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SOME RESULTS ON STRONGLY NONLINEAR ANISOTROPIC DIFFERENTIAL EQUATIONS

Abstract. The paper concerns the existence of weak solutions to nonlinear elliptic equations of the form $A(u) + g(x, u, \nabla u) = f$, where A is an operator from an appropriate anisotropic function space to its dual and the right hand side term is in L^{1+m} with $0 < m < 1$. We assume a sign condition on the nonlinear term g, but no growth restrictions on u.

1. Introduction. Let Ω be an open bounded subset of $\mathbb{R}^N, N \geq 3$. Let $p_1, \dots, p_N$ be N real numbers with $p_i > 2$, $i = 1, \dots, N$. Let X be the Banach space, called an anisotropic Sobolev space, obtained as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\| = \|u\|_{1+1/m} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$$

where m is a real number such that $1/(p_i - 1) < m < 1$.

Let A be the nonlinear operator from X into the dual X^* defined as

$$Au = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a(x, s, \xi) = \{a_i(x, s, \xi)\} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, \dots, N$, is a Carathéodory vector-valued function, that is, measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω. Suppose that A satisfies the following conditions of strict monotonicity, coerciveness and growth: there exist positive constants α and β and a nonnegative function $k \in L^1(\Omega)$ such that

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(A1) for a.e. $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and all $i = 1, \dots, N$,

$$|a_i(x, s, \xi)| \le \beta \Big[k(x) + |s|^{1+1/m} + \sum_{j=1}^{N} |\xi_j|^{p_j} \Big]^{1-1/p_i}$$

(A2) for a.e. $x \in \Omega$ and every $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$,

$$[a(x, s, \xi) - a(x, s, \xi^*)] \cdot [\xi - \xi^*] > 0,$$

(A3) for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$a(x, s, \xi) \cdot \xi \ge \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}.$$

Consider the nonlinear elliptic equation

$$Au + g(x, u, \nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

where g is a nonlinear lower-order term having no growth conditions with respect to $|u|$ and satisfying the following assumption:

(G) $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(x, s, \xi)| \le h(|s|) \Big[c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i} \Big]^{\frac{1-m}{2(1+m)}}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function such that $h(u) \in L^2(\Omega)$ for all $u \in X$, c is a positive function in $L^1(\Omega)$, and m satisfies $1/(p_i - 1) < m < 1$. We also assume the "sign condition" $g(x, s, \xi)s \ge 0$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

Let us mention that in the isotropic case ($p_i = p$, $i = 1, \dots, N$), the problem (1.1) has been investigated by Bensoussan, Boccardo and Murat [4]. In particular, they proved the existence of solutions using different ideas based essentially on the strong convergence of the positive and negative parts of the approximate solution.

Note that some results have been proved for the problem (1.1) in the isotropic case in the framework of weighted Sobolev spaces; for more details, we refer the reader to [2].

The purpose of this paper is to study the above problem in the anisotropic case. More precisely, under the hypotheses (A1)–(A3) and (G), we prove an existence result for the anisotropic problem (1.1). Our study is motivated by an anisotropic Sobolev inequality due to Troisi [18].

Let us point out that interesting work in the anisotropic case has been done e.g. in [7], [10] and [14]. Finally, when g does not depend on ∇u, we refer the reader to the recent works [3] and [8], dealing with elliptic equations for a general class of operators of finite and infinite order, and proving the existence of solutions in anisotropic spaces.

As a prototype example, one can consider the problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) + g(x,u,\nabla u) = f \quad \text{in } \Omega,$$

where

$$g(x,s,\xi) = \operatorname{sgn}(s)(1+|s|)\Big[\sum_{i=1}^{N}|\xi_i|^{p_i}\Big]^{\frac{1-m}{2(1+m)}}$$

with $p_i > 2$ and $1/(p_i-1) < m < 1$ for $i = 1,\dots,N$.

