APPLICATIONES MATHEMATICAE
29,1 (2002), pp. 75–95

Ivan Hlaváček (Praha)
Ján Lovíšek (Bratislava)

CONTROL IN OBSTACLE-PSEUDOPLATE PROBLEMS WITH FRICTION ON THE BOUNDARY. APPROXIMATE OPTIMAL DESIGN AND WORST SCENARIO PROBLEMS

Abstract. In addition to the optimal design and worst scenario problems formulated in a previous paper [3], approximate optimization problems are introduced, making use of the finite element method. The solvability of the approximate problems is proved on the basis of a general theorem of [3]. When the mesh size tends to zero, a subsequence of any sequence of approximate solutions converges uniformly to a solution of the continuous problem.

Introduction. The optimal design problems and reliable solution (worst scenario) problems, which have been introduced and studied in [3], have to be solved approximately. To this end, we employ the simplest kind of finite elements, namely piecewise linear functions over triangulations. In this way the space of state functions and the sets of admissible design variables are discretized in Section 1. To simplify the calculations we also use some quadrature formulae. We define an approximate state problem (variational inequality), optimal design and penalized weight minimization problems. We prove that these problems have at least one solution on the basis of the general Theorem 2.2 of [3]. In Section 2 we study the convergence of approximate solutions when the mesh size tends to zero.

Section 3 is devoted to approximate reliable solutions, i.e., to approximations of the worst scenario method, which has been formulated in [3, Section 4]. We prove the solvability of approximate problems. The convergence of some subsequence of approximate solutions is justified in Section 4.

2000 *Mathematics Subject Classification*: 65N30, 74K20, 74P05, 93C30.

Key words and phrases: control of variational inequalities, optimal design, weight minimization, worst scenario, uncertain data.

1. Approximate optimal design. In the following we use the notation of [3]. Assume that the domain Ω has a polygonal boundary $\partial\Omega$. Consider a regular family of triangulations $\{\mathcal{T}_h\}$, $h \to 0+$, of Ω which are consistent with all subdomains Ω_i^* and G_j and with $\partial\Omega_C$. We introduce the finite element space of piecewise linear functions

$$X_h = \{v_h \in C(\overline{\Omega}) : v_h|T \in P_1(T) \text{ for all triangles } T \in \mathcal{T}\}$$

and the following sets:

$$V_h = X_h \cap V,$$
$$U_{\text{ad}}^{Hh} = U_{\text{ad}}^{H} \cap X_h, \quad U_{\text{ad}}^{\mathcal{Z}h} = U_{\text{ad}}^{\mathcal{Z}} \cap X_h^0, \quad U_{\text{ad}}^{\mathcal{F}h} = U_{\text{ad}}^{\mathcal{F}} \cap X_h^c,$$

where

$$X_h^0 = X_h|_{\bar{\Omega}\setminus\bar{\Omega}^*}, \quad X_h^c = X_h|_{\partial\Omega_C},$$

$$U_{\text{ad}}^h = U_{\text{ad}}^{Hh} \times U_{\text{ad}}^{\mathcal{Z}h} \times U_{\text{ad}}^{\mathcal{F}h},$$

$$\mathcal{K}_h(H_h) = \{v_h \in V_h : v_h(P) \geq H_h(P) - O_i,\ i = 1, \ldots, N, \text{ for all nodes } P \in \Sigma_h\},$$

where Σ_h denotes the set of all vertices of triangles $T \in \mathcal{T}_h$, $T \subset \bar{\Omega}^*$ and $H_h \in U_{\text{ad}}^{Hh}$. Note that $\mathcal{K}_h(H_h) \subset \mathcal{K}(H_h)$ (cf. [3, (1.3)]).

Let $[p, \xi]_h$ be a suitable quadrature formula such that

$$(1.1) \qquad [p, \cdot]_h \in (V_h)^*, \quad |\langle p, \xi\rangle_0 - [p, \xi]_h| \leq Ch\|\xi\|_1$$

for all $\xi \in V_h$. We define

$$(1.2) \qquad b_h(\mathcal{Z}_h; u_h, w_h) = \sum_{T\subset\bar{\Omega}\setminus\Omega^*} \int_T \mathcal{Z}_h[u_h]^-(\gamma) w_h\, dx,$$

where γ is the centroid of the triangle T and

$$(1.3) \qquad \Phi_h(e_h)(v_h) = \sum_{E\subset\partial\Omega_C} \int_E \mathcal{F}_h|v_h(\gamma)|\, ds + I_{\mathcal{K}_h(H_h)}(v_h),$$

where E denotes the edge of a triangle $T \in \mathcal{T}_h$ adjacent to $\partial\Omega_C$ and γ is the midpoint of E.

Now we may define the following

Approximate State Problem. Given any $e_h \equiv \{H_h, \mathcal{Z}_h, \mathcal{F}_h\} \in U_{\text{ad}}^h$, find $u_h(e_h) \in \mathcal{K}_h(H_h)$ such that

$$(1.4) \qquad a(H_h; u_h(e_h), v_h - u_h(e_h)) + b_h(\mathcal{Z}_h; u_h(e_h), v_h - u_h(e_h)) + \Phi_h(e_h)(v_h) - \Phi_h(e_h)(u_h(e_h)) \geq [p, v_h - u_h(e_h)]_h - 2\omega\langle H_h, v_h - u_h(e_h)\rangle_0$$

for all $v_h \in \mathcal{K}_h(H_h)$.

Finally, let us define the functionals

$$
\begin{aligned}
\mathcal{L}^h_{\mathrm{DD}} &= \mathcal{L}_{\mathrm{DD}}, \qquad \mathcal{L}^h_{\mathrm{ISS}} = \mathcal{L}_{\mathrm{ISS}},\\
\mathcal{L}^h_{\mathrm{COM}}(e_h, v_h) &= [p, v_h]_h - 2\omega\langle H_h, v_h\rangle_0,\\
\mathcal{L}^h_{\mathrm{TR}}(e_h, v_h) &= \langle H_h \operatorname{grad} v_h, \operatorname{grad}\theta\rangle_0 + 2\omega\langle H_h, \theta\rangle_0\\
&\quad + \sum_{T\subset\bar{\Omega}\setminus\Omega^*} \langle \mathcal{Z}_h[v_h(\gamma)]^-, \theta\rangle_{0,T} - [p,\theta]_h.
\end{aligned}
\tag{1.5}
$$

(Note that the auxiliary function θ can be chosen in $X(\Omega)\cap V_h$.)

We introduce

Approximate Optimal Design Problems. Given a fixed triangulation $\mathcal{T}_h$, find

$$
e^{*h}_J = \arg\min_{e_h\in U^h_{\mathrm{ad}}} \mathcal{L}^h_J(e_h, u_h(e_h)) \tag{1.6}
$$

where $J = \mathrm{DD}, \mathrm{ISS}, \mathrm{COM}, \mathrm{TR}$ and $u_h(e_h)$ is the solution of the Approximate State Problem (1.4).

Theorem 1.1. (i) *The Approximate State Problem* (1.4) *has a unique solution $u_h(e_h)$ for any $e_h \in U^h_{\mathrm{ad}}$ and any h sufficiently small.*

(ii) *The Approximate Optimal Design Problem* (1.6) *has at least one solution for any* $J = \mathrm{DD}, \mathrm{ISS}, \mathrm{COM}, \mathrm{TR}$ *and for any h sufficiently small.*

Proof. Let us verify the assumptions of [3, Theorem 2.2], where we set $U_{\mathrm{ad}} := U^h_{\mathrm{ad}}$, $e := e_h$, $V := V_h$, $\mathcal{K}(e) := \mathcal{K}_h(H_h)$ and define $A^h(e_h) : V_h \to (V_h)^*$ by the relation

$$
\langle A^h(e_h)v_h, w_h\rangle := a(H_h; v_h, w_h) + b_h(\mathcal{Z}_h; v_h, w_h), \tag{1.7}
$$

and

$$
\Phi(e) := \Phi_h(e_h), \qquad \langle f, v\rangle := [p, v]_h, \qquad \langle B(e), v\rangle := -2\omega\langle H_h, v\rangle_0.
$$

Lemma 1.2. *For any $H_h \in U^{Hh}_{\mathrm{ad}}$ the set $\mathcal{K}_h(H_h)$ is a closed convex subset of V_h. If $H_{hn} \in U^{Hh}_{\mathrm{ad}}$ and $H_{hn} \to H_h$ as $n \to \infty$, then*

$$
\mathcal{K}_h(H_h) = \operatorname*{Lim}_{n\to\infty} \mathcal{K}_h(H_{hn}).
$$

Proof. The argument is nearly the same as that for [3, Lemma 2.2]. Instead of the function ϑ we may take $\vartheta_h \in V_h$ such that $0 \le \vartheta_h \le 1$ in Ω and $\vartheta_h = 1$ in Ω^*. ∎

Lemma 1.3. *For any $\mathcal{Z}_h \subset U^{\mathcal{Z}h}_{\mathrm{ad}}$, $u_h, w_h \in V_h$,*

$$
b_h(\mathcal{Z}_h; u_h, u_h - w_h) - b_h(\mathcal{Z}_h; w_h, u_h - w_h) \ge -C_1 h\|u_h - w_h\|_1^2, \tag{1.8}
$$

where C_1 does not depend on h and $\mathcal{Z}_h$.

