B_0 \in H$ is said to be *uniformly absorbing* for the family of processes $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}^+$ and for every $B \in \mathcal{B}(H)$ there exists $t_0 = t_0(\tau, B)$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subseteq B_0$ for all $t \ge t_0$. If the set B_0 is compact, we call the family of processes *uniformly compact.*

Definition 3.3 (see [1, Ch. 4, Definition 3.4]). A set P belonging to H is said to be *uniformly attracting* for the family of processes $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma\in\Sigma$, if for any fixed $\tau\in\mathbb{R}^+$,

$$\lim_{t\to\infty}(\sup_{\sigma\in\Sigma}\operatorname{dist}_E(U_\sigma(t,\tau)B,P)) = 0.$$

If the set P is compact, we call the family of processes *uniformly asymptotically compact.*

Definition 3.4 (see [1, Ch. 4, Definition 3.5]). A closed set $\mathcal{A}_\Sigma\subset H$ is said to be the *uniform attractor* of the family of processes $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma\in\Sigma$, if it is uniformly attracting set and it is contained in any closed uniformly attracting set $\mathcal{A}'$ of $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma\in\Sigma$.

The existence of the global attractor follows from the following theorem

Theorem 3.1 (see [1, Ch. 4, Theorem 3.1]). *If a family of processes $\{U_\sigma(t,\tau)\}_{t\ge\tau\ge0}$, $\sigma\in\Sigma$ is uniformly asymptotically compact, then it has the uniform global attractor $\mathcal{A}_\Sigma$. The sef $\mathcal{A}_\Sigma$ is compact in H.*

In [4] we have proved that there exists a bounded, compact and absorbing set B in H. Now we will show that this set is in fact compact in V. We need to bound this set in $H^2(\Omega)$.

Lemma 3.1. *Assume that $f_{,t}\in L_2(kT,(k+1)T;L_{6/5}(\Omega))\cap L_\infty(kT,(k+1)T;L_2(\Omega))$. Then there exists a bounded and absorbing set in $H^2(\Omega)$ for the semiprocess $\{S(t)\}_{t\ge0}$.*

Proof. Let $b=\alpha\bar e_3$, where $\bar e_3$ is directed along the x_3-axis. Let u be defined by

$$\begin{aligned} &\operatorname{div} u = -\operatorname{div} b && \text{in } \Omega,\\ &u\cdot\bar n = 0 && \text{on } S.\end{aligned}$$

Next we define a function φ as the solution to the problem

$$\begin{aligned} &\Delta\varphi = -\operatorname{div} b && \text{in } \Omega,\\ &\nabla\varphi\cdot\bar n = 0 && \text{on } S,\\ &\int_\Omega \varphi\,dx = 0. \end{aligned}$$

Therefore, we introduce the new function

$$w = u-\nabla\varphi = v-(b+\nabla\varphi)\equiv v-\delta, \tag{3.2}$$

which is the solution to the problem

$$
(3.3)\qquad \begin{aligned}
&(w+\delta)_{,t} + (w+\delta)\cdot\nabla(w+\delta) - \operatorname{div}\mathbb{T}(w+\delta,p) = f && \text{in } \Omega,\\
&\operatorname{div} w = 0 && \text{in } \Omega,\\
&w\cdot\bar n = 0 && \text{on } S_1,\\
&\nu\bar n\cdot\mathbb{D}(w+\delta)\cdot\bar\tau_\alpha + \gamma(w+\delta)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1,\\
&w\cdot\bar n = 0 && \text{on } S_2,\\
&\bar n\cdot\mathbb{D}(w+\delta)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_2.
\end{aligned}
$$

We differentiate $(3.3)_1$ with respect to time and take the inner product in $L_2(\Omega)$ with $w_{,t}$ to obtain

$$
\begin{aligned}
\frac12\frac{d}{dt}\|w_{,t}\|^2_{L_2(\Omega)} + \|w_{,t}\|^2_{H^1(\Omega)} + \gamma\int_{S_1}|w_{,t}\cdot\bar\tau_\alpha|^2\,dS_1 &\le \varepsilon\|w_{,t}\|^2_{H^1(\Omega)}\\
&\quad + \frac{c}{\varepsilon}(\|\delta_{,t}\|^2_{H^1(\Omega)} + \|f_{,t}\|^2_{L_{6/5}(\Omega)})\\
+\Big|\int_\Omega (w_{,t}+\delta_{,t})\cdot\nabla(w+\delta)w_{,t}\,dx\Big| &+ \Big|\int_\Omega (w+\delta)\cdot\nabla(w_{,t}+\delta_{,t})w_{,t}\,dx\Big|.
\end{aligned}
$$