2. Preliminaries

Anisotropic Sobolev spaces. We start by recalling that anisotropic Sobolev spaces were introduced and studied by Nikol'skiĭ [12], Slobodeckiĭ [17], and Troisi [18], and later by Trudinger [19] in the framework of Orlicz spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ and let $p_0, p_1, \dots, p_N$ be real numbers with $1 < p_i < \infty$, $i = 0,\dots,N$. We denote by $W^{1,p_i}(\Omega)$ the *anisotropic Sobolev space* consisting of all real-valued functions $u \in L^{p_0}(\Omega)$ whose derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \quad \text{for all } i = 1,\dots,N\,.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p_i} = \Big(\int_\Omega |u(x)|^{p_0}\,dx\Big)^{1/p_0} + \sum_{i=1}^{N}\Big(\int_\Omega \Big|\frac{\partial u(x)}{\partial x_i}\Big|^{p_i}\,dx\Big)^{1/p_i}.$$

The space $W_0^{1,p_i}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p_i}$. The theory of anisotropic Sobolev spaces was developed in [13], [15], [16] and [18]. It was proved that $(W_0^{1,p_i}(\Omega), \|\cdot\|_{1,p_i})$ is a reflexive Banach space for any $p_0,\dots,p_N$ with $1 < p_i < \infty$, $i = 0,\dots,N$. The dual space of $W_0^{1,p_i}(\Omega)$ is equivalent to $W^{-1,p_i'}(\Omega)$, where p_i' is the conjugate of p_i, i.e., $p_i' = p_i/(p_i-1)$, $i = 0,\dots,N$.

In the following, we assume

$$p_i > 2 \quad \text{for all } i = 1,\dots,N \quad \text{and} \quad \sum_{i=1}^{N}\frac{1}{p_i} > 1.$$

Let m be a real number such that $1/(p_i-1) < m < 1$ for all $i = 1,\dots,N$.

We define X to be the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\| = \|u\|_{1+1/m} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}. \tag{2.1}$$

REMARK 2.1. X endowed with the norm (2.1) is a reflexive Banach space. This can be easily deduced by constructing an isometric isomorphism from X to a closed subspace of $L^{1+1/m}(\Omega) \times \prod_{i=1}^{N} L^{p_i}(\Omega)$.

We prove the existence of distributional solutions for the nonlinear elliptic equation

$$(P)\begin{cases} Au + g(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{cases}$$

where $f \in L^{1+m}(\Omega)$. Herein, the operator A and the function g satisfy (A1)–(A3) and (G) respectively.

We use the following anisotropic Sobolev inequality given by Troisi [18].

THEOREM 2.1. *Let $q_i \geq 1$, $i = 1, \ldots, N$, and $u \in C_0^\infty(\Omega)$. If $\sum_{i=1}^{N} 1/q_i > 1$, then*

$$\|u\|_{\bar{q}^*} \leq C \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{q_i}^{1/N}$$

where

$$\frac{1}{\bar{q}^*} = \frac{1}{N}\left(-1 + \sum_{i=1}^{N} \frac{1}{q_i}\right)$$

and C depends only on q_i and N.

Now we state the main result of the paper.

THEOREM 2.2. *Let m be a real number with $1/(p_i - 1) < m < 1$, $i = 1, \ldots, N$. Assume that* (A1)–(A3) *and* (G) *hold and $f \in L^{1+m}(\Omega)$. Then the problem (P) has at least one solution $u \in X$ such that*

$$\begin{cases} g(x,u,\nabla u) \in L^1(\Omega) \quad \text{and} \quad g(x,u,\nabla u)u \in L^1(\Omega), \\ \langle Au, v\rangle + \int_\Omega g(x,u,\nabla u)v\,dx = \langle f, v\rangle \quad \forall v \in X. \end{cases}$$

REMARK 2.2. Note that $v \in X$, therefore $v \in L^{1+1/m}(\Omega)$. Then the condition (G) guarantees that $\int_\Omega g(x,u,\nabla u)v\,dx$ is well defined since $h(u) \in L^2(\Omega)$ and $m < 1$.

