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ON AN INEQUALITY FOR A FREE BOUNDARY PROBLEM FOR EQUATIONS OF A VISCOUS COMPRESSIBLE HEAT-CONDUCTING CAPILLARY FLUID

Abstract. We derive an inequality for a local solution of a free boundary problem for a viscous compressible heat-conducting capillary fluid. This inequality is crucial to proving the global existence of solutions belonging to certain anisotropic Sobolev–Slobodetskiĭ spaces and close to an equilibrium state.

1. Introduction. In the paper we obtain some crucial inequality for a local solution of the equations of motion of a viscous compressible heat-conducting capillary fluid bounded by a free surface. The motion of such a fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ (which depends on time $t \in \mathbb{R}_+$) is described by the following system with the boundary and initial conditions (see [6], [7]):

$$
\begin{aligned}
&\varrho[v_t + (v \cdot \nabla)v] - \operatorname{div} \mathbb{T}(v,p) = 0 &&\text{in } \widetilde{\Omega}^T, \\
&\varrho_t + \operatorname{div}(\varrho v) = 0 &&\text{in } \widetilde{\Omega}^T, \\
&\varrho c_v(\theta_t + v \cdot \nabla \theta) + \theta p_\theta \operatorname{div} v - \varkappa \Delta \theta \\
&\quad - \frac{\mu}{2} \sum_{i,j=1}^{3} (v_{ix_j} + v_{jx_i})^2 - (\nu - \mu)(\operatorname{div} v)^2 = \varrho r &&\text{in } \widetilde{\Omega}^T, \\
&\mathbb{T}\overline{n} - \sigma H \overline{n} = -p_0 \overline{n} &&\text{on } \widetilde{S}^T, \\
&v \cdot \overline{n} = -\varphi_t/|\nabla \varphi| &&\text{on } \widetilde{S}^T,
\end{aligned}
$$

(1.1)

$$\partial\theta/\partial n = \overline{\theta} \qquad\qquad\qquad\qquad \text{on } \widetilde{S}^T,$$
$$\varrho|_{t=0} = \varrho_0, \quad \theta|_{t=0} = \theta_0, \quad v|_{t=0} = v_0 \quad \text{in } \Omega,$$

where $\widetilde{\Omega}^T \equiv \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, $\Omega_0 = \Omega$ is the initial domain, $\varphi(x,t) = 0$ describes S_t (at least locally), $\widetilde{S}^T \equiv \bigcup_{t\in(0,T)} S_t \times \{t\}$, $\overline{n}$ is the unit outward vector normal to the boundary, i.e. $\overline{n} = \nabla\varphi/|\nabla\varphi|$. Moreover, $v = v(x,t)$ is the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $r = r(x,t)$ the heat sources per unit mass, $\overline{\theta} = \overline{\theta}(x,t)$ the heat flow per unit surface, $p = p(\varrho,\theta)$ the pressure, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, μ and ν the viscosity coefficients, $\varkappa$ the coefficient of heat-conductivity, σ the constant coefficient of surface tension, and p_0 the external (constant) pressure.

From the thermodynamic considerations we have

$$\nu > \frac{1}{3}\mu > 0, \quad \varkappa > 0, \quad c_v > 0, \quad \sigma > 0.$$

Further, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{\mu S_{ij}(v) + (\nu-\mu)\delta_{ij}\operatorname{div} v - p\delta_{ij}\}_{i,j=1,2,3},$$

where $\mathbb{S}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3}$ is the velocity deformation tensor.

Finally, we denote by H the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\overline{n} = \Delta_{S_t}(t)x, \quad x = (x_1, x_2, x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be determined by $x = x(s_1, s_2, t)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$. Then we have

$$\Delta_{S_t}(t) = g^{-1/2}\frac{\partial}{\partial s_\gamma}\left(g^{1/2}g^{\gamma\delta}\frac{\partial}{\partial s_\delta}\right) \quad (\gamma, \delta = 1, 2),$$

where the convention summation over repeated indices is assumed, $g = \det\{g_{\gamma\delta}\}_{\gamma,\delta=1,2}$, $g_{\gamma\delta} = \frac{\partial x}{\partial s_\gamma}\cdot\frac{\partial x}{\partial s_\delta}$, and $\{g^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\gamma\delta}\}$.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi, t), \xi \in \Omega\}$, where $x = x(\xi, t)$ is the solution of the Cauchy problem

$$(1.2) \qquad \frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Integrating (1.2) we obtain

$$x = \xi + \int_0^t u(\xi, t')\,dt' \equiv X_u(\xi, t),$$

where $u(\xi,t) = v(X_u(\xi,t))$ and $x = X_u(\xi,t)$ describes the relation between the Eulerian x and Lagrangian ξ coordinates. Moreover, by $(1.1)_5$, $S_t = \{x : x = x(\xi,t), \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_5$ the total mass is conserved, i.e.

$$\int\limits_{\Omega_t} \varrho(x,t)\, dx = \int\limits_{\Omega} \varrho_0(\xi)\, d\xi = M.$$

Now, assume that $p_\varrho > 0$, $p_\theta > 0$ for $\varrho, \theta \in \mathbb{R}_+$ and consider the equation

$$(1.3) \qquad p\left(\frac{M}{\frac{4}{3}\pi R_e^3}, \theta_e\right) = p_0 + \frac{2\sigma}{R_e}.$$

We assume that there exist $R_e > 0$ and $\theta_e > 0$ satisfying (1.3). Then we have the following definition.

DEFINITION 1.1. Let $r = \overline{\theta} = 0$. By an *equilibrium state* we mean a solution $(v, \theta, \varrho, \Omega_t)$ of problem (1.1) such that $v = 0$, $\theta = \theta_e$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \geq 0$, where $\varrho_e = M/(\frac{4}{3}\pi R_e^3)$, Ω_e is a ball of radius R_e, and $R_e > 0$ and $\theta_e > 0$ satisfy equation (1.3).

Next, we introduce

$$(1.4) \qquad \varrho_\sigma = \varrho - \varrho_e, \quad \theta_\sigma = \theta - \theta_e, \quad p_\sigma = p - \frac{2\sigma}{R_e} - p_0.$$

We can write p_σ in the form

$$(1.5) \qquad p_\sigma(\varrho, \theta) = p_1 \varrho_\sigma + p_2 \theta_\sigma,$$

where

$$p_1(\varrho, \theta) = \int\limits_0^1 p_\varrho(\varrho_e + s(\varrho - \varrho_e), \theta)\, ds, \quad p_2(\theta) = \int\limits_0^1 p_\theta(\varrho_e, \theta_e + s(\theta - \theta_e))\, ds.$$

Then by using (1.4) problem (1.1) can be rewritten in Lagrangian coordinates as follows:

$$\eta u_t - \operatorname{div}_u \mathbb{T}_u(u, p_\sigma) = 0 \qquad\qquad \text{in } \Omega^T \equiv \Omega \times (0, T),$$

$$\eta_{\sigma t} + \eta \operatorname{div}_u u = 0 \qquad\qquad \text{in } \Omega^T,$$

$$\eta c_v \vartheta_{\sigma t} + \vartheta p_\vartheta \operatorname{div}_u u - \varkappa \nabla_u^2 \vartheta_\sigma$$

$$(1.6) \qquad = \frac{\mu}{2} \sum_{i,j=1}^{3} (\xi_{x_i} \cdot \nabla_\xi u_j + \xi_{x_j} \cdot \nabla_\xi u_i)^2$$

$$\qquad\quad - (\nu - \mu)(\operatorname{div}_u u)^2 + \eta k \qquad\qquad \text{in } \Omega^T,$$

$$\mathbb{T}_u(u, p_\sigma)\overline{n}_u - \sigma(H + 2/R_e)\overline{n}_u = 0 \qquad\qquad \text{on } S^T \equiv S \times (0, T),$$

$$\overline{n}_u \cdot \nabla_u \vartheta_\sigma = \overline{\vartheta} \qquad\qquad \text{on } S^T,$$

$$\eta_\sigma|_{t=0} = \varrho_{\sigma 0}, \quad \vartheta_\sigma|_{t=0} = \theta_{\sigma 0}, \quad u|_{t=0} = v_0 \quad \text{in } \Omega,$$

where $\eta(\xi, t) = \varrho(X_u(\xi, t), t)$, $\vartheta(\xi, t) = \theta(X_u(\xi, t), t)$, $\eta_\sigma = \eta - \varrho_e$, $\vartheta_\sigma = \vartheta - \theta_e$, $k(\xi, t) = r(X_u(\xi, t), t)$, $\overline{\vartheta}(\xi, t) = \overline{\theta}(X_u(\xi, t), t)$, $\overline{n}_u(\xi, t) = \overline{n}(X_u(\xi, t), t)$,

$$\nabla_u = \xi_{ix}\partial_{\xi_i} = \{\xi_{ix_j}\partial_{\xi_i}\}_{j=1,2,3}, \ \mathbb{T}_u(u,p) = -pI + \mathbb{D}_u(u), \ I = \{\delta_{ij}\}_{i,j=1,2,3},$$

$$\mathbb{D}_u(u) = \{\mathbb{D}_{uij}(u)\}_{i,j=1,2,3}$$
$$= \{\mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu - \mu)\delta_{ij}\,\mathrm{div}_u\,u\}_{i,j=1,2,3},$$

$\mathrm{div}_u u = \nabla_u \cdot u = \partial_{x_i}\xi_k\partial_{\xi_k}u_i$, $\mathrm{div}_u\,\mathbb{T}_u(u,p) = \{\partial_{x_j}\xi_k\partial_{\xi_k}\mathbb{T}_{uij}(u,p)\}_{i=1,2,3}$
and $\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x which is inverse to the matrix $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$.

The aim of our considerations is to prove the global existence of solutions to problem (1.1) which are sufficiently close to the equilibrium solution (see Definition 1.1) for all $t > 0$. The result is stated as follows:

THEOREM (see [12]). *Let*

$$(1.7) \quad \begin{aligned} \|v_0\|_{1+\alpha,\Omega} + \|\varrho_{\sigma0}\|_{1+\alpha,\Omega} + \|\theta_{\sigma0}\|_{1+\alpha,\Omega} &\le \delta_1, \\ \|r\|_{C_B^2(\mathbb{R}^3\times(0,\infty))} + \|\overline{\theta}\|_{C_B^3(\mathbb{R}^3\times(0,\infty))} &\le \delta_2, \\ \int_\Omega \varrho_0\,d\xi = M, \quad \int_\Omega \varrho_0\xi\,d\xi = 0, \quad \int_\Omega \varrho_0 v_0\,d\xi &= 0. \end{aligned}$$

Assume that the initial domain Ω is close to a ball and S is described by $|x| = R(\omega)$, where $\omega \in S^1$ (S^1 is the unit sphere). Let

$$\|R(\omega) - R_e\|_{1,S^1} \le \delta_3, \qquad \|H(\cdot,0) + 2/R_e\|_{1+\alpha/2,S} \le \delta_4.$$

Let $\alpha \in (3/4,1)$ and assume that the data satisfy some other conditions which are implied by conservation laws and which are described in [12]. *Then for sufficiently small δ_i ($i = 1,\ldots,4$) the local solution to problem (1.1) (see* [13]) *can be prolonged infinitely in time.*

The main role in the process of prolonging the local solution in time is played by inequality (3.1) of Section 3. The paper is devoted to showing it.

To prove the theorem we also need a global energy estimate and an estimate showing the increase of regularity of the solution after some time. Both have been derived in [14]. However, the latter estimate has been obtained in the case of $r = \overline{\theta} = 0$. For nonvanishing r and $\overline{\theta}$ such an estimate is shown in Section 4 of this paper (see Theorem 4.1).

The global existence result is proved in spaces such that $(u,\vartheta_\sigma,\eta_\sigma) \in W_2^{2+\alpha,1+\alpha/2}(\Omega_{kT} \times (kT,(k+1)T)) \times W_2^{2+\alpha,1+\alpha/2}(\Omega_{kT} \times (kT,(k+1)T)) \times C([kT,(k+1)T];W_2^{1+\alpha}(\Omega_{kT})) \cap W_2^{1+\alpha,1+\alpha/2}(\Omega_{kT} \times (kT,(k+1)T))$, where $k \in \mathbb{N} \cup \{0\}$; $u,\eta_\sigma,\vartheta_\sigma$ denote $v,\varrho_\sigma,\theta_\sigma$ written in Lagrangian coordinates $\xi \in \Omega_{kT}$; and T is the time of local existence.

We use the spaces with fractional derivatives to prove existence of solutions with the lowest possible regularity. In the compressible case (see [9], [10], [5]) it was possible to prove the existence in classes of functions such that $\alpha \in (1/2,1)$. However, in the general heat-conducting case we need

to assume that $\alpha \in (3/4, 1)$. This restriction is connected with the strong nonlinearity of equation $(1.1)_3$.

We derive (3.1) for the local solution $(u, \vartheta_\sigma, \eta_\sigma) \in W_2^{2+\alpha, 1+\alpha/2}(\Omega^T) \times W_2^{2+\alpha, 1+\alpha/2}(\Omega^T) \times W_2^{1+\alpha, 1/2+\alpha/2}(\Omega^T) \cap C([0, T]; W_2^{1+\alpha}(\Omega))$, $\alpha \in (3/4, 1)$ (see definitions in Section 2) in which the norms of $u, \vartheta_\sigma, \eta_\sigma$ are estimated by the sum of the L_2-norms of these functions, the nonlinear terms consisting of the products of the highest order norms of $u, \vartheta_\sigma, \eta_\sigma$ and the norms of data. The most important feature of the inequality is that on the r.h.s. of (3.1) the L_2-norms of $u, \vartheta_\sigma, \eta_\sigma$ enter only linearly. This is crucial for the proof of global existence (see [12]). The main result of the paper is formulated in Theorem 3.1.

2. Notation and auxiliary results. In Section 3 we use a partition of unity $(\{\widetilde{\Omega}_i\}, \{\zeta_i\})$, $\Omega \subset \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and $\zeta(\xi) = \zeta_i(\xi)$ be the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain then let $\widetilde{\omega} \subset \widetilde{\Omega}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$. Otherwise we assume that $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$. Let $\beta \in \overline{\widetilde{\omega}} \cap S \subset \overline{\widetilde{\Omega}} \cap S$, $\widetilde{S} \equiv S \cap \widetilde{\Omega}$.

Introduce local coordinates connected with $\{\xi\}$ by

(2.1) $\qquad y_k = \alpha_{kl}(\xi_l - \beta_l), \qquad \alpha_{3k} = n_k(\beta), \qquad k = 1, 2, 3,$

where α_{kl} is a constant orthogonal matrix such that $\widetilde{S}$ is determined by

$$y_3 = F(y_1, y_2), \qquad F \in W_2^{5/2+\alpha},$$

and

(2.2) $\quad \widetilde{\Omega} = \{y : |y_i| < d, \ i = 1, 2, \ F(y') < y_3 < F(y') + d, \ y' = (y_1, y_2)\}.$

Next, we introduce functions u', ϑ' and η' by

$$u_i'(y) = \alpha_{ij} u_j(\xi)|_{\xi=\xi(y)}, \qquad \vartheta'(y) = \vartheta(\xi)|_{\xi=\xi(y)}, \qquad \eta'(y) = \eta(\xi)|_{\xi=\xi(y)},$$

where $\xi = \xi(y)$ is the inverse transformation to (2.1). Further we introduce new variables by

(2.3) $\qquad z_i = y_i, \qquad i = 1, 2, \qquad z_3 = y_3 - \widetilde{F}(y), \qquad y \in \widetilde{\Omega},$

which will be denoted by $z = \Phi(y)$, where $\widetilde{F}$ is an extension of F onto $\widetilde{\Omega}$.

Let

(2.4) $\qquad \widehat{\Omega} = \Phi(\widetilde{\Omega}) = \{z : |z_i| < d, \ i = 1, 2, \ 0 < z_3 < d\}$

and $\widehat{S} = \Phi(\widetilde{S})$.