Proof. The left-hand side of (1.8) is equal to

$$\sum_T \int_T \mathcal{Z}_h(u_h - w_h)([u_h]^-(\gamma) - [w_h]^-(\gamma))\,dx \tag{1.9}$$
$$= \sum_T \Big\{ \int_T \mathcal{Z}_h(u_h(\gamma) - w_h(\gamma))([u_h]^-(\gamma) - [w_h(\gamma)]^-)\,dx$$
$$+ \int_T \mathcal{Z}_h((u_h - w_h) - (u_h - w_h)(\gamma))([u_h]^-(\gamma) - [w_h]^-(\gamma))\,dx \Big\}$$
$$= \sum_T M_T + \sum_T \mathcal{R}_T.$$

Since

$$(a^- - b^-)(a - b) \ge (a^- - b^-)^2, \tag{1.10}$$

the terms M_T are non-negative. For brevity, set $v_h := u_h - w_h$. Using the estimate $|a^- - b^-| \le |a - b|$, which follows from (1.10), we may write

$$|\mathcal{R}_T| = |[u_h]^-(\gamma) - [w_h]^-(\gamma)| \cdot |F(v_h)| \le |v_h(\gamma)| \cdot |F(v_h)|, \tag{1.11}$$

where

$$F(v_h) = \int_T \mathcal{Z}_h(v_h - v_h(\gamma))\,dx,$$

$$|F(v_h)| \le \mathcal{Z}_{\max} \int_T |v_h - v_h(\gamma)|\,dx \le \mathcal{Z}_{\max} \int_T h_T |\operatorname{grad} v_h|\,dx \tag{1.12}$$
$$\le \mathcal{Z}_{\max} h_T^2 |v_h|_{1,T}.$$

Moreover, a standard affine transformation to the reference triangle $\widehat{T}$ yields

$$|v(\gamma)| \le C\|\widehat{v}\|_{1,p,\widehat{T}} \le C h_T^{-2/p} \|v\|_{1,p,T}$$

for any $p > 2$ and

$$\|v_h\|_{1,p,T} \le C h_T^{-1+2/p} \|v_h\|_{1,2,T}.$$

Combining the previous estimates, we obtain

$$|\mathcal{R}_T| \le C_1 h_T \|v_h\|^2_{1,2,T}$$

so that

$$\Big|\sum_T \mathcal{R}_T\Big| \le C_1 h \|v_h\|^2_{1,2,\Omega} = C_1 h \|u_h - w_h\|^2_1.$$

Substituting this estimate into (1.9), we arrive at (1.8). ■

Using [3, (2.16)] and Lemma 1.3, we may write

$$a(H_h; u_h - w_h, u_h - w_h) + b_h(\mathcal{Z}_h; u_h, u_h - w_h)$$
$$- b_h(\mathcal{Z}_h; w_h, u_h - w_h) \ge (C_F H_{\min} - C_1 h)\|u_h - w_h\|^2_1. \tag{1.13}$$

As a consequence, the strong monotonicity of [3, (2.1)(iii)] is satisfied for sufficiently small mesh size h.

Next, we have

$$\begin{aligned}(1.14)\quad &|b_h(\mathcal{Z}_h; u_h, w) - b_h(\mathcal{Z}_h; v_h, w)| \\ &= \Big|\sum_T \int_T \mathcal{Z}_h([u_h]^-(\gamma) - [v_h]^-(\gamma)) w\,dx\Big| \\ &\le \sum_T \mathcal{Z}_{\max} \int_T |u_h(\gamma) - v_h(\gamma)| \cdot |w|\,dx \le \sum_T \mathcal{Z}_{\max} \|u_h - v_h\|_{0,T} \|w\|_{0,T} \\ &\le \mathcal{Z}_{\max} \|u_h - v_h\|_{0,\Omega} \|w\|_{0,\Omega}.\end{aligned}$$

Here we employed the estimate

$$\|\varphi_h\|_{\infty,T} \le C h_T^{-1} \|\varphi_h\|_{0,T} \tag{1.15}$$

for all $\varphi_h \in V_h$ [1, Thm. 3.2.6].

Using [3, (2.18)] and (1.14), we deduce that the mapping $A^h(e_h)$ from (1.7) is Lipschitz-continuous in V_h, uniformly in U_{ad}.

Next, let $e_{hn} \to e_h$ as $n \to \infty$, $e_{hn} \in U^h_{\mathrm{ad}}$. We may write

$$\begin{aligned}(1.16)\quad &|\langle A^h(e_{hn}) v_h - A^h(e_h) v_h, w\rangle| \\ &\le \|H_{hn} - H_h\|_\infty \|v_h\|_1 \|w\|_1 + \sum_T \|\mathcal{Z}_{hn} - \mathcal{Z}_h\|_\infty \|v_h\|_{\infty,T} h_T \|w\|_{0,2,T} \\ &\le C\|e_{hn} - e_h\|_\infty \|v_h\|_1 \|w\|_1,\end{aligned}$$

arguing as in the derivation of (1.14). As a consequence,

$$A^h(e_{hn}) v_h \to A^h(e_h) v_h \quad \text{in } (V_h)^* \text{ for all } v_h \in V_h.$$

LEMMA 1.4. *The system of functionals* $\{\Phi_h(e_h)\}$, $e_h \in U^h_{\mathrm{ad}}$, *defined by* (1.3), *satisfies the assumptions* [3, (2.2), (2.3)].

Proof. We can proceed as in the proof of [3, Lemma 2.4]. Write

$$\phi_h(e_h) = \phi_h^{(1)}(e_h) + \phi_h^{(2)}(e_h),$$

where

$$\phi_h^{(1)}(e_h) v = \sum_{E \subset \partial\Omega_C} \int_E \mathcal{F}_h |v(\gamma)|\,ds, \qquad \phi_h^{(2)}(e_h) v = I_{\mathcal{K}_h(H_h)}(v).$$

We shall verify the condition [3, (2.2)] by means of [3, Definition 2.2]. Consider a sequence $\{e_{hn}\}$, $e_{hn} \to e_h$ as $n \to \infty$, $e_{hn} \in U^h_{\mathrm{ad}}$.

(i) Let $v_h \in \mathcal{K}_h(H_h)$. By Lemma 1.2 there exists a sequence $\{v_{hn}\}$, $v_{hn} \in \mathcal{K}_h(H_{hn})$, such that $v_{hn} \to v_h$ as $n \to \infty$. Then

$$|\phi_h(e_{hn})v_{hn} - \phi_h(e_h)v_h| \le |\lambda_{1n}| + |\lambda_{2n}|,$$

where

$$\begin{aligned}|\lambda_{1n}| &= |\phi_h^{(1)}(e_{hn})v_{hn} - \phi_h^{(1)}(e_h)v_h| \\ &\le |\phi_h^{(1)}(e_{hn})v_{hn} - \phi_h^{(1)}(e_h)v_{hn}| + |\phi_h^{(1)}(e_h)v_{hn} - \phi_h^{(1)}(e_h)v_h| \\ &\le \sum_E \Big(|v_{hn}(\gamma)| \int_E |\mathcal{F}_{hn} - \mathcal{F}_h|\, ds + |v_{hn}(\gamma) - v_h(\gamma)| \int_E \mathcal{F}_h\, ds\Big) \to 0 \\ &\qquad\qquad \text{as } n \to \infty,\end{aligned}$$

$$|\lambda_{2n}| = |I_{\mathcal{K}_h(H_{hn})}(v_{hn}) - I_{\mathcal{K}_h(H_h)}(v_h)| = 0 \quad \text{for all } n.$$

Altogether, we have

$$\lim_{n\to\infty} \Phi_h(e_{hn})v_{hn} = \Phi_h(e_h)v_h. \tag{1.17}$$

Second, let $v_h \notin \mathcal{K}_h(H_h)$. Setting $v_{hn} = v_h$ for all $n = 1, 2, \ldots$, we have

$$\begin{aligned}\limsup \Phi_h(e_{hn})v_{hn} &\le \limsup \sum_E \int_E \mathcal{F}_{hn}|v_h(\gamma)|\, ds + \infty \\ &= \sum_E \int_E \mathcal{F}_h |v_h(\gamma)|\, ds + \Phi_h^{(2)}(e_h)v_h = \Phi_h(e_h)v_h.\end{aligned} \tag{1.18}$$

Combining (1.17) and (1.18), we obtain

$$\limsup_{n\to\infty} \Phi_h(e_{hn})v_{hn} \le \Phi_h(e_h)v_h.$$

(ii) Let $v_{hn} \to v_h$ as $n \to \infty$. We have

$$\liminf \Phi_h(e_{hn})v_{hn} \ge \liminf \phi_h^{(1)}(e_{hn})v_{hn} + \liminf \phi_h^{(2)}(e_{hn})v_{hn}.$$

Arguing as in the case of λ_{1n}, we obtain

$$\lim \Phi_h^{(1)}(e_{hn})v_{hn} = \Phi_h^{(1)}(e_h)v_h.$$

Next, we may write

$$\liminf I_{\mathcal{K}_h(H_{hn})}(v_{hn}) = a,$$

where a is either $+\infty$ or zero. If $a = +\infty$, then obviously

$$a \ge I_{\mathcal{K}_h(H_h)}(v_h). \tag{1.19}$$

If $a = 0$, there exists a subsequence $\{v_{hk}\} \subset \{v_{hn}\}$ such that $v_{hk} \in \mathcal{K}_h(H_{hk})$ for all $k \to \infty$. By Lemma 1.2 the limit v_h belongs to $\mathcal{K}_h(H_h)$, so that $I_{\mathcal{K}_h(H_h)}(v_h) = 0$ and (1.19) holds again. As a consequence,

$$\liminf \Phi_h^{(2)}(e_{hn})v_{hn} \ge \Phi_h^{(2)}(e_h)v_h$$

and the condition [3, (2.2)(ii)] is fulfilled.

