APPLICATIONES MATHEMATICAE
29,1 (2002), pp. 51–63

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ON IMBEDDING THEOREMS FOR WEIGHTED ANISOTROPIC SOBOLEV SPACES

Abstract. Using the Il'in integral representation of functions, imbedding theorems for weighted anisotropic Sobolev spaces in $\mathbb{E}^n$ are proved. By the weight we assume a power function of the distance from an $(n-2)$-dimensional subspace passing through the domain considered.

1. Introduction. The aim of this paper is to show some imbedding theorems for weighted Sobolev spaces. We introduce the weighted Sobolev spaces $W_{p,\mu}^{l,l/2}(\Omega^T)$, $l=2k$, $k\in\mathbb{N}\cup\{0\}$, $\Omega\subset\mathbb{E}^n$, $\Omega^T=\Omega\times(0,T)$, $\mu\in\mathbb{R}$, $p\ge 1$, with the norm

$$\|u\|_{W_{p,\mu}^{l,l/2}(\Omega^T)}=\Big(\sum_{|\alpha|+2\alpha_0\le l}\int_{\Omega^T}|D_x^\alpha\partial_{x_0}^{\alpha_0}u|^p\varrho^{p\mu}(x)\,dx\Big)^{1/p}, \tag{1.1}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multiindex, $|\alpha|=\alpha_1+\dots+\alpha_n$, $D_x^\alpha=\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$ and $\varrho=\varrho(x)$ is the distance from x to either a subspace of $\mathbb{R}^n$ or a point. In this paper we assume that $\varrho(x)=\operatorname{dist}(x,M)$ where M is an $(n-2)$-dimensional subspace of $\mathbb{E}^n$. To simplify considerations we assume that M is defined by $x_1=x_2=0$. Finally $\mathbb{E}^n$ is the n-dimensional Euclidean space. Therefore, we can assume that $\varrho(x)=\sqrt{x_1^2+x_2^2}$.

More precisely we define $W_{p,\mu}^{l,l/2}(\Omega^T)$ as the closure of the set $C_0^\infty(\Omega^T\setminus M)$ in the norm (1.1).

We use the standard anisotropic notation (see [1]). We consider more general anisotropic Sobolev spaces $W_{p,\mu}^{\overline{l}}(\mathbb{E}^{n+1})$, where $\overline{l}=(l_0,l_1,\dots,l_n)$,

2000 *Mathematics Subject Classification*: Primary 46E35.

Key words and phrases: weighted Sobolev spaces, imbedding theorems, anisotropic Sobolev spaces.

Research supported by the Polish KBN Grant No. 2 P03A-038-16.

with the norm defined as follows:

$$\|u\|_{W^{\overline{l}}_{p,\mu}(\mathbb{E}^{n+1})} = \|u\|_{L_{p,\mu}(\mathbb{E}^{n+1})} + \sum_{i=0}^{n} \|\partial^{l_i}_{x_i} u\|_{L_{p,\mu}(\mathbb{E}^{n+1})} \equiv \|u\|_{L_{p,\mu}(\mathbb{E}^{n+1})} + \|u\|_{L^{\overline{l}}_{p,\mu}(\mathbb{E}^{n+1})}, \tag{1.2}$$

where $\|u\|_{L_{p,\mu}(\mathbb{E}^{n+1})} = (\int_{\mathbb{E}^{n+1}} |u|^p \varrho^{p\mu}\, dx)^{1/p}$. Comparing (1.1) and (1.2) we see that for a norm equivalent to (1.1) we have

$$l_0 = l/2, \quad l_i = l, \quad i = 1, \ldots, n.$$

We also introduce $\overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ and $\sigma_i = 1/l_i$, $i = 0, 1, \ldots, n$. In the case of the norm (1.1) we have

$$\sigma_0 = 2/l, \quad \sigma_i = 1/l, \quad i = 1, \ldots, n.$$

We use the following integral representation of a function f with integrable $\overline{l}$-derivative (see [1, 3]):

$$f(\overline{x}) = f_{r^{\overline{\sigma}}}(\overline{x}) + \int_0^r \sum_{i=0}^{n} \int_{\mathbb{E}^{n+1}} h^{-|\overline{\sigma}|} \Phi_i(\overline{x}/h^{\overline{\sigma}}) D_i^{l_i} f(\overline{x} + \overline{y})\, d\overline{y}\, dh, \tag{1.3}$$

where $\overline{x} = (x_0, x_1, \ldots, x_n)$, $\overline{y} = (y_0, y_1, \ldots, y_n)$, $x_0 = t$, $y_0 = \tau$, $|\overline{\sigma}| = \sigma_0 + \sum_{i=1}^n \sigma_i$, $h^{\overline{\sigma}} = (h^{\sigma_0}, h^{\sigma_1}, \ldots, h^{\sigma_n})$, $\overline{y}/h^{\overline{\sigma}} = (y_0/h^{\sigma_0}, y_1/h^{\sigma_1}, \ldots, y_n/h^{\sigma_n})$, D_i denotes the derivative with respect to the ith argument, and

$$f_{r^{\overline{\sigma}}}(\overline{x}) = r^{-|\overline{\sigma}|} \int_{\mathbb{E}^{n+1}} \Phi_*(\overline{y}/r^{\overline{\sigma}}) f(\overline{x} + \overline{y})\, d\overline{y}. \tag{1.4}$$

We assume that $\Phi_*, \Phi_i \in C_0^\infty(\mathbb{E}^{n+1})$, $i = 0, 1, \ldots, n$, have compact supports in the first coordinate angle and for any α,

$$\int D^\alpha \Phi_i(\overline{x})\, d\overline{x} = 0, \quad i = 0, 1, \ldots, n, \quad \int D^\alpha \Phi_*(\overline{x})\, d\overline{x} = 0. \tag{1.5}$$

