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# APPLICATION OF MAZUR–ORLICZ'S THEOREM IN AMISE CALCULATION

*Abstract.* An approximation error and an asymptotic formula are given for shift invariant operators of polynomial order $\varrho$. Density estimators based on shift invariant operators are introduced and AMISE is calculated.

**1. Asymptotic formulas.** We assume that $F, G : \mathbb{R}^d \to \mathbb{R}$ are functions such that there are constants $C > 0$ and $0 < q < 1$ such that for all $x \in \mathbb{R}^d$,

$$|F(x)| < Cq^{|x|} \quad \& \quad |G(x)| < Cq^{|x|}, \tag{1}$$

where $|x|^2 = x \cdot x$ and $x \cdot x$ is the scalar product in $\mathbb{R}^d$. Consider the operator given by

$$Qf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \tag{2}$$

where

$$K(x,y) = \sum_{\alpha \in \mathbb{Z}^d} F(y-\alpha)G(x-\alpha). \tag{3}$$

For $h > 0$, define

$$Q_h = \sigma_h \circ Q \circ \sigma_{1/h}, \tag{4}$$

where

$$\sigma_h f(x) = f(x/h).$$

We call the operators with kernel of type (3) *shift invariant*. Examples of such operators are:

• spline operators: the Ciesielski–Durrmeyer operator (see [C]), a quasi-projection (see [Dz1]), an orthogonal projection (see [BD2], [BHR]),

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• an orthogonal projection based on multiresolution approximation [M], operators based on shift invariant spaces (see [JZ] and [BDR]; in particular shift invariant spaces constructed by a function which satisfies the Strang–Fix conditions, see [SF]).

Let $W_p^r$ be a Sobolev space (for details see [M]). Let $C_0^r$ be the space of $r$-differentiable functions with compact support. Set

$$|f|_{r,p} = \sum_{|\beta|=r} \|D^\beta f\|_p, \qquad \|f\|_p = \Big(\int_{\mathbb{R}^d} |f|^p\Big)^{1/p},$$

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1}\ldots\partial x_d^{\beta_d}}, \quad \beta = (\beta_1,\ldots,\beta_d), \quad |\beta| = \beta_1+\ldots+\beta_d.$$

Assume the operator $Q$ *reproduces* all polynomials of degree less than $r$, i.e. $Q(P) = P$ provided $\deg P < r$. We then say that $Q$ has *polynomial order* $r$. The following theorem is a generalization of [BHR, Proposition 4, p. 63].

THEOREM 1.1. *Let $1 \le p < \infty$. Assume that $Q$ has polynomial order $r$. Then there is a constant $C(p) > 0$ such that for all $f \in W_p^r(\mathbb{R}^d)$,*

$$\|Q_h f - f\|_p \le C(p) h^r |f|_{r,p}, \tag{5}$$

*Proof.* Since the operators $Q_h$ are bounded from $L^p$ to $L^p$ it is sufficient to prove (5) for $f \in C_0^r$. Let $f \in C_0^r$. Let $P_x$ be the Taylor polynomial of $f$ of degree $r-1$ at $x$. Note that $f(x) = P_x(x)$ and $Q_h f(x) - f(x) = Q_h(f - P_x)(x)$. Now Lemma 1.1 below yields (5) for $1 \le p < \infty$. ∎

An easy computation shows the assertion for $p = \infty$ (see proof of [Dz4, Theorem 9.7]).

In statistics we need an asymptotic formula for the error in shift invariant operators. Such a formula was proved in [BD3], [BD4] for an interpolation operator and an orthogonal projection. Those proofs are based on a generalization of Mazur–Orlicz's theorem (see [BD3]). This theorem goes back to L. Fejér. Recall that a function $g$ defined on $\mathbb{R}^d$ is called $\mathbb{Z}^d$-*periodic* if for all $x \in \mathbb{R}^d$,

$$g(x) = g(x+\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^d. \tag{6}$$

THEOREM 1.2 (Mazur–Orlicz [MO]). *If for $j = 1,\ldots,m$, $g_j$ are measurable, bounded, $\mathbb{Z}^d$-periodic functions and $f_j$ are measurable functions with*

$$\int_{\mathbb{R}^d} |f_j(x)|^p\,dx < \infty$$

*for some $1 \le p < \infty$, then*

$$\int_{\mathbb{R}^d} \Big|\sum_{j=1}^m f_j(x) g_j(x/h)\Big|^p dx \to \int_{[0,1)^d}\int_{\mathbb{R}^d} \Big|\sum_{j=1}^m f_j(t) g_j(x)\Big|^p dt\,dx \quad \textit{as } h \to 0. \tag{7}$$

Earlier results concerning the asymptotic formula can be found in [C], [Dz1] for spline operators, and in [DU], [BD2] for an orthogonal projection in $L^2$. See also [DLP]. Let $[]^\beta(x) = x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d}$. We present a new and simpler proof of the asymptotic formula for the error in shift invariant operators.