Define

$$\widehat{f}(z) = f'(y)|_{y=\Phi^{-1}(z)}, \qquad f \in \{u, \vartheta, \eta\}.$$

Introduce $\widehat{\nabla}_k = \xi_{l x_k} z_{i\xi_l} \nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is defined by (2.1). Introduce also the notation

$$
\begin{aligned}
&\widetilde{u}(\xi) = u(\xi)\zeta(\xi), \qquad \widetilde{\vartheta}_\sigma(\xi) = \vartheta_\sigma(\xi)\zeta(\xi), \\
(2.5) \quad &\widetilde{\eta}_\sigma(\xi) = \eta_\sigma(\xi)\zeta(\xi), \qquad \xi \in \widetilde{\Omega},\ \widetilde{\Omega} \cap S = \emptyset, \\
&\widetilde{u}(z) = \widehat{u}(z)\widehat{\zeta}(z), \qquad \widetilde{\vartheta}_\sigma(z) = \widehat{\vartheta}_\sigma(z)\widehat{\zeta}(z), \\
&\widetilde{\eta}_\sigma(z) = \widehat{\eta}_\sigma(z)\widehat{\zeta}(z), \qquad z \in \widehat{\Omega} = \Phi(\widetilde{\Omega}),\ \overline{\widetilde{\Omega}} \cap S \neq \emptyset,
\end{aligned}
$$

where $\widehat{\zeta}(z) = \zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

Under the above notation problem (1.6) has the following form in an interior subdomain:

$$
\begin{aligned}
\eta\widetilde{u}_{it} &- \nabla_j T_{ij}(\widetilde{u}, \widetilde{p}_\sigma) = -\nabla_{u_j} B_{uij}(u, \zeta) - T_{uij}(u, p_\sigma)\nabla_{u_j}\zeta \\
&- (\nabla_j T_{ij}(\widetilde{u}, \widetilde{p}_\sigma) - \nabla_{u_j} T_{uij}(\widetilde{u}, \widetilde{p}_\sigma)) \equiv k_{1i}, \quad i = 1, 2, 3, \\
\widetilde{\eta}_{\sigma t} &+ \varrho_e \nabla \cdot \widetilde{u} = \varrho_e u \cdot \nabla_u \zeta - \eta_\sigma \nabla_u \cdot u\zeta + \varrho_e(\nabla \cdot \widetilde{u} - \nabla_u \cdot \widetilde{u}) \equiv k_2, \\
\eta c_v&(\eta, \vartheta)\widetilde{\vartheta}_{\sigma t} - \varkappa \nabla^2 \widetilde{\vartheta}_\sigma + \theta_e p_\vartheta(\varrho_e, \theta_e)\nabla \cdot \widetilde{u}
\end{aligned}
$$

$$
\begin{aligned}
(2.6) \quad &= \eta\widetilde{k} + \left[\frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{k x_i}\partial_{\xi_k} u_j + \xi_{k x_j}\partial_{\xi_k} u_i)^2 + (\nu - \mu)(\nabla_u \cdot u)^2 \right]\zeta \\
&+ \theta_e p_\vartheta(\varrho_e, \theta_e)u \cdot \nabla_u \zeta + (\theta_e p_\vartheta(\varrho_e, \theta_e) - \vartheta p_\vartheta(\varrho, \vartheta))\nabla_u \cdot u\zeta \\
&+ \theta_e p_\vartheta(\varrho_e, \theta_e)(\nabla \cdot \widetilde{u} - \nabla_u \cdot \widetilde{u}) - \varkappa(\nabla_u^2 \zeta \vartheta_\sigma + 2\nabla_u \zeta \cdot \nabla_u \vartheta_\sigma) \\
&- \varkappa(\nabla^2 \widetilde{\vartheta}_\sigma - \nabla_u^2 \widetilde{\vartheta}_\sigma) \equiv \eta\widetilde{k} + k_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{B}_u(u, \zeta) &= \{B_{uij}(u, \zeta)\}_{i,j=1,2,3} \\
&= \{\mu(u_i\nabla_{u_i}\zeta + u_j\nabla_{u_i}\zeta) + (\nu - \mu)\delta_{ij}u \cdot \nabla_u\zeta\}_{i,j=1,2,3}, \\
\nabla_{u_j} &= \xi_{i x_j}\partial_{\xi_i}, \qquad \widetilde{p}_\sigma = p_\sigma\zeta,
\end{aligned}
$$

and in a boundary subdomain:

$$
\begin{aligned}
\widehat{\eta}\widetilde{u}_{it} &- \nabla_j T_{ij}(\widetilde{u}, \widetilde{p}_\sigma) = -\widehat{\nabla}_j \widehat{B}_{ij}(\widehat{u}, \widehat{\zeta}) - \widehat{T}_{ij}(\widehat{u}, \widehat{p}_\sigma)\widehat{\nabla}_j\widehat{\zeta} \\
&- (\nabla_j T_{ij}(\widetilde{u}, \widetilde{p}_\sigma) - \widehat{\nabla}_j \widehat{T}_{ij}(\widetilde{u}, \widetilde{p}_\sigma)) \equiv k_{4i}, \quad i = 1, 2, 3, \\
\widetilde{\eta}_{\sigma t} &+ \varrho_e \nabla \cdot \widetilde{u} = \varrho_e\widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} - \widehat{\eta}_\sigma \widehat{\nabla} \cdot \widehat{u}\widehat{\zeta} + \varrho_e(\nabla \cdot \widetilde{u} - \widehat{\nabla} \cdot \widetilde{u}) \equiv k_5, \\
(2.7) \quad \widehat{\eta}c_v&(\widehat{\eta}, \widehat{\vartheta})\widetilde{\vartheta}_{\sigma t} - \varkappa\Delta\widetilde{\vartheta}_\sigma + \theta_e p_{\widehat{\vartheta}}(\varrho_e, \theta_e)\nabla \cdot \widetilde{u}
\end{aligned}
$$

$$
\begin{aligned}
&= \widehat{\eta}\widetilde{k} + \left[\frac{\mu}{2} \sum_{i,j=1}^3 (\widehat{\nabla}_i\widehat{u}_j + \widehat{\nabla}_j\widehat{u}_i)^2 + (\nu - \mu)(\widehat{\nabla} \cdot \widehat{u})^2 \right]\widehat{\zeta} \\
&+ \theta_e p_{\widehat{\vartheta}}(\varrho_e, \theta_e)\widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} + (\theta_e p_{\widehat{\vartheta}}(\varrho_e, \theta_e) - \widehat{\vartheta}p_{\widehat{\vartheta}}(\widehat{\eta}, \widehat{\vartheta}))\widehat{\nabla} \cdot \widehat{u}\widehat{\zeta}
\end{aligned}
$$

$$+ \theta_e p_{\widehat{\vartheta}}(\varrho_e, \theta_e)(\nabla \cdot \widetilde{u} - \widehat{\nabla} \cdot \widetilde{u}) - \varkappa(\widehat{\nabla}^2 \widehat{\zeta} \widetilde{\vartheta}_\sigma + 2\widehat{\nabla}\widehat{\zeta} \cdot \nabla\widetilde{\vartheta}_\sigma)$$
$$- \varkappa(\nabla^2 \widetilde{\vartheta}_\sigma - \widehat{\nabla}^2 \widetilde{\vartheta}_\sigma) \equiv \widehat{\eta}\widetilde{k} + k_6,$$

$$\mathbb{T}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n}' = \sigma \Delta_{\widehat{S}}\widehat{\widetilde{\xi}\zeta} \cdot \widehat{n}\widehat{n} + \sigma \Delta_{\widehat{S}} \int_0^t \widetilde{u}\, dt' \cdot \widehat{n}\widehat{n}$$

$$+ \mathbb{T}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n}' - \widehat{\mathbb{T}}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n} + \widehat{\mathbb{B}}(\widehat{u}, \widehat{\zeta})\widehat{n}$$

$$- \sigma \left(2\widehat{\nabla} \int_0^t \widehat{u}\, dt' \, \widehat{\nabla}\widehat{\zeta} + \int_0^t \widehat{u}\, dt' \, \widehat{\nabla}^2 \widehat{\zeta} \right) \cdot \widehat{n}\widehat{n} + \frac{2\sigma}{R_e} \widehat{n}\widehat{\zeta}$$

$$\equiv \frac{2\sigma}{R_e} \widehat{n}\widehat{\zeta} + k_7,$$

$$\widehat{n}' \cdot \nabla\widetilde{\vartheta}_\sigma = \widetilde{\widetilde{\vartheta}} + \widehat{n} \cdot \widehat{\nabla}\widehat{\zeta}\widetilde{\vartheta}_\sigma + \widehat{n}' \cdot \nabla\widetilde{\vartheta}_\sigma - \widehat{n} \cdot \widehat{\nabla}\widetilde{\vartheta}_\sigma \equiv \widetilde{\widetilde{\vartheta}} + k_8,$$

where $\widehat{\mathbb{T}}$ and $\widehat{\mathbb{B}}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$; $\widehat{n}' = (0, 0, 1)$, $\widehat{n}$ is the vector $\overline{n}_u = \overline{n}(x(\xi, t), t)$ written in z coordinates and

$$(2.8) \qquad \Delta_{\widehat{S}}(t) = \frac{1}{\sqrt{g_{\widehat{u}}}} \frac{\partial}{\partial z_\gamma} \left(\sqrt{g_{\widehat{u}}}\, g_{\widehat{u}}^{\gamma\delta} \frac{\partial}{\partial z_\delta} \right),$$

$\{g_{\widehat{u}}^{\gamma\delta}\}$ is the inverse matrix to $\{(g_{\widehat{u}})_{\gamma\delta}\}$,

$$(g_{\widehat{u}})_{\gamma\delta} = \frac{\partial x}{\partial z_\gamma} \cdot \frac{\partial x}{\partial z_\delta}, \qquad x = \widehat{\xi} + \int_0^t \widehat{u}(z, t')\, dt', \qquad g_{\widehat{u}} = \det\{(g_{\widehat{u}})_{\gamma\delta}\}.$$

From (2.8) we have

$$\Delta_{\widehat{S}}(0) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial z_\gamma} \left(\sqrt{g}\, g^{\gamma\delta} \frac{\partial}{\partial z_\delta} \right),$$

where by (2.1) and (2.3), $g = \det\{g_{\gamma\delta}\} = \det\left\{ \frac{\partial\xi}{\partial z_\gamma} \cdot \frac{\partial\xi}{\partial z_\delta} \right\} = 1 + F_{z_1}^2 + F_{z_2}^2$ and

$$g^{11} = g^{-1}(1 + F_{z_2}^2), \qquad g^{22} = g^{-1}(1 + F_{z_1}^2), \qquad g^{12} = g^{21} = -g^{-1} F_{z_1} F_{z_2}.$$

We assume that the d from (2.2) and (2.4) is so small that

$$(2.9) \qquad\qquad |F_{z_i}| \le 1/4, \qquad i = 1, 2.$$

Now, we introduce the notation and the spaces used in the paper. Let $f = f(x_1, x_2, x_3)$ be a scalar-valued function defined on a domain $\Omega \subset \mathbb{R}^3$. The gradient of f is denoted by ∇f or f_x. We also use the notation $\partial_x f$. By f_{xx} we denote the matrix $\{f_{x_i x_j}\}_{i,j=1,2,3}$ and we set $|f_{xx}| = (\sum_{i,j=1}^3 f_{x_i x_j}^2)^{1/2}$.

Next, let $f = f(x_1, x_2, x_3)$ be a vector-valued function defined on $\Omega \subset \mathbb{R}^3$, i.e. $f : \Omega \to \mathbb{R}^3$. By f_x we denote the matrix $\{f_{i x_j}\}_{i,j=1,2,3}$. Moreover,

we use the notation:

$$f_x^2 = \sum_{i,j=1}^{3} f_{ix_j}^2, \quad |f_{xx}| = \Big(\sum_{i,j,k=1}^{3} f_{ix_j x_k}^2 \Big)^{1/2}.$$

Let now $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary S and let $T > 0$. By $W_2^{k+\alpha, k/2+\alpha/2}(\Omega^T)$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1)$, we denote the Sobolev–Slobodetskiĭ space with the norm

$$\|u\|_{W_2^{k+\alpha, k/2+\alpha/2}(\Omega^T)} = \Big[\sum_{|\beta|+2i \le k} \|\partial_x^\beta \partial_t^i u\|_{L_2(\Omega^T)}^2$$

$$+ \sum_{|\beta|=k} \int_0^T \int_\Omega \int_\Omega \frac{|\partial_x^\beta u(x,t) - \partial_{x'}^\beta u(x',t)|^2}{|x-x'|^{3+2\alpha}} \, dx \, dx' \, dt$$

$$+ \int_\Omega \int_0^T \int_0^T \frac{|\partial_t^{[k/2]} u(x,t) - \partial_{t'}^{[k/2]} u(x,t')|^2}{|t-t'|^{1+\alpha+k-2[k/2]}} \, dx \, dt \, dt' \Big]^{1/2},$$

where $\partial_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3}$, $\beta = (\beta_1, \beta_2, \beta_3)$ is a multi-index, $|\beta| = \beta_1 + \beta_2 + \beta_3$. Similarly we define the norms in $W_2^{k+\alpha}(\Omega)$ and $W_2^{k+\alpha, k/2+\alpha/2}(S^T)$.

Moreover, we shall use the notation:

$$\|u\|_{W_2^{k+\alpha, k/2+\alpha/2}(\Omega^T)} = \|u\|_{k+\alpha, \Omega^T},$$

$$\|u\|_{W_2^{k+\alpha}(Q)} = \|u\|_{k+\alpha, Q}, \quad Q \in \{\Omega, S, S^1\} \ (S^1 \text{ is the unit sphere});$$

$$\|u\|_{L_p(Q)} = |u|_{p,Q}, \quad \text{where } p \in [1, \infty], \ Q \in \{\Omega, S\};$$

$$\|u\|_{L_2(Q)} = \|u\|_{0,Q}, \quad \text{where } q \in \{\Omega, S, \Omega^T, S^T\};$$

$$\|u\|_{\Omega^T}^{(\alpha+2, \alpha/2+1)} = \Big[\|u\|_{2+\alpha, \Omega^T}^2 + T^{-\alpha} \Big(\|u_t\|_{0,\Omega^T}^2 + \sum_{|\beta|=2} \|\partial_x^\beta u\|_{0,\Omega^T}^2 \Big)$$

$$+ \sup_{t \le T} \|u(\cdot, t)\|_{\alpha+1, \Omega}^2 \Big]^{1/2};$$

$$\|u\|_{Q^T}^{(\alpha, \alpha/2)} = (\|u\|_{\alpha, Q^T}^2 + T^{-\alpha} \|u\|_{0,Q^T}^2)^{1/2}, \quad Q \in \{\Omega, S\}.$$

Next, we define the isotropic Besov spaces by introducing the norm (see [3, Sect. 18])

$$(2.10) \quad \|u\|_{B_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^n \Big(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx \, \frac{|\Delta_i^m(h) \partial_{x_i}^k u|^p}{h^{1+(l-k)p}} \Big)^{1/p},$$

where $p \in [1, \infty]$,

$$\Delta_i^m(h) f(x) = \sum_{j=0}^m (-1)^{m-j} c_{jm} f(x + jhe_i),$$

$c_{jm} = \binom{m}{j} = m!/(j!(m-j)!)$, $x \in \mathbb{R}^n$, e_i is the ith unit vector, $i = 1, \ldots, n$, and $m > l - k$, $m, k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{R}_+$, $l \notin \mathbb{Z}$.

It is proved in [4] that for all m, k satisfying $m > l - k$ the norms defined by (2.10) are equivalent.

Now, we define the Sobolev–Slobodetskiĭ spaces by introducing the norm

$$\|u\|_{W_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^{n} \left(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx \frac{|\Delta_i(h)\partial_{x_i}^{[l]} u|^p}{h^{1+p(l-[l])}} \right)^{1/p},$$

where $\Delta_i(h) = \Delta_i^1(h)$, $l \notin \mathbb{Z}$, $[l]$ is the integer part of l.

By the Golovkin theorem [4] the norms of $B_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ are equivalent.

We also define the spaces $\widetilde{B}_p^l(\mathbb{R}^n)$ and $\widetilde{W}_p^l(\mathbb{R}^n)$ by introducing the norms

$$\|u\|_{\widetilde{B}_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|\Delta^m(x-y)\partial_y^k u(y)|^p}{|x-y|^{n+p(l-k)}} \right)^{1/p}$$

and

$$\|u\|_{\widetilde{W}_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|\Delta(x-y)\partial_y^{[l]} u(y)|^p}{|x-y|^{n+p(l-[l])}} \right)^{1/p},$$

where $\Delta^m(x-y)f(y) = \sum_{i=1}^{m} \Delta_i^m(h)f(y)$, $(x-y) \cdot e_i = h$, $\Delta(x-y)f(y) = \Delta^1(x-y)f(y)$ and $m > l - k$, $m, k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{R}_+$, $l \notin \mathbb{Z}$.

By Lemma 7.44 of [1] the norms of $\widetilde{B}_p^l(\mathbb{R}^n)$, $B_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$), $\widetilde{W}_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$) are equivalent.

Now, we recall the following imbedding for the Besov spaces (2.10) (see [3, Sect. 18]):

$$(2.11) \qquad \partial_x^\sigma B_p^l(\mathbb{R}^n) \subset B_q^\varrho(\mathbb{R}^n) \quad \text{for } n/p - n/q + |\sigma| + \varrho \leq l.$$

Moreover, for

$$\varkappa = \frac{1}{l}\left(\frac{n}{p} - \frac{n}{q} + |\sigma| + \varrho \right) > 0 \quad \text{and} \quad \varkappa_1 = \frac{3}{p} - \frac{3}{r}$$

we have the interpolation inequality

$$\|\partial_x^\sigma u\|_{B_q^\varrho(\mathbb{R}^n)} \leq (\varepsilon^{1-\varkappa} + c\varepsilon^{-\varkappa}\varepsilon_1^{1-\varkappa_1})\|u\|_{B_p^l(\mathbb{R}^n)} + c\varepsilon^{-\varkappa}\varepsilon_1^{-\varkappa_1}\|u\|_{L_r(\mathbb{R}^n)}$$

with $r \geq 1$.

In the above notation $B_p^l(\mathbb{R}^n)$ with $l \in \mathbb{Z}_+$ is the Sobolev space.

By $C_B^k(Q)$ ($Q \subset \mathbb{R}^n$ is a domain) we denote the space of functions $u \in C^k(Q)$ such that $D^\sigma u$, $0 \leq |\sigma| \leq k$, is bounded on Q, with the norm $\|u\|_{C_B^k(Q)} = \max_{0 \leq |\sigma| \leq k} \sup_{x \in Q} |D^\sigma u(x)|$.

All the above remarks apply to spaces of functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$, and by using a partition of unity we can also define spaces of traces on the boundary of Ω and formulate the corresponding trace theorems.

3. Inequality for the global existence. In order to derive global estimates we assume the following condition: Ω_t is diffeomorphic to a ball and S_t can be described by

$$|x| = r = R(\omega, t), \quad \omega \in S^1.$$

The main result of the paper is the following theorem.