Moreover, we assume that the supports of Φ_*, Φ_i, $i = 0, 1, \ldots, n$, belong to the "horn" (see [1, 3])

$$R(\overline{l}, r, \varepsilon) = \{\overline{y} : y_i > 0,\ a_i > 0,\ 0 < a_i h < y_i^{l_i} < (a_i + \varepsilon)h,\ i = 0, 1, \ldots, n,\ 0 < h < r < \infty\}, \tag{1.6}$$

where $\varepsilon > 0$. If $l_i = l$, $i = 0, 1, \ldots, n$, the horn $R(\overline{l}, r, \varepsilon)$ becomes the cone $V(r, \varepsilon) = \{y : y_i > 0,\ a_i > 0,\ 0 < a_i h < y_i < (a_i + \varepsilon)h,\ i = 0, 1, \ldots, n,\ 0 < h < r < \infty\}$. For simplicity we shall omit the ε in $R(\overline{l}, r, \varepsilon)$ and $V(r, \varepsilon)$.

We define

$$f_\varepsilon(\overline{x}) = f_{r^{\overline{\sigma}}}(\overline{x}) + \int_\varepsilon^r dh \sum_{i=0}^{n} \int_{\mathbb{E}^{n+1}} h^{-|\overline{\sigma}|} \Phi_i(\overline{y}/h^{\overline{\sigma}}) D_i^{l_i} f(\overline{x} + \overline{y})\, d\overline{y}. \tag{1.7}$$

From [3] we have the estimate

$$\int_{\varepsilon}^{r} h^{\tau-1}\Phi_i^{(\overline{\nu}(k))}(y/h)\,dh \le c\chi(y, R(\bar{l}, r))\Big(\sum_{i=0}^{n} |y_i|^{l_i}\Big)^{\tau} \tag{1.8}$$

where $\chi(y, R(\bar{l}, r))$ is the characteristic function of the horn $R(\bar{l}, r)$ and $\Phi_i^{(\overline{\nu}(k))}$ is defined below (2.3).

For the reader's convenience we recall some results. From [3] we recall the following extension of the Calderón–Zygmund theorem.

Lemma 1.1. *Let the support of* $\Phi \in C_0^\infty(\mathbb{E}^n)$ *be in the first coordinate cube and* $\int_{\mathbb{E}^n} \Phi(x)\,dx = 0$. *Let* $1 < p < \infty$ *and let*

$$v_{\varepsilon r}(x) = \int_{\varepsilon}^{r} dh \int_{\mathbb{E}^n} h^{-1-|\sigma|}\Phi(y/h^{\sigma})u(x+y)\,dy$$

for $u \in L_p(\mathbb{E}^n)$, $x, y \in \mathbb{E}^n$, $\sigma \in \mathbb{R}_+$. *Then*

$$\|v_{\varepsilon r}\|_{L_p(\mathbb{E}^n)} \le c_p \|u\|_{L_p(\mathbb{E}^n)}$$

and

$$v_{\varepsilon r} \to v_{0r} \quad \text{in } L_p(\mathbb{E}^n) \text{ as } \varepsilon \to 0.$$

We also need the Hardy inequality

$$\|\,|x|^{-\gamma}F(x)\|_{L_p(\mathbb{E}_+)} \le c\|\,|x|^{-\gamma+1}f(x)\|_{L_p(\mathbb{E}_+)}, \quad 1 \le p \le \infty, \tag{1.9}$$

where $\gamma \ne 1/p$, $F(x) = \int_x^\infty f(\xi)\,d\xi$ for $\gamma < 1/p$, $F(x) = \int_0^x f(\xi)\,d\xi$ for $\gamma > 1/p$ and $\mathbb{E}_+ = \{x \in \mathbb{E} : x > 0\}$.

In the case of isotropic weighted Sobolev spaces similar results are proved in [4].

From [1, Ch. 2, Sect. 8] we recall

Definition 1.2. We say that a domain Q satisfies that the $R(\bar{l}, r)$-*horn condition* if there exist K open subdomains Q_k and horns $R_k(\bar{l}, r)$ such that

$$Q = \bigcup_{k=1}^{K} Q_k = \bigcup_{k=1}^{K} (Q_k + R_k(\bar{l}, r)).$$

In [1] this property is called the weak $R(\bar{l}, r)$-horn condition.

2. Imbedding theorems for $p \ne q$. First we prove

Theorem 2.1. *Assume that* $f \in W_{p,\alpha}^{\bar{l}}(Q)$, $Q \subset \mathbb{E}^{n+1}$, $\bar{l} = (l_0, l_1, \ldots, l_n)$, $1 < p < q < \infty$, $\alpha, \beta \in \mathbb{R}_+$, $0 < l_i \in \mathbb{Z}$, $0 \le \nu_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$, $l_1 = l_2 = l_*$,

$$\varkappa = 1 - \Big(\frac{1}{p} - \frac{1}{q}\Big)\sum_{i=0}^{n} \frac{1}{l_i} - \sum_{i=0}^{n} \frac{\nu_i}{l_i} - \frac{1}{l_*}(\alpha - \beta) \ge 0, \tag{2.1}$$

where Q satisfies the $R(\bar{l},r)$-horn condition. Assume

$$\alpha > \beta. \tag{2.1'}$$

Then $D^{\overline{\nu}} f \in L_{q,\beta}(Q)$ and

$$\|D^{\overline{\nu}} f\|_{L_{q,\beta}(Q)} \le c_1 \delta^{\varkappa} \|f\|_{L^{\bar{l}}_{p,\alpha}(Q)} + c_2 \delta^{\varkappa-1} \|f\|_{L_{p,\alpha}(Q)}, \tag{2.2}$$

where the constants c_1, c_2 do not depend on f, $\delta \in (0,h_0)$, $h_0 = h_0(Q)$.

