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A. Benkirane (Atlas Fès)
A. Elmahi (Fès)
D. Meskine (Atlas Fès)

AN EXISTENCE THEOREM FOR A CLASS OF ELLIPTIC PROBLEMS IN L^1

Abstract. We prove an existence result for solutions of some class of nonlinear elliptic problems having natural growth terms and L^1 data.

1. Introduction. Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property, and let $f \in L^1(\Omega)$. Consider the following nonlinear Dirichlet problem:

$$A(u) + g(x, u, \nabla u) = f \tag{1.1}$$

where $A(u) = -\operatorname{div}(a(x, u, \nabla u))$ is a Leray–Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, with M an N-function which satisfies the Δ_2-condition, and g is a nonlinearity having "natural growth" and satisfies the classical "sign condition" with respect to u.

In the variational case (i.e. where $f \in W^{-1}E_{\overline{M}}(\Omega)$), it is well known that Gossez solved (1.1) in the case where g depends only on x and u. If g does depend also on ∇u, an existence theorem has recently been proved by Benkirane and Elmahi in [4] and [5].

In the case where $f \in L^1(\Omega)$, they also give an existence result in [6] if the nonlinearity g satisfies further the following coercivity condition:

$$|g(x, s, \zeta)| \geq \beta M(|\zeta|/\mu) \quad \text{for } |s| \geq \gamma. \tag{1.2}$$

It is our purpose, in this paper, to prove an existence result for some class of problems of the kind (1.1), without assuming the coercivity condition (1.2). The technical method used here allows us not only to generalize the result of [21], but also to give a different proof for it.

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For the classical variational case in Orlicz spaces, the reader is referred, for example, to [3–5, 7, 15–18], and for some results in the L^p case, to [2, 10–14, 20, 21].

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-*function*, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$. Equivalently, M admits a representation $M(t) = \int_0^t a(s)\,ds$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M} = \int_0^t \overline{a}(s)\,ds$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \le t\}$ (see [1]).

The N-function M is said to satisfy the Δ_2-*condition*, written $M \in \Delta_2$, if for some $k > 0$,

$$M(2t) \le kM(t), \qquad \forall t \ge 0; \tag{2.1}$$

when (2.1) holds only for $t \ge$ some $t_0 > 0$ then M is said to satisfy the Δ_2-condition *near infinity*.

We will extend these N-functions to even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, i.e. for each $\varepsilon > 0$, $P(t)/Q(\varepsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} Q^{-1}(t)/P^{-1}(t) = 0$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The *Orlicz class* $K_M(\Omega)$ (resp. the *Orlicz space* $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\,dx < \infty \qquad \Big(\text{resp. } \int_\Omega M(u(x)/\lambda)\,dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M(u(x)/\lambda)\,dx \le 1\Big\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz–Sobolev space. $