THEOREM 1.3. *Assume that $Q$ has maximal polynomial order $\varrho$. Let $1 \le p < \infty$ and $f \in W_p^\varrho(\mathbb{R}^d)$. Then*

$$(8)\qquad \lim_{h\to 0^+} \left\| \frac{Q_h f - f}{h^\varrho} \right\|_p^p = \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t)(Q([]^\beta)(x) - x^\beta) \right|^p dx \right) dt.$$

*Proof.* It is sufficient to prove (8) for the dense subset $C_0^{\varrho+1}$ of $W_p^\varrho(\mathbb{R}^d)$ since

$$\left\| \frac{Q_h f - f}{h^\varrho} \right\|_p \le C|f|_{\varrho,p}.$$

Fix $f \in C_0^{\varrho+1}$. Let $P_x$ be the Taylor polynomial of degree $\varrho$ of $f$ at $x$. By the triangle inequality (we take $F(x) = Q_h(P_x)(x) \ne P_x(x)$)

$$\left\| \frac{Q_h f - f}{h^\varrho} \right\|_p \le \left\| \frac{Q_h (f - P.)}{h^\varrho} \right\|_p + \left\| \frac{Q_h P. - P.}{h^\varrho} \right\|_p$$

and

$$\left\| \frac{Q_h P. - P.}{h^\varrho} \right\|_p \le \left\| \frac{Q_h f - f}{h^\varrho} \right\|_p + \left\| \frac{Q_h (f - P.)}{h^\varrho} \right\|_p.$$

If we prove that there is $C$ such that for all $f \in C_0^{\varrho+1}$,

$$(9)\qquad \|Q_h(f - P.)\|_p \le C h^{\varrho+1} |f|_{\varrho+1,p},$$

then the proof of (8) is completed by showing that

$$(10)\qquad \lim_{h\to 0^+} \left\| \frac{Q_h P. - P.}{h^\varrho} \right\|_p^p = \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t)(Q([]^\beta)(x) - x^\beta) \right|^p dx \right) dt.$$

The technical proof of (9) is postponed to Lemma 1.1. Let

$$P_x = T_x + R_x,$$

where $T_x$ is homogeneous of degree $\varrho$ and $\deg R_x < \varrho$. Since $Q(R_x) = R_x$ we have

$$(11)\qquad \frac{Q_h(P_x)(t) - P_x(t)}{h^\varrho} = \frac{Q_h(T_x)(t) - T_x(t)}{h^\varrho} = Q(T_x)(t/h) - T_x(t/h)$$
$$= \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(x)(Q([]^\beta)(t/h) - (t/h)^\beta).$$

Consequently, from (11) we get

$$\left\|\frac{Q_h P_\cdot - P_\cdot}{h^\varrho}\right\|_p^p = \int\limits_{\mathbb{R}^d} \Big|\sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(x)(Q([]^\beta)(x/h) - (x/h)^\beta)\Big|^p \, dx.$$

An easy calculation shows (cf. [Dz3, Lemma 3.3]) that the functions

$$Q([]^\beta)(x) - x^\beta = (-1)^{|\beta|} \sum_{\alpha\in\mathbb{Z}^d} \int\limits_{\mathbb{R}^d} (x-y)^\beta F(y-\alpha)\, dy\, G(x-\alpha)$$

are $\mathbb{Z}^d$-periodic. Now the Mazur–Orlicz Theorem (7) implies (10).

Lemma 1.1. *Let $1 \le p < \infty$. Let $P_x$ be the Taylor polynomial of degree $k-1$ of a function $f$. There is $C$ such that for all $f \in C_0^k$,*

$$\|Q_h(f - P_\cdot)\|_p \le C h^k |f|_{k,p}. \tag{12}$$

*Proof.* By Taylor's formula,

$$\|Q_h(f-P_\cdot)\|_p^p = \int\limits_{\mathbb{R}^d} \Big| \sum_{\alpha\in\mathbb{Z}^d} \int\limits_{\mathbb{R}^d} \int\limits_0^1 \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta f(x + s(hy-x))(1-s)^{k-1}\, ds$$
$$\times (hy-x)^\beta F(y-\alpha)\, dy\, G(x/h-\alpha)\Big|^p \, dx.$$