Proof. Let $x = (x_1,\dots,x_n)$, $x' = (x_1,x_2)$, $x'' = (x_3,\dots,x_n)$. Then we introduce the cylindrical coordinates $(\varrho_x, \varphi_x, x'')$ connected with x, where $\varrho_x = |x'|$, $x_1 = \varrho_x \cos\varphi_x$, $x_2 = \varrho_x \sin\varphi_x$.

Let $k \in \mathbb{N}_0$. Then integrating by parts and using the compactness of the supports of Φ_*, Φ_i, $i = 0,1,\dots,n$, we obtain from (1.7) the expression

$$\begin{aligned} D^{\overline{\nu}} f_\varepsilon(\overline{x}) &= c \int_{\mathbb{E}^{n+1}} \Phi_*^{(\overline{\nu}(k))}(\overline{y}/r^{\overline{\sigma}}) \int_{\varrho_y}^{\infty} \int_{t_{k-1}}^{\infty} \dots \int_{t_1}^{\infty} f(\overline{x}+\overline{y}(t_0))\, dt_0\, dt_1 \dots dt_{k-1}\, d\overline{y} \\ &\quad + \int_\varepsilon^r dh \sum_{i=0}^n \int_{\mathbb{E}^{n+1}} h^{-|\overline{\sigma}|-k\sigma_*-(\overline{\sigma},\overline{\nu})} \Phi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) \\ &\quad \times \int_{\varrho_y}^{\infty} \int_{t_{k-1}}^{\infty} \dots \int_{t_1}^{\infty} D_i^{l_i} f(\overline{x}+\overline{y}(t_0))\, dt_0 \dots dt_{k-1}\, d\overline{y}, \end{aligned} \tag{2.3}$$

where $\sigma_* = \sigma_1 = \sigma_2$ so that $l_1 = l_2 = l_*$, $\overline{\nu} = (\nu_0,\nu_1,\dots,\nu_n)$, $\overline{\nu}(k) = (\nu_0, \nu_1+k_1, \nu_2+k_2, \nu_3,\dots,\nu_n)$, $k_1+k_2 = k$,

$$\Phi^{(\overline{\nu}(k))}(x) = \sum_{k_1+k_2=k} c_{k_1k_2} (\cos\varphi_x)^{k_1} (\sin\varphi_x)^{k_2} \partial_{x_0}^{\nu_0} \partial_{x_1}^{\nu_1+k_1} \partial_{x_2}^{\nu_2+k_2} \partial_{x_3}^{\nu_3} \dots \partial_{x_n}^{\nu_n} \Phi(x),$$

$\overline{y} = (y_0,y_1,\dots,y_n)$, $\overline{y}(t_0) = (y_0, t_0\cos\varphi_y, t_0\sin\varphi_y, y_3,\dots,y_n)$, and $(\overline{\sigma},\overline{\nu}) = \sigma_0\nu_0 + \sigma_1\nu_1 + \dots + \sigma_n\nu_n$. Finally $c_{k_1k_2}$ are determined by the relation

$$\partial_{\varrho_x}^k \Phi = \sum_{k_1+k_2=k} c_{k_1k_2} \left(\frac{\partial x_1}{\partial \varrho_x}\right)^{k_1} \left(\frac{\partial x_2}{\partial \varrho_x}\right)^{k_2} \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \Phi.$$

Let us introduce the notation

$$F(\overline{y}) = \int_{\varrho_y}^{\infty} \int_{t_{k-1}}^{\infty} \dots \int_{t_1}^{\infty} f(\overline{y}(t_0))\, dt_0\, dt_1 \dots dt_{k-1}, \tag{2.4}$$

$$F_i(\overline{y}) = \int_{\varrho_y}^{\infty} \int_{t_{k-1}}^{\infty} \dots \int_{t_1}^{\infty} D_i^{l_i} f(\overline{y}(t_0))\, dt_0\, dt_1 \dots dt_{k-1}. \tag{2.5}$$

Then we can write (2.3) in the following shorter form:

$$D^{\overline{\nu}} f_\varepsilon(\overline{x}) = c \int\limits_{\mathbb{E}^{n+1}} \Phi^{(\overline{\nu}(k))}(\overline{y}/r^{\overline{\sigma}}) F(\overline{x}+\overline{y})\, d\overline{y} + c\int\limits_{\varepsilon}^{r} dh \sum_{i=0}^{n} \int\limits_{\mathbb{E}^{n+1}} h^{-|\overline{\sigma}|-k\sigma_*-(\overline{\sigma},\overline{\nu})} \Phi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) F_i(\overline{x}+\overline{y})\, d\overline{y}. \tag{2.6}$$

Now we examine the case $\varkappa = 0$ (see (2.1)). Consider

$$\|D^{\overline{\nu}} f_\varepsilon(\overline{x})\|_{L_{q,\beta}(Q)} \le c\Big\| \int\limits_{\mathbb{E}^{n+1}} \Phi_*^{(\overline{\nu}(k))}(\overline{y}/r^{\overline{\sigma}}) F(\overline{x}+\overline{y})\, d\overline{y}\Big\|_{L_{q,\beta}(Q)} + c\sum_{i=0}^{n} \Big\| \int\limits_{\varepsilon}^{r} dh \int\limits_{\mathbb{E}^{n+1}} h^{-|\overline{\sigma}|-k\sigma_*-(\overline{\sigma},\overline{\nu})} \Phi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) \times \frac{|x'|^\beta}{|x'+y'|^{\alpha-k}} F_i'(\overline{x}+\overline{y})\, d\overline{y}\Big\|_{L_q(Q)} \equiv M + N, \tag{2.7}$$

where $F_i'(x) = |x'|^{\alpha-k} F_i(x)$ and $|x'| = \sqrt{x_1^2+x_2^2}$.