REMARK 2.3. To enlarge the class of operators A for which the conclusion of Theorem 2.2 remains true, we can also assume instead of (A1) the following condition:

(A1)$'$ for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, all $i = 1,\ldots,N$ and some function $k \in L^{p_i'}(\Omega)$,

$$|a_i(x,s,\xi)| \le \beta\Big[k(x) + |s|^{(1+1/m)/p_i'} + \Big(\sum_{j=1}^N |\xi_j|^{p_j}\Big)^{1/p_i'}\Big],$$

Proof of Theorem 2.2.

STEP 1: *Existence for the approximate problem.* Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 < \varphi < 1$ and $\varphi = 1$ in some neighbourhood of 0. Set

$$g_k(x,u,\nabla u) = \varphi(x/k)T_k g(x,u,\nabla u), \qquad b_k(u,v) = \int_\Omega g_k(x,u,\nabla u)v\,dx$$

for a.e. $x \in \Omega$ for all $u, v \in X$, where T_k is the usual truncation given by

$$T_k\mu = \begin{cases} \mu & \text{if } |\mu| \le k, \\ k\mu/|\mu| & \text{if } |\mu| > k. \end{cases}$$

Observe that $b_k(u,v)$ is well defined since $g_k(x,u,\nabla u)$ is bounded with compact support. Define the following operator:

$$G_k u : X \to \mathbb{R}, \qquad v \mapsto \int_\Omega g_k(x,u,\nabla u)v\,dx.$$

Since $v \in L^{1+1/m}(\Omega)$ it is easy to see that $G_k : X \to X^*$ is well defined.

PROPOSITION 2.1. *Under the assumptions* (A1)–(A3), *the operator* $A+G_k$ *is coercive, strictly monotone, hemicontinuous and bounded. Precisely:*

(i) $\lim_{\|u\|\to+\infty} \langle (A+G_k)u,u\rangle/\|u\| = +\infty$.
(ii) $\langle (A+G_k)u - (A+G_k)v, u-v\rangle > 0$ *if* $u \ne v$, $u, v \in X$.
(iii) *The map* $\lambda \in \mathbb{R} \mapsto \langle (A+G_k)(u+\lambda v), w\rangle$ *is continuous for each* $u,v,w \in X$.
(iv) *If* $Y \subset X$ *is bounded, then* $(A+G_k)(Y)$ *is bounded.*

Proof. (i) Let i_0 be such that

$$\Big\|\frac{\partial u}{\partial x_{i_0}}\Big\|_{p_{i_0}} = \max\Big\{\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p_i} : i = 1,\ldots,N\Big\}.$$

Since $1/(p_i-1) < m < 1$, we have $1+1/m < p_i$ for all $i = 1,\ldots,N$. This easily implies that

$$1+\frac{1}{m} \le \bar{p}^* \quad \text{where} \quad \frac{1}{\bar{p}^*} = \frac{1}{N}\Big(-1+\sum_{i=1}^N \frac{1}{p_i}\Big).$$

Then by Theorem 2.1 we have

$$\|u\|_{1+1/m} \le \tilde{K}\|u\|_{\bar{p}^*} \le K\Big\|\frac{\partial u}{\partial x_{i_0}}\Big\|_{p_{i_0}},$$

where $\tilde{K}, K$ are positive constants. Hence, thanks to (A3) and (G) we obtain

$$\begin{aligned}\frac{\langle (A+G_k)u,u\rangle}{\|u\|} &\geq \frac{\int_\Omega \alpha \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx}{\|u\|}\\ &\geq \frac{\int_\Omega \left(\frac{\alpha}{2}\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i} + \frac{\alpha}{2}\left|\frac{\partial u}{\partial x_{i_0}}\right|^{p_{i_0}}\right) dx}{\|u\|_{1+1/m} + \left\|\frac{\partial u}{\partial x_{i_0}}\right\|_{p_0}}\\ &\geq K' \frac{\|u\|_{1+1/m}^{p_{i_0}} + \left\|\frac{\partial u}{\partial x_{i_0}}\right\|_{p_{i_0}}^{p_{i_0}}}{\|u\|_{1+1/m} + \left\|\frac{\partial u}{\partial x_{i_0}}\right\|_{p_{i_0}}},\end{aligned}$$

where K' is a suitable positive constant. Then the coercivity follows immediately since the powers in the numerator are greater than those of the denominator in the above inequality.