THEOREM 3.1. *Let $k \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\overline{\vartheta} \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$, $\alpha \in (3/4, 1)$ and assume that (u, ϑ, η) is the local solution of problem (1.6) determined by Theorem 3.4 of [13]. Moreover, let $\rho^* = \sup_{\Omega^T} \eta$, $\rho_* = \inf_{\Omega^T} \eta$, $\theta^* = \sup_{\Omega^T} \vartheta$, $\theta_* = \inf_{\Omega^T} \vartheta$, $0 < \rho_* < \rho^* < \infty$, $0 < \theta_* < \theta^* < \infty$. Then*

$$(3.1) \quad \phi(t, \Omega) + \overline{c}(\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2)$$

$$\leq c(t, Z_1, Z_2)[\|u\|_{0,\Omega^t}^2 + \|\eta_\sigma\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{\alpha,\Omega^t}^2 + \|\overline{\vartheta}\|_{\alpha+1/2,S^t}^2$$

$$+ \varepsilon_1 Z_4 + Z_1 Z_3] + \phi(0, \Omega),$$

where $t \leq T$, T is the time of local existence; $\phi(t, \Omega)$ is defined by (3.86); $c = c(t, Z_1, Z_2)$ is a positive continuous function nondecreasing with respect to its arguments; $\overline{c} \leq 1$ is a positive constant; $\varepsilon_1 \in (0, 1)$ is a sufficiently small constant and

$$Z_1 = \|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2,$$

$$Z_2 = \sup_{0 \leq t' \leq t} \|u\|_{1+\alpha,\Omega}^2 + \sup_{0 \leq t' \leq t} \|\vartheta_\sigma\|_{1+\alpha,\Omega}^2 + \sup_{0 \leq t' \leq t} \|\eta_\sigma\|_{1+\alpha,\Omega}^2,$$

$$Z_3 = [\|u\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} + \|\vartheta_\sigma\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)}]^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2 + \sup_{0 \leq t' \leq t} \|\eta_\sigma\|_{1+\alpha,\Omega}^2,$$

$$Z_4 = t\|H(\cdot, 0) + 2/R_e\|_{\alpha+1/2,S}^2 + \int_0^t \|R(\cdot, t') - R(\cdot, 0)\|_{5/2+\alpha,S^1}^2 \, dt'.$$

Inequality (3.1) is derived in several steps. First, we consider problem (2.6) to obtain an inequality similar to (3.1) in an interior subdomain $\widetilde{\Omega}$. In order to get this inequality one has to estimate separately the norms $\int_0^t \|\widetilde{u}\|_{2+\alpha,\widetilde{\Omega}}^2 \, dt'$, $\int_0^t \|\widetilde{\theta}_\sigma\|_{2+\alpha,\widetilde{\Omega}}^2 \, dt'$, $\int_0^t \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 \, dt'$ and the seminorms $\int_{\widetilde{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$, $\int_{\widetilde{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$.

The resulting inequality (see (3.37)) contains terms similar to (3.1) except for the seminorm $\int_{\widetilde{\Omega}} \|\eta_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$ on the left-hand side.

Next, we consider problem (2.7) in order to derive an inequality in a boundary subdomain $\widehat{\Omega}$. Now, to estimate the norms $\int_0^t \|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}}^2 \, dt'$, $\int_0^t \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2 dt'$ and the seminorms $\int_{\widehat{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)} \, dz$, $\int_{\widehat{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha/2,0,t)}^2 \, dz$ the boundary conditions $(2.7)_4$–$(2.7)_5$ have to be taken into account. First we get estimates of the derivatives of $\widetilde{u}$ and $\widetilde{\vartheta}_\sigma$ in the directions tangent to $\widehat{S}$ and then using these estimates we are able to estimate the derivatives of $\widetilde{u}$ and $\widetilde{\vartheta}_\sigma$ in the normal direction to $\widehat{S}$. Finally, by using equation $(1.6)_2$ we obtain an estimate of $\int_\Omega \|\eta_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$.

It should be underlined that the form of inequality (3.1) is precisely connected with the method of proof of the global existence theorem for problem (1.1), presented in [12]. Therefore, in the long process of obtaining the successive estimates it is essential to control their right-hand sides in order to obtain forms similar to (3.1).

Since the proof of Theorem 3.1 is very long, its consecutive steps will be presented in separate lemmas.

First, we will obtain estimates which hold in interior subdomains $\widetilde{\Omega}_i$ (see Section 2).

Let us consider an interior subdomain $\widetilde{\Omega}$ and let $\zeta \in C_0^\infty(\widetilde{\Omega})$ be the corresponding function from the partition of unity. We can assume that

$$\widetilde{\Omega} = B_{2\lambda}(\xi_0) = \{\zeta \in \mathbb{R}^3 : |\xi - \xi_0| < 2\lambda\}$$

and that $\zeta(\xi) = 1$ in $B_{\lambda+\varepsilon_0}(\xi_0) = \{\xi \in \mathbb{R}^3 : |\xi - \xi_0| < \lambda + \varepsilon_0\}$, for some $-\lambda < \varepsilon_0 < \lambda$, while $\zeta(\xi) = 0$ for $\xi \in \mathbb{R}^3 \setminus B_{2\lambda}(\xi_0)$.

Denote by $\Delta^s(z)f(\xi)$ the sth finite difference of f such that

$$\Delta^s(z)f(\xi) = \sum_{k=0}^s \binom{s}{k}(-1)^{s-k} f(\xi + kz).$$

We start with the following lemma.

LEMMA 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Then for the functions* $\widetilde{u}, \widetilde{\eta}_\sigma, \widetilde{\vartheta}_\sigma$ *defined by* (2.5) *the following estimate holds:*

$$(3.2) \quad \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\widetilde{\eta}_\sigma|^2}{\varrho_e |z|^{3+2(1+\alpha)}} \right.$$

$$\left. + \frac{p_2\eta c_v}{\theta_e p_\vartheta(\varrho_e,\theta_e)} \frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}} \right) + c_1 \int_{\mathbb{R}^3} dz \frac{\|\Delta^2(z)\widetilde{u}\|_{1,B_{2\lambda}(\xi_0)}^2}{|z|^{3+2(1+\alpha)}}$$

$$+ \varkappa \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{p_2|\nabla(\Delta^2(z)\widetilde{\vartheta}_\sigma)|^2}{|z|^{3+2(1+\alpha)}}$$

$$\leq \varepsilon(\|u\|^2_{2+\alpha,\widetilde\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\widetilde\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde\Omega})$$

$$+ \psi_{14}(1/\varepsilon,a)\Big[X_1 X_2 + X_1(1+X_2^2)\int_0^t \|u\|^2_{2+\alpha,\Omega}\,dt' + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde u_t\|^2_{\alpha,\widetilde\Omega}\Big]$$

$$+ \psi_{15}(1/\varepsilon,a)(\|u\|^2_{1+\alpha,\widetilde\Omega} + \|\widetilde\eta_\sigma\|^2_{\alpha,\widetilde\Omega} + \|\widetilde\vartheta_\sigma\|^2_{\alpha,\widetilde\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\widetilde\Omega})$$

$$+ \psi_{16}(1/\varepsilon,a)\|\widetilde k\|^2_{\alpha,\widetilde\Omega},$$

where $t \leq T$, and

$$(3.3) \qquad X_1 = \|u\|^2_{2+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega},$$

$$(3.4) \qquad X_2 = \|u\|^2_{1+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega}.$$

Proof. Similarly to [5] we apply $\Delta^2(z)$ to $(2.6)_1$–$(2.6)_3$ and multiply the resulting equations by $\Delta^2(z)\widetilde u_i$, $\frac{p_1}{\varrho_e}\Delta^2(z)\widetilde\eta_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\Delta^2(z)\widetilde\vartheta_\sigma$, respectively. Then integrating each of the equations over $B_{2\lambda}(\xi_0)$ and with respect to z over $\mathbb{R}^3$ with the weight $1/|z|^{3+2(\alpha+1)}$ and using (1.5), $(1.6)_2$ and the Korn inequality we obtain

$$(3.5) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta|\Delta^2(z)\widetilde u|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\widetilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}\right.$$

$$\left. + \frac{p_2\eta c_v}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\widetilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right)$$

$$+ c_1\int_{\mathbb{R}^3} dz \frac{\|\Delta^2(z)\widetilde u\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}} + \varkappa \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{p_2|\Delta^2(z)\nabla\widetilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}$$

$$\leq I_1 + \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{p_2\Delta^2(z)k_3\Delta^2(z)\widetilde\vartheta_\sigma}{|z|^{3+2(1+\alpha)}},$$

where $c_1 > 0$ is a constant and

$$(3.6) \quad I_1 \leq \varepsilon\left(\|\widetilde u\|^2_{2+\alpha,\widetilde\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde\Omega} + \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\widetilde\eta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right)$$

$$+ \psi_1(1/\varepsilon,a)(\|\widetilde u\|^2_{1+\alpha,\widetilde\Omega} + \|\eta_\sigma\|^2_{\alpha,\widetilde\Omega} + \|\vartheta_\sigma\|^2_{\alpha,\widetilde\Omega})$$

$$+ \psi_2(1/\varepsilon,a)\Big[\int_0^t \|u\|^2_{2+\alpha,\Omega}\,dt'\,(\|\widetilde u\|^2_{2+\alpha,\widetilde\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\widetilde\Omega}$$

$$+ \|\vartheta_\sigma\|^2_{1+\alpha,\widetilde\Omega} + \|\widetilde\eta_\sigma\|^2_{1+\alpha,\widetilde\Omega}\|\widetilde u\|^2_{2+\alpha,\widetilde\Omega} + \|\widetilde\vartheta_\sigma\|^2_{1+\alpha,\widetilde\Omega}\|\widetilde u\|^2_{2+\alpha,\widetilde\Omega})$$

$$+ \|u\|^2_{2+\alpha,\Omega}\|\widetilde u\|^2_{2+\alpha,\widetilde\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde u_t\|^2_{\alpha,\widetilde\Omega}$$

$$+ (\|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + \|\widetilde{\vartheta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})(\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega})$$

$$+ (\|\vartheta_{\sigma t}\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega})\|\widetilde{\vartheta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde{k}\|^2_{\alpha,\widetilde{\Omega}} \Big],$$

where $a = t^{1/2}(\int_0^t \|u\|^2_{2+\alpha,\Omega}\, dt')^{1/2}$, and ψ_i $(i = 1, 2)$ and all functions ψ_i occurring below are positive continuous nondecreasing functions of their arguments.

Next, we obtain

$$(3.7) \quad \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{p_2\Delta^2(z)k_3\Delta^2(z)\widetilde{\vartheta}_\sigma}{|z|^{3+2(1+\alpha)}}$$

$$= - \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\Delta(z)k_3\Delta(z)p_2\Delta^2(z)\widetilde{\vartheta}_\sigma}{|z|^{3+2(1+\alpha)}}$$

$$- \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{p_2\Delta(z)k_3\Delta^3(z)\widetilde{\vartheta}_\sigma}{|z|^{3+2(1+\alpha)}}$$

$$\leq \varepsilon \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta^3(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(2+\alpha)}} + c(\varepsilon) \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta(z)k_3|^2}{|z|^{3+2\alpha}}$$

$$+ c\Bigg[\int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta(z)\eta_\sigma|^{p_1}}{|z|^{3+\frac{p_1}{2}(1+\alpha)}} + \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta(z)\vartheta_\sigma|^{p_1}}{|z|^{3+\frac{p_1}{2}(1+\alpha)}} \Bigg]^{1/p_1}$$

$$\cdot \Bigg(\int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta(z)k_3|^{p_2}}{|z|^{3+\frac{p_2}{2}(1+\alpha)}} \Bigg)^{1/p_2}$$

$$\cdot \Bigg(\int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^{p_3}}{|z|^{3+p_3(1+\alpha)}} \Bigg)^{1/p_3} \leq \varepsilon\|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}} + I_2 + I_3$$

with

$$(3.8) \quad |I_2| \leq c \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{1}{|z|^{3+2\alpha}} (|\Delta(z)\xi_x^2|^2|u_\xi^2|^2 + |\Delta(z)u\cdot\nabla_u\zeta|^2$$

$$+ |\Delta(z)(\theta_e p_\vartheta(\varrho_e, \theta_e) - \vartheta p_\vartheta(\eta, \vartheta))\nabla_u\cdot u\zeta|^2$$

$$+ |\Delta(z)(\nabla\cdot\widetilde{u} - \nabla_u\cdot\widetilde{u})|^2$$

$$+ |\Delta(z)(\nabla_u^2\zeta\vartheta_\sigma + 2\nabla_u\zeta\cdot\nabla_u\vartheta_\sigma)|^2$$

$$+ |\Delta(z)(\nabla^2\widetilde{\vartheta}_\sigma - \nabla_u^2\widetilde{\vartheta}_\sigma)|^2) \equiv \sum_{i=1}^{6} I_2^i$$

and

$$(3.9) \qquad |I_3| \leq \varepsilon \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}} + c(\varepsilon)[\|k_3\|^2_{\alpha,\widetilde{\Omega}}(\|\eta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|\vartheta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})],$$

where $1/p_1 + 1/p_2 + 1/p_3 = 1$ and we have used the imbeddings $B_2^{1+\alpha}(\Omega) \subset B_{p_1}^{(1+\alpha)/2}(\Omega)$, $B_2^\alpha(\Omega) \subset B_{p_2}^{(1+\alpha)/2}(\Omega)$, $B_2^{2+\alpha}(\Omega) \subset B_{p_3}^{1+\alpha}(\Omega)$ valid for $3/2 - 3/p_1 + 1 + \alpha/2 \leq 1 + \alpha$, $3/2 - 3/p_2 + 1 + \alpha/2 \leq \alpha$, $3/2 - 3/p_3 + 1 + \alpha \leq 2 + \alpha$, which all hold for suitable p_i if $\alpha > 1/2$.

Now, we estimate the terms on the right-hand side of (3.8). We have

$$(3.10) \quad I_2^1 \leq c \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)u_\xi|^2|u_\xi|^2}{|z|^{3+2\alpha}}$$

$$+ c \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)\int_0^t u_\xi \, dt'|^2|\psi_3(b)u_\xi^2|^2}{|z|^{3+2\alpha}}$$

$$+ c \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\int_0^t u_\xi \, dt'|^2|\Delta(z)\psi_3(b)u_\xi^2|^2}{|z|^{3+2\alpha}}$$

$$+ c \int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)\int_0^t u_\xi \, dt'|^2|\psi_3(b)u_\xi^2|^2}{|z|^{3+2\alpha}}$$

$$\leq c|u_\xi|^2_{\infty,\Omega} \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)u_\xi|^2}{|z|^{3+2\alpha}}$$

$$+ c\psi_4(a)|u_\xi|^2_{\infty,\Omega} \left[\left(\int\limits_{B_{2\lambda}(0)} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|u_\xi|^{2p}}{|z|^{3-\delta p}} \right)^{1/(2p)} \right]^2$$

$$\times \left[\left(\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)\int_0^t u_\xi \, dt'|^{2p'}}{|z|^{3+2p'\alpha+\delta p'}} \right)^{1/(2p')} \right]^2$$

$$+ c\psi_5(a)\left| \int_0^t u_\xi(\xi,t') \, dt' \right|^2_{\infty,\Omega} |u_\xi|^2_{\infty,\Omega} \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \, \frac{|\Delta(z)u_\xi|^2}{|z|^{3+2\alpha}},$$

where $\delta > 0$ is sufficiently small and $1/p + 1/p' = 1$, $b = \int_0^t u_\xi \, dt'$ and we have used the fact that $\xi_x = I + b\psi_3(b)$, I is the unit matrix and $\psi_3(0) \neq 0$, ψ_3 is a positive continuous increasing function.