To satisfy condition [3, (2.3)], we can choose $a_n = 0$ for all n, since $0 \in \mathcal{K}_h(H_{hn})$ for all $H_{hn} \in U_{\mathrm{ad}}^{Hh}$, due to [3, (1.1)]. Then

$$\Phi_h(e_{hn})a_n = 0 \quad \text{for all } n. \ \blacksquare$$

LEMMA 1.5. *The functionals* $\mathcal{L}^h_{\mathrm{DD}}$, $\mathcal{L}^h_{\mathrm{ISS}}$, $\mathcal{L}^h_{\mathrm{COM}}$, $\mathcal{L}^h_{\mathrm{TR}}$ *satisfy condition* [3, (2.5)].

Proof. The proof of the cases $\mathcal{L}^h_{\mathrm{DD}}$ and $\mathcal{L}^h_{\mathrm{ISS}}$ is the same as that for [3, Lemma 2.5]. Let $e_{hn} \to e_h$ and $v_{hn} \to v_h$ as $n \to \infty$. We may write

$$\mathcal{L}^h_{\mathrm{COM}}(e_{hn}, v_{hn}) = [p, v_{hn}]_h - 2\omega\langle H_h, v_{hn}\rangle_0 + \psi_n,$$
$$|\psi_n| = 2\omega|\langle H_{hn} - H_h, v_{hn}\rangle_0| \le C\|H_{hn} - H_h\|_\infty \|v_{hn}\|_0 \to 0.$$

Then

$$\lim_{n\to\infty} \mathcal{L}^h_{\mathrm{COM}}(e_{hn}, v_{hn}) = [p, v_h]_h - 2\omega\langle H_h, v_h\rangle_0 = \mathcal{L}^h_{\mathrm{COM}}(e_h, v_h).$$

Next, we may write

$$\mathcal{L}^h_{\mathrm{TR}}(e_{hn}, v_{hn}) = \mathcal{L}^h_{\mathrm{TR}}(e_h, v_{hn}) + M_n, \tag{1.20}$$

$$\begin{aligned} |M_n| = \Big|&\langle (H_{hn} - H_h) \operatorname{grad} v_{hn}, \operatorname{grad}\theta\rangle_0 + 2\omega\langle H_{hn} - H_h, \theta\rangle_0 \\ &+ \sum_T \langle (\mathcal{Z}_{hn} - \mathcal{Z}_h)[v_{hn}(\gamma)]^-, \theta\rangle_{0,T}\Big| \\ &\le (\|H_{hn} - H_h\|_\infty + \|\mathcal{Z}_{hn} - \mathcal{Z}_h\|_\infty)(C_1\|v_{hn}\|_1 + C_2) \to 0, \end{aligned} \tag{1.21}$$

using also estimate (1.15) in the last inequality. Making use of (1.15) again, we obtain

$$\begin{aligned} |\mathcal{L}^h_{\mathrm{TR}}(e_h, v_{hn}) - \mathcal{L}^h_{\mathrm{TR}}(e_h, v_h)| &\le H_{\max}\|v_{hn} - v_h\|_1\|\theta\|_1 \\ &+ \mathcal{Z}_{\max}\sum_T \|v_{hn} - v_h\|_{0,T}\|\theta\|_{0,T} \le C\|v_{hn} - v_h\|_1 \to 0. \end{aligned} \tag{1.22}$$

Combining (1.20)–(1.22), we arrive at

$$\lim_{n\to\infty} \mathcal{L}^h_{\mathrm{TR}}(e_{hn}, v_{hn}) = \mathcal{L}^h_{\mathrm{TR}}(e_h, v_h). \ \blacksquare$$

We define the following

APPROXIMATE WEIGHT MINIMIZATION PROBLEM. Find

$$e^\varepsilon_h = \arg\min_{e_h \in U^h_{\mathrm{ad}}} \mathcal{L}_W(\varepsilon; e_h, u_h(e_h)), \tag{1.23}$$

where $\mathcal{L}_W$ is the penalized cost functional, defined in [3, Section 3].

THEOREM 1.6. *The Approximate Weight Minimization Problem* (1.23) *has at least one solution for any positive* ε *and any* h *sufficiently small.*

Proof. In proving Theorem 1.1 we have verified all assumptions of the abstract [3, Theorem 2.1], so that

$$u_h(e_{hn}) \to u_h(e_h)$$

provided h is sufficiently small, $e_{hn} \in U^h_{\mathrm{ad}}$ and $e_{hn} \to e_h$ as $n \to \infty$. Then the functions

$$e_h \mapsto [F_j(u_h(e_h)]^+, \quad j = 1, \ldots, M,$$

are continuous in U_{ad}^h (cf. the analogous proof of [3, Lemma 3.1]). Since the weight $\langle \omega, H_h\rangle_0$ is continuous in U_{ad}^h as well, we find that

$$e_h \mapsto \mathcal{L}_W(\varepsilon; e_h, u_h(e_h))$$

is continuous in the compact set U_{ad}^h. As a consequence, a minimizer e_h^{ε} exists. ∎

2. Convergence results. In the present section we will study the convergence of finite element approximations when the mesh size tends to zero. To this end we establish the crucial

PROPOSITION 2.1. *Let $e_h \in U_{\mathrm{ad}}^h$ with $e_h \to e$ in U as $h \to 0+$. Then*

$$u_h(e_h) \to u(e) \quad \text{in } V \text{ as } h \to 0+.$$

Proof. For brevity, set $u_h := u_h(e_h)$. Substituting $v_h = 0$ in the inequality (1.4) and using (1.1), (1.7), (1.13), we obtain

$$\begin{aligned}(C_F H_{\min} - C_1 h)\|u_h\|_1^2 \le \langle A^h(e_h)u_h, u_h\rangle &\le -[p, u_h]_h + 2\omega\langle H_h, u_h\rangle_0 \\ &\le C_2\|u_h\|_1,\end{aligned}$$

so that $\|u_h\|_1 \le C$ for all h sufficiently small. As a consequence, there exist $u \in V$ and a subsequence of $\{u_h\}$ (denoted by the same symbol) such that

$$u_h \rightharpoonup u \quad \text{(weakly) in } V. \tag{2.1}$$

One can prove that $u \in \mathcal{K}(H)$. Indeed, following [2, pp. 33–34], consider any function $\varphi \in C_0^\infty(\Omega_i^*)$ with $\varphi \ge 0$ and define a piecewise constant function

$$\varphi_h = \sum_{T \subset \bar{\Omega}_i^*} \varphi(\gamma)\chi_T,$$

where χ_T is the characteristic function of the triangle T and γ is the centroid of T. Define $\psi = H - O_i$ on Ω_i^* and $\psi_h = H_h - O_i$. Then

$$\lim_{h\to 0} \int_{\Omega_i^*} (u_h - \psi_h)\varphi_h\, dx = \int_{\Omega_i^*} (u - \psi)\varphi\, dx, \tag{2.2}$$

since $u_h \to u$ in $L^2(\Omega_i^*)$ by Rellich's Theorem and $\varphi_h \to \varphi$, $\psi_h \to \psi$ in $L^2(\Omega_i^*)$.

On the other hand, we have

$$\int_{\Omega_i^*} (u_h - \psi_h)\varphi_h\, dx = \sum_{T\subset \bar{\Omega}_i^*} \varphi(\gamma) \int_T (u_h - \psi_h)\, dx. \tag{2.3}$$

By definition of $\mathcal{K}_h(H_h)$, we obtain

$$\int_T (u_h - \psi_h)\, dx = \frac{1}{3}(\operatorname{meas} T)\sum_{j=1}^{3}(u_h - \psi_h)(a_j) \ge 0, \tag{2.4}$$

where a_j are the vertices of T. Combining (2.3) and (2.4), we arrive at

$$\int_{\Omega_i^*} (u_h - \psi_h)\varphi_h \, dx \geq 0.$$

Then (2.2) yields

$$\int_{\Omega_i^*} (u - \psi)\varphi \, dx \geq 0,$$

which in turn implies that $u \geq \psi$ a.e. in Ω_i^*, i.e., $u \in \mathcal{K}(H)$.