To prove (12), using assumption (1), it is sufficient to estimate

$$J_\beta = \int\limits_{\mathbb{R}^d} \Big| \sum_{\alpha\in\mathbb{Z}^d} \int\limits_{\mathbb{R}^d} \int\limits_0^1 |D^\beta f(x+s(hy-x))|\, ds$$
$$\times |hy-x|^k q^{|y-\alpha|}\, dy\, q^{|x/h-\alpha|}\Big|^p \, dx.$$

We apply Jensen's inequality three times:

$$\Big(\int\limits_0^1 g(s)\, ds\Big)^p \le \int\limits_0^1 |g(x)|^p dx,$$

$$\Big(\sum_{\alpha\in\mathbb{Z}^d} |a_\alpha| q^{|x-\alpha|}\Big)^p \le C_1 \sum_{\alpha\in\mathbb{Z}^d} |a_\alpha|^p q^{|x-\alpha|},$$

where $C_1$ is independent of $x$, i.e. $C_1 = \max_x(\sum_{\alpha\in\mathbb{Z}^d} q^{|x-\alpha|})^{p-1}$,

$$\Big(\int\limits_{\mathbb{R}^d} |g(y)| q^{|y-\alpha|}\, dy\Big)^p \le C_2 \int\limits_{\mathbb{R}^d} |g(y)|^p q^{|y-\alpha|}\, dy,$$

where $C_2$ is independent of $\alpha$. Consequently,

$$J_\beta \le C \sum_{\alpha\in\mathbb{Z}^d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \int\limits_0^1 |D^\beta f(x+s(hy-x))|^p\, ds$$
$$\times |hy-x|^{pk} q^{|y-\alpha|}\, dy\, q^{|x/h-\alpha|}\, dx.$$

Letting $x/h - \alpha = u$ yields

$$J_\beta \le Ch^d \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + s(hy - hu - h\alpha))|^p \, ds \times |hy - hu - h\alpha|^{pk} q^{|y-\alpha|} \, dy \, q^{|u|} \, du$$

and by obvious changes of variables

$$\begin{aligned} J_\beta &\le Ch^d \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + sh(z-u))|^p \, ds \\ &\quad \times h^{pk} |z-u|^{pk} q^{|z|} \, dz \, q^{|u|} \, du \\ &= Ch^{d+pk} \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p \, ds |v|^{pk} q^{|u+v|} \, dv \, q^{|u|} \, du. \end{aligned}$$

Let us split the integrals:

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^\infty \sum_{\alpha\in\mathbb{Z}^d} \int_{n-1<|u|<n} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p \, ds \\ &\quad \times |v|^{pk} q^{|u+v|} \, dv \, q^{n-1} \, du \\ &\le Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{\alpha\in\mathbb{Z}^d} \int_{n-1<|u|<n} \sum_{j=1}^\infty q^{j-1} \\ &\quad \times \int_{j-1<|u+v|<j} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p \, ds \, |v|^{pk} \, dv \, du. \end{aligned}$$

Note that if $|v+u| < j$ and $|u| < n$ then

$$|v| < |v+u| + |u| < j + n.$$

Thus

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} \sum_{j=1}^\infty q^{j-1} \\ &\quad \times \int_{|v|<j+n} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p \, ds \, |v|^{pk} \, dv \, du. \end{aligned}$$

Changing the order of the integrations we get

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{j=1}^\infty q^{j-1} \\ &\quad \times \int_{|v|<j+n} \int_0^1 \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} |D^\beta f(hu + h\alpha + shv)|^p \, du \, ds \, |v|^{pk} \, dv. \end{aligned}$$

Note that if $|v| < j + n$ then

$$h^d \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} |D^\beta f(hu + h\alpha + shv)|^p \, du$$
$$\le \sum_{\alpha\in\mathbb{Z}^d} \int_{|x-h\alpha|<h(2n+j)} |D^\beta f(x)|^p \, dx \le (4n+2j)^d \int_{\mathbb{R}^d} |D^\beta f(x)|^p \, dx$$

and moreover

$$\int_{|v|<j+n} |v|^{pk} \, dv = C(j+n)^{pk+d}.$$

Consequently,

$$J_\beta \le C \sum_{n=1}^{\infty} q^{n-1} h^{pk} \sum_{j=1}^{\infty} q^{j-1} (4n+2j)^d (j+n)^{pk+d} \int_{\mathbb{R}^d} |D^\beta f(x)|^p \, dx$$
$$\le Ch^{pk} \int_{\mathbb{R}^d} |D^\beta f|^p.$$