Using the imbeddings $B_2^\alpha(\Omega) \subset L_{2p}(\Omega)$ and $B_2^{1+\alpha}(\Omega) \subset B_{2p'}^{\alpha+\delta/2}(\Omega)$ which hold simultaneously for $\alpha > 1/2$ and some δ, p, p' such that $1/p + 1/p' = 1$, $3/2 - 3/(2p) \leq \alpha$ and $3/2 - 3/(2p') + \alpha + \delta/2 \leq 1 + \alpha$ (see imbedding (2.11)) we get

(3.11) $I_2^1 \le c\|u\|_{2+\alpha,\Omega}^2\|u\|_{1+\alpha,\Omega}^2 + \psi_6(a)\|u\|_{2+\alpha,\Omega}^2\|u\|_{1+\alpha,\Omega}^2\int\limits_0^t\|u\|_{2+\alpha,\Omega}^2\,dt'.$

In the same way we obtain

(3.12) $I_2^2 + I_2^5 \le c(\|u\|_{\alpha,\Omega}^2 + \|\vartheta_\sigma\|_{1+\alpha,\Omega}^2)$

$$+ \psi_7(a)\int\limits_0^t\|u\|_{2+\alpha,\Omega}^2\,dt'\,(\|u\|_{1+\alpha,\Omega}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega}^2).$$

Next, we estimate

(3.13) $I_2^6 \le c\int\limits_{B_{2\lambda}(0)}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\Delta(z)\int_0^t u_\xi\,dt'|^2|\psi_3(b)\widetilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}$

$$+ c\int\limits_{B_{2\lambda}(0)}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\int_0^t u_\xi\,dt'|^2|\Delta(z)\psi_3(b)|^2|\widetilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}$$

$$+ c\int\limits_{B_{2\lambda}(0)}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\int_0^t u_\xi\,dt'|^2|\psi_3(b)\Delta\widetilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}} = I_4^1 + I_4^2 + I_4^3,$$

where

$$I_4^1 \le \psi_8(a)\Bigg[\Bigg(\int\limits_{B_{2\lambda}(0)}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\widetilde\vartheta_{\sigma\xi\xi}|^{2p}}{|z|^{3-\delta p}}\Bigg)^{1/(2p)}\Bigg]^2$$

$$\times\Bigg[\Bigg(\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\Delta(z)\int_0^t u_\xi\,dt'|^{2p'}}{|z|^{3+2p'\alpha+\delta p'}}\Bigg)^{1/(2p')}\Bigg]^2,$$

$\delta > 0$ is sufficiently small and $1/p + 1/p' = 1$. Hence, we get

(3.15) $I_4^1 \le \psi_9(a)\|\widetilde\vartheta_\sigma\|_{2+\alpha,\widetilde\Omega}^2\int\limits_0^t\|u\|_{2+\alpha,\Omega}^2\,dt'.$

I_4^2 is estimated in the same way, and

(3.16) $I_4^3 \le \psi_{10}(a)\Big|\int\limits_0^t u_\xi\,dt'\Big|_{\infty,\Omega}^2\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\dfrac{|\Delta(z)\widetilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}$

$$\le \psi_{11}(a)\int\limits_0^t\|u\|_{2+\alpha,\Omega}^2\,dt'\,\|\widetilde\vartheta_\sigma\|_{2+\alpha,\widetilde\Omega}^2.$$

Hence

(3.17) $I_2^6 \le \psi_{12}(a)\int\limits_0^t\|u\|_{2+\alpha,\Omega}^2\,dt'\,\|\widetilde\vartheta_\sigma\|_{2+\alpha,\widetilde\Omega}^2.$

Finally,

$$(3.18) \qquad I_2^3 \le \psi_{13}(a)(\|\vartheta_\sigma\|_{\alpha,\widetilde\Omega}^2 + \|\eta_\sigma\|_{\alpha,\widetilde\Omega}^2 + 1)$$

$$\cdot (\|\eta_\sigma\|_{1+\alpha,\widetilde\Omega}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde\Omega}^2)\|u\|_{2+\alpha,\widetilde\Omega}^2 \int_0^t \|u\|_{2+\alpha,\Omega}^2 \, dt'.$$

From estimates (3.5)–(3.18) we obtain (3.2). ∎

Our aim is to estimate the sum $\|\widetilde u\|_{2+\alpha,\widetilde\Omega}^2 + \|\widetilde\vartheta_\sigma\|_{2+\alpha,\widetilde\Omega}^2 + \|\widetilde\eta_\sigma\|_{1+\alpha,\widetilde\Omega}^2$ but we see that on the left-hand side of inequality (3.2) only the seminorms

$$\int_{\mathbb{R}^3} dz \, \frac{\|\Delta^2(z)\widetilde u\|_{1,B_{2\lambda}(\xi_0)}^2}{|z|^{3+2(1+\alpha)}} \quad \text{and} \quad \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\zeta_0)} d\xi \, \frac{p_2|\nabla(\Delta^2(z)\widetilde\vartheta_\sigma)|^2}{|z|^{3+2(1+\alpha)}}$$

occur. Therefore we have to complete (3.2) by deriving an estimate for $\|\widetilde\eta_\sigma\|_{1+\alpha,\widetilde\Omega}^2 + \|\widetilde u\|_{1,\widetilde\Omega}^2 + \|\nabla\widetilde\vartheta_\sigma\|_{0,\widetilde\Omega}^2$.

Thus, we obtain

LEMMA 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied. Then*

$$(3.19) \qquad \frac{1}{2}\frac{d}{dt}\int_{\widetilde\Omega}\left(\eta\widetilde u^2 + \frac{p_1}{\varrho_e}\widetilde\eta_\sigma^2 + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\widetilde\vartheta_\sigma^2\right)d\xi$$

$$+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta|\Delta^2(z)\widetilde u|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\widetilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}\right.$$

$$\left. + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\widetilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right)$$

$$+ c_2(\|\widetilde u\|_{2+\alpha,\widetilde\Omega}^2 + \|\widetilde\vartheta_\sigma\|_{2+\alpha,\widetilde\Omega}^2 + \|\widetilde\eta_\sigma\|_{1+\alpha,\widetilde\Omega}^2)$$

$$\le \varepsilon\|\widetilde u_t\|_{\alpha,\widetilde\Omega}^2 + \varepsilon(\|u\|_{2+\alpha,\widetilde\Omega}^2 + \|\eta_\sigma\|_{1+\alpha,\widetilde\Omega}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde\Omega}^2)$$

$$+ c\|\widetilde k\|_{\alpha,\widetilde\Omega}^2 + c(\varepsilon)(\|u\|_{0,\widetilde\Omega}^2 + \|\widetilde\eta_\sigma\|_{0,\widetilde\Omega}^2 + \|\widetilde\vartheta_\sigma\|_{0,\widetilde\Omega}^2)$$

$$+ cX_1\left[X_2(1 + X_2) + (1 + X_2^2)\int_0^t \|u\|_{2+\alpha,\Omega}^2 \, dt'\right] + c\|\eta_\sigma\|_{1+\alpha,\Omega}^2\|\widetilde u_t\|_{\alpha,\widetilde\Omega}^2,$$

where $c_2 > 0$ *is a constant.*

Proof. Multiply equations $(2.6)_1$ by $\widetilde u_i$ $(i = 1, 2, 3)$ and equations $(2.6)_2$–$(2.6)_3$ by $\frac{p_1}{\varrho_e}\widetilde\eta_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\widetilde\vartheta_\sigma$, respectively. Then integrating the resulting equations over $\widetilde\Omega$, using the boundary condition $\widetilde u|_{\partial\widetilde\Omega} = 0$, the equation of continuity, the Korn inequality and relation (1.5), and next adding these

inequalities we get

$$(3.20) \quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}} \left(\eta\widetilde{u}^2 + \frac{p_1}{\varrho_e}\widetilde{\eta}_\sigma^2 + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\widetilde{\vartheta}_\sigma^2 \right) d\xi$$

$$+ c_3(\|\widetilde{u}\|_{1,\widetilde{\Omega}}^2 + \|\nabla\widetilde{\vartheta}_\sigma\|_{0,\widetilde{\Omega}}^2)$$

$$\leq \varepsilon(\|\widetilde{u}\|_{1,\widetilde{\Omega}}^2 + \|\eta_\sigma\|_{0,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{0,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{1,\widetilde{\Omega}}^2)$$

$$+ \psi_{17}(1/\varepsilon,a)\Big\{(\|\widetilde{u}\|_{2,\Omega}^2 + \|\eta_\sigma\|_{1,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{2,\widetilde{\Omega}}^2)\int\limits_0^t \|u\|_{2,\Omega}^2\,dt'$$

$$+ (\|\widetilde{\vartheta}_\sigma\|_{2,\widetilde{\Omega}}^2 + 1)\|\widetilde{u}\|_{1,\Omega}^2\int\limits_0^t \|u\|_{2+\alpha,\Omega}^2\,dt'$$

$$+ \|\eta_\sigma\|_{1,\widetilde{\Omega}}^2[\|\vartheta_{\sigma t}\|_{1,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{2,\widetilde{\Omega}}^2 + \|u\|_{2,\widetilde{\Omega}}^2(1 + \|\widetilde{\eta}_\sigma\|_{1,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{1,\widetilde{\Omega}}^2)]$$

$$+ \|\vartheta_\sigma\|_{1,\widetilde{\Omega}}^2(\|u\|_{2,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{2,\widetilde{\Omega}}^2 + \|\vartheta_{\sigma t}\|_{1,\widetilde{\Omega}}^2)$$

$$+ \|u\|_{1,\widetilde{\Omega}}^2\|\widetilde{u}\|_{1,\widetilde{\Omega}}^2 + \|u\|_{1,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{0,\widetilde{\Omega}}^2 + \|\widetilde{k}\|_{0,\widetilde{\Omega}}^2\Big\},$$

where $c_3 > 0$ is a constant.

Now, using the equivalence of norms for fractional spaces (see [4]) and estimates (3.2) and (3.20) we obtain

$$(3.21) \quad \frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}} \left(\eta\widetilde{u}^2 + \frac{p_1}{\varrho_e}\widetilde{\eta}_\sigma^2 + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\widetilde{\vartheta}_\sigma^2 \right) d\xi$$

$$+ \frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\widetilde{\eta}_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}} \right.$$

$$\left. + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}} \right) + c_4(\|\widetilde{u}\|_{2+\alpha,\widetilde{\Omega}}^2 + \|\nabla\widetilde{\vartheta}_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2)$$

$$\leq \varepsilon(\|u\|_{2+\alpha,\widetilde{\Omega}}^2 + \|\eta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{2+\alpha,\widetilde{\Omega}}^2) + c\|\widetilde{k}\|_{\alpha,\widetilde{\Omega}}^2$$

$$+ c(\|u\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\eta}_\sigma\|_{\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{\alpha,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\eta_\sigma\|_{0,\widetilde{\Omega}}^2)$$

$$+ cX_1\Big[X_2(1 + X_2) + (1 + X_2^2)\int\limits_0^t \|u\|_{2+\alpha,\Omega}^2\,dt'\Big]$$

$$+ c\|\eta_\sigma\|_{1+\alpha,\Omega}^2\|u_t\|_{\alpha,\widetilde{\Omega}}^2,$$

where X_1 and X_2 are given by (3.3) and (3.4) respectively, and $c_4 > 0$ is a constant.

To examine the third term on the r.h.s. of (3.21) we apply the interpolation inequality

$$(3.22) \quad \|u\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\eta}_\sigma\|_{\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{\alpha,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2$$
$$\leq \varepsilon(\|u\|_{2+\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2)$$
$$+ c(\varepsilon)(\|u\|_{0,\widetilde{\Omega}}^2 + \|\widetilde{\eta}_\sigma\|_{0,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{0,\widetilde{\Omega}}^2).$$

From $(2.6)_1$ we have

$$(3.23) \quad \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 \leq \varepsilon(\|u\|_{2+\alpha,\widetilde{\Omega}}^2 + \|\eta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2)$$
$$+ c(\|\widetilde{u}\|_{2+\alpha,\widetilde{\Omega}}^2 + \|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}^2 + \|u\|_{0,\widetilde{\Omega}}^2 + \|\eta_\sigma\|_{0,\widetilde{\Omega}}^2 + \|\vartheta_\sigma\|_{0,\widetilde{\Omega}}^2)$$
$$+ c\int_0^t \|u\|_{2+\alpha,\Omega}^2 \, dt' \, (X_1 + X_2^2) + cX_2^2.$$

Inequalities (3.21)–(3.23) imply (3.19). ∎

Let us notice that all the terms on the right-hand side of (3.19) different from $\|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}$ contain only spatial derivatives of $\widetilde{u}$, $\widetilde{\eta}_\sigma$, $\widetilde{\vartheta}_\sigma$, u, η_σ and ϑ_σ. In what follows we will integrate (3.19) with respect to time over $(0,t)$. Therefore, we have to consider $\int_0^t \|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}^2 \, dt'$. To do this we use the following interpolation inequality (see [8]):

$$(3.24) \quad \int_0^t \|\widetilde{u}_{t'}\|_{\alpha,\widetilde{\Omega}}^2 \, dt \leq \varepsilon \int_0^t \|\widetilde{u}\|_{2+\alpha,\widetilde{\Omega}}^2 \, dt + c(\varepsilon) \int_{\widetilde{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)}^2 \, d\xi,$$

where $c(\varepsilon)$ does not depend on t.

We see that in order to obtain an inequality of the form (3.1) for an interior subdomain $\widetilde{\Omega}$ we should estimate the second term on the right-hand side of (3.24). To do this we consider the time differences

$$\Delta_t^k(h)f(\xi,t) = \sum_{j=0}^{k} c_k^j (-1)^{k-j} f(\xi, t+jh), \quad k > \alpha.$$

The estimate for $\int_{\widetilde{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$ is established in the following lemma.

LEMMA 3.4. *Let the assumptions of Theorem* 3.1 *be satisfied. Then*

$$(3.25) \quad c_5 \int_{\widetilde{\Omega}} \|u_{t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$$
$$+ \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde{\Omega}} \left(\frac{\mu}{4} |\mathbb{S}(\Delta_{t'}\widetilde{u})|^2 + \frac{\nu-\mu}{2} |\operatorname{div}\Delta_{t'}\widetilde{u}|^2 \right) d\xi \bigg|_{t'=t-h_0}$$

$$\leq \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t}) + d\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t}$$

$$+ c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t}$$

$$+ \|u\|^2_{2+\alpha,\Omega^t} \sup_{0\leq t'\leq t} \|u\|^2_{1+\alpha,\Omega}$$

$$+ \|u\|^2_{2+\alpha,\widetilde{\Omega}^t}(\|u\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t})(1 + \|u\|^2_{1+\alpha,\widetilde{\Omega}^t})$$

$$+ \sup_{0\leq t'\leq t} \|u\|^2_{1+\alpha,\widetilde{\Omega}}(\|\vartheta_\sigma\|^2_{\alpha,\widetilde{\Omega}^t} + \|\eta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t})$$

$$+ \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t}(\|\widetilde{\vartheta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t})],$$

where $t \leq T$, $c_5 > 0$ *is a constant, and* $c(\varepsilon,t)$ *is a positive continuous function increasing with respect to* t.

Proof. Applying $\Delta_t(h)$ to $(2.6)_1$ we obtain

$$(3.26) \qquad \eta\Delta_t\widetilde{u}_{it} - \nabla_j T_{ij}(\Delta_t\widetilde{u}, \Delta_t\widetilde{p}_\sigma) = \eta\Delta_t\widetilde{u}_{it} - \Delta_t(\eta\widetilde{u}_{it}) + \Delta_t k_{1i}.$$

In the same way as in [5] from (3.26) we get

$$(3.27) \qquad \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t')\,dt' \int_{\widetilde{\Omega}} \eta|\Delta_{t'}\widetilde{u}_{t'}|^2 \, d\xi$$

$$+ \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde{\Omega}} \left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\widetilde{u})|^2 + \frac{\nu-\mu}{2}|\operatorname{div}\Delta_{t'}\widetilde{u}|^2 \right) d\xi \bigg|_{t'=t-h_0}$$

$$- \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{t_0}^{2t_0} \psi'(t')\,dt' \int_{\widetilde{\Omega}} \left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\widetilde{u})|^2 + \frac{\nu-\mu}{2}|\operatorname{div}\Delta_{t'}\widetilde{u}|^2 \right) d\xi$$

$$= \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t')\,dt' \int_{\widetilde{\Omega}} [\eta\Delta_{t'}\widetilde{u}_{t'} - \Delta_{t'}(\eta\widetilde{u}_{t'})] \cdot \Delta_{t'}\widetilde{u}_{t'} \, d\xi$$

$$+ \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t')\,dt' \int_{\widetilde{\Omega}} \Delta_{t'}k_1 \cdot \Delta_{t'}\widetilde{u}_{t'} \, d\xi$$

$$- \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t')\,dt' \int_{\widetilde{\Omega}} \nabla(\Delta_{t'}\widetilde{p}_\sigma) \cdot \Delta_t\widetilde{u}_t \, d\xi \equiv I_5 + I_6 + I_7,$$

where $\mathbb{S}(v) = \{v_{i\xi_j} + v_{j\xi_i}\}_{i,j=1,2,3}$, $0 < h_0 < t$; $\psi(t')$ *is a smooth function vanishing for* $t' \leq t_0$ *and equal to 1 for* $t' \geq 2t_0$.