Next, let us verify that u coincides with a solution $u(e)$ of the variational inequality [3, (1.7)]. Consider an arbitrary $v \in \mathcal{K}(H)$. There exists a function $\psi \in C^{(0),1}(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega_D$, $\psi = H - O_i$ on Ω_i^* for all $i = 1, \ldots, N$. Then

$$\omega := v - \psi \in \mathcal{K}_0 = \{w \in V : w \geq 0 \text{ a.e. in } \Omega^*\}.$$

Let us employ a regularization operator ϱ_κ (see e.g. [4]). Let $\varrho_\kappa E\psi$ and $\varrho_\kappa E\omega$ denote the regularization applied to a proper extension of the functions ψ and ω to a larger domain $\widetilde{\Omega} \supset \overline{\Omega}$, so that

$$\varrho_\kappa E\omega \geq 0 \quad \text{and} \quad \varrho_\kappa EO_i = O_i \quad \text{on } \Omega_i^*, \ i = 1, \ldots, N.$$

We define

$$v_h = \pi_h(\varrho_\kappa E\psi + \varrho_\kappa E\omega + (\|\varrho_\kappa EH - H\|_{\infty,\Omega^*} + \|H - H_h\|_{\infty,\Omega^*})\vartheta), \tag{2.5}$$

where $\vartheta \in C_0^\infty(\Omega)$ is such that $0 \leq \vartheta \leq 1$ in Ω and $\vartheta = 1$ for $x \in \Omega^*$ and π_h denotes the Lagrange linear interpolation over $\mathcal{T}_h$. Consequently, $v_h \in V_h$ and for any node $P \in \Sigma_h$ we have

$$v_h(P) \geq \varrho_\kappa E\psi(P) + |\varrho_\kappa EH(P) - H(P)| + |H(P) - H_h(P)| \geq H_h(P) - O_i,$$

so that $v_h \in \mathcal{K}_h(H_h)$. Furthermore, we may write

$$\begin{aligned} \|v_h - v\|_1 &= \|\pi_h(\varrho_\kappa E\psi) - \psi + \pi_h(\varrho_\kappa E\omega) - \omega \\ &\qquad + (\|\varrho_\kappa EH - H\|_{\infty,\Omega^*} + \|H - H_h\|_{\infty,\Omega^*})\pi_h\vartheta\|_1 \\ &\leq \|\pi_h(\varrho_\kappa E\psi) - \varrho_\kappa E\psi\|_1 + \|\varrho_\kappa E\psi - \psi\|_1 \\ &\qquad + \|\pi_h(\varrho_\kappa E\omega) - \varrho_\kappa E\omega\|_1 + \|\varrho_\kappa E\omega - \omega\|_1 \\ &\qquad + (\|\varrho_\kappa EH - H\|_{\infty,\Omega^*} + \|H - H_h\|_{\infty,\Omega^*})\|\pi_h\vartheta\|_1 \to 0 \end{aligned} \tag{2.6}$$

as $\kappa \to 0+$ and $h \to 0+$.

Here we have used the fact that $H \in W^{1,p}(\Omega)$ for any $p > 2$ and

$$\|\varrho_\kappa EH - H\|_{\infty,\Omega^*} \leq C\|\varrho_\kappa EH - H\|_{1,p,\Omega} \to 0$$

as $\kappa \to 0+$ (see [4, Thms. 2.1 and 3.1]).

For any $e_h \in U_{\mathrm{ad}}^h$, $u_h \in V_h$ and $v \in V$ the following estimate holds (see [3, (2.15)] for the definition of $A(e_h)$):

$$|\langle A^h(e_h)u_h, v\rangle - \langle A(e_h)u_h, v\rangle| \leq Ch\|u_h\|_1\|v\|_0. \tag{2.7}$$

Indeed, we have

$$\text{(2.7a)}\qquad \begin{aligned}|\langle A^h(e_h)u_h, v\rangle - \langle A(e_h)u_h, v\rangle| &= |b_h(\mathcal{Z}_h; u_h, v) - b(\mathcal{Z}_h; u_h, v)| \\ &= \Big|\sum_{T\subset\bar{\Omega}\setminus\Omega^*}\int_T \mathcal{Z}_h([u_h]^- - [u_h]^-(\gamma))v\,dx\Big| \\ &\le \sum_T\int_T \mathcal{Z}_h|u_h - u_h(\gamma)|\cdot|v|\,dx \le Ch\|u_h\|_1\|v\|_0,\end{aligned}$$

arguing as in the proof of Lemma 1.3.

Let us substitute v_h in the inequality (1.4) and pass to lim inf as $h \to 0+$. It is easy to see that

$$\text{(2.8)}\qquad \liminf\,\langle A(e)u_h, u_h\rangle \ge \langle A(e)u, u\rangle.$$

In fact, the functional $u \mapsto a(H; u, u)$ is weakly lower semicontinuous, being convex and differentiable. Second, we may write

$$\text{(2.9)}\qquad \begin{aligned}|b(\mathcal{Z}; u_h, u_h) - b(\mathcal{Z}; u, u)| &\le \int_{\Omega\setminus\Omega^*} \mathcal{Z}|u_h[u_h]^- - u[u]^-|\,dx \\ &\le C\mathcal{Z}_{\max}(\|u_h\|_0 + \|u\|_0)\|u_h - u\|_0 \to 0\end{aligned}$$

due to Rellich's Theorem. Hence,

$$\text{(2.10)}\qquad \begin{aligned}\liminf\{a(H; u_h, u_h) + b(\mathcal{Z}; u_h, u_h)\} &\ge a(H; u, u) + b(\mathcal{Z}; u, u) \\ &= \langle A(e)u, u\rangle.\end{aligned}$$

Making use of [3, (2.1)(iv) and Lemma 2.3], we derive that

$$\text{(2.11)}\qquad |\langle A(e_h)u_h, u_h\rangle - \langle A(e)u_h, u_h\rangle| \le \|A(e_h)u_h - A(e)u_h\|_*\|u_h\|_1 \to 0.$$

Therefore,

$$\liminf\,\langle A(e_h)u_h, u_h\rangle \ge \liminf\,\langle A(e)u_h, u_h\rangle \ge \langle A(e)u, u\rangle$$

by (2.10) and (2.11).

Making also use of (2.7), we obtain

$$\text{(2.12)}\qquad \begin{aligned}\liminf\,\langle A^h(e_h), u_h, u_h\rangle &\ge \liminf\,\langle A(e_h), u_h, u_h\rangle \\ &+ \liminf\,(\langle A^h(e_h)u_h, u_h\rangle - \langle A(e_h)u_h, u_h\rangle) \ge \langle A(e)u, u\rangle.\end{aligned}$$

Next, we prove that

$$\text{(2.13)}\qquad \lim\,\langle A^h(e_h)u_h, v\rangle = \langle A(e)u, v\rangle$$

for all $v \in V$. Indeed, if we employ (2.7), it suffices to show that

$$\text{(2.14)}\qquad \lim\,\langle A(e_h)u_h, v\rangle = \langle A(e)u, v\rangle.$$

First, we may write

$$\text{(2.15)}\qquad |\langle A(e_h)u_h, v\rangle - \langle A(e)u_h, v\rangle| \le \|A(e_h)u_h - A(e)u_h\|_*\|v\|_1 \to 0$$

by [3, (2.1)(iv) and Lemma 2.3]. Second,

$$a(H; u_h, v) \to a(H; u, v) \tag{2.16}$$

by the weak convergence (2.1). Third, we have

$$\begin{aligned} |b(\mathcal{Z}; u_h, v) - b(\mathcal{Z}; u, v)| &= \Big| \int_{\Omega \setminus \Omega^*} \mathcal{Z}([u_h]^- - [u]^-) v \, dx \Big| \\ &\le \mathcal{Z}_{\max} \|u_h - u\|_0 \|v\|_0 \to 0. \end{aligned} \tag{2.17}$$

Then (2.14) follows from (2.15)–(2.17).

Next, using the Lipschitz continuity of $A^h(e_h)$ in V_h (see (1.14)), $A^h(e_h)0 = 0$, (2.6) and (2.13), we obtain

$$\begin{aligned} &|\langle A^h(e_h)u_h, v_h\rangle - \langle A(e)u, v\rangle| \\ &\quad \le |\langle A^h(e_h)u_h, v_h - v\rangle| + |\langle A^h(e_h)u_h, v\rangle - \langle A(e)u, v\rangle| \to 0. \end{aligned} \tag{2.18}$$

Consider the estimate

$$\begin{aligned} |\Phi_h(e_h)v_h - \Phi(e_h)v_h| &= \Big| \sum_{E \subset \partial\Omega_C} \int_E \mathcal{F}_h(|v_h(\gamma)| - |v_h|) \, ds \Big| \\ &\le \mathcal{F}_{\max} \sum_E \int_E |v_h(\gamma) - v_h| \, ds. \end{aligned}$$

We may write

$$\begin{aligned} |v_h(\gamma) - v_h(s)| &\le \tfrac{1}{2} \ell_E |\partial v_h / \partial s| \le \tfrac{1}{2} \ell_E \|\operatorname{grad} v_h(T_E)\| \\ &\le \ell_E C \varrho_T^{-1} |v_h|_{1,T_E} \le \widetilde{C} |v_h|_{1,T_E}, \end{aligned}$$

where $\ell_E = \operatorname{meas} E$, T_E is the triangle adjacent to the edge E and ϱ_T is the radius of the largest circle inscribed in T_E.