(ii) The strict monotonicity of $A+G_k$ follows easily from (A2) and (G).

(iii) To show that $A+G_k$ is hemicontinuous, we will prove that

$$\langle (A+G_k)(u+\lambda v),w\rangle \to \langle (A+G_k)(u+\lambda_0 v),w\rangle$$

as $\lambda \to \lambda_0$ for all $u,v,w \in X$. Since for a.e. $x \in \Omega$,

$$a_i(x,u+\lambda v,\nabla(u+\lambda v)) \to a_i(x,u+\lambda_0 v,\nabla(u+\lambda_0 v))$$

as $\lambda \to \lambda_0$, thanks to the growth condition (A1) we have

$$a_i(x,u+\lambda v,\nabla(u+\lambda v)) \to a_i(x,u+\lambda_0 v,\nabla(u+\lambda_0 v)) \quad \text{weakly in } \prod_{i=1}^N L^{p_i}(\Omega)$$

as $\lambda \to \lambda_0$. Therefore

$$\langle A(u+\lambda v),w\rangle \to \langle A(u+\lambda_0 v),w\rangle \qquad \text{as } \lambda \to \lambda_0.$$

On the other hand,

$$g_k(x,u+\lambda v,\nabla(u+\lambda v)) \to g(x,u+\lambda_0 v,\nabla(u+\lambda_0 v))$$

as $\lambda \to \lambda_0$ for a.e. $x \in \Omega$. Then

$$g_k(x,u+\lambda v,\nabla(u+\lambda v)) \to g(x,u+\lambda_0 v,\nabla(u+\lambda_0 v)) \qquad \text{in } L^1(\Omega)$$

as $\lambda \to \lambda_0$ since $(g_k(x,u+\lambda v,\nabla(u+\lambda v)))_\lambda$ is bounded in $L^1(\Omega)$. Therefore

$$\langle G_k(u+\lambda v),w\rangle \to \langle G_k(u+\lambda_0 v),w\rangle \qquad \text{as } \lambda \to \lambda_0.$$

(iv) For the boundedness we have

$$
\begin{aligned}
&|\langle (A+G_k)u, v\rangle| = \langle Au, v\rangle + \int_\Omega g_k(x,u,\nabla u)v\,dx \\
&\le \beta \sum_{i=1}^N \bigg[\bigg(\int_\Omega \bigg(k(x) + |u|^{1+1/m} + \sum_{j=1}^N \left|\frac{\partial u}{\partial x_j}\right|^{p_j}\bigg)\,dx\bigg)^{1/p_i'} \bigg(\int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx\bigg)^{1/p_i}\bigg] \\
&\quad + c_1\|v\|_{1+1/m} \\
&\le c_2\|v\|(c_3 + \|u\|)^{\gamma_1}
\end{aligned}
$$

where c_1 and c_2 are positive constants and γ_1 is a positive number. This implies the boundedness of $A + G_k$.

Therefore, thanks to Theorem 2.1 of [11], there exists a solution $u_k \in X$ of the problem

$$Au_k + g_k(x, u_k, \nabla u_k) = f,$$

or variationally

$$\sum_{i=1}^N \int_\Omega a_i(x,u_k,\nabla u_k)\frac{\partial v}{\partial x_i}\,dx + \int_\Omega g_k(x,u_k,\nabla u_k)v\,dx = \langle f, v\rangle \tag{2.2}$$

for all $v \in X$.