First we estimate N. Using (1.8) we obtain

$$N \le c\sum_{i=0}^{n} \Big\| \int\limits_{\mathbb{E}^{n+1}} \chi(y, R(\overline{l}, h_0)) \Big(\sum_{i=0}^{n} |y_i|^{l_i}\Big)^{\tau} \times \frac{|x'|^\beta}{|x'+y'|^{\alpha-k}} F_i'(\overline{x}+\overline{y})\, d\overline{y}\Big\|_{L_q(\mathbb{E}^{n+1})} \equiv N_1,$$

where $\tau = -(1-1/p+1/q)|\overline{\sigma}| + (\alpha-\beta-k)\sigma_*$.

Since $q > p > 1$ we can assume that $1/p - 1/q = 1 - 1/s$ and we also assume also that $\alpha - \beta - k \le 1/s$. To estimate N_1 we apply the one-dimensional Young inequality

$$\|f * g\|_{L_q} \le \|g\|_{L_s} \|f\|_{L_p} \quad \text{for } \frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{s},\ 1 \le p \le q \le \infty, \tag{2.8}$$

with respect to the variables $x_0, x_3, \ldots, x_n$.

Using (2.8) with respect to x_0 we find that the kernel in N_1 is estimated as follows:

$$\Big(\int\limits_0^\infty \Big(\sum_{i=0}^{n} |y_i|^{l_i}\Big)^{\tau s} dy_0\Big)^{1/s} = \Big(\int\limits_0^\infty (|y_0|^{l_0} + a_0)^{\tau s}\, dy_0\Big)^{1/s} \qquad (a_0 = \textstyle\sum_{i=1}^{n} |y_i|^{l_i})$$

$$\le \Big(\int\limits_0^\infty (y_0 + a_0^{1/l_0})^{\tau s l_0}\, dy_0\Big)^{1/s} = \Big(\frac{1}{\tau s l_0 + 1}(y_0 + a_0^{1/l_0})^{\tau s l_0 + 1}\Big|_{y_0=0}^{y_0=\infty}\Big)^{1/s}$$

$$= \Big(-\frac{1}{\tau s l_0 + 1}\Big)^{1/s} a_0^{\tau + 1/(s l_0)} \equiv K_1,$$

where we have used the fact that

$$\begin{aligned}\tau s l_0 + 1 &= \Big[-\frac{1}{s}\Big(\frac{1}{l_0}+\frac{1}{l_1}+\frac{1}{l_2}+\ldots+\frac{1}{l_n}\Big)+(\alpha-\beta-k)\frac{1}{l_*}\Big] s l_0 + 1\\ &= -l_0\Big(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}+\ldots+\frac{1}{l_n}\Big)+\frac{s l_0}{l_*}(\alpha-\beta-k)\\ &\le -\frac{2l_0}{l_*}+\frac{s l_0}{l_*}(\alpha-\beta-k)-l_0\Big(\frac{1}{l_3}+\ldots+\frac{1}{l_n}\Big)\\ &\le -\frac{l_0}{l_*}-l_0\Big(\frac{1}{l_3}+\ldots+\frac{1}{l_n}\Big)<0,\end{aligned}$$

since $l_1 = l_2 = l_*$, $\alpha-\beta-k \le 1/s$.

Using (2.8) with respect to x_3 we see that the kernel K_1 is estimated by

$$\begin{aligned}\Big(\int_0^\infty \Big(\sum_{i=1}^n |y_i|^{l_i}\Big)^{\tau s+1/l_0} dy_3\Big)^{1/s} &\le \Big(\int_0^\infty (y_3 + a_3^{1/l_3})^{(\tau s+1/l_0)l_3}\, dy_3\Big)^{1/s}\\ &= \Big(-\frac{1}{\tau s l_3 + l_3/l_0 + 1}\Big)^{1/s} a_3^{[(\tau s+1/l_0)l_3+1]/(s l_3)},\end{aligned}$$

where $a_3 = |y_1|^{l_1} + |y_2|^{l_2} + \sum_{i=4}^n |y_i|^{l_i}$,

$$\begin{aligned}\tau s l_3 + \frac{l_3}{l_0} + 1 &= \Big(\tau s + \frac{1}{l_0} + \frac{1}{l_3}\Big) l_3\\ &= \Big[-\Big(\frac{1}{l_0}+\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}+\ldots+\frac{1}{l_n}\Big)+(\alpha-\beta-k)\frac{s}{l_*}+\frac{1}{l_0}+\frac{1}{l_3}\Big] l_3\\ &\le \Big[-\frac{1}{l_*}-\Big(\frac{1}{l_4}+\frac{1}{l_5}+\ldots+\frac{1}{l_n}\Big)\Big] l_3 < 0,\end{aligned}$$

where we have used the fact that $l_1 = l_2 = l_*$, $\alpha-\beta-k\le 1/s$.

Continuing the calculations we obtain

$$N_1 \le c\sum_{i=0}^n \Big\| \int_{\mathbb{E}^2} \chi(y'-x', R(l_*, h_0))|x'-y'|^{-2/s+\alpha-\beta-k} \times \frac{|x'|^\beta}{|y'|^{\alpha-k}} G_i(y')\, dy' \Big\|_{L_q(\mathbb{E}^2)} \equiv N_2,$$

where $G_i(y') = (\int_Q |F_i'(\overline{y})|^p\, dy_0\, dy_3 \ldots dy_n)^{1/p}$.