Thus we obtain

$$\begin{aligned} &\sum_E \int_E |v_h(\gamma) - v_h| \, ds \le \widetilde{C} \sum_E \ell_E |v_h|_{1,T_E} \\ &\quad \le \widetilde{C} h^{1/2} \Big(\sum_E \ell_E \Big)^{1/2} \Big(\sum_E |v_h|_{1,T_E}^2 \Big)^{1/2} \le \widetilde{C} h^{1/2} (\operatorname{meas} \partial\Omega_C)^{1/2} |v_h|_{1,\Omega} \to 0. \end{aligned}$$

As a consequence,

$$\Lambda_{1h} := |\Phi_h(e_h)v_h - \Phi(e_h)v_h| \to 0. \tag{2.19}$$

Since $v \in \mathcal{K}(H)$, we have

$$\begin{aligned} \Lambda_{2h} := |\Phi(e_h)v_h - \Phi(e_h)v| &= \Big| \int_{\partial\Omega_C} \mathcal{F}_h(|v_h| - |v|) \, ds \Big| \\ &\le \mathcal{F}_{\max} (\operatorname{meas} \partial\Omega_C)^{1/2} \|v_h - v\|_{0,\partial\Omega_C} \to 0. \end{aligned} \tag{2.20}$$

Finally, we may write

$$\begin{aligned} \Lambda_{3h} &:= |\Phi(e_h)v - \Phi(e)v| \\ &= \Big| \int_{\partial\Omega_C} (\mathcal{F}_h - \mathcal{F}) |v| \, ds \Big| \le \|\mathcal{F}_h - \mathcal{F}\|_\infty \int_{\partial\Omega_C} |v| \, ds \to 0. \end{aligned} \tag{2.21}$$

Combining (2.19)–(2.21), we arrive at

$$|\Phi_h(e_h)v_h - \Phi(e)v| \le \Lambda_{1h} + \Lambda_{2h} + \Lambda_{3h} \to 0 \quad \text{as } h \to 0+. \tag{2.22}$$

In a parallel way, we can deduce that

$$|\Phi_h(e_h)u_h - \Phi(e)u| \to 0 \quad \text{as } h \to 0+, \tag{2.23}$$

using the boundedness of $\{u_h\}$ in V and the compactness of the trace operator (cf. (2.20)).

On the basis of (1.1), (2.6) and the weak convergence (2.1), we obtain

$$\begin{aligned} &|[p, v_h - u_h]_h - \langle p, v - u\rangle_0| \\ &\quad \le |[p, v_h - u_h]_h - \langle p, v_h - u_h\rangle_0| + |\langle p, (v_h - u_h) - (v - u)\rangle_0| \to 0. \end{aligned} \tag{2.24}$$

Finally, it is easy to see that

$$\begin{aligned} &|\langle H_h, v_h - u_h\rangle_0 - \langle H, v - u\rangle_0| \\ &\quad \le |\langle H_h - H, v_h - u_h\rangle_0| + |\langle H, (v_h - u_h) - (v - u)\rangle_0| \\ &\quad \le C\|H_h - H\|_\infty \|v_h - u_h\|_0 + \|H\|_0 \|v_h - v\|_0 + |\langle H, u - u_h\rangle_0| \to 0. \end{aligned} \tag{2.25}$$

Coming back to the variational inequality (1.4) and passing to limes inferior or limes superior as $h \to 0+$, we employ (2.12), (2.18), (2.22)–(2.25) to get

$$\begin{aligned} \langle A(e)u, u\rangle &\le \liminf \langle A^h(e_h)u_h, u_h\rangle \le \limsup \langle A^h(e_h)u_h, u_h\rangle \\ &\le \langle A(e)u, v\rangle + \Phi(e)v - \Phi(e)u + \langle p - 2\omega H, u - v\rangle_0 \end{aligned} \tag{2.26}$$

for all $v \in \mathcal{K}(H)$.

Thus u is a solution of the inequality [3, (1.7)]. From the uniqueness of $u(e)$ we conclude that $u = u(e)$ and the whole sequence $\{u_h(e_h)\}$ tends to $u(e)$ weakly in V as $h \to 0+$.

It remains to prove the strong convergence. We may set $v := u$ in (2.26) to obtain

$$\lim \langle A^h(e_h)u_h, u_h\rangle = \langle A(e)u, u\rangle. \tag{2.27}$$

Next, we have

$$\begin{aligned} &|\langle A(e)u_h, u_h\rangle - \langle A^h(e_h)u_h, u_h\rangle| \\ &\quad \le |\langle A(e)u_h, u_h\rangle - \langle A(e_h)u_h, u_h\rangle| + |\langle A(e_h)u_h, u_h\rangle - \langle A^h(e_h)u_h, u_h\rangle| \\ &\quad \le \|A(e)u_h - A(e_h)u_h\|_* \|u_h\|_1 + Ch\|u_h\|_1 \|u_h\|_0 \to 0, \end{aligned} \tag{2.28}$$

making use of (2.15) and (2.7). Thus

$$\lim \langle A(e)u_h, u_h\rangle = \langle A(e)u, u\rangle \tag{2.29}$$

follows from (2.27) and (2.28).

Using (2.29), [3, (2.15)] and (2.9), we arrive at

$$\begin{aligned}(2.30)\qquad \lim a(H;u_h,u_h) &= \lim\langle A(e)u_h,u_h\rangle - \lim b(\mathcal{Z};u_h,u_h)\\ &= \langle A(e)u,u\rangle - b(\mathcal{Z};u,u) = a(H;u,u).\end{aligned}$$

The bilinear form $a(H;\cdot,\cdot)$ can be taken for a scalar product in V (see [3, (2.16)]). From (2.30) and the weak convergence of (u_h) we conclude that

$$\lim a(H;u_h-u,u_h-u)=0,$$

which in turn implies that $u_h\to u$ in V. ∎

PROPOSITION 2.2. *Let $e_h\in U_{\mathrm{ad}}^h$ with $e_h\to e$ in U as $h\to 0+$. Then*

$$\lim_{h\to 0+}\mathcal{L}_J^h(e_h,u_h(e_h))=\mathcal{L}_J(e,u(e))$$

for $J=\mathrm{DD},\mathrm{ISS},\mathrm{COM},\mathrm{TR}$, and

$$\lim_{h\to 0+}\mathcal{L}_W(\varepsilon;e_h,u_h(e_h))=\mathcal{L}_W(\varepsilon;e,u(e))$$

for any $\varepsilon>0$.

Proof. Define $u:=u(e)$, $u_h:=u_h(e_h)$. It is readily seen that

$$\begin{aligned}|\mathcal{L}_{\mathrm{DD}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{DD}}(e,u)| &= \Big|\int_\Omega((u_h-z)^2-(u-z)^2)\,dx\Big|\\ &\le \|u_h-u\|_0\|u_h+u-2z\|_0\to 0,\\ |\mathcal{L}_{\mathrm{ISS}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{ISS}}(e,u)| &= \big||u_h|_1^2-|u|_1^2\big| \le \big||u_h|_1-|u|_1\big|\cdot(|u_h|_1+|u|_1)\\ &\le C|u_h-u|_1\to 0.\end{aligned}$$

Next, we have

$$\begin{aligned}&|\mathcal{L}_{\mathrm{COM}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{COM}}(e,u)|\\ &\quad\le|\mathcal{L}_{\mathrm{COM}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{COM}}(e_h,u)|+|\mathcal{L}_{\mathrm{COM}}(e_h,u)-\mathcal{L}_{\mathrm{COM}}(e,u)|\equiv L_1+L_2,\end{aligned}$$

where

$$\begin{aligned}L_1&\le|[p,u_h]_h-\langle p,u_h\rangle_0|+|\langle p,u_h-u\rangle_0|+2\omega|\langle H_h,u-u_h\rangle_0|\\ &\le Ch\|u_h\|_1+C_1\|u-u_h\|_0\to 0\end{aligned}$$

by (1.1) and Proposition 2.1.