First, we estimate I_5. Using the continuity equation $(1.6)_2$ we have

$$(3.28) \qquad |I_5| \leq \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} |\Delta_{t'}(\eta_\sigma)\widetilde{u}_{t'}| \, |\Delta_{t'}\widetilde{u}_{t'}| \, d\xi$$

$$\leq \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + c(\varepsilon) \int\limits_0^{h_0} dh \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \, \frac{|\int_{t'}^{t'+h} u_\xi \, d\tau|^2 |\widetilde{u}_{t'}|^2}{h^{1+\alpha}}$$

$$\leq \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + c(\varepsilon) t^{1-\alpha} \int\limits_0^t |u_\xi|^2_{\infty,\widetilde{\Omega}} \, dt' \int\limits_0^t |\widetilde{u}_{t'}|^2_{2,\widetilde{\Omega}} \, dt'$$

$$\leq \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + c(\varepsilon) t^{1-\alpha} \|u\|^4_{2+\alpha,\widetilde{\Omega}^t}.$$

Next, we get

$$(3.29) \qquad |I_6| \leq \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \, (|\Delta_{t'}u|^2 + |\Delta_{t'}\eta_\sigma|^2$$

$$+ |\Delta_{t'}\vartheta_\sigma|^2 + |\Delta_t u_\xi|^2)$$

$$+ c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \left| \int\limits_{t'}^{t'+h} u_\xi \, d\tau \right|^2 (|u|^2 + |\eta_\sigma|^2 + |\vartheta_\sigma|^2$$

$$+ |u_\xi|^2 + |\widetilde{u}_{\xi\xi}|^2 + |\widetilde{\eta}_{\sigma\xi}|^2 + |\widetilde{\vartheta}_{\sigma\xi}|^2)$$

$$+ c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \left| \int\limits_0^{t'} u_\xi \, d\tau \right|^2$$

$$\cdot (|\Delta_t \widetilde{u}_{\xi\xi}|^2 + |\Delta_t \widetilde{\eta}_{\sigma\xi}|^2 + |\Delta_t \widetilde{\vartheta}_{\sigma\xi}|^2)$$

$$+ c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \left| \int\limits_{t'}^{t'+h} u_\xi \, d\tau \right|^2 \left| \int\limits_0^t u_{\xi\xi} \, d\tau \right|^2$$

$$\cdot (|\widetilde{u}_\xi|^2 + |\widetilde{\eta}_\sigma|^2 + |\widetilde{\vartheta}_\sigma|^2)$$

$$+ c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \left| \int\limits_{t'}^{t'+h} u_{\xi\xi} \, d\tau \right|^2 (|\widetilde{u}_\xi|^2 + |\widetilde{\eta}_\sigma|^2 + |\widetilde{\vartheta}_\sigma|^2)$$

$$+ c(\varepsilon) \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} dt' \int\limits_{\widetilde{\Omega}} d\xi \left| \int\limits_0^{t'} u_{\xi\xi} \, d\tau \right|^2$$

$$\cdot (|\Delta_t \widetilde{u}_\xi|^2 + |\Delta_t \widetilde{\eta}_\sigma|^2 + |\Delta_t \widetilde{\vartheta}_\sigma|^2)$$

$$\equiv \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \sum_{i=1}^{6} I_6^i.$$

Let us estimate the terms on the r.h.s. of (3.29). We obtain

$$(3.30) \quad I_6^1 \leq c \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} dt' \int_{\widetilde{\Omega}} d\xi \left(\left| \int_{t'}^{t'+h} u_\tau \, d\tau \right|^2 + \left| \int_{t'}^{t'+h} \vartheta_{\sigma\tau} \, d\tau \right|^2 \right)$$

$$\leq ct^{1-\alpha} (\|u_{t'}\|_{0,\widetilde{\Omega}^t}^2 + \|\vartheta_{\sigma t'}\|_{0,\widetilde{\Omega}^t}^2).$$

From $(1.6)_1$ and $(1.6)_3$ we get respectively

$$(3.31) \quad \|u_{t'}\|_{0,\Omega}^2 \leq \varepsilon_1^{1-\varkappa} (\|u\|_{2+\alpha,\Omega}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega}^2 + \|\vartheta_\sigma\|_{1+\alpha,\Omega}^2)$$

$$+ c\varepsilon_1^{-\varkappa} (\|u\|_{0,\Omega}^2 + \|\eta_\sigma\|_{0,\Omega}^2 + \|\vartheta_\sigma\|_{0,\Omega}^2)$$

$$+ c\|u\|_{2+\alpha,\Omega^t}^2 \|u\|_{1+\alpha,\Omega}^2$$

and

$$(3.32) \quad \|\vartheta_{\sigma t'}\|_{0,\Omega}^2 \leq \varepsilon_1^{1-\varkappa} (\|u\|_{1+\alpha,\Omega}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega}^2)$$

$$+ c\varepsilon_1^{-\varkappa} (\|u\|_{0,\Omega}^2 + \|\vartheta_\sigma\|_{0,\Omega}^2)$$

$$+ c(\|k\|_{0,\Omega}^2 + \|u\|_{2+\alpha,\Omega}^2 \|u\|_{1,\Omega}^2).$$

Using (3.31)–(3.32) in (3.30) and assuming $\varepsilon_1 = (\varepsilon/(ct^{1-\alpha}))^{1/(1-\varkappa)}$ yields

$$I_6^1 \leq \varepsilon (\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2)$$

$$+ c(\varepsilon) t^{(1-\alpha)/(1-\varkappa)} (\|u\|_{0,\Omega^t}^2 + \|\eta_\sigma\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{0,\Omega^t}^2$$

$$+ \|u\|_{2+\alpha,\Omega^t}^2 \sup_{0 \leq t' \leq t} \|u\|_{1+\alpha,\Omega}^2),$$

where $\varepsilon \in (0,1)$ is sufficiently small.

Next, we have

$$I_6^2 \leq ct^{1-\alpha} \int_0^t |u|_{\infty,\widetilde{\Omega}}^2 \, dt' \left(\int_0^t |u|_{2,\widetilde{\Omega}}^2 \, dt' + \int_0^t |\eta_\sigma|_{2,\widetilde{\Omega}}^2 \, dt' + \int_0^t |\vartheta_\sigma|_{2,\widetilde{\Omega}}^2 \, dt' \right.$$

$$+ \int_0^t |u_\xi|_{2,\widetilde{\Omega}}^2 \, dt' + \int_0^t |\widetilde{u}_{\xi\xi}|_{2,\widetilde{\Omega}}^2 \, dt' + \int_0^t |\widetilde{\eta}_{\sigma\xi}|_{2,\widetilde{\Omega}}^2 \, dt' + \int_0^t |\widetilde{\vartheta}_{\sigma\xi}|_{2,\widetilde{\Omega}}^2 \, dt' \right)$$

$$\leq ct^{1-\alpha} \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 (\|u\|_{2,\widetilde{\Omega}^t}^2 + \|\eta_\sigma\|_{1,\widetilde{\Omega}^t}^2 + \|\vartheta_\sigma\|_{1,\widetilde{\Omega}^t}^2).$$

I_6^3 is estimated in the following way:

$$I_6^3 \leq ct \int_0^t |u_\xi|_{\infty,\widetilde{\Omega}}^2 \, dt' \int_0^{h_0} \frac{dh}{h^{1+\alpha}}$$

$$\cdot \int_0^{t-h_0} dt' (|\Delta_t \widetilde{u}_{\xi\xi}|_{2,\widetilde{\Omega}}^2 + |\Delta_t \widetilde{\eta}_{\sigma\xi}|_{2,\widetilde{\Omega}}^2 + |\Delta_t \widetilde{\vartheta}_{\sigma\xi}|_{2,\widetilde{\Omega}}^2)$$

$$\leq ct \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 (\|\widetilde{u}\|_{2+\alpha,\Omega^t}^2 + \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widetilde{\Omega}^t}^2 + \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\Omega^t}^2).$$

Next, we get

$$
I_6^4 \leq c t^{2-\alpha} \int_0^t |u_\xi|^2_{\infty,\widetilde{\Omega}}\, dt' \int_0^t |u_{\xi\xi}|^2_{2,\widetilde{\Omega}}\, dt'
$$

$$
\cdot \left(\int_0^t |\widetilde{u}_\xi|^2_{\infty,\widetilde{\Omega}}\, dt' + \int_0^t |\widetilde{\eta}_\sigma|^2_{\infty,\widetilde{\Omega}}\, dt' + \int_0^t |\widetilde{\vartheta}_\sigma|^2_{\infty,\widetilde{\Omega}}\, dt' \right)
$$

$$
\leq c t^{2-\alpha} \|u\|^4_{2+\alpha,\widetilde{\Omega}^t} (\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t}).
$$

Estimating I_6^5 in the same way we obtain

$$
I_6^5 \leq c t^{1-\alpha} \|u\|^2_{2+\alpha,\widetilde{\Omega}^t} (\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t}).
$$

Finally, we have

$$
I_6^6 \leq c t \int_0^t |u_{\xi\xi}|^2_{4,\widetilde{\Omega}}\, dt' \int_0^{h_0} \frac{dh}{h^{1+\alpha}}
$$

$$
\cdot \int_0^{t-h_0} dt'\, (|\Delta_t \widetilde{u}_\xi|^2_{4,\widetilde{\Omega}} + |\Delta_t \widetilde{\eta}_\sigma|^2_{4,\widetilde{\Omega}} + |\Delta_t \widetilde{\vartheta}_\sigma|^2_{4,\widetilde{\Omega}})
$$

$$
\leq c t \|u\|^2_{2+\alpha,\widetilde{\Omega}^t} \int_0^{h_0} \frac{dh}{h^{1+\alpha}}
$$

$$
\cdot \int_0^{t-h_0} dt\, [\varepsilon_1(|\Delta_t \widetilde{u}_{\xi\xi}|^2_{2,\widetilde{\Omega}} + |\Delta_t \widetilde{\eta}_{\sigma\xi}|^2_{2,\widetilde{\Omega}} + |\Delta_t \widetilde{\vartheta}_{\sigma\xi}|^2_{2,\widetilde{\Omega}})
$$

$$
+ c(\varepsilon_1)(|\Delta_t \widetilde{u}|^2_{2,\widetilde{\Omega}} + |\Delta_t \widetilde{\eta}_\sigma|^2_{2,\widetilde{\Omega}} + |\Delta_t \widetilde{\vartheta}_\sigma|^2_{2,\widetilde{\Omega}})]
$$

$$
\leq c t \|u\|^2_{2+\alpha,\widetilde{\Omega}^t} (\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t}).
$$

Summarizing the above considerations we get

$$
\begin{aligned}
(3.33) \quad |I_6| \leq\ & \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t}) \\
& + c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t} \\
& + \|u\|^2_{2+\alpha,\Omega^t} \sup_{0 \leq t' \leq t} \|u\|^2_{1+\alpha,\Omega} + \|u\|^2_{2+\alpha,\widetilde{\Omega}^t}(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} \\
& + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t})(1 + \|u\|^2_{1+\alpha,\widetilde{\Omega}^t})],
\end{aligned}
$$

where $c(\varepsilon,t)$ is a positive continuous function increasing with respect to t.

In the same way we estimate I_7 and we get

$$
\begin{aligned}
(3.34) \quad |I_7| \leq\ & \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}) + d\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t} \\
& + c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t}
\end{aligned}
$$

$$+ \sup_{0 \le t' \le t} \|u\|_{1+\alpha,\widetilde{\Omega}}^2 (\|\vartheta_\sigma\|_{\alpha,\widetilde{\Omega}^t}^2 + \|\eta_\sigma\|_{1+\alpha,\widetilde{\Omega}^t}^2)$$

$$+ \sup_{0 \le t' \le t} \|u\|_{1+\alpha,\Omega^t}^2 \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2$$

$$+ \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 (\|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 + \|\widetilde{\vartheta}_\sigma\|_{2,\widetilde{\Omega}^t}^2 + \|\widetilde{\eta}_\sigma\|_{1,\widetilde{\Omega}^t}^2)$$

$$+ \|\vartheta_\sigma\|_{2+\alpha,\widetilde{\Omega}^t}^2 (\|\widetilde{\vartheta}_\sigma\|_{1+\alpha,\widetilde{\Omega}^t}^2 + \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widetilde{\Omega}^t}^2)],$$

where d is given in (2.4).

Thus, taking into account (3.27), (3.28), (3.33) and (3.34) we get (3.25). This completes the proof. ∎

REMARK 3.5. In [5], where the motion of a compressible barotropic viscous fluid in a fixed domain was considered, an estimate analogous to (3.25) was obtained. However, in contrast to the estimate obtained in [5] inequality (3.25) is derived with a function $c(\varepsilon, t)$ on the right-hand side increasing with respect to t. This is essential to the proof of the global existence of solutions to problem (1.1) (see [12]) and insignificant for the proof of the global existence for the problem considered in [5].

We also have to estimate $\int_{\widetilde{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, d\xi$. To do this we apply $\Delta_{t'}(h)$ to $(2.6)_3$ and then multiply the equation by $\Delta_{t'}\widetilde{\vartheta}_{\sigma t'}$. The result of the calculations similar to those of Lemma 3.4 is the lemma below.

LEMMA 3.6. *Let the assumptions of Theorem* 3.1 *be satisfied. Then*

$$(3.35) \quad c_6 \int_{\widetilde{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, d\xi + \frac{\varkappa}{2} \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde{\Omega}} |\nabla(\Delta_{t'}\widetilde{\vartheta}_\sigma)|^2 \, d\xi \Big|_{t'=t-h_0}$$

$$\le \varepsilon(\|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|u\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2)$$

$$+ d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widetilde{\Omega}^t}^2 + c(\varepsilon,t)[\|u\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2$$

$$+ \|\eta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{\alpha,\Omega^t}^2 + \|\overline{\vartheta}\|_{\alpha+1/2,S^t}^2$$

$$+ \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 (\|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 + \sup_{0 \le t' \le t} \|\vartheta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 + \sup_{0 \le t' \le t} \|\eta_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2$$

$$+ \sup_{0 \le t' \le t} \|u\|_{1+\alpha,\widetilde{\Omega}}^2 + \|u\|_{1+\alpha,\widetilde{\Omega}^t}^2 \sup_{0 \le t' \le t} \|u\|_{1+\alpha,\widetilde{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widetilde{\Omega}^t}^2$$

$$+ \|u\|_{2+\alpha,\widetilde{\Omega}^t}^2 \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\Omega^t}^2) + \|u\|_{2+\alpha,\Omega^t}^2 \|u\|_{1+\alpha,\Omega^t}^2$$

$$+ \sup_{0 \le t' \le t} \|\eta\|_{1+\alpha,\widetilde{\Omega}}^2 \|\widetilde{k}\|_{\alpha,\widetilde{\Omega}^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\widetilde{\Omega}^t}^2 \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widetilde{\Omega}^t}^2],$$

where $c_6 > 0$ is a constant and $c(\varepsilon, t)$ is a positive continuous function increasing with respect to t.

Now let us introduce the quantity

$$
(3.36) \qquad \phi_1(t, \widetilde{\Omega}) = \int\limits_{\widetilde{\Omega}} \left(\eta \widetilde{u}^2 + \frac{p_1}{\varrho_e} \widetilde{\eta}_\sigma^2 + \frac{p_2 \eta c_v(\eta, \vartheta)}{\theta_e p_\vartheta(\varrho_e, \theta_e)} \widetilde{\vartheta}_\sigma^2 \right) d\xi
$$

$$
+ \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta |\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1 |\Delta^2(z)\widetilde{\eta}_\sigma|^2}{\varrho_e |z|^{3+2(1+\alpha)}} \right.
$$

$$
\left. + \frac{p_2 \eta c_v(\eta, \vartheta)}{\theta_e p_\vartheta(\varrho_e, \theta_e)} \frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}} \right).
$$

We see that

$$
\overline{c}_1 (\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{\vartheta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})
$$

$$
\leq \phi_1(t, \widetilde{\Omega}) \leq \overline{c}_2 (\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{\vartheta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}),
$$

and $\overline{c}_1, \overline{c}_2$ depend on the bound from the estimate for the local solution.

Integrating (3.19) with respect to time and using (3.24) (for $\widetilde{u}$ and $\widetilde{\vartheta}_\sigma$), (3.25) and (3.35) we finally obtain

LEMMA 3.7. *Let the assumptions of Theorem 3.1 be satisfied. Then, for* $t \leq T$,

$$
(3.37) \quad \phi_1(t, \widetilde{\Omega}) + c_7 \Big(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t} + \int\limits_0^t \|\widetilde{\eta}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} \, dt' \Big)
$$

$$
\leq \varepsilon (\|u\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\eta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t})
$$

$$
+ c(\varepsilon, t) \{ \|u\|^2_{0,\widetilde{\Omega}^t} + \|\vartheta_\sigma\|^2_{0,\widetilde{\Omega}^t} + \|\eta_\sigma\|^2_{0,\widetilde{\Omega}^t}
$$

$$
+ (1 + \sup_{t'} \|\eta\|^2_{1+\alpha,\Omega}) \|k\|^2_{\alpha,\widetilde{\Omega}^t} + \|\overline{\vartheta}\|^2_{1/2+\alpha,S^t}
$$

$$
+ X_3 [X_4(1 + X_4) + (1 + X_4^2) X_3] \} + \phi_1(0, \widetilde{\Omega}),
$$

where

$$
(3.38) \qquad X_3 = \|u\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t},
$$

$$
(3.39) \qquad X_4 = \sup_{t' \leq t} \|u\|^2_{1+\alpha,\Omega} + \sup_{t' \leq t} \|\eta_\sigma\|^2_{1+\alpha,\Omega} + \sup_{t' \leq t} \|\vartheta_\sigma\|^2_{1+\alpha,\Omega},
$$

T *is the time of local existence,* $c(\varepsilon, t)$ *is a positive continuous function increasing with respect to* t, *and* $c_7 > 0$ *is a constant.*

Now, we shall derive an estimate analogous to (3.37) which holds in a boundary subdomain $\widehat{\Omega}(\xi_0)$. First, we get an estimate similar to (3.2) for differences of $\widetilde{u}$, $\widetilde{\vartheta}_\sigma$ and $\widetilde{\eta}_\sigma$ and their derivatives in directions tangent to $\widehat{S}(\xi_0)$. Then we complete the resulting inequality by estimates which hold for differences in the normal direction to $\widehat{S}(\xi_0)$.

By $\Delta^s(\tau)f(z)$ we denote the sth finite difference of f in the tangent direction such that

$$\Delta^s(\tau)f(z) = \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}f(z'+k\tau, z_3),$$

where $z' = (z_1, z_2)$, $\tau = (\tau_1, \tau_2)$.

Similarly $\Delta^s(n)f(z) = \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}f(z', z_3 + kn)$ is the sth finite difference of f in the normal direction.