To estimate N_2 we use Remark 3.1 from [2], which we formulate in the following form. Assume that

$$(2.9)\qquad p_1 \ge 1,\quad q_1\ge 1,\quad 1/p_1 + 1/q_1 \ge 1,\quad \delta < 2/q_1',\quad \lambda < 2/p_1' + 2/q_1',$$

where r' is dual to r, so $1/r + 1/r' = 1$. Then

$$\left(\int_{\mathbb{E}^2}\left|\frac{1}{|x|^{2/p_1'+2/q_1'-(\lambda+\delta)}}\int_{|y|\le|x|}\frac{g(y)\,dy}{|x-y|^\lambda|y|^\delta}\right|^{p_1'}dx\right)^{1/p_1'} \le c\|g\|_{L_{q_1}(\mathbb{E}^2)}, \tag{2.10}$$

$$\left(\int_{\mathbb{E}^2}\left|\frac{1}{|y|^\delta}\int_{|y|\le|x|}\frac{f(x)\,dx}{|x-y|^\lambda|x|^{2/p_1'+2/q_1'-(\lambda+\delta)}}\right|^{q_1'}dy\right)^{1/q_1'} \le c\|f\|_{L_{p_1}(\mathbb{E}^2)}. \tag{2.11}$$

Inserting $p_1' := q$, $q_1 := p$, $\delta := \delta_1$ into (2.10) yields

$$\left(\int_{\mathbb{E}^2}\left|\frac{1}{|x|^{2/s-(\lambda+\delta_1)}}\int_{|y|\le|x|}\frac{g(y)\,dy}{|x-y|^\lambda|y|^{\delta_1}}\right|^{q}dx\right)^{1/q} \le c\|g\|_{L_{p}(\mathbb{E}^2)}. \tag{2.12}$$

Inserting $p_1 := p$, $q_1' := q$, $x := y$, $y := x$, $f := g$, $\delta := \delta_2$ in (2.11) implies

$$\left(\int_{\mathbb{E}^2}\left|\frac{1}{|x|^{\delta_2}}\int_{|x|\le|y|}\frac{g(y)\,dy}{|x-y|^\lambda|y|^{2/s-(\lambda+\delta_2)}}\right|^{q}dx\right)^{1/q} \le c\|g\|_{L_{p}(\mathbb{E}^2)}. \tag{2.13}$$

The conditions (2.9) imply for (2.12) the restrictions

$$\delta_1 < 2(1-1/p), \qquad \lambda < 2/s. \tag{2.14}$$

For case (2.13) conditions (2.9) yield

$$\delta_2 < 2/q, \qquad \lambda < 2/s. \tag{2.15}$$

Comparing (2.12) with N_2 we see that

$$\lambda = \frac{2}{s} - (\alpha-\beta-k), \qquad \delta_1 = \alpha - k, \qquad -\beta = \frac{2}{s} - (\lambda - \delta_1), \tag{2.16}$$

where the last condition follows from the first two, and comparing (2.13) with N_2 we obtain

$$\lambda = \frac{2}{s} - (\alpha-\beta-k), \quad \delta_2 = -\beta, \quad \alpha - k = \frac{2}{s} - (\lambda + \delta_2), \tag{2.17}$$

where the last condition also follows from the first two.

However to estimate N_2 we need estimate (2.13). Therefore we have to impose the following restrictions:

$$-\beta < 2/q, \quad 2/s - (\alpha-\beta-k) < 2/s, \tag{2.18}$$

where the last inequality implies

$$\alpha - k > \beta. \tag{2.19}$$

For $k = 0$ the condition gives

$$\alpha > \beta, \tag{2.20}$$

so the case $\alpha = \beta$ cannot be considered. Since the first condition of (2.18) is trivial and (2.19) with $k = 0$ is less restrictive we see that (2.20) is exactly (2.1′).

Now using Remark 3.1 from [2] we obtain the estimate

$$N_1 \le c\sum_{i=0}^{n} \|F_i'\|_{L_p(\mathbb{E}^{n+1})} \le c\sum_{i=0}^{n} \|D_i^{l_i} f\|_{L_{p,\alpha}(\mathbb{E}^{n+1})}, \tag{2.21}$$

where in the second inequality we exploited the Hardy inequality.

Similarly using (1.8) we have

$$M \le c\|F'\|_{L_p(\mathbb{E}^{n+1})} \le c\|f\|_{L_{p,\alpha}(\mathbb{E}^{n+1})}. \tag{2.22}$$

From (2.21) and (2.22) we obtain (2.2) with $\delta = c$, $\varkappa = 0$ and $Q = \mathbb{E}^{n+1}$ after letting $\varepsilon \to 0$ (see [3, p. 139]).

To show (2.2) with parameter δ and $\varkappa > 0$ we exploit the considerations from [1, Ch. 3].

To obtain (2.2) for Q bounded we apply the standard considerations with a partition of unity. This concludes the proof.

It seems that condition (2.1′) is artificial. It follows from applying [2] to estimate the integral N_2. However we do not know how to estimate N_2 in a different way.

From (2.1)′ we see that the case

$$0 \ge \alpha \ge \beta \tag{2.23}$$

is not included in Theorem 1. Hence we need

Corollary 2. *Assume that Q is bounded and satisfies the $R(\bar{l}, h_0)$-horn condition, $f \in W^{\bar{l}}_{p,\alpha}(Q)$, and*

$$\alpha \le \beta. \tag{2.24}$$

Take $\alpha' > \beta$ such that

$$\varkappa' = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\sum_{i=0}^{n} \frac{1}{l_i} - \sum_{i=0}^{n} \frac{\nu_i}{l_i} - \frac{1}{l_*}(\alpha' - \beta) \ge 0. \tag{2.25}$$

Then $D^{\bar{\nu}} f \in L_{q,\beta}(Q)$ and

$$\|D^{\bar{\nu}} f\|_{L_{q,\beta}(Q)} \le c\varepsilon^{\varkappa'}\|f\|_{L^{\bar{l}}_{p,\alpha}(Q)} + c\varepsilon^{\varkappa'-1}\|f\|_{L_{p,\alpha}(Q)} \tag{2.26}$$

for all $\varepsilon \in (0, h_0)$, where c does not depend on f and ε.