Moreover,

$$L_2=2\omega|\langle H-H_h,u_h\rangle_0|\le C\|H-H_h\|_\infty\|u_h\|_0\to 0,$$

so that

$$\mathcal{L}_{\mathrm{COM}}^h(e_h,u_h)\to\mathcal{L}_{\mathrm{COM}}(e,u).$$

Next, we may write

$$\begin{aligned}&|\mathcal{L}_{\mathrm{TR}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{TR}}(e,u)|\\ &\qquad\le|\mathcal{L}_{\mathrm{TR}}^h(e_h,u_h)-\mathcal{L}_{\mathrm{TR}}(e_h,u)|+|\mathcal{L}_{\mathrm{TR}}(e_h,u)-\mathcal{L}_{\mathrm{TR}}(e,u)|\equiv M_1+M_2.\end{aligned}$$

Using also (2.7a) in the final step, we derive that

$$M_1 \le |\langle H_h \operatorname{grad}(u_h - u), \operatorname{grad}\theta\rangle_0| + \Big|\sum_{T\subset\bar\Omega\setminus\Omega^*} \langle \mathcal{Z}_h([u_h(\gamma)]^- - [u]^-), \theta\rangle_{0,T}\Big|$$
$$\le H_{\max} C\|u_h - u\|_1 + \mathcal{Z}_{\max}\sum_T \int_T (|u_h(\gamma) - u_h| + |u_h - u|)|\theta|\,dx$$
$$\le H_{\max} C\|u_h - u\|_1 + C_1\Big(h\|u_h\|_1\|\theta\|_0 + \sum_T \|u_h - u\|_{0,T}\|\theta\|_{0,T}\Big) \to 0.$$

Next, we also have

$$M_2 \le |\langle (H_h - H)\operatorname{grad} u, \operatorname{grad}\theta\rangle_0| + 2\omega|\langle H - H_h, \theta\rangle_0|$$
$$+ |\langle(\mathcal{Z}_h^0 - \mathcal{Z}^0)[u]^-, \theta\rangle_0| \le C(\|H_h - H\|_\infty + \|\mathcal{Z}_h^0 - \mathcal{Z}^0\|_\infty)\|u\|_1\|\theta\|_1 \to 0,$$

so that

$$\mathcal{L}^h_{\mathrm{TR}}(e_h, u_h) \to \mathcal{L}_{\mathrm{TR}}(e, u).$$

Finally, we may write

$$|\mathcal{L}_W(\varepsilon, e_h, u_h) - \mathcal{L}_W(\varepsilon; e, u)|$$
$$\le |\langle \omega, H_h - H\rangle_0| + \varepsilon^{-1}\sum_{j=1}^M |[F_j(u_h)]^+ - [F_j(u)]^+|$$
$$\le C\{\|H_h - H\|_\infty + \|u_h - u\|_1(\|u_h\|_1 + \|u\|_1)\} \to 0,$$

using an argument analogous to the proof of [3, Lemma 3.1]. ∎

Lemma 2.3. *For any* $e \equiv \{H, \mathcal{Z}, \mathcal{F}\} \in U_{\mathrm{ad}}$ *and any sequence* $\{h\}$ *with* $h \to 0+$, *there exists a sequence* $\{e_h\}$ *such that*

$$e_h \equiv \{H_h, \mathcal{Z}_h, \mathcal{F}_h\} \in U^h_{\mathrm{ad}}, \quad e_h \to e \quad \text{in } U \equiv C(\bar\Omega) \times C(\bar\Omega \setminus \Omega^*) \times C(\overline{\partial\Omega}_C).$$

Proof. Let $\pi_h H$ denote the Lagrange linear interpolate of H over the triangulation $\mathcal{T}_h$. Since $H \in W^{1,\infty}(\Omega)$, interpolation theory (see e.g. [1]) yields

$$\|H - \pi_h H\|_{0,\infty} \le Ch\|H\|_{1,\infty}.$$

Obviously, $H_{\min} \le \pi_h H \le H_{\max}$ everywhere. For any straight-line segment $\overline{PQ} \in T$ parallel to the x_i-axis and any triangle $T \subset \mathcal{T}_h$ we have

$$|\partial \pi_h H/\partial x_i| = \ell^{-1}|H(Q) - H(P)| \le \ell^{-1}\int_P^Q |\partial H/\partial x_i| \le C_i^H,$$

where $\ell = |\overline{PQ}|$.

Analogous arguments hold for $\pi_h \mathcal{Z}$ and for $\pi_h^0\mathcal{F} \in X_h^C$, i.e., the Lagrange linear interpolate of $\mathcal{F}$ over the partition of $\partial\Omega_C$, generated by $\mathcal{T}_h$.

Now $e_h = \{\pi_h H, \pi_h\mathcal{Z}, \pi_h^0\mathcal{F}\}$ satisfies the conditions of the lemma. ∎

THEOREM 2.4. *Let* $\{e_J^{*h}\}$, $h \to 0+$, *be a sequence of solutions to the Approximate Optimal Design Problem* (1.6), $J = \mathrm{DD}, \mathrm{ISS}, \mathrm{COM}, \mathrm{TR}$. *Then there exists a subsequence* $\{e_J^{*\widehat{h}}\} \subset \{e_J^{*h}\}$ *such that*

$$e_J^{*\widehat{h}} \to e_J^* \quad \text{in } U \equiv C(\overline{\Omega}) \times C(\overline{\Omega} \setminus \Omega^*) \times C(\overline{\partial\Omega_C}), \tag{2.31}$$

$$u_{\widehat{h}}(e_J^{*\widehat{h}}) \to u(e_J^*) \quad \text{in } V, \tag{2.32}$$

where e_J^* *is a solution of the Optimal Design Problem* [3, one of (1.14)–(1.17)]. *The limit of each subsequence of* $\{e_J^{*h}\}$, *converging in* U, *is a solution of the latter problem and an analogue of* (2.32) *holds.*

Proof. Since each $U_{\mathrm{ad}}^h \subset U_{\mathrm{ad}}$ and U_{ad} is compact in U, there exists a subsequence $\{e_J^{*\widehat{h}}\}$, $\widehat{h} \to 0+$, such that (2.31) holds. Consider an $e \in U_{\mathrm{ad}}$. By Lemma 2.3, there exists a sequence of $e_{\widehat{h}} \in U_{\mathrm{ad}}^{\widehat{h}}$ such that $e_{\widehat{h}} \to e$ in U as $\widehat{h} \to 0+$. By definition, we have

$$\mathcal{L}_J^{\widehat{h}}(e_J^{*\widehat{h}}, u_{\widehat{h}}(e_J^{*\widehat{h}})) \le \mathcal{L}_J^{\widehat{h}}(e_{\widehat{h}}, u_{\widehat{h}}(e_{\widehat{h}})).$$

Letting $\widehat{h} \to 0+$ and applying Proposition 2.2 to both sides of this inequality, we arrive at

$$\mathcal{L}_J(e_J^*, u(e_J^*)) \le \mathcal{L}_J(e, u(e)),$$

so that e_J^* is a solution of the original Optimal Design Problem. Making use of Proposition 2.1, we obtain (2.32). This line of thought may be repeated for any uniformly convergent subsequence of $\{e_J^{*h}\}$. ■

THEOREM 2.5. *Let* $\{e_h^\varepsilon\}$, $h \to 0+$, *be a sequence of solutions of the Approximate Weight Minimization Problem* (1.23). *Then there exists a subsequence* $\{e_{\widehat{h}}^\varepsilon\} \subset \{e_h^\varepsilon\}$ *such that*

$$e_{\widehat{h}}^\varepsilon \to e_\varepsilon \quad \text{in } U,$$

where e_ε *is a solution of the penalized optimization problem* [3, (3.1)].

Proof. Analogous to that of Theorem 2.4. ■

3. Approximate reliable solutions. We shall introduce approximations of the method of reliable solution (alias worst scenario method), which has been introduced in [3, Section 4] for problems with some uncertain input data. In contrast with the previous sections, we keep the half-thickness $H(x)$ fixed, $H \in C^{(0),1}(\overline{\Omega})$, $H > 0$ everywhere and $O_i \ge \max_{x \in \overline{\Omega}} H(x)$, $1 \le i \le N$ (see [3, (1.1)]). On the other hand, we allow the loading function p to vary in the set U_{ad}^p.

Here we use again the finite element spaces X_h, V_h, and the sets $U_{\mathrm{ad}}^{\mathcal{Z}h}$, $U_{\mathrm{ad}}^{\mathcal{F}h}$, but we introduce a new set $U_{\mathrm{ad}}^{ph} = U_{\mathrm{ad}}^p \cap X_h$. Assume that $p_0 \in X_{h_0}$ for some triangulation $\mathcal{T}_{h_0}$.

Hence we have to assume that the triangulations $\mathcal{T}_h$ are consistent also with the boundaries $\partial\Omega_m$, $m = 1, \dots, M$, which play a role in the definition of U^p_{ad}, and with the boundaries of G_j, which appear in the definition of Φ_1 and Φ_2. Then we define

$$U^h_{\mathrm{ad}} = U^{ph}_{\mathrm{ad}} \times U^{\mathcal{Z}h}_{\mathrm{ad}} \times U^{\mathcal{F}h}_{\mathrm{ad}}$$

and consider approximate input data $e_h = \{p_h, \mathcal{Z}_h, \mathcal{F}_h\} \in U^h_{\mathrm{ad}}$.