This finishes the proof of the lemma. ∎

Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f \in W_2^\varrho$. We define a density estimator based on the kernel $K$ by

$$f_{h,n}(x) = \frac{1}{n} \sum_{j=1}^{n} K_h(x, X_j), \tag{13}$$

where

$$K_h(x,y) = (1/h)^d K(x/h, y/h).$$

Note that

$$Ef_{h,n} = Q_h f. \tag{14}$$

As usual we consider the estimation error given by

$$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^d} [f_{h,n} - f]^2\Big]. \tag{15}$$

It is known that

$$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^d} [f_{h,n} - Q_h f]^2\Big] + \int_{\mathbb{R}^d} [Q_h f - f]^2. \tag{16}$$

The asymptotic formula for the second factor in (16) is given in (8). We prove that

THEOREM 1.4. *Assume that $Q$ has maximal polynomial order $\varrho > 0$. If $nh^d \to \infty$, $h \to 0$ then*

$$\lim_{nh^d\to\infty} nh^d E\Big[\int_{\mathbb{R}^d} [f_{h,n} - Q_h f]^2\Big] = \int_{\mathbb{R}^d} \Big[\int_{[0,1]^d} K^2(x,y)\, dy\Big] dx, \tag{17}$$

*where*

$$\int_{\mathbb{R}^d} \Big[ \int_{[0,1]^d} K^2(x,y)\,dy \Big]\,dx = \sum_{\alpha \in \mathbb{Z}^d} \eta(\alpha)\xi(\alpha) \tag{18}$$

*and*

$$\eta = G * \breve{G}, \qquad \xi = F * \breve{F}, \qquad \breve{G}(x) = G(-x), \qquad \breve{F}(x) = F(-x).$$

*Proof.* Note that

$$E\Big[ \int_{\mathbb{R}^d} [f_{h,n} - Ef_{h,n}]^2 \Big] = \frac{1}{n^2} \sum_{j=1}^{n} \int_{\mathbb{R}^d} E[K_h(x,X_j) - EK_h(x,X_j)]^2\,dx$$
$$= \frac{1}{n^2} \sum_{j=1}^{n} \int_{\mathbb{R}^d} (E[K_h^2(x,X_j)] - [EK_h(x,X_j)]^2)\,dx.$$

If $h \to 0$ then by (5),

$$\int_{\mathbb{R}^d} [EK_h(x,X_j)]^2\,dx = \int_{\mathbb{R}^d} (Q_h f)^2 \to \int_{\mathbb{R}^d} f^2.$$

On the other hand

$$\int_{\mathbb{R}^d} EK_h^2(x,X_j)\,dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h^2(x,y) f(y)\,dy\,dx.$$

From Fubini's theorem

$$\int_{\mathbb{R}^d} EK_h^2(x,X_j)\,dx = \frac{1}{h^d} \int_{\mathbb{R}^d} \Big[ \int_{\mathbb{R}^d} K^2(u, y/h)\,du \Big] f(y)\,dy.$$

Note that for all $\alpha \in \mathbb{Z}^d$,

$$\int_{\mathbb{R}^d} K^2(x, y+\alpha)\,dx = \int_{\mathbb{R}^d} K^2(x,y)\,dx. \tag{19}$$

From Mazur–Orlicz's theorem we get

$$\lim_{h\to 0} \int_{\mathbb{R}^d} \Big[ \int_{\mathbb{R}^d} K^2(u, y/h)\,du \Big] f(y)\,dy = \int_{[0,1]^d} \Big[ \int_{\mathbb{R}^d} K^2(u,y)\,du \Big]\,dy \int_{\mathbb{R}^d} f(y)\,dy.$$

We thus get (17). A simple calculation leads to (18). ∎

REMARKS. 1. From (16)–(8) we get

$$\mathrm{MISE}(f,h) \sim \mathrm{AMISE} := \frac{1}{nh^d} \int_{\mathbb{R}^d} \Big[ \int_{[0,1]^d} K^2(x,y)\,dy \Big]\,dx$$
$$+ h^{2\varrho} \int_{\mathbb{R}^d} \Big( \int_{[0,1]^d} \Big| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t) Q([]^\beta)(x) - x^\beta \Big|^2\,dx \Big)\,dt.$$

So the best choice of $h > 0$ which minimizes (16) is

$$h \sim n^{-1/(2\varrho+d)}.$$

2. Using the methods of [Dz2] one can prove the central limit theorem. This theorem generalizes the results for wavelet estimators [DL1]–[DL2] in $\mathbb{R}^d$ and box spline estimators [Dz2]. These results are motivated by the result for the Rosenblatt–Parzen estimator [H].

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