Proof. Since Q is bounded we have $f \in W^{\bar{l}}_{p,\alpha'}(Q)$ and

$$\|f\|_{W^{\bar{l}}_{p,\alpha'}(Q)} \le c\|f\|_{W^{\bar{l}}_{p,\alpha}(Q)}. \tag{2.27}$$

Using Theorem 1 we obtain (2.26). This concludes the proof.

The results of this paper, especially Corollary 2, are necessary for the proof of the existence of global regular special solutions to Navier–Stokes equations (see [5]).

3. Imbedding theorems for $p = q$. First we prove

THEOREM 3.1. *Assume that $f \in W^{\overline{l}}_{p,\alpha}(Q)$, $Q \subset \mathbb{E}^{n+1}$ and Q satisfies the $R(\overline{l}, h_0)$-horn condition, $\overline{l} = (l_0, l_1, \ldots, l_n)$, $1 < p < \infty$, $\alpha, \beta \in \mathbb{R}_+$, $\alpha \geq \beta$, $0 < l_i \in \mathbb{Z}$, $0 \leq \nu_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$, $l_1 = l_2 = l_*$ and*

$$\varkappa \equiv 1 - \sum_{i=0}^{n} \frac{\nu_i}{l_i} - \frac{1}{l_*}(\alpha - \beta) \geq 0. \tag{3.1}$$

Then $D^{\overline{\nu}} f \in L_{p,\beta}(Q)$ and

$$\|D^{\overline{\nu}} f\|_{L_{p,\beta}(Q)} \leq c_1 h^{\varkappa} \|f\|_{L^{\overline{l}}_{p,\alpha}(Q)} + c_2 h^{\varkappa - 1} \|f\|_{L_{p,\alpha}(Q)}, \tag{3.2}$$

where c_1, c_2 do not depend on f and $h \in (0, h_0)$, $h_0 = h_0(Q)$.

Proof. We consider the case $\alpha - \beta = k + \gamma$, $k \in \mathbb{N}_0$, $\gamma \in [0, 1)$.

First we examine the case $\varkappa = 0$ and $\gamma = 0$. Then we write (2.6) in the form

$$\begin{aligned} D^{\overline{\nu}} f_\varepsilon(\overline{x}) = c \int\limits_{\mathbb{E}^{n+1}} \phi_*^{(\overline{\nu}(k))}(\overline{y}/r^{\overline{\sigma}}) F(\overline{x} + \overline{y})\, d\overline{y} \\ + c \int\limits_{\varepsilon}^{r} dh \sum_{i=0}^{n} \int\limits_{\mathbb{E}^{n+1}} h^{-1-|\overline{\sigma}|} \varPhi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) F_i(\overline{x} + \overline{y})\, d\overline{y}. \end{aligned} \tag{3.3}$$

Estimating (3.3) implies

$$\begin{aligned} \|D^{\overline{\nu}} f_\varepsilon\|_{L_{p,\beta}(Q)} \leq c \Big\| \int\limits_{\mathbb{E}^{n+1}} \varPhi_*^{(\overline{\nu}(k))}(\overline{y}/r^{\overline{\sigma}}) F(\overline{x} + \overline{y})\, d\overline{y} \Big\|_{L_{p,\beta}(Q)} \\ + c \sum_{i=0}^{n} \Big\| \int\limits_{\varepsilon}^{r} dh \int\limits_{\mathbb{E}^{n+1}} h^{-1-|\overline{\sigma}|} \varPhi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) F_i(\overline{x} + \overline{y})\, d\overline{y} \Big\|_{L_{p,\beta}(Q)} \equiv M_0 + M_1. \end{aligned} \tag{3.4}$$

Using estimates for integral operators we have

$$M_0 \leq c \|f\|_{L_{p,\alpha}(Q)}.$$

Next we examine

$$\begin{aligned} M_1 \leq c \sum_{i=0}^{n} \Big\| \int\limits_{\varepsilon}^{r} h^{-1-|\overline{\sigma}|} \varPhi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) F_i'(\overline{x} + \overline{y})\, d\overline{y}\, dh \Big\|_{L_p(Q)} \\ + c \sum_{i=0}^{n} \Big\| \int\limits_{\varepsilon}^{r} h^{-1-|\overline{\sigma}|} \varPhi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) \frac{|x'|^{\beta} - |x' + y'|^{\alpha - k}}{|x' + y'|^{\alpha - k}} F_i'(\overline{x} + \overline{y})\, d\overline{y}\, dh \Big\|_{L_p(Q)} \\ \equiv M_3 + M_4, \end{aligned}$$

where $F_i'(x) = |x'|^{\alpha - k} F_i(x)$.

Lemma 1.1 and the Hardy inequality (1.9) yield

$$M_3 \leq c \sum_{i=0}^{n} \|F_i'\|_{L_p(Q)} \leq c \sum_{i=0}^{n} \|D_i^{l_i} f\|_{L_{p,\alpha}(Q)}.$$

Using (1.8) in M_4 implies

$$M_4 \le c\sum_{i=0}^{n}\bigg\|\int\limits_{\mathbb{E}^{n+1}}\chi(y,R(\bar{l},h_0))\Big(\sum_{j=0}^{n}|y_j|^{l_j}\Big)^{-|\overline{\sigma}|} \times\bigg|\frac{|x'|^{\beta}}{|x'+y'|^{\alpha-k}}-1\bigg|F_i'(\overline{x}+\overline{y})\,d\overline{y}\bigg\|_{L_p(Q)} \equiv M_5.$$