Instead of the criterions Φ_i, $i = 1, 2, 3$, we introduce

$$\Phi^h_1(v_h) = \max_{1 \le j \le J} (\operatorname{meas} G_j)^{-1} \sum_{T \subset G_j} |v_h(\gamma)| \operatorname{meas} T,$$

$$\Phi^h_2(v_h) = \Phi_2(v_h) = \max_{1 \le j \le J} (\operatorname{meas} G_j)^{-1} \int_{G_j} |\operatorname{grad} v_h|^2 \, dx,$$

$$\Phi^h_3(e_h, v_h) = \langle H \operatorname{grad} v_h, \operatorname{grad} \varphi\rangle_0 + \langle 2\omega H - p_h, \varphi\rangle_0 + \sum_{T \subset \bar{\Omega} \setminus \Omega^*} \langle \mathcal{Z}_h [v_h(\gamma)]^-, \varphi\rangle_{0,T}, \qquad \varphi \in H^1_0(\Omega) \cap X_h.$$

We solve the following *approximate maximization problems*: find

$$e^{*h}_i = \arg \max_{e_h \in U^h_{\mathrm{ad}}} \Phi^h_i(e_h, u_h(e_h)), \quad i = 1, 2, 3, \tag{3.1$_i$}$$

where $u_h(e_h)$ denotes the solution of the Approximate State Problem (1.4) for the input data $e_h \equiv \{p_h, \mathcal{Z}_h, \mathcal{F}_h\} \in U^h_{\mathrm{ad}}$, i.e., $u_h(e_h) \in \mathcal{K}_h(H)$ such that

$$a(H; u_h(e_h), v_h - u_h(e_h)) + b_h(\mathcal{Z}_h; u_h(e_h), v_h - u_h(e_h)) + \Phi_h(e_h)(v_h) - \Phi_h(e_h)(u_h(e_h)) \ge \langle p_h - 2\omega H, v_h - u_h(e_h)\rangle_0 \tag{3.2}$$

for all $v_h \in \mathcal{K}_h(H)$.

THEOREM 3.1. (i) *The problem* (3.2) *has a unique solution* $u_h(e_h)$ *for any* $e_h \in U^h_{\mathrm{ad}}$ *and any* h *sufficiently small.*

(ii) *The approximate maximization problem* (3.1$_i$), $i = 1, 2, 3$, *has at least one solution for any* h *sufficiently small.*

Proof. The argument is analogous to that of Theorem 1.1. Let us verify the assumptions of [3, Theorem 2.1], where we set $\mathcal{K}(e) := \mathcal{K}_h(H)$, $\langle f, v_h\rangle = -2\omega\langle H, v_h\rangle_0$, $\langle Be, v_h\rangle = \langle p_h, v_h\rangle_0$, $U_{\mathrm{ad}} := U^h_{\mathrm{ad}}$, $e := e_h$, $V := V_h$, $A(e) := A^h(e_h)$,

$$\langle A^h(e_h) v_h, w_h\rangle := a(H; v_h, w_h) + b_h(\mathcal{Z}_h; v_h, w_h),$$

$$\Phi(e)(v_h) := \Phi_h(e_h)(v_h) = \sum_{E \subset \partial\Omega_C} \int_E \mathcal{F}_h |v_h(\gamma)| \, ds + I_{\mathcal{K}_h(H)}(v_h).$$

Then Lemma 1.3 holds and Lemma 1.4 can be proved by nearly the same (simpler) argument. Instead of Lemma 1.5 we prove the following

LEMMA 3.2. *Let* $e_{hn} \to e_h$, $e_{hn} \in U_{\mathrm{ad}}^h$ *and* $v_{hn} \to v_h$, $v_{hn} \in V_h$, *as* $n \to \infty$. *Then*

$$\lim_{n\to\infty} \Phi_i^h(e_{hn}, v_{hn}) = \Phi_i^h(e_h, v_h), \qquad i = 1, 2, 3.$$

Proof. We may write

$$\lim_{n\to\infty} \Phi_1^h(v_{hn}) = \lim \max_{1\le j\le J} (\operatorname{meas} G_j)^{-1} \sum_{T\subset G_j} |v_{hn}(\gamma)| \operatorname{meas} T$$
$$= \max_j \lim_{n\to\infty} \psi_j(v_{hn}) = \max_j \psi_j(v_h) = \Phi_1^h(v_h),$$

since

$$\Big| \sum_{T\subset G_j} |v_{hn}(\gamma)| \operatorname{meas} T - \sum_{T\subset G_j} |v_h(\gamma)| \operatorname{meas} T \Big|$$
$$\le \sum_{T\subset G_j} |v_{hn}(\gamma) - v_h(\gamma)| \operatorname{meas} T \le \sum_{T\subset G_j} \|v_{hn} - v_h\|_{\infty,T} \operatorname{meas} T$$
$$\le \sum_{T\subset G_j} C\|v_{hn} - v_h\|_{0,T} (\operatorname{meas} T)^{1/2} \le C\|v_{hn} - v_h\|_{0,G_j} \operatorname{meas} G_j \to 0.$$

Here we have used the inequality (1.15) in the final step.

Second, we have

$$\Big| \int_{G_j} (|\operatorname{grad} v_{hn}|^2 - |\operatorname{grad} v_h|^2)\, dx \Big| \le \|v_{hn} - v_h\|_1 (\|v_{hn}\|_1 + \|v_h\|_1) \to 0,$$

so that

$$\lim_{n\to\infty} \Phi_2^h(v_{hn}) = \lim_{n\to\infty} \max_{1\le j\le J} (\operatorname{meas} G_j)^{-1} \int_{G_j} |\operatorname{grad} v_{hn}|^2\, dx$$
$$= \max_j \lim_{n\to\infty} (\ldots) = \Phi_2^h(v_h).$$

Third, we may write

$$|\Phi_3^h(e_{hn}, v_{hn}) - \Phi_3^h(e_h, v_h)| \le |\Phi_3^h(e_{hn}, v_{hn}) - \Phi_3^h(e_h, v_{hn})|$$
$$+ |\Phi_3^h(e_h, v_{hn}) - \Phi_3^h(e_h, v_h)| \equiv L_1 + L_2,$$

and using (1.15) again,

$$L_1 \le |\langle p_{hn} - p_h, \varphi\rangle_0| + \sum_T |\langle (\mathcal{Z}_{hn} - \mathcal{Z}_h)[v_{hn}(\gamma)]^-,\ \varphi\rangle_{0,T}|$$
$$\le C(\|p_{hn} - p_h\|_0 + \|\mathcal{Z}_{hn} - \mathcal{Z}_h\|_\infty \|v_{hn}\|_0) \to 0,$$
$$L_2 \le |\langle H \operatorname{grad}(v_{hn} - v_h), \operatorname{grad}\varphi\rangle_0| + \sum_T |\langle \mathcal{Z}_h([v_{hn}(\gamma)]^- - [v_h(\gamma)]^-), \varphi\rangle_{0,T}|$$
$$\le C\|v_{hn} - v_h\|_1 + \mathcal{Z}_{\max} \sum_T \|v_{hn} - v_h\|_{0,T} \cdot \|\varphi\|_{0,T} \to 0.$$

As a consequence,

$$\lim_{n\to\infty} \Phi_3^h(e_{hn}, v_{hn}) = \Phi_3^h(e_h, v_h).$$

Finally, the existence of solutions of the problems (3.1_i) follows if we set $\mathcal{L} \equiv -\Phi_i^h$. ∎

4. Convergence results. Let us study the convergence of finite-element approximations when the mesh size tends to zero. First of all, we have to establish the following

PROPOSITION 4.1. *Let $e_h \in U_{\mathrm{ad}}^h$ with $e_h \to e$ in U as $h \to 0+$. Then*

$$u_h(e_h) \to u(e) \quad \text{in } V \text{ as } h \to 0+.$$

Proof. The proof is analogous to that of Proposition 2.1. We can insert $v_h = 0$ in the inequality (3.2) to get the boundedness of $u_h := u_h(e_h)$ as $h \to 0+$. In proving that the weak limit of a subsequence $\{u_h\}$ belongs to $\mathcal{K}(H)$, we substitute $H_h := \pi_h H$, i.e., the linear Lagrange interpolate of H over the triangulation $\mathcal{T}_h$, and use the fact that

$$\|\pi_h H - H\|_{0,\Omega_i^*} \to 0 \quad \text{as } h \to 0+$$

(cf. the proof of Lemma 2.3).

We derive (2.7)–(2.18), (2.22), (2.23). Instead of (2.24), (2.25) we obtain

$$\begin{aligned}(4.1)\quad & |\langle p_h - 2\omega H, v_h - u_h\rangle_0 - \langle p - 2\omega H, v - u\rangle_0| \\ &\qquad \le C\{\|v_h - v\|_0 + \|u_h - u\|_0 + \|p_h - p\|_\infty\} \to 0.\end{aligned}$$

Passing to limes inferior or limes superior in the inequality (3.2) and employing (2.12), (2.18), (2.22), (2.23) and (4.1), we arrive at (2.26), so that u satisfies the inequality [3, (1.7)]. As a consequence, the whole sequence $\{u_h(e_h)\}$ tends to $u(e)$ weakly in V as $h \to 0+$.