We estimate the last two integrals:

$$
\begin{aligned}
\Big|\int_\Omega &(w_{,t}+\delta_{,t})\cdot\nabla(w+\delta)w_{,t}\,dx\Big|\\
&\le \int_\Omega |w_{,t}|^2(|\nabla w|+|\nabla\delta|)\,dx + \int_\Omega |\delta_{,t}|\,|w_{,t}|(|\nabla w|+|\nabla\delta|)\,dx\\
&\le c\|w_{,t}\|^{1/2}_{L_2(\Omega)}\|w_{,t}\|^{3/2}_{H^1(\Omega)}(\|w\|_{H^1(\Omega)} + \|\delta\|_{H^1(\Omega)})\\
&\quad + \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c\|\delta_{,t}\|^2_{L_4(\Omega)}(\|w\|^2_{H^1(\Omega)} + \|\delta\|^2_{H^1(\Omega)})\\
&\le \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c\|w_{,t}\|^2_{L_2(\Omega)}(\|w\|^4_{H^1(\Omega)} + \|\delta\|^4_{H^1(\Omega)})\\
&\quad + c\|\delta_{,t}\|^2_{L_4(\Omega)}(\|w\|^2_{H^1(\Omega)} + \|\delta\|^2_{H^1(\Omega)})
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_\Omega (w+\delta)\cdot\nabla(w_{,t}+\delta_{,t})w_{,t}\,dx\Big| &\le \Big|\int \delta\cdot\nabla(w_t+\delta_t)\cdot w_t\,dx\Big| + \Big|\int w\cdot\nabla\delta_t\cdot w_t\,dx\Big|\\
\le \frac{c}{\varepsilon}(\|w_{,t}\|^2_{L_2(\Omega)}\|\delta\|_{L_\infty(\Omega)} &+ \|\delta\|^2_{L_3(\Omega)}\|\delta_{,t}\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)}\|\delta_t\|^2_{H^1(\Omega)})\\
&\quad + \varepsilon\|w_{,t}\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\frac{d}{dt}\|w_{,t}\|^2_{L_2(\Omega)} &\le c(\|w_{,t}\|^2_{L_2(\Omega)}(\|w\|^4_{H^1(\Omega)} + \|\delta\|^4_{H^1(\Omega)} + \|\delta\|_{L_\infty(\Omega)})\\
&\quad + \|\delta_{,t}\|^2_{H^1(\Omega)}(\|\delta\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)} + 1) + \|f_{,t}\|^2_{L_{6/5}(\Omega)}).
\end{aligned}
$$

In view of the uniform Gronwall inequality (Lemma 2.4), we get

$$\|w_{,t}(t)\|_{L_2(\Omega)}^2 \le (b_3/T + b_2)e^{b_1}, \quad t \ge T, \tag{3.4}$$

where

$$\int\limits_{kT}^{(k+1)T} (\|w(t)\|_{H^1(\Omega)}^4 + \|\delta(t)\|_{H^1(\Omega)}^4 + \|\delta(t)\|_{L_\infty(\Omega)})\,dt \le b_1,$$

$$c\int\limits_{kT}^{(k+1)T} (\|\delta_{,t}(t)\|_{H^1(\Omega)}^2(\|\delta(t)\|_{H^1(\Omega)}^2 + \|w(t)\|_{H^1(\Omega)}^2 + 1) + \|f_{,t}(t)\|_{L_{6/5}(\Omega)}^2)\,dt \le b_2,$$

$$\int\limits_{kT}^{(k+1)T} \|w_{,t}(t)\|_{L_2(\Omega)}^2\,dt \le b_3.$$

Now we multiply $(1.1)_1$ by $\operatorname{div}\mathbb{T}(v,p)$, integrate over Ω and use the Hölder inequality to get

$$\begin{aligned}\|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)}^2 \le{}& \|v_{,t}\|_{L_2(\Omega)}\|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)} \\ &+ \|v\|_{L_6(\Omega)}\|\nabla v\|_{L_3(\Omega)}\|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)} \\ &+ \|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)}\|f\|_{L_2(\Omega)}.\end{aligned} \tag{3.5}$$

We estimate the second term on the r.h.s. of (3.5).