LEMMA 3.8. *Let the assumptions of Theorem* 3.1 *be satisfied. Then for functions* $\widetilde{u}$, $\widetilde{\eta}_\sigma$, $\widetilde{\vartheta}_\sigma$ *defined by* (2.5) *the following estimate holds*:

$$(3.40)\qquad \phi_2(t,\widehat{\Omega}) + c_8\int_0^t \Phi_1(t',\widehat{\Omega})\,dt' \le c\int_0^t Y\,dt' + \phi_2(0,\widehat{\Omega}) \quad \text{for } t \le T,$$

where

$$(3.41)\quad \phi_2(t,\widehat{\Omega})$$

$$= \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \left(\frac{\widehat{\eta}|\Delta^3(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e}\frac{|\Delta^3(\tau)\eta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_2\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})}{\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)}\frac{|\Delta^3(\tau)\widetilde{\vartheta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}} \right)$$

$$+ \int_{\widehat{\Omega}(\xi_0)} \left(\widehat{\eta}\widetilde{u}^2 + \frac{p_1}{\varrho_e}\widetilde{\eta}_\sigma^2 + \frac{p_2\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})}{\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)}\widetilde{\vartheta}_\sigma^2 \right) dz$$

$$+ \frac{2(\mu+\nu)}{\varrho_e}\left[\int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta^2(\tau)\widehat{\nabla}_3\widetilde{\eta}_\sigma|^2}{|\tau|^{2+2\alpha}} + \int_{\mathbb{R}_+} dn \int_{\widehat{\Omega}(\xi_0)} \frac{|\Delta_3^2(n)\widehat{\nabla}_3\widetilde{\eta}_\sigma|^2}{|n|^{1+2\alpha}} \right]$$

$$+ \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \left(\widehat{\eta}\frac{|\Delta^2(\tau)\widetilde{u}_z|^2}{|\tau|^{2+2\alpha}} + \frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\widetilde{\eta}_{\sigma z}|^2}{|\tau|^{2+2\alpha}} \right)$$

$$+ \sigma \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \left(4\sqrt{\widehat{c}}\,\frac{\Delta^2(\tau)((H(\cdot,0)+2/R_e)\widehat{\zeta})}{|\tau|^{1+\alpha}} \right.$$

$$+ \frac{1}{4\sqrt{\widehat{c}}}\frac{\Delta(\tau)(\int_0^t \Delta^3(\tau)\widetilde{u}\,dt' \cdot \widehat{n})}{|\tau|^{1+2+\alpha}} \Big)^2$$

$$+ \sigma \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \left(\frac{4\Delta^2(\tau)((H(\cdot,0)+2/R_e)\widehat{\zeta})}{|\tau|^{1+\alpha}} \right.$$

$$+ \frac{1}{4}\frac{\sum_{\gamma,\delta=1}^2 \int_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\,dt' \cdot \widehat{n}}{|\tau|^{1+\alpha}} \Big)^2$$

$$+ \frac{\sigma}{4} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \, \frac{\sum_{\gamma,\delta=1}^{2} (\int_0^t \Delta(\tau) \widetilde{u}_{z_\gamma z_\delta} \, dt' \cdot \widehat{n})^2}{|\tau|^{2+2\alpha}}$$

$$+ \frac{\sigma}{4} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \, \frac{\sum_{\gamma=1}^{2} (\int_0^t \Delta^3(\tau) \widetilde{u}_{z_\gamma} \, dt' \cdot \widehat{n})^2}{|\tau|^{2+2(1+\alpha)}},$$

Φ_1 and Y are given by (3.63) and (3.64) respectively, and $c_8 > 0$ is a constant.

Proof. We apply $\Delta^3(\tau)$ to $(2.7)_1$–$(2.7)_3$ and multiply the resulting equations by $\Delta^3(\tau) \widetilde{u}_i$, $\frac{p_1}{\varrho_e} \Delta^3(\tau) \widetilde{\eta}_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e, \theta_e)} \Delta^3(\tau) \widetilde{\vartheta}_\sigma$, respectively. Then integrating each of the equations over $\widehat{\Omega}(\xi_0)$ and with respect to τ over $\mathbb{R}^2$ with the weight $1/|\tau|^{2+2(1+\alpha)}$ and using (1.5), the Korn inequality and the equation of continuity in $\widehat{\Omega}$, i.e. $\widehat{\eta}_t + \widehat{\eta} \widehat{\nabla} \cdot \widehat{u} = 0$, we get

$$(3.42) \quad \frac{1}{2} \frac{d}{dt} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz \left(\frac{\widehat{\eta} |\Delta^3(\tau) \widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e} \frac{|\Delta^3(\tau) \widetilde{\eta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}} \right.$$

$$\left. + \frac{p_2 \widehat{\eta} c_v}{\theta_e p_\vartheta(\varrho_e, \theta_e)} \frac{|\Delta^3(\tau) \widetilde{\vartheta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}} \right) + c_9 \int\limits_{\mathbb{R}^2} d\tau \, \frac{\|\Delta^3(\tau) \widetilde{u}\|_{1, \widehat{\Omega}(\xi_0)}^2}{|\tau|^{2+2(1+\alpha)}}$$

$$+ \varkappa \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz \, \frac{p_2 |\Delta^3(\tau) \nabla \widetilde{\vartheta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}$$

$$- \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \, \frac{\mathbb{T}(\Delta^3(\tau) \widetilde{u}, \Delta^3(\tau) \widetilde{p}_\sigma) \widehat{n}' \cdot \Delta^3(\tau) \widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$- \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \, \frac{\widehat{n}' \cdot \nabla(\Delta^3(\tau) \widetilde{\vartheta}_\sigma) \cdot \Delta^3(\tau) \widetilde{\vartheta}_\sigma}{|\tau|^{2+2(1+\alpha)}}$$

$$\leq \varepsilon (\|\widehat{u}\|_{2+\alpha, \widehat{\Omega}}^2 + \|\widehat{\eta}_\sigma\|_{1+\alpha, \widehat{\Omega}}^2 + \|\widehat{\vartheta}_\sigma\|_{2+\alpha, \widehat{\Omega}}^2)$$

$$+ \psi_{18}(1/\varepsilon, a) \Big[X_1 X_2 + X_1 (1 + X_2^2) \int\limits_0^t \|u\|_{2+\alpha, \Omega}^2 \, dt'$$

$$+ \|\widehat{\eta}_\sigma\|_{1+\alpha, \Omega}^2 \|\widetilde{u}_t\|_{\alpha, \widehat{\Omega}}^2 \Big]$$

$$+ \psi_{19}(1/\varepsilon, a)(\|\widetilde{u}\|_{1+\alpha, \widehat{\Omega}}^2 + \|\widetilde{\eta}_\sigma\|_{\alpha, \widehat{\Omega}}^2$$

$$+ \|\widetilde{\vartheta}_\sigma\|_{\alpha, \widehat{\Omega}}^2 + \|\vartheta_\sigma\|_{1+\alpha, \widehat{\Omega}}^2) + \psi_{20}(1/\varepsilon, a) \|\widetilde{k}\|_{\alpha, \widehat{\Omega}}^2,$$

where X_1 and X_2 are given by (3.3) and (3.4) respectively.

Using the boundary condition $(2.7)_4$, definition (2.4) and imbedding (2.11) we obtain

$$(3.43) \quad \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\mathbb{T}(\Delta^3(\tau)\widetilde{u}, \Delta^3(\tau)\widetilde{p}_\sigma)\widehat{n}' \cdot \Delta^3(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$= -\frac{\sigma}{2}\frac{d}{dt} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{g_{\widetilde{u}}^{\gamma\delta} \int_0^t \Delta^3(\tau)\frac{\partial \widetilde{u}}{\partial z_\gamma}\,dt' \cdot \widehat{n} \int_0^t \Delta^3(\tau)\frac{\partial \widetilde{u}}{\partial z_\delta}\,dt' \cdot \widehat{n}}{|\tau|^{2+2(1+\alpha)}}$$

$$+ \sigma \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\Delta^3(\tau)((\Delta_{\widehat{S}}(0)\widehat{\xi} \cdot \widehat{n} + (2/R_e)\widehat{n})\widehat{\zeta})\widehat{n} \cdot \Delta^3(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} + I_8$$

$$\equiv J_1 + J_2 + I_8,$$

where

$$(3.44) \quad |I_8| \leq \psi_{21}(a, \|F\|_{5/2+\alpha,\widehat{S}})\Big[(\varepsilon + d)\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}}^2$$

$$+ \varepsilon(\|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2)$$

$$+ \varepsilon_1 \Big\|\int\limits_0^t \widehat{u}\,dt'\Big\|_{5/2+\alpha,\widehat{S}}^2 + \varepsilon_1\|H(\cdot,0) + 2/R_e\|_{1/2+\alpha,S}^2$$

$$+ \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}}^2 \int\limits_0^t \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}}^2\,dt' + \int\limits_0^t \|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}}^2\,dt'\Big\|\int\limits_0^t \widehat{u}\,dt'\Big\|_{2+\alpha,\widehat{\Omega}}^2 + \|\widetilde{u}\|_{0,\widehat{\Omega}}^2\Big],$$

ε_1 and ε are sufficiently small positive constants, and d is given in (2.4).

Now, we estimate the second term on the right-hand side of (3.43):

$$(3.45) \quad J_2 = \sigma \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\Delta^3(\tau)((H(\cdot,0) + 2/R_e)\widehat{\zeta}) \cdot \Delta^3(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$+ \frac{2\sigma}{R_e} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\Delta^3(\tau)\Delta_{\widehat{S}}(0)\widehat{\zeta} \cdot (\widehat{n} - \widehat{n}_0) \cdot \Delta^3(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} \equiv J_2^1 + J_2^2,$$

where $\widehat{n}_0(z') = \widehat{n}(z_1'0)$ and

$$(3.46) \quad |J_2^2| \leq \varepsilon_1 \Big\|\int\limits_0^t \widehat{u}\,dt'\Big\|_{5/2+\alpha,\widehat{S}}^2 + c(\varepsilon_1)(\varepsilon\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}}^2 + c(\varepsilon)\|\widetilde{u}\|_{0,\widehat{\Omega}}^2).$$

Next, we obtain

$$(3.47)$$

$$J_2^1 = -\frac{\sigma}{2}\frac{d}{dt} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} \frac{2\Delta^2(\tau)((H(\cdot,0) + 2/R_e)\widehat{\zeta}) \cdot \Delta(\tau)(\int_0^t \Delta^3(\tau)\widetilde{u}\,dt' \cdot \widehat{n})}{|\tau|^{2+2+(1+\alpha)}}$$

$$
= -\frac{\sigma}{2}\frac{d}{dt}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\left(4\sqrt{\widehat{c}}\,\frac{\Delta^2(\tau)((H(\cdot,0)+2/R_e)\widehat{\zeta})}{|\tau|^{1+\alpha}}\right.
$$

$$
+\frac{1}{4\sqrt{\widehat{c}}}\frac{\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n})}{|\tau|^{1+2+\alpha}}\Bigg)^2
$$

$$
+\frac{\sigma}{32\overline{c}}\frac{d}{dt}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{(\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n}))^2}{|\tau|^{2+2(2+\alpha)}},
$$

where $\widehat{c}>0$ is a constant such that (see [3], [4])

$$
(3.48)\qquad \int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi)}dz_{\widehat{S}}\frac{(\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n}))^2}{|\tau|^{2+2(2+\alpha)}}
$$

$$
\le\widehat{c}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{\sum_{\gamma=1}^2((\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n})_{z_\gamma})^2}{|\tau|^{2+2(1+\alpha)}}.
$$

By (3.43), (3.45), (3.47) we get

$$
(3.49)\qquad -\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{\mathbb{T}(\Delta^3(\tau)\widetilde{u},\Delta^3(\tau)\widetilde{p}_\sigma)\widehat{n}'\cdot\Delta^3(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}
$$

$$
=\frac{\sigma}{2}\frac{d}{dt}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{g_{\widehat{u}}^{\gamma\delta}\int_0^t\Delta^3(\tau)\widetilde{u}_{z_\gamma}\,dt'\cdot\widehat{n}\int_0^t\Delta^3(\tau)\widetilde{u}_{z_\delta}\,dt'\cdot\widehat{n}}{|\tau|^{2+2(1+\alpha)}}
$$

$$
+\frac{\sigma}{2}\frac{d}{dt}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\left(4\sqrt{\widehat{c}}\,\frac{\Delta^2(\tau)((H(\cdot,0)+2/R_e)\widehat{\zeta})}{|\tau|^{1+\alpha}}\right.
$$

$$
+\frac{1}{4\sqrt{\widehat{c}}}\frac{\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n})}{|\tau|^{1+2+\alpha}}\Bigg)^2
$$

$$
-\frac{\sigma}{32\widehat{c}}\frac{d}{dt}\int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{(\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n})^2}{|\tau|^{2+2(2+\alpha)}}+I_8+J_2^2,
$$

where estimates for I_8 and J_2^2 are given by (3.44) and (3.46), respectively.

Next, using the boundary condition $(2.7)_5$ we have

$$
(3.50)\qquad \int_{\mathbb{R}^2}d\tau\int_{\widehat{S}(\xi_0)}dz_{\widehat{S}}\frac{\widehat{n}'\cdot\nabla(\Delta^3(\tau)\widetilde{\vartheta}_\sigma)\cdot\Delta^3(\tau)\widetilde{\vartheta}_\sigma}{|\tau|^{2+2(1+\alpha)}}
$$

$$
\le\psi_{22}(a,\|F\|_{2+\alpha,\widehat{S}})\Big[(\varepsilon+d)\|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2+\|\overline{\widetilde{\vartheta}}\|_{\alpha+1/2,\widehat{S}}
$$

$$
+\|\widehat{\vartheta}_\sigma\|_{0,\widehat{\Omega}}^2+\|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2\Big\|\int_0^t\widehat{u}\,dt'\Big\|_{2+\alpha,\widehat{\Omega}}^2\Big].
$$

Taking into account (3.42), (3.44), (3.46), (3.49), (3.50) and the considerations for interior subdomains we get

$$(3.51) \qquad \frac{d}{dt}\phi_3(t,\widehat{\Omega}) + c_{10}\Phi_2(t,\widehat{\Omega}) \leq cX,$$

where

$$(3.52) \quad \phi_3(t,\widehat{\Omega}) = \frac{1}{2}\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\left(\frac{\widehat{\eta}|\Delta^3(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e}\frac{|\Delta^3(\tau)\widetilde{\eta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\right.$$

$$\left. + \frac{p_2\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})}{\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)}\frac{|\Delta^3(\tau)\widetilde{\vartheta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\right) + \frac{1}{2}\int\limits_{\widehat{\Omega}(\xi_0)}\left(\widehat{\eta}\widetilde{u}^2 + \frac{p_1}{\varrho_e}\widetilde{\eta}_\sigma^2 + \frac{p_2\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})}{\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)}\widetilde{\vartheta}_\sigma^2\right)dz$$

$$+ \frac{\sigma}{2}\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}}\frac{g_{\widetilde{u}}^{\gamma\delta}\int_0^t \Delta^3(\tau)\frac{\partial\widetilde{u}}{\partial z_\gamma}dt'\cdot\widehat{n}\int_0^t\Delta^3(\tau)\frac{\partial\widetilde{u}}{\partial z_\delta}dt'\cdot\widehat{n}}{|\tau|^{2+2(1+\alpha)}}$$

$$+ \frac{\sigma}{2}\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}}\left(4\sqrt{\widehat{c}}\frac{\Delta^2(\tau)((H(\cdot,0)+2/R_e)\widehat{\zeta})}{|\tau|^{1+\alpha}}\right.$$

$$\left. + \frac{1}{4\sqrt{\widehat{c}}}\frac{\Delta(\tau)(\int_0^t\Delta^3(\tau)\widetilde{u}\,dt'\cdot\widehat{n})}{|\tau|^{1+2+\alpha}}\right)^2$$

$$- \frac{\sigma}{32\overline{c}}\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}}\frac{(\Delta(\tau)(\int_0^t\Delta^3(\tau')\widetilde{u}\,dt'\cdot\widehat{n}))^2}{|\tau|^{2+2(2+\alpha)}},$$

$$(3.53) \qquad \Phi_2(t,\widehat{\Omega}) = c\left(\|\widetilde{u}\|_{1,\widehat{\Omega}}^2 + \|\widetilde{\vartheta}_{\sigma x}\|_{0,\widehat{\Omega}}^2 + \int\limits_{\mathbb{R}^2} d\tau\frac{\|\Delta^3(\tau)\widetilde{u}\|_{1,\widehat{\Omega}(\xi_0)}^2}{|\tau|^{2+2(1+\alpha)}}\right.$$

$$\left. + \int\limits_{\mathbb{R}^2} d\tau\int\limits_{\widehat{\Omega}(\xi_0)} dz\frac{p_2|\nabla(\Delta^3(\tau)\widetilde{\vartheta}_\sigma)|^2}{|\tau|^{2+2(1+\alpha)}}\right),$$

$$(3.54) \quad X = \psi_{23}(a,\|F\|_{5/2+\alpha,\widehat{S}})\left\{d(\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}}^2 + \|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2)\right.$$

$$+ \varepsilon(\|\widehat{u}\|_{2+\alpha,\widehat{\Omega}}^2 + \|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2) + \varepsilon\|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 + \varepsilon_1\left\|\int_0^t\widehat{u}\,dt'\right\|_{5/2+\alpha,\widehat{S}}^2$$

$$+ A + \|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2\left\|\int_0^t\widehat{u}\,dt'\right\|_{2+\alpha,\widehat{\Omega}}^2 + \varepsilon_1\|H(\cdot,0)+2/R_e\|_{1/2+\alpha,S}^2$$

$$\left. + X_1 X_2 + X_1(1+X_2^2)\int_0^t\|\widehat{u}\|_{2+\alpha,\Omega}^2\,dt' + \|\eta_\sigma\|_{1+\alpha,\Omega}^2\|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}^2\right\},$$

$$(3.55) \quad A = \|\widehat{u}\|_{0,\widehat{\Omega}}^2 + \|\widehat{\eta}_\sigma\|_{0,\widehat{\Omega}}^2 + \|\widehat{\vartheta}_\sigma\|_{0,\widehat{\Omega}}^2 + \|\widehat{k}\|_{\alpha,\widehat{\Omega}}^2 + \|\widetilde{\overline{\vartheta}}\|_{\alpha+1/2,\widehat{S}}^2,$$

and X_1 and X_2 are given by (3.3) and (3.4), respectively.