Step 2: *A priori estimates.* Substituting $v = u_k$ in (2.2) and using (A3) and (G) results in

$$\alpha \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_k}{\partial x_i}\right|^{p_i} dx \le c\|f\|_{L^{1+m}(\Omega)}\|u_k\|,$$

where c is a positive constant. Then similarly to the proof of (i) in Proposition 2.1, we get

$$\frac{\|u_k\|_{1+1/m}^{p_{i_0}} + \left\|\frac{\partial u_k}{\partial x_{i_0}}\right\|_{p_{i_0}}^{p_{i_0}}}{\|u_k\|_{1+1/m} + \left\|\frac{\partial u_k}{\partial x_{i_0}}\right\|_{p_{i_0}}} \le c\|f\|_{L^{1+m}(\Omega)}.$$

If we suppose towards a contradiction that $\|u_k\|$ is not bounded, the left hand side of the above inequality becomes unbounded since

$$\lim_{x+y\to+\infty} \frac{x^t + y^t}{x+y} = +\infty \quad \text{for } x, y \in \mathbb{R}_+ \text{ and } t > 1.$$

Thus,

$$\|u_k\| \le C, \tag{2.3}$$

$$\int_\Omega g_k(x,u_k,\nabla u_k)u_k\,dx \le C \tag{2.4}$$

for some constant $C > 0$ independent of k. By a similar argument, we can prove that A is a bounded operator, and we get

$$\|Au_k\|_{X^*} \le C' \tag{2.5}$$

for some constant $C' > 0$ independent of k.

Step 3: *Convergence of u_k.* In view of Remark 2.1, X is reflexive, and we deduce from (2.3) and (2.5) that

$$\begin{aligned} u_k &\to u && \text{weakly in } X,\\ \frac{\partial u_k}{\partial x_i} &\to \frac{\partial u}{\partial x_i} && \text{weakly in } L^{p_i'}(\Omega),\\ Au_k &\to \chi && \text{weakly in } X^*. \end{aligned}$$

This implies that we can take a subsequence still denoted by u_k such that

$$u_k \to u \quad \text{a.e. in } \Omega. \tag{2.6}$$

This is not sufficient to pass to the limit in g_k. We need for instance

$$\nabla u_k \to \nabla u \quad \text{a.e. in } \Omega. \tag{2.7}$$

We will not give the proof of this since it is identical to one in [4]. In fact, we can prove as in [4] that

$$u_k^+ \to u^+, \quad u_k^- \to u^- \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e.}$$

Therefore, since g is continuous, we get the conclusions

$$\begin{aligned} g_k(x,u_k,\nabla u_k) &\to g(x,u,\nabla u) && \text{a.e. in } \Omega,\\ g_k(x,u_k,\nabla u_k)u_k &\to g(x,u,\nabla u)u && \text{a.e. in } \Omega. \end{aligned}$$

From (2.4) and in view of Fatou's lemma, we obtain

$$\int_\Omega g(x,u,\nabla u)u\,dx \le \lim_{k\to+\infty}\int_\Omega g_k(x,u_k,\nabla u_k)u_k\,dx \le C,$$

which implies that $g(x,u,\nabla u)u \in L^1(\Omega)$.

Now let $\delta > 0$. Since $|g_k(x,s,\xi)|\delta \le |g_k(x,s,\xi)s|$ for $|s| \ge \delta$, we obtain $|g_k(x,s,\xi)| \le \delta^{-1}|g_k(x,s,\xi)s|$ for $|s| \ge \delta$. In view of Hölder's inequality, for any measurable subset E of Ω,

$$\int_E |g_k(x,u_k,\nabla u_k)|\,dx \tag{2.8}$$
$$\le \int_{|u|\le\delta} |g_k(x,u_k,\nabla u_k)|\,dx + \delta^{-1}\int_{|u|>\delta} |g_k(x,u_k,\nabla u_k)u_k|\,dx$$
$$\le h(\delta)\int_E \left[c(x)+\sum_{j=1}^N \left|\frac{\partial u_k}{\partial x_i}\right|^{p_i}\right]^{\frac{1-m}{2(1+m)}} dx + \delta^{-1}C$$
$$\le h(\delta)c_3(c_4+c_5\|u_k\|^{\gamma_2})|E| + \delta^{-1}C$$

where c_3, c_4 and c_5 are positive constants, γ_2 is a positive number and C is the constant of (2.4). Thanks to (2.3), the above inequality implies the equiintegrability of $g_k(x,u_k,\nabla u_k)$.