Applying the one-dimensional Young inequality

$$\|f*g\|_{L_p}\le\|g\|_{L_1}\|f\|_{L_p} \tag{3.5}$$

to M_5 with respect to the variables $x_0,x_3,\dots,x_n$, we obtain

$$\begin{aligned}M_5 &\le c\sum_{i=0}^{n}\bigg\|\int\limits_{\mathbb{E}^2}\chi(y',R(l_*,h_0))|y'|^{-2} \times\bigg|\frac{|x'|^{\beta}}{|x'+y'|^{\alpha-k}}-1\bigg|\|F_i'(x'+y')\|_{L_p(\mathbb{E}^{n-1})}\,dy'\bigg\|_{L_p(Q)}\\ &\le c\sum_{i=0}^{n}\bigg\|\int\limits_{\mathbb{E}^2}\chi(y'-x',R(l_*,h_0))|x'-y'|^{-2} \times\bigg|\frac{|x'|^{\beta}}{|y'|^{\alpha-k}}-1\bigg|\|F_i'(y')\|_{L_p(\mathbb{E}^{n-1})}\,dy'\bigg\|_{L_p(Q)} \equiv M_6.\end{aligned}$$

Introducing the polar coordinates $x'=(\varrho\cos\varphi_x,\varrho\sin\varphi_x)$, $y'=(\eta\cos\varphi_y,\eta\sin\varphi_y)$ we obtain

$$M_6 \le c\sum_{i=0}^{n}\bigg(\int_0^{\infty}\varrho\,d\varrho\int_0^{2\pi}d\varphi_x\bigg|\int_{\varrho}^{\infty}\eta\,d\eta\int_0^{2\pi}d\varphi_y\,\frac{1}{\varrho^2+\eta^2-2\varrho\eta\cos(\varphi_x-\varphi_y)} \times\bigg|\frac{\varrho^{\beta}}{\eta^{\alpha-k}}-1\bigg|\overline{F}_i'(\eta\cos\varphi_y,\eta\sin\varphi_y)\bigg|^p\bigg)^{1/p} \equiv M_7,$$

where $\overline{F}_i'=\|F_i'(y')\|_{L_p(\mathbb{E}^{n-1})}$.

Using the Young inequality (3.5) with respect to φ_x and the expression

$$\int_{-\pi}^{\pi}\frac{d\varphi}{\varrho^2+\eta^2-2\varrho\eta\cos\varphi}=\frac{2\pi}{\varrho^2-\eta^2}$$

we obtain

$$M_7\le c\sum_{i=0}^{n}\bigg(\int_0^{\infty}\varrho\,d\varrho\bigg|\int_{\varrho}^{\infty}\frac{1}{|\varrho^2-\eta^2|}\bigg|1-\frac{\varrho^{\beta}}{\eta^{\alpha-k}}\bigg|\widetilde{F}_i'(\eta)\eta\,d\eta\bigg|^p\bigg)^{1/p}\equiv M_8,$$

where $\widetilde{F}_i'(\eta)=(\int_0^{2\pi}|\overline{F}_i'(\eta\cos\varphi_y,\eta\sin\varphi_y)|^p\,d\varphi_y)^{1/p}$.

Introducing a new variable λ by $\eta = \lambda\varrho$ in the inner integral in M_8 and using the generalized Minkowski inequality (see [1, Ch. 1]) we obtain

$$M_8 \le c \int\limits_1^\infty \frac{\lambda^{1-2/p}}{(\lambda+1)(\lambda-1)}(1-\lambda^{-(\alpha-k)})\,d\lambda \sum_{i=0}^n \|F_i'\|_{L_p(\mathbb{E}^{n+1})}$$
$$\le c\sum_{i=0}^n \|D_i^{l_i} f\|_{L_{p,\alpha}(\mathbb{E}^{n+1})},$$

where in the last inequality the Hardy inequality (1.9) was also used.

Let us consider the case $\varkappa = 0$ and $\alpha - \beta = k + \gamma$, $\gamma > 0$. Then M_1 takes the form

$$M_1' = c\sum_{i=0}^n \bigg\| \int\limits_\varepsilon^r dh \int\limits_{\mathbb{E}^{n+1}} h^{-1-|\overline{\sigma}|+\gamma/l_*} \varPhi_i^{(\overline{\nu}(k))}(\overline{y}/h^{\overline{\sigma}}) F_i(\overline{x}+\overline{y})\,d\overline{y}\bigg\|_{L_{p,\beta}(Q)}.$$

In view of (1.8) we have

$$M_1' \le c\sum_{i=0}^n \bigg\| \int\limits_{\mathbb{E}^{n+1}} \chi(y, R(\overline{l}, h_0)) \Big(\sum_{j=0}^n |y_j|^{l_j}\Big)^{-|\overline{\sigma}|+\gamma/l_*}$$
$$\times \frac{|x'|^\beta}{|x'+y'|^{\alpha-k}} F_i'(\overline{x}+\overline{y})\,d\overline{y}\bigg\|_{L_p(Q)} \equiv M_2'.$$

Applying the one-dimensional Young inequality (3.4) with respect to $x_0, x_3, \ldots, x_n$ gives

$$M_2' \le c\sum_{i=0}^n \bigg\| \int\limits_{\mathbb{E}^2} \chi(y', R(l_*, h_0))|y'|^{-2+\gamma}$$
$$\times \frac{|x'|^\beta}{|x'+y'|^{\alpha-k}} \|F_i'(x'+y')\|_{L_p(\mathbb{E}^{n-1})}\,dy'\bigg\|_{L_p(\mathbb{E}^2)} \equiv M_3'.$$