From Lemma 2.3 we get

$$\begin{aligned}&\|v\|_{L_6(\Omega)}\|\nabla v\|_{L_3(\Omega)}\|\operatorname{div}\mathbb{T}(v,p)\|_{L_2(\Omega)} \\ &\quad\le \nu\|v\|_{L_6(\Omega)}\|\nabla v\|_{L_3(\Omega)}(\|\operatorname{div}\mathbb{D}(v)\|_{L_2(\Omega)} + \|\nabla p\|_{L_2(\Omega)}) \\ &\quad\le c\|v\|_{L_6(\Omega)}\|\nabla v\|_{L_3(\Omega)}(\|\operatorname{div}\mathbb{D}(v)\|_{L_2(\Omega)} \\ &\qquad + \|d\|_{3/2,2,S_2} + \|f\|_{L_2(\Omega)} + \|v\|_{L_2(\Omega)} + \gamma\|v\cdot\bar\tau_\alpha\|_{1/2,2,S_1}) \equiv I.\end{aligned}$$

The interpolation inequality, the Young inequality and Lemma 2.2 yield

$$\begin{aligned}I &\le c|v|_{6,\Omega}|\nabla v|_{2,\Omega}^{1/2}|\nabla v|_{6,\Omega}^{1/2}(|\operatorname{div}\mathbb{D}(v)|_{2,\Omega} + \|d\|_{3/2,2,S_2} \\ &\quad + |f|_{2,\Omega} + |v|_{2,\Omega} + \gamma\|v\cdot\bar\tau_\alpha\|_{1,2,2,S_1}) \\ &\le c\|v\|_{1,\Omega}^{3/2}\|v\|_{2,\Omega}^{1/2}(|\operatorname{div}\mathbb{D}(v)|_{2,\Omega} + \|d\|_{3,2,2,S_2} + |f|_{2,\Omega} + |v|_{2,\Omega} \\ &\quad + \gamma\|v\cdot\bar\tau_\alpha\|_{1/2,2,S_1}) \\ &\le c\|v\|_{1,\Omega}^{3/2}(|\operatorname{div}\mathbb{D}(v)|_{2,\Omega}^{3/2} + \|d\|_{3/2,2,S_2}^{3/2} + |f|_{2,\Omega}^{3/2} + |v|_{2,\Omega}^{3/2} \\ &\quad + (\gamma\|v\cdot\bar\tau_\alpha\|_{1/2,2,S_1})^{3/2}) \\ &\le c(\varepsilon^{-1}\|v\|_{1,\Omega}^6 + \varepsilon|\operatorname{div}\mathbb{D}(v)|_{2,\Omega}^2 \\ &\quad + \varepsilon|f|_{2,\Omega}^2 + \varepsilon|v|_{2,\Omega}^2 + \varepsilon\|d\|_{3/2,2,S_2}^2 + \varepsilon\gamma^2\|v\cdot\bar\tau_\alpha\|_{1/2,2,S_1}^2).\end{aligned}$$

Then from (3.5) we obtain

$$\begin{aligned}(3.6)\qquad \|\operatorname{div}\mathbb{T}(v,p)\|^2_{L_2(\Omega)} \le c(\|v_{,t}\|^2_{L_2(\Omega)} + \|v\|^6_{H^1(\Omega)} + \varepsilon\|\operatorname{div}\mathbb{D}(v)\|^2_{L_2(\Omega)} \\ + \|f\|^2_{L_2(\Omega)} + \|d\|^2_{3/2,2,S_2}),\end{aligned}$$

where the first two norms on the r.h.s. are estimated by (2.8).

Finally from (2.5), (3.5) and (3.6) we conclude that

$$\|v(t)\|_{H^2(\Omega)} < \infty \quad \text{for almost all } t > T.$$

Hence there exists a ball $\mathcal{B}(0,\rho_3) \subset H^2(\Omega)$ centered at 0 with radius ρ_3 sufficiently large so that $v(t) \in \mathcal{B}(0,\rho_3)$ for almost all $t > t_0 = t_0(v_0)$.

In view of Theorem 2.1, there exists a global attractor in V, namely we have proved the following

THEOREM 3.2. *There exists a global attractor $\mathcal{A}$ in V for the semiprocess $\{S(t)\}_{t\ge 0}$ defined by* (3.1). *The attractor is bounded in $H^2(\Omega)$ and compact and connected in V. It attracts bounded sets in V.*

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