Now, repeating the argument from Lemma 4.4 of [15] we obtain the estimate

$$(3.56) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \left(\widehat{\eta}\frac{|\Delta^2(\tau)\widetilde{u}_z|^2}{|\tau|^{2+2\alpha}} + \frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\widetilde{\eta}_{\sigma z}|^2}{|\tau|^{2+2\alpha}}\right)$$

$$+ \frac{\sigma}{2}\frac{d}{dt}\int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{g_{\widehat{u}}^{\gamma\delta}\int_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\gamma}\,dt' \cdot \widehat{n}\int_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\delta}\,dt' \cdot \widehat{n}}{|\tau|^{2+2\alpha}}$$

$$+ \frac{\sigma}{2}\frac{d}{dt}\int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \left(\frac{4\Delta^2(\tau)(H(\cdot,0)+2/R_e)\widehat{\zeta}}{|\tau|^{1+\alpha}}\right.$$

$$+ \frac{1}{4}\frac{\sum_{\gamma,\delta=1}^2 \int_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\,dt' \cdot \widehat{n}}{|\tau|^{1+\alpha}}\bigg)^2$$

$$- \frac{\sigma}{32}\frac{d}{dt}\int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\sum_{\gamma,\delta=1}^2(\int_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\,dt')^2}{|\tau|^{2+2\alpha}}$$

$$+ \frac{\mu}{2}\int_{\mathbb{R}^2}\frac{\|\Delta^2(\tau)\widetilde{u}\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\,d\tau + \int_{\mathbb{R}^2}\frac{\|\Delta^2(\tau)\widetilde{\eta}_{\sigma z}\|_{0,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\,d\tau$$

$$\leq \varepsilon\Big(\Big\|\int_0^t \widetilde{u}\,dt'\Big\|_{5/2+\alpha,\widehat{S}}^2 + \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}}^2$$

$$+ \|H(\cdot,0)+2/R_e\|_{\alpha+1/2,S}^2 + \|R(\cdot,t)-R(\cdot,0)\|_{5/2+\alpha,S^1}^2\Big)$$

$$+ c(\|\widehat{u}\|_{0,\widehat{\Omega}}^2 + \|\widehat{\eta}_\sigma\|_{0,\widehat{\Omega}}^2 + \|\widehat{\vartheta}_\sigma\|_{0,\widehat{\Omega}}^2 + cX_2\Big(X_1 + \int_0^t \|u\|_{2+\alpha,\Omega}^2\,dt'\Big).$$

Next,

$$(3.57) \quad \Big\|\int_0^t \widetilde{u}\,dt'\Big\|_{5/2+\alpha,\widehat{S}}^2 \leq c\|R(\cdot,t)-R(\cdot,0)\|_{5/2+\alpha,S^1}.$$

Now, we consider the problem

$$(3.58) \quad \begin{aligned} &\varkappa\Delta\widetilde{\vartheta}_\sigma = \widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})\widetilde{\vartheta}_{\sigma t} + \theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)\nabla\cdot\widetilde{u} - \widehat{\eta}\widetilde{k} - k_6 \equiv K_3 &&\text{in } \widehat{\Omega}, \\ &\widehat{n}\cdot\nabla\widetilde{\vartheta}_\sigma = \widetilde{\overline{\vartheta}} + k_8 \equiv K_4 &&\text{on } \widehat{S}. \end{aligned}$$

From Agmon–Douglis–Nirenberg theory (see [2]) we get

$$(3.59) \qquad \int_{\mathbb{R}^2} \frac{\|\Delta(\tau)\widetilde{\vartheta}_\sigma\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}} \, d\tau \le c \int_{\mathbb{R}^2} \frac{\|\Delta(\tau)\nabla \cdot \widetilde{u}\|_{1,\widehat{\Omega}}}{|\tau|^{2+2\alpha}} \, d\tau$$

$$+ c \int_{\mathbb{R}^2} \frac{\|\Delta(\tau)K_3\|_{0,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}} \, d\tau + c \int_{\mathbb{R}^2} \frac{\|\Delta(\tau)K_4\|_{1/2,\widehat{S}}^2}{|\tau|^{2+2\alpha}} \, d\tau$$

$$\le c(X + \|\widetilde{\vartheta}_{\sigma t}\|_{\alpha,\widehat{\Omega}}^2) \equiv c\overline{X}.$$

Next, applying $\Delta_3(n)$ to $(3.58)_1$ we obtain the estimate

$$(3.60) \qquad \int_{\mathbb{R}_+} dn \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3(n)\widetilde{\vartheta}_{\sigma z_3 z_3}|^2}{|n|^{1+2\alpha}} \le c \int_{\mathbb{R}_+} dn \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3(n)\widetilde{\vartheta}_{\sigma zz'}|^2}{|n|^{1+2\alpha}} + C\overline{X},$$

where the first term on the right-hand side of (3.60) is estimated using the following interpolation inequality:

$$(3.61) \qquad \int_{\mathbb{R}_+} dn \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3(n)\widetilde{\vartheta}_{\sigma zz'}|^2}{|n|^{1+2\alpha}} \le c \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} \frac{|\Delta(\tau)\widetilde{\vartheta}_{\sigma zz'}|^2}{|\tau|^{2+2\alpha}} \, dz$$

$$+ c \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} \frac{|\Delta(\tau)\widetilde{\vartheta}_{\sigma zz}|^2}{|\tau|^{2+2\alpha}} \, dz.$$

Repeating the considerations from [5] (see the proof of Lemma 3.2) and taking into account (3.51), (3.56), (3.57), (3.59)–(3.61) we get

$$(3.62) \qquad \frac{d}{dt}\phi_4(t,\widehat{\Omega}) + c_{11}\Phi_1(t,\widehat{\Omega}) \le cY,$$

where

$$\phi_4(t,\widehat{\Omega}) = \phi_3(t,\widehat{\Omega}) + \frac{\mu+\nu}{\varrho_e} \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta^2(\tau)\widehat{\nabla}_3\widetilde{\eta}_\sigma|^2}{|\tau|^{2+2\alpha}}$$

$$+ \frac{\mu+\nu}{\varrho_e} \int_{\mathbb{R}_+} dn \int_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3^2(n)\widehat{\nabla}_3\widetilde{\eta}_\sigma|^2}{|n|^{1+2\alpha}}$$

$$+ \frac{1}{2} \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz \left(\widehat{\eta} \frac{|\Delta^2(\tau)\widetilde{u}_z|^2}{|\tau|^{2+2\alpha}} + \frac{p_1}{\varrho_e} \frac{|\Delta^2(\tau)\widetilde{\eta}_{\sigma z}|^2}{|\tau|^{2+2\alpha}} \right)$$

$$+ \frac{\sigma}{2} \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{g_{\widehat{u}}^{\gamma\delta} \int_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\gamma} \, dt' \cdot \widehat{n} \int_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\delta} \, dt' \cdot \widehat{n}}{|\tau|^{2+2\alpha}}$$

$$+ \frac{\sigma}{2} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \left(\frac{4\Delta^2(\tau)(H(\cdot,0)+2/R_2)\widehat{\zeta}}{|\tau|^{1+\alpha}} \right.$$

$$+ \left. \frac{1}{4} \frac{\sum_{\gamma,\delta=1}^{2} \int_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\, dt' \cdot \widehat{n}}{|\tau|^{1+\alpha}} \right)^2$$

$$- \frac{\sigma}{32} \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}} \frac{\sum_{\gamma,\delta=1}^{2}(\int_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\, dt')^2}{|\tau|^{2+2\alpha}},$$

$$(3.63) \quad \Phi_1(t,\widehat{\Omega}) = \Phi_2(t,\widehat{\Omega}) + \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta^2(\tau)\widehat{\nabla}_3\widehat{\eta}_\sigma|^2}{|\tau|^{2+2\alpha}}$$

$$+ \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta^2(\tau)\widehat{\nabla}_3^2\widetilde{u}_3|^2}{|\tau|^{2+2\alpha}} + \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3^2(n)\widetilde{u}_{zz'}|^2}{|n|^{1+2\alpha}}$$

$$+ \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3^2(n)\widehat{\nabla}_3\widehat{\eta}_\sigma|^2}{|n|^{1+2\alpha}} + \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3^2(n)\widetilde{u}_{z_3 z_3}|^2}{|n|^{1+2\alpha}}$$

$$+ \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3^2(n)\widetilde{\eta}_{\sigma z'}|^2}{|n|^{1+2\alpha}} + \mu \int\limits_{\mathbb{R}^2} \frac{\|\Delta^2(\tau)\widetilde{u}\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\, d\tau$$

$$+ \int\limits_{\mathbb{R}^2} \frac{\|\Delta^2(\tau)\widetilde{\eta}_{\sigma z}\|_{0,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\, d\tau + \int\limits_{\mathbb{R}^2} \frac{\|\Delta(\tau)\widetilde{\vartheta}_\sigma\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\, d\tau$$

$$+ \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3(n)\widetilde{\vartheta}_{\sigma z_3 z_3}|^2}{|n|^{1+2\alpha}} + \int\limits_{\mathbb{R}_+} dn \int\limits_{\widehat{\Omega}(\xi_0)} dz \frac{|\Delta_3(n)\widetilde{\vartheta}_{\sigma zz'}|^2}{|n|^{1+2\alpha}},$$

$$(3.64) \quad Y = \psi_{24}(a, \|F\|_{5/2+\alpha,\widehat{S}})\{X + X_2\|u\|_{2+\alpha,\Omega^t}^2 + \|\widetilde{u}_t\|_{\alpha,\widehat{\Omega}}^2 + \|\vartheta_{\sigma t}\|_{\alpha,\widehat{\Omega}}^2$$

$$+ \varepsilon(\|H(\cdot,0)+2/R_e\|_{\alpha+1/2,S}^2 + \|R(\cdot,t)-R(\cdot,0)\|_{5/2+\alpha,S^1}^2)\},$$

and ϕ_3, Φ_2, X_2, X are given by (3.52), (3.53), (3.4), (3.54), respectively.

Now, using (2.9) and assuming that the δ from the local existence theorem (see Theorem 3.4 of [13]) such that $T^{1/2}\|u\|_{2+\alpha,\Omega^T} \leq \delta$ is sufficiently small we get

$$(3.65) \quad g_{\widehat{u}}^{\gamma\delta} \int\limits_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\gamma}\, dt' \cdot \widehat{n} \int\limits_0^t \Delta^2(\tau)\widetilde{u}_{z_\beta z_\delta}\, dt' \cdot \widehat{n}$$

$$\geq \frac{3}{4} \sum_{\gamma,\delta=1}^{2} \left(\int\limits_0^t \Delta^2(\tau)\widetilde{u}_{z_\gamma z_\delta}\, dt' \cdot \widehat{n} \right)^2.$$

Moreover,

$$(3.66) \quad g_{\widehat{u}}^{\gamma\delta} \int\limits_0^t \Delta^3(\tau) \frac{\partial \widetilde{u}}{\partial z_\gamma} \, dt' \cdot \widehat{n} \int\limits_0^t \Delta^3(\tau) \frac{\partial \widetilde{u}}{\partial z_\delta} \, dt' \cdot \widehat{n}$$

$$\geq \frac{3}{4} \sum_{\gamma=1}^2 \left(\int\limits_0^t \Delta^3(\tau) \frac{\partial \widetilde{u}}{\partial z_\gamma} \, dt' \cdot \widehat{n} \right)^2.$$

Therefore integrating (3.62) with respect to time and using (3.65), (3.66), (3,48) we obtain inequality (3.40) for $t \leq T$.

This completes the proof of the lemma. ∎

Now, similarly to the case of interior subdomains we will estimate the integrals $\int_0^t \|\widetilde{u}_{t'}\|_{\alpha,\widehat{\Omega}}^2 \, dt'$ and $\int_0^t \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha,\widehat{\Omega}}^2 \, dt'$. First, we use inequality (3.24). Then to estimate $\int_{\widehat{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)}^2 \, dz$ and $\int_{\widehat{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha/2,(0,t)}^2 \, dz$ we apply the method used to obtain (3.25) and (3.35). However, in this case we have to consider some additional terms which appear in connection with the boundary conditions $(2.7)_4$ and $(2.7)_5$.

LEMMA 3.9. *Let the assumptions of Theorem* 3.1 *be satisfied. Then*

$$(3.67) \quad c_{12} \int\limits_{\widehat{\Omega}} \|\widetilde{u}_{t'}\|_{\alpha/2,(0,t)}^2 \, dz$$

$$+ \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_{\widehat{\Omega}} \left(\frac{\mu}{4} |\mathbb{S}(\Delta_{t'}\widetilde{u})|^2 + \frac{\nu-\mu}{2} |\operatorname{div} \Delta_{t'}\widetilde{u}|^2 \right) dz \Bigg|_{t'=t-h_0}$$

$$\leq \varepsilon(\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2) + c(\varepsilon,t)[d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2$$

$$+ \|u\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{0,\Omega^t}^2 + (1+X_3)X_3(X_3+X_4)],$$

where $t \leq T$, $c_{12} > 0$ *is a constant,* $c(\varepsilon,t)$ *is a positive continuous function nondecreasing with respect to* t, *and* X_3 *and* X_4 *are given by* (3.38) *and* (3.39), *respectively.*

Proof. The calculations of Lemma 3.4 and the boundary condition $(2.7)_4$ yield

$$(3.68) \quad \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{t-h_0} \psi(t') \, dt' \int\limits_{\widehat{\Omega}} \widehat{\eta} |\Delta_{t'}\widetilde{u}_{t'}|^2 dz$$

$$+ \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_{\widehat{\Omega}} \left(\frac{\mu}{4} |\mathbb{S}(\Delta_{t'}\widetilde{u})|^2 + \frac{\nu-\mu}{2} |\operatorname{div} \Delta_{t'}\widetilde{u}|^2 \right) dz \Bigg|_{t'=t-h_0}$$

$$\leq I + \sum_{i=1}^5 |K_i|,$$

where

$$(3.69) \quad I \leq \varepsilon(\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2)$$
$$+ d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + c(\varepsilon,t)[\|u\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2$$
$$+ \|\eta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{0,\Omega^t}^2 + \|u\|_{2+\alpha,\Omega^t}^2 \sup_{0 \leq t' \leq t} \|u\|_{1+\alpha,\Omega}^2$$
$$+ \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 (\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega^t}^2)(1 + \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}^t}^2)$$
$$+ \sup_{0 \leq t' \leq t} \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\widehat{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}^t}^2)$$
$$+ \|\widehat{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 (\|\widetilde{\vartheta}_\sigma\|_{1+\alpha,\widehat{\Omega}^t}^2 + \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}^t}^2)$$
$$+ \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 (\sup_{0 \leq t' \leq t} \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 + \sup_{0 \leq t' \leq t} \|\widetilde{\vartheta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2)].$$

and

$$(3.70) \quad K_1 = \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t') \, dt' \int_{\widehat{S}} \Delta_{t'} \widetilde{p}_\sigma I \widehat{n}' \Delta_{t'} \widetilde{u}_{t'} \, dz_{\widehat{S}},$$

$$(3.71) \quad K_2 = -\sigma \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t') \, dt' \int_{\widehat{S}} \Delta_{t'}(\Delta_{\widehat{S}}) \int_0^{t'} \widetilde{u} \, d\tau \cdot \Delta_{t'} \widetilde{u}_{t'} \, dz_{\widehat{S}},$$

$$(3.72) \quad K_3 = -\sigma \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t') \, dt' \, \Delta_{\widehat{S}} \Delta_{t'} \int_0^{t'} \widetilde{u} \, d\tau \cdot \Delta_{t'} \widetilde{u}_{t'} \, dz_{\widehat{S}},$$

$$(3.73) \quad K_4 = \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} \psi(t') \, dt' \int_{\widehat{S}} \widehat{\mathbb{T}}(\widetilde{u}, \widetilde{p}_\sigma) \Delta_{t'} \widehat{n} \cdot \Delta_{t'} \widetilde{u}_t \, dz_{\widehat{S}}.$$

Notice that the sum of the first two terms on the right-hand side of (3.68) has the same form as the right-hand side of (3.25). Thus, it remains to estimate K_i ($i = 1, \ldots, 5$).