Changing variables implies

$$M_3' = c\sum_{i=0}^n \bigg\| \int\limits_{\mathbb{E}^2} \chi(y'-x', R(l_*, h_0))|x'-y'|^{-2+\gamma} \frac{|x'|^\beta}{|y'|^{\alpha-k}} \overline{F}_i'(y')\,dy'\bigg\|_{L_p(\mathbb{E}^2)},$$

where $\overline{F}_i'(y') = \|F_i'(y')\|_{L_p(\mathbb{E}^{n-1})}$. Introducing the polar coordinates $x' = (\varrho\cos\varphi_x, \varrho\sin\varphi_x)$, $y' = (\eta\cos\varphi', \eta\sin\varphi')$, $\varphi = \varphi_x - \varphi'$, in M_3' yields

$$M_3' \le c\sum_{i=0}^n \bigg(\int\limits_0^\infty \varrho\,d\varrho \int\limits_0^{2\pi} d\varphi' \bigg| \int\limits_\varrho^\infty \eta\,d\eta \int\limits_0^{2\pi} \frac{d\varphi_x}{(\varrho^2+\eta^2-2\varrho\eta\cos(\varphi_x-\varphi'))^{1-\gamma/2}}$$
$$\times \frac{\varrho^\beta}{\eta^{\alpha-k}} F_i'(\eta\cos\varphi', \eta\sin\varphi')\bigg|^p\bigg)^{1/p} \equiv M_4'.$$

Using the fact that the integral over φ_x can be made independent of φ' and then applying the Minkowski inequality (see [1, Ch. 1]) we obtain

$$M_4' \le c\sum_{i=0}^n\bigg(\int_0^\infty \varrho\,d\varrho\bigg|\int_\varrho^\infty \eta\,d\eta\int_0^{2\pi}\frac{d\varphi}{(\varrho^2+\eta^2-2\varrho\eta\cos\varphi)^{1-\gamma/2}}\frac{\varrho^\beta}{\eta^{\alpha-k}} \times\bigg(\int_0^{2\pi}|F_i'(\eta\cos\varphi',\eta\sin\varphi')|^p\,d\varphi'\bigg)^{1/p}\bigg|^p\bigg)^{1/p} \equiv M_5'.$$

Changing variables $\eta=\lambda\varrho$ gives

$$M_5' = c\sum_{i=0}^n\bigg(\int_0^\infty \varrho\,d\varrho\bigg|\int_1^\infty \lambda\,d\lambda\int_0^{2\pi}\frac{d\varphi}{(\lambda^2+1-2\lambda\cos\varphi)^{1-\gamma/2}}\,\lambda^{-(\alpha-k)} \times\bigg(\int_0^{2\pi}|F_i'(\lambda\varrho\cos\varphi',\lambda\varrho\sin\varphi')|^p\,d\varphi'\bigg)^{1/p}\bigg|^p\bigg)^{1/p}.$$

Applying the Minkowski inequality (see [1, Ch. 1]) yields

$$M_5' \le c\sum_{i=0}^n\int_1^\infty \lambda\,d\lambda\int_0^{2\pi}\frac{d\varphi}{(\lambda^2+1-2\lambda\cos\varphi)^{1-\gamma/2}}\,\lambda^{-(\alpha-k)} \times\bigg(\int_0^\infty \varrho\,d\varrho\int_0^{2\pi}d\varphi'\,|F_i'(\lambda\varrho\cos\varphi',\lambda\varrho\sin\varphi')|^p\bigg)^{1/p} \equiv M_6'.$$

Introducing a new variable $\sigma=\lambda\varrho$, $d\sigma=\lambda\,d\varrho$, implies

$$\begin{aligned} M_6' &= c\sum_{i=0}^n\int_1^\infty \lambda\,d\lambda\int_0^{2\pi}\frac{d\varphi}{(\lambda^2+1-2\lambda\cos\varphi)^{1-\gamma/2}}\,\lambda^{-(\alpha-k)-2/p} \\ &\quad\times\bigg(\int_0^\infty \sigma\,d\sigma\int_0^{2\pi}d\varphi'\,|F_i'(\sigma\cos\varphi',\sigma\sin\varphi')|^p\bigg)^{1/p} \\ &\le c\sum_{i=0}^n\int_1^\infty \lambda\,d\lambda\int_0^{2\pi}\frac{d\varphi}{(\lambda^2+1-2\lambda\cos\varphi)^{1-\gamma/2}}\,\lambda^{-(\alpha-k)-2/p}\|F_i\|_{L_p(\mathbb{E}^{n+1})} \equiv M_7'. \end{aligned}$$

Passing to the Cartesian coordinates $z_1=\lambda\cos\varphi$, $z_2=\lambda\sin\varphi$, $\overline{z}_0=(1,0)$, $\overline{z}=(z_1,z_2)$ we write M_7' in the form

$$M_7' = c\sum_{i=0}^n\|F_i\|_{L_p(\mathbb{E}^{n+1})}\int_{|\overline{z}|\ge 1}d\overline{z}\,\frac{|\overline{z}|^{-(\alpha-k)-2/p}}{|\overline{z}-\overline{z}_0|^{2-\gamma}}.$$

For $|\overline{z}|$ large the integral converges for $\alpha-k+2/p>\gamma$.

For $|\overline{z}|$ in a neighbourhood of 1 we can show convergence by passing to the coordinates with origin at $\overline{z}_0$. Finally we have

$$M_7' \le c \sum_{i=0}^{n} \|F_i\|_{L_p(\mathbb{E}^{n+1})}.$$

Applying now the remarks from [3] we can let $\varepsilon \to 0$ to obtain (3.2).

Let $\varkappa > 0$. Using the considerations from [1, Ch. 3] we obtain (3.2) with a parameter $\delta > 0$.

For Q bounded we apply the standard considerations with a partition of unity. This concludes the proof.

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Received on 12.10.2000;
revised version on 7.11.2001 (1554)