The proof of strong convergence is the same as in the proof of Proposition 2.1. ∎

PROPOSITION 4.2. *Let $e_h \in U_{\mathrm{ad}}^h$ with $e_h \to e$ in U as $h \to 0+$. Then*

$$\lim_{h\to 0+} \Phi_i^h(e_h, u_h(e_h)) = \Phi_i(e, u(e)), \quad i = 1, 2, 3.$$

Proof. For $u_h := u_h(e_h)$ and $u := u(e)$ we may write

$$\begin{aligned}\Big| \sum_{T\subset G_j} |u_h(\gamma)| \operatorname{meas} T - \int_{G_j} |u|\,dx \Big| &\le \sum_{T\subset G_j} \int_T (|u_h(\gamma) - u_h| + |u_h - u|)\,dx \\ &\le (h\|u_h\|_{1,G_j} + \|u_h - u\|_{0,G_j}) \operatorname{meas} G_j \to 0 \quad \text{as } h \to 0,\end{aligned}$$

by Proposition 4.1. As a consequence,

$$\lim_{h\to 0+} \Phi_1^h(u_h) = \lim_{h\to 0} \max_{1\le j\le J} (\operatorname{meas} G_j)^{-1} \sum_{T\subset G_j} |u_h(\gamma)| \operatorname{meas} T$$
$$= \max_{j\le J} \lim_{h\to 0} (\ldots) = \max_{j\le J} (\operatorname{meas} G_j)^{-1} \int_{G_j} |u|\, dx = \Phi_1(u).$$

Since

$$\Big| \int_{G_j} |\operatorname{grad} u_h|^2\, dx - \int_{G_j} |\operatorname{grad} u|^2\, dx \Big| \le C|u_h - u|_{1,G_j} \to 0,$$

we have

$$\lim_{h\to 0+} \Phi_2^h(u_h) = \lim_{h\to 0+} \max_{1\le j\le J} (\operatorname{meas} G_j)^{-1} \int_{G_j} |\operatorname{grad} u_h|^2\, dx$$
$$= \max_{j\le J} \lim_{h\to 0+} (\ldots) = \Phi_2(u).$$

Third, we may write

$$|\Phi_3^h(e_h, u_h) - \Phi_3(e, u)| \le |\Phi_3^h(e_h, u_h) - \Phi_3(e_h, u)| + |\Phi_3(e_h, u) - \Phi_3(e, u)|$$
$$= M_1 + M_2,$$

where

$$M_1 \le |\langle H \operatorname{grad}(u_h - u), \operatorname{grad} \varphi\rangle_0| + \sum_{T\subset \bar{\Omega}\setminus \Omega^*} |\langle \mathcal{Z}_h([u_h(\gamma)]^- - [u]^-), \varphi\rangle_{0,T}|$$
$$\le C\|u_h - u\|_1 + C_1(h\|u_h\|_1 + \|u_h - u\|_0)\|\varphi\|_0 \to 0$$

(cf. the proof of Proposition 2.2 for $\mathcal{L}_{\mathrm{TR}}^h$), and

$$M_2 \le \Big| \int_{\Omega\setminus\Omega^*} (\mathcal{Z}_h - \mathcal{Z})[u]^- \varphi\, dx \Big| + |\langle p - p_h, \varphi\rangle_0|$$
$$\le C(\|\mathcal{Z}_h - \mathcal{Z}\|_\infty + \|p - p_h\|_\infty) \to 0.$$

As a consequence, we obtain $\lim_{h\to 0} \Phi_3^h(e_h, u_h) = \Phi_3(e, u)$. ■

LEMMA 4.3. *For any $e \equiv \{p, \mathcal{Z}, \mathcal{F}\} \in U_{\mathrm{ad}}$ and any sequence $\{h\}$, $h \to 0+$, there exists a sequence $\{e_h\}$ such that $e_h \equiv \{p_h, \mathcal{Z}_h, \mathcal{F}_h\} \in U_{\mathrm{ad}}^h$ and $e_h \to e$ in $U = (\prod_{m=1}^M C(\bar{\Omega}_m)) \times C(\bar{\Omega} \setminus \Omega^*) \times C(\overline{\partial\Omega}_C)$.*

Proof. Consider the restriction $p_m = p|\Omega_m$ of any $p \in U_{\mathrm{ad}}^p$ and define $p_h = \pi_h p_\varepsilon$, where π_h is the linear Lagrange interpolation over $\mathcal{T}_h$ and

$$p_\varepsilon = \varepsilon p_0 + (1-\varepsilon)p_m, \quad x \in \Omega_m,$$

where ε is a real parameter, $0 < \varepsilon < 1$. We have

$$\|\partial p_\varepsilon/\partial x_i\|_{\infty,\Omega_m} \le \varepsilon\|\partial p_0/\partial x_i\|_\infty + (1-\varepsilon)\|\partial p_m/\partial x_i\|_\infty \tag{4.2}$$
$$\le C_2, \quad i = 1, 2,$$

by definitions of $U_{\rm ad}$ and p_0. Since

$$\|p_\varepsilon\|_{\infty,\Omega_m} \le \varepsilon\|p_0\|_\infty + (1-\varepsilon)\|p_m\|_\infty \le \max\{\|p_0\|_\infty, \|p_m\|_\infty\} \equiv C_3,$$

we obtain

$$\|p_\varepsilon\|_{1,\infty,\Omega_m} \le C_3 + 2C_1 \equiv C_4$$

for all ε. Using the estimate

$$\|q - \pi_h q\|_{\infty,\Omega_m} \le Ch\|q\|_{1,\infty,\Omega_m}$$

we may write

$$\begin{aligned} \|p_h - p_0\|_{\infty,\Omega_m} &\le \|\pi_h p_\varepsilon - p_\varepsilon\|_\infty + \|p_\varepsilon - p_0\|_\infty \\ &\le CC_4 h + (1-\varepsilon)\|p_m - p_0\|_\infty \\ &\le CC_4 h + (1-\varepsilon)C_1 \le C_1 \end{aligned} \tag{4.3}$$

if

$$CC_4 h \le C_1 \varepsilon. \tag{4.4}$$

Let $\overline{PQ} \subset T \subset \overline{\Omega}_m$ be a straight-line segment of length ℓ, parallel to the x_i-axis. Then

$$|\partial \pi_h p_\varepsilon/\partial x_i| = \left|\ell^{-1}\int_P^Q \frac{\partial p_\varepsilon}{\partial x_i}\,dx_i\right| \le \ell^{-1}\int_P^Q |\partial p_\varepsilon/\partial x_i|\,dx_i \le C_2$$

by (4.2), so that

$$\|\partial p_h/\partial x_i\|_{\infty,\Omega_m} \le C_2, \quad i = 1,2. \tag{4.5}$$

Next, we have

$$\begin{aligned} \|p_h - p_m\|_{\infty,\Omega_m} &\le \|\pi_h p_\varepsilon - p_\varepsilon\|_\infty + \|p_\varepsilon - p_m\|_\infty \\ &\le CC_4 h + \varepsilon\|p_0 - p_m\|_\infty \\ &\le CC_4 h + \varepsilon C_1 \to 0 \quad \text{ as } h \to 0+ \text{ and } \varepsilon \to 0+. \end{aligned} \tag{4.6}$$

Combining (4.3)–(4.6), we can find a sequence $\{p_h\}$, $h \to 0+$, such that $p_h \in U_{\rm ad}^{ph}$ and $p_h \to p$ in $\prod_{m=1}^M C(\overline{\Omega}_m)$.

The components $\mathcal{Z}_h$ and $\mathcal{F}_h$ can be defined as linear Lagrange interpolates of $\mathcal{Z}$ and $\mathcal{F}$, respectively (cf. the proof of Lemma 2.3). ■

THEOREM 4.4. *Let $\{e_i^{*h}\}$, $h \to 0+$, be a sequence of solutions of the approximate maximization problem* (3.1_i), $i = 1,2,3$. *Then there exists a subsequence $\{e_i^{*\widehat{h}}\} \subset \{e_i^{*h}\}$ such that*

$$e_i^{*\widehat{h}} \to e_i^* \quad \text{in } U, \tag{4.7}$$

$$u_{\widehat{h}}(e_i^{*\widehat{h}}) \to u(e_i^*) \quad \text{in } V, \tag{4.8}$$

$$\Phi_i^{\widehat{h}}(e_i^{*\widehat{h}}, u_{\widehat{h}}(e_i^{*\widehat{h}})) \to \Phi_i(e_i^*, u(e_i^*)), \tag{4.9}$$

where e_i^* *is a solution of the maximization problem* $(4.1)_i$ *of* [3]*. The limit of each subsequence of* $\{e_i^{*h}\}$*, converging in* U*, is a solution of the problem* $(4.1)_i$ *and the analogues of* (4.8), (4.9) *hold.*

Proof. Analogous to that of Theorem 2.4. Instead of Proposition 2.2 and Lemma 2.3, we employ Proposition 4.2 and Lemma 4.3. Proposition 2.1 is replaced by Proposition 4.1. ■

Acknowledgments. The first author thankfully acknowledges the support of the Grant Agency of the Czech Republic under grant 201/98/0528 and of the Ministry of Education, Youth and Sports under grant OK-407.

References

[1] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

[2] R. Glowinski, *Lectures on Numerical Methods for Non-linear Variational Problems*, Tata Inst. Fund. Res. and Springer, 1980.

[3] I. Hlaváček and J. Lovíšek, *Control in obstacle-pseudoplate problems with friction on the boundary. Optimal design and problems with uncertain data*, Appl. Math. (Warsaw) 28 (2001), 407–426.

[4] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, 1967, and Masson, 1967.

Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25
CZ-115 67 Praha 1, Czech Republic
E-mail: hlavacek@math.cas.cz

Faculty of Civil Engineering
Slovak Technical University
Radlinského 11
SK-813 68 Bratislava, Slovak Republic
E-mail: bock@kmat.elf.stuba.sk

Institute of Computer Science
Pod vodárenskou věží 2
CZ-182 07 Praha 8, Czech Republic

Received on 4.10.2001 (1594)