First, we have

$$(3.74) \quad |K_1| \leq \varepsilon \int_{\widehat{S}} dz_{\widehat{S}} \int_0^{h_0} dh \int_0^{t-h_0} dt' \frac{|\Delta_{t'} \widetilde{u}_{t'}|^2}{h^{1+2(\alpha/2-1/4)}}$$
$$+ c(\varepsilon) \int_{\widehat{S}} dz_{\widehat{S}} \int_0^{h_0} dh \int_0^{t-h_0} dt' \frac{|\Delta_{t'}(p_1 \widetilde{\eta}_\sigma + p_2 \widetilde{\vartheta}_\sigma)|^2}{h^{3/2+\alpha}}$$
$$\leq \varepsilon \|\widetilde{u}\|_{3/2+\alpha,\widehat{S}^t}^2 + c(\varepsilon) \int_{\widehat{S}} dz_{\widehat{S}} \int_0^{h_0} dh \int_0^{t-h_0} dt' \frac{|\Delta_{t'} p_1 \widetilde{\eta}_\sigma|^2}{h^{3/2+\alpha}}$$

$$+ c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_{0}^{h_0} dh \int\limits_{0}^{t-h_0} dt' \frac{|\Delta_{t'} p_2 \widetilde{\vartheta}_\sigma|^2}{h^{3/2+\alpha}} + c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_{0}^{h_0} dh \int\limits_{0}^{t-h_0} dt' \frac{|\Delta_t \widetilde{\eta}_\sigma|^2}{h^{3/2+\alpha}}$$

$$+ c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_{0}^{h_0} dh \int dt' \frac{|\Delta_t \widetilde{\vartheta}_\sigma|^2}{h^{3/2+\alpha}} \equiv \varepsilon \|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + \sum_{i=1}^{4} K_1^i.$$

We have to estimate the terms K_1^i ($i = 1, \ldots, 4$). First, we obtain

$$(3.75) \qquad K_1^4 \le c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_{0}^{h_0} dh \int\limits_{0}^{t-h_0} dt' \frac{|\int_{t'}^{t'+h} \widetilde{\vartheta}_{\sigma\tau} \, d\tau|^2}{h^{3/2+\alpha}}$$

$$\le c(\varepsilon) h_0^{3/2-\alpha} \int\limits_{0}^{t-h_0} dt' \int\limits_{\widehat{S}} dz_{\widehat{S}} \left(\sup_{h>0} \frac{1}{h} \int\limits_{t'}^{t'+h} |\widetilde{\vartheta}_{\sigma\tau}| \, d\tau \right)^2$$

$$\le c(\varepsilon) t^{3/2-\alpha} \int\limits_{0}^{t} \|\widetilde{\vartheta}_{\sigma t'}\|_{0,\widehat{S}}^2 \, dt'$$

$$\le c(\varepsilon) t^{3/2-\alpha} \left(\varepsilon_1^{1-\varkappa} \int\limits_{0}^{t} \|\widetilde{\vartheta}_{\sigma t'}\|_{\alpha,\widehat{\Omega}}^2 \, dt' + c\varepsilon_1^{-\varkappa} \int\limits_{0}^{t} \|\widetilde{\vartheta}_{\sigma t'}\|_{0,\widehat{\Omega}}^2 \, dt' \right).$$

From equation $(2.7)_3$ we calculate

$$(3.76) \quad \|\widetilde{\vartheta}_{\sigma t}\|_{0,\widehat{\Omega}}^2 \le \varepsilon_2^{1-\varkappa_2} (\|u\|_{2+\alpha,\Omega}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega}^2) + d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}}^2$$

$$+ c\varepsilon_2^{-\varkappa_2} [\|u\|_{0,\Omega}^2 + \|\vartheta_\sigma\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2$$

$$+ \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widehat{u}\|_{1+\alpha,\widehat{\Omega}}^2 + \|\widehat{\vartheta}_\sigma\|_{\alpha,\widehat{\Omega}}^2 + \|\widehat{\eta}_\sigma\|_{\alpha,\widehat{\Omega}}^2 + \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}^t}^2)$$

$$+ \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 \|\widetilde{\vartheta}_\sigma\|_{2,\widehat{\Omega}}^2].$$

Using (3.76) in (3.75) with appropriate choice of ε_1 and ε_2 yields

$$(3.77) \qquad K_1^4 \le \varepsilon (\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2)$$

$$+ c(\varepsilon,t)(d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|u\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{0,\Omega^t}^2$$

$$+ \sup_{0 \le t' \le t} \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widehat{u}\|_{1+\alpha,\widehat{\Omega}^t}^2 + \|\widehat{\vartheta}_\sigma\|_{\alpha,\widehat{\Omega}^t}^2 + \|\widehat{\eta}_\sigma\|_{\alpha,\widehat{\Omega}^t}^2)$$

$$+ \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 \|\widetilde{\vartheta}_\sigma\|_{2,\widehat{\Omega}^t}^2],$$

where $c(\varepsilon,t)$ is a positive continuous function increasing with respect to t. Now, we estimate

$$(3.78) \qquad K_1^3 \le c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_{0}^{h_0} dh \int\limits_{0}^{t-h_0} dt' \frac{|\int_{t'}^{t'+h} \widehat{u}_z \, d\tau|^2}{h^{3/2+\alpha}}$$

$$\leq c(\varepsilon)h_0^{3/2-\alpha} \int\limits_0^{t-h_0} dt' \int\limits_{\widehat{S}} dz_{\widehat{S}} \left(\sup_{h>0} \frac{1}{h} \int\limits_{t'}^{t'+h} |\widehat{u}_z| \, d\tau \right)^2$$

$$\leq c(\varepsilon)t^{3/2-\alpha} \int\limits_0^t \|\widehat{u}_z\|_{0,\widehat{S}}^2 \, dt' \leq c(\varepsilon)t^{3/2-\alpha}(\varepsilon_1^{1-\varkappa}\|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + c\varepsilon_1^{-\varkappa}\|\widehat{u}\|_{0,\widehat{\Omega}^t}^2)$$

$$\leq \varepsilon\|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + c(\varepsilon,t)\|\widehat{u}\|_{0,\widehat{\Omega}^t}^2,$$

where we assumed $\varepsilon_1 = (\varepsilon/(c(\varepsilon)t^{3/2-\alpha}))^{1/(1-\varkappa)}$.

Next, we have

$$(3.79) \qquad K_1^1 \leq c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_0^{h_0} dh \int\limits_0^{t-h_0} dt' \frac{|\Delta_{t'}\widehat{\eta}_\sigma|^2|\widetilde{\eta}_\sigma|^2}{h^{3/2+\alpha}}$$

$$+ c(\varepsilon) \int\limits_{\widehat{S}} dz_{\widehat{S}} \int\limits_0^{h_0} dh \int\limits_0^{t-h_0} dt' \frac{|\Delta_{t'}\widehat{\vartheta}_\sigma|^2|\widetilde{\eta}_\sigma|^2}{h^{3/2+\alpha}}$$

$$\leq c(\varepsilon) \sup_{0\leq t'\leq t} \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 (|K_1^4| + |K_1^3|)$$

$$\leq c(\varepsilon)t^{3/2-\alpha} \sup_{0\leq t'\leq t} \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2).$$

In the same way we estimate $|K_1^2|$.

Thus, taking into account (3.74)–(3.79) we get

$$(3.80) \quad |K_1| \leq \varepsilon(\|u\|_{2+\alpha,\Omega^t}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^t}^2) + c(\varepsilon,t)[d\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2$$

$$+ \|u\|_{0,\Omega^t}^2 + \|\vartheta_\sigma\|_{0,\Omega^t}^2 + \|k\|_{0,\Omega^t}^2 + \sup_{0\leq t'\leq t} \|\widehat{u}\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widehat{u}\|_{1+\alpha,\widehat{\Omega}^t}^2$$

$$+ \|\widehat{\vartheta}_\sigma\|_{\alpha,\widehat{\Omega}^t}^2 + \|\widehat{\eta}_\sigma\|_{\alpha,\widehat{\Omega}^t}^2) + \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 \|\widetilde{\vartheta}_\sigma\|_{2,\widehat{\Omega}^t}^2$$

$$+ \sup_{0\leq t'\leq t} \|\widetilde{\eta}_\sigma\|_{1+\alpha,\widehat{\Omega}}^2 (\|\widetilde{\vartheta}_\sigma\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2)].$$

In the same way we estimate K_2, K_3, K_4 obtaining

$$(3.81) \quad |K_2| \leq \varepsilon\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + c(\varepsilon)t^{3/2-\alpha}\|\widetilde{u}\|_{2+\alpha,\Omega^t}^4(1 + \|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2),$$

$$(3.82) \quad |K_3| \leq \varepsilon\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + c(\varepsilon)t^{3/2-\alpha}(\|\widehat{u}\|_{2+\alpha,\Omega^t}^2 \|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + \|\widetilde{u}\|_{0,\widehat{\Omega}^t}^2)$$

$$(3.83) \quad |K_4| \leq \varepsilon\|\widetilde{u}\|_{2+\alpha,\widehat{\Omega}^t}^2 + c(\varepsilon)t^{3/2-\alpha}(\sup_{0\leq t'\leq t} \|\widetilde{u}\|_{1+\alpha,\widehat{\Omega}}^2$$

$$+ \sup_{0\leq t'\leq t} \|\widetilde{\eta}_\sigma\|_{\alpha,\widehat{\Omega}}^2 + \sup_{0\leq t'\leq t} \|\widetilde{\vartheta}_\sigma\|_{\alpha,\widehat{\Omega}}^2)\|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2(1 + \|\widehat{u}\|_{2+\alpha,\widehat{\Omega}^t}^2).$$

Now, inequalities (3.68), (3.69), (3.74), (3.80)–(3.83) yield (3.67).

This completes the proof of the lemma. ∎

The lemma below can be proved in the same way as Lemma 3.9.

LEMMA 3.10. *Let the assumptions of Theorem* 3.1 *be satisfied. Then*

$$(3.84) \quad c_{13} \int_{\widehat{\Omega}} \|\widetilde{\vartheta}_{\sigma t'}\|^2_{\alpha/2,(0,t)} \, dz + \frac{\varkappa}{2} \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widehat{\Omega}} (|\nabla(\Delta_{t'}\widetilde{\vartheta}_\sigma)|^2)|_{t'=t-h_0} dz$$

$$\leq \varepsilon(\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|u\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t})$$

$$+ d\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t} + c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t}$$

$$+ \|\eta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{\alpha,\Omega^t} + \|\overline{\vartheta}\|^2_{\alpha+1/2,S^t}$$

$$+ \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}(\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t} + \sup_{0\leq t'\leq t}\|\widehat{\vartheta}_\sigma\|^2_{1+\alpha,\widehat{\Omega}}$$

$$+ \sup_{0\leq t'\leq t}\|\widehat{\eta}_\sigma\|^2_{1+\alpha,\widehat{\Omega}} + \sup_{0\leq t'\leq t}\|\widehat{u}\|^2_{1+\alpha,\widehat{\Omega}}$$

$$+ \|\widehat{u}\|^2_{1+\alpha,\widehat{\Omega}^t}\sup_{0\leq t'\leq t}\|\widehat{u}\|^2_{1+\alpha,\widehat{\Omega}} + \|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t}$$

$$+ \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t}) + \|u\|^2_{2+\alpha,\Omega^t}\|u\|^2_{1+\alpha,\Omega^t}$$

$$+ \sup_{0\leq t'\leq t}\|\widehat{\eta}\|^2_{1+\alpha,\widehat{\Omega}}\|\widetilde{k}\|^2_{\alpha,\widehat{\Omega}^t} + \|\widehat{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t}\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t}]$$

where $t \leq T$, $c_{13} > 0$ is a constant, and $c(\varepsilon,t)$ is a positive continuous function nondecreasing with respect to t.

REMARK 3.11. As in the proofs of Lemmas 3.4 and 3.6, we had to derive inequalities (3.67) and (3.84) in such a way as to obtain functions $c(\varepsilon,t)$ on the right-hand sides nondecreasing with respect to t.

Now, we can finish the proof of Theorem 3.1.

Proof of Theorem 3.1. In addition to the estimates derived above we need one more estimate for $\int_\Omega \|\eta_{\sigma t'}\|^2_{\alpha/2,(0,t)} \, d\xi$, where $t \leq T$.

Using $(1.6)_2$ and (3.31) we get

$$(3.85) \quad \int_\Omega \|\eta_{\sigma t'}\|^2_{\alpha/2,(0,t)} \, d\xi$$

$$\leq \varepsilon\Big(\|u\|^2_{2+\alpha,\Omega^t} + \int_0^t \|\eta_\sigma\|^2_{1+\alpha,\Omega} \, dt' + \int_0^t \|\vartheta_\sigma\|^2_{1+\alpha,\Omega} \, dt'\Big)$$

$$+ c(\varepsilon,t)(\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|u\|^4_{2+\alpha,\Omega^t}).$$

Now let us introduce a partition of unity $(\{\widetilde{\Omega}_i\}, \{\zeta_i\})$, $\Omega \subset \bigcup_{i \in \mathfrak{M} \cup \mathfrak{N}} \widetilde{\Omega}_i$, such that $\widetilde{\Omega}_i$, $i \in \mathfrak{M}$, is an interior subdomain, and $\widetilde{\Omega}_i$, $i \in \mathfrak{N}$, is a boundary

subdomain. Then we introduce the quantity

$$(3.86) \qquad \phi(t, \Omega) = \sum_{i \in \mathfrak{M}} \phi_1(t, \widetilde{\Omega}_i) + \sum_{i \in \mathfrak{N}} \phi_2(t, \widehat{\Omega}_i),$$

where $\widehat{\Omega}_i = \Phi(\widetilde{\Omega}_i)$ is given by (2.4), ϕ_1 and ϕ_2 are given by (3.36) and (3.41), respectively.

Now taking into account estimates (3.85), (3.37), (3.40), (3.67) and (3.84) (where in all terms of (3.40), (3.67), (3.84) except ϕ_2 and ϕ_5 we return to the old variables ξ) we obtain (3.1).

This completes the proof of the theorem. ∎

4. A remark on an estimate derived in [14]. In [14] we obtained an estimate which showed the increase of the regularity of the solution. Since we omitted r and $\overline{\theta}$ in our considerations, the estimate derived in [14] had the form

$$(4.1) \qquad \sup_{t_0 + \lambda \le t \le T} \|u\|^2_{2+\alpha, \Omega} + \sup_{t_0 + \lambda \le t \le T} \|\vartheta_\sigma\|^2_{2+\alpha, \Omega} \le c(K)\overline{K},$$

where $\overline{K} = \|n\|^2_{2+\alpha, \Omega^T} + \|\vartheta_\sigma\|^2_{2+\alpha, \Omega^T}$,

$$K = \overline{K} + \sup_{0 \le t \le T} \|u\|^2_{1+\alpha, \Omega} + \sup_{0 \le t \le T} \|\vartheta_\sigma\|^2_{1+\alpha, \Omega}$$
$$+ \|\eta_\sigma\|^2_{1+\alpha, \Omega^T} + \sup_{0 \le t \le T} \|\eta_\sigma\|^2_{1+\alpha, \Omega};$$

$c(K)$ is a positive nondecreasing function of K depending also on T.

To obtain the analogous inequality for nonvanishing r and $\overline{\theta}$ we have to estimate additionally $\|\eta' k^{(s)}\|^2_{\alpha, Q_{\lambda/2}}$, $\|\eta^{(s)}_* k_\lambda\|^2_{\alpha, Q_{\lambda/2}}$ and $\|\overline{\vartheta}^{(s)}\|^2_{\alpha+1/2, G_{\lambda/2}}$ (see [14]), where $\eta'(\xi, t) = \eta(\xi, t - s)$, $k^{(s)}(\xi, t) = k_\lambda(\xi, t) - k_\lambda(\xi, t - s)$, $\eta^{(s)}_*(\xi, t) = \eta(\xi, t) - \eta(\xi, t - s)$, $\overline{\vartheta}^{(s)}(\xi, t) = \overline{\vartheta}_\lambda(\xi, t) - \overline{\vartheta}_\lambda(\xi, t - s)$, $k_\lambda = \zeta_\lambda k$, $\overline{\vartheta}_\lambda = \zeta_\lambda \overline{\vartheta}$; $\zeta_\lambda \in C^\infty$ is a function such that $\zeta_\lambda(t) = 1$ for $t \ge t_0 + \lambda$, $\zeta_\lambda(t) = 0$ for $t \le t_0 + \lambda/2$; $Q_{\lambda/2} = \Omega \times (t_0 + \lambda/2, T)$, $G_{\lambda/2} = S \times (t_0 + \lambda/2, T)$, $0 < s < t_0$.

By simple calculations we get

$$(4.2) \quad \|\eta' k^{(s)}\|^2_{\alpha, Q_{\lambda/2}} + \|\eta^{(s)}_* k_\lambda\|^2_{\alpha, Q_{\lambda/2}}$$

$$\le c\Big(\int_{t_0+\lambda/2}^{T} \|\eta' k^{(s)}\|^2_{1, \Omega}\, dt + \int_{\Omega} \|\eta' k^{(s)}\|^2_{1, (t_0+\lambda/2, T)}\, d\xi$$

$$+ \int_{t_0+\lambda/2}^{T} \|\eta^{(s)}_* k_\lambda\|^2_{1, \Omega}\, dt + \int_{\Omega} \|\eta^{(s)}_* k_\lambda\|^2_{1, (t_0+\lambda/2, T)}\, d\xi \Big)$$

$$\le c(K) s^{1+\alpha} \|r\|^2_{C^2_B(\mathbb{R}^3 \times \mathbb{R}_+)}$$

and

$$(4.3) \qquad \|\overline{\vartheta}^{(s)}\|^2_{\alpha+1/2,G_{\lambda/2}} \le c\|\overline{\vartheta}^{(s)}\|^2_{\alpha+1,Q_{\lambda/2}}$$

$$\le c\Big(\int\limits_{t_0+\lambda/2}^{T} \|\overline{\vartheta}^{(s)}\|^2_{2,\Omega}\, dt + \int\limits_{\Omega} \|\overline{\vartheta}^{(s)}\|^2_{1,(t_0+\lambda/2,T)}\, d\xi \Big)$$

$$\le c(K)s^{1+\alpha}\|\overline{\theta}\|^2_{C^3_B(\mathbb{R}^3\times\mathbb{R}_+)}.$$

Taking into account (4.2)–(4.3) and the considerations from [14] we obtain

THEOREM 4.1. *Let* $(u,\vartheta,\eta) \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times C([0,T];W_2^{2+\alpha}(\Omega))\cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$ $(\alpha \in (3/4,1))$ *be the local solution of problem* (1.1). *Then for any* $0 < t_0 < T$ *and* $\lambda > 0$ *the function* u *is in* $C([t_0+\lambda,T];W_2^{2+\alpha}(\Omega))$ *and*

$$(4.4) \qquad \sup_{t_0+\lambda\le t\le T} \|u\|^2_{2+\alpha,\Omega} + \sup_{t_0+\lambda\le t\le T} \|\vartheta_\sigma\|^2_{2+\alpha,\Omega}$$

$$\le c(K)(\overline{K} + \|r\|_{C^2_B(\mathbb{R}^3\times\mathbb{R}_+)} + \|\overline{\theta}\|^2_{C^3_B(\mathbb{R}^3\times\mathbb{R}_+)}).$$

REMARK 4.2. By Theorem 4.1 the global existence theorem proved in [12] holds for r and $\overline{\theta}$ satisfying assumption (1.7).

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