For $|E|$ sufficiently small and $\delta = 2C/\varepsilon$ with $\varepsilon > 0$ we obtain

$$\int_E |g_k(x,u_k,\nabla u_k)|\,dx \le \varepsilon.$$

Thanks to (2.6), (2.7), (2.8) and Vitali's theorem we get

$$g_k(x,u_k,\nabla u_k) \to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega).$$

Hence $g(x,u,\nabla u) \in L^1(\Omega)$. Passing to the limit, we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x,u,\nabla u)v\,dx = \langle f,v\rangle \quad \text{for all } v\in X.$$

It remains to show that $Au = \chi$. For this purpose, note that since A is bounded, hemicontinuous and monotone, it is pseudomonotone (see Proposition 2.5 of [11]). Now, substituting $v = u_k$ in (2.2), in view of Fatou's lemma we get

$$\limsup_{k\to+\infty} \langle Au_k, u_k\rangle \le \langle f,u\rangle - \int_\Omega g(x,u,\nabla u)u\,dx.$$

This implies

$$\limsup_{k\to+\infty} \langle Au_k, u_k\rangle \le \langle \chi, v\rangle.$$

Since A is pseudomonotone, we obtain $\chi = Au$.

Finally, we conclude that

$$\begin{cases} g(x,u,\nabla u)\in L^1(\Omega),\, g(x,u,\nabla u)u \in L^1(\Omega), \\ \langle Au, v\rangle + \int_\Omega g(x,u,\nabla u)v\,dx = \langle f,v\rangle \quad \text{for all } v\in X. \end{cases}$$

This completes the proof.

3. Data in the dual case. Let p_i, $i = 1, \ldots, N$, be N real numbers such that

$$p_i > 1 \quad \text{for all } i = 1, \ldots, N \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1.$$

Set $p = \min\{p_i : i = 1, \ldots, N\}$ and suppose further that the nonlinear lower-order term g satisfies the following condition instead of (G):

(G′) $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(x, s, \xi)| \le h(|s|)\Big[c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i}\Big]^{\frac{1}{1-1/p-1/q}}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function such that $h(u) \in L^q(\Omega)$ for all $u \in X$, c is a positive function in $L^1(\Omega)$, and q satisfies $1 < q < p$ with $1/p + 1/q < 1$. We also assume that $g(x, s, \xi)s \ge 0$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

Define the anisotropic space E to be the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\| = \|u\|_p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}.$$

It has been shown in [14] that E is a reflexive Banach space. Consider the following nonlinear elliptic equation with Dirichlet boundary condition:

$$(P') \begin{cases} Au + g(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{cases}$$

where $f \in E^*$, with E^* denoting the dual of E.

The existence result is the following.

THEOREM 3.1. *Assume* (A1)–(A3) *and* (G′) *hold and* $f \in E^*$. *Then the problem* (P') *has at least one solution, i.e., there exists* $u \in E$ *such that*

$$\begin{cases} g(x, u, \nabla u) \in L^1(\Omega) \quad \text{and} \quad g(x, u, \nabla u)u \in L^1(\Omega), \\ \langle Au, v\rangle + \int_\Omega g(x, u, \nabla u)v\, dx = \langle f, v\rangle \quad \forall v \in E. \end{cases}$$

The conclusion of Theorem 3.1 follows by adapting the techniques used in the proof of Theorem 2.2.

REMARK 3.1. As in [14], consider the operator A of the form

$$Au = -\mathrm{div}(b(\nabla u) + a(x, u, \nabla u)),$$

where $a(x, s, \xi) = \{a_i(x, s, \xi)\}$, $i = 1, \ldots, N$, are as in the introduction and $b(\xi) = (b_1\xi_1, \ldots, b_N\xi_N)$ for $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ with

$$b_i \in L^\infty(\Omega), \qquad b_i(x) > 0,$$

for a.e. $x \in \Omega$ and all $i = 1, \dots, N$. Further, consider the weighted space associated to b_i on Ω which is given by

$$L^2(\Omega, b_i) = \{u = u(x) : b_i^{1/2} u \in L^2(\Omega)\}$$

In this space we define the norm

$$\|u\|_{b_i} = \Big(\int_\Omega b_i |u|^2 \, dx \Big)^{1/2}.$$

We define X to be the anisotropic Banach space obtained as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\| = \|u\|_{1+1/m} + \max_{i=1}^{N} \bigg[\bigg\| \frac{\partial u}{\partial x_i} \bigg\|_{p_i} \vee \Big(\int_\Omega b_i |u|^2 \, dx \Big)^{1/2} \bigg]. \tag{3.1}$$

The space X endowed with the norm (3.1) is a reflexive Banach space. This can be deduced as in [14] by constructing an isometric isomorphism from X to a closed subspace of

$$L^{1+1/m}(\Omega) \times \prod_{i=1}^{N} (L^{p_i}(\Omega) \cap L^2(\Omega, b_i)).$$

The authors of [14] have shown the reflexivity of the space $L^{p_i}(\Omega) \cap L^2(\Omega, b_i)$ endowed with the norm $\|\cdot\|_{p_i} \vee \|\cdot\|_{b_i}$ for $i = 1, \dots, N$.

Note that as in the previous section, Theorems 2.2 and 3.1 remain true for such operators.

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References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Y. Akdim, E. Azroul and A. Benkirane, *Existence of solutions for quasilinear degenerate elliptic equations*, Electron. J. Differential Equations 2001, no. 71, 19 pp.

[3] A. Benkirane, M. Chrif and S. El Manouni, *Existence results for strongly nonlinear elliptic equations of infinite order*, Z. Anal. Anwend. 26 (2007), 303–312.

[4] A. Bensoussan, L. Boccardo and F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Ann. Inst. H. Poincaré 5 (1988), 347–364.

[5] L. Boccardo and T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Anal. 19 (1992), 573–579.

[6] —, —, *Nonlinear elliptic equations with right hand side measures*, Comm. Partial Differential Equations 14 (1992), 641–655.

[7] L. Boccardo, T. Gallouët and P. Marcellini, *Anisotropic equations in L^1*, Differential Integral Equations 9 (1996), 209–212.

[8] M. Chrif and S. El Manouni, *On a strongly anisotropic equation with L^1 data*, Appl. Anal. 87 (2008), 865–871.

[9] T. Gallouët and J.-M. Morel, *Resolution of a semilinear equation in L^1*, Proc. Roy. Soc. Edinburgh 96 (1984), 275–288.
[10] F. Q. Li, *Anisotropic elliptic equations in L^m*, J. Convex Anal. 8 (2001), 417–422.
[11] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.
[12] S. M. Nikol'skiĭ, *An imbedding theorem for functions with partial derivatives considered in different metrics*, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 321–336 (in Russian); English transl.: Amer. Math. Soc. Transl. 90 (1970), 27–44.
[13] —, *On imbedding, continuation and approximation theorems for differentiable functions of several variables*, Russian Math. Surveys 16 (1961), 55–104.
[14] P. Oppezzi and A. M. Rossi, *Anisotropic equations with measure data*, Riv. Mat. Univ. Parma 7 (2002), 73–89.
[15] J. Rákosník, *Some remarks to anisotropic Sobolev spaces I*, Beiträge Anal. 13 (1979), 55–68.
[16] —, *Some remarks to anisotropic Sobolev spaces II*, ibid. 15 (1981), 127–140.
[17] L. N. Slobodeckiĭ, *Generalized Sobolev spaces and their application to boundary problems for partial differential equations*, Leningrad. Gos. Ped. Inst. Uchen. Zap. 197 (1958), 54–112 (in Russian); English transl.: Amer. Math. Soc. Transl. 57 (1966), 207–275.
[18] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. 18 (1969), 3–24.
[19] N. S. Trudinger, *An imbedding theorem for $H_0(G,\Omega)$ spaces*, Studia Math. 50 (1974), 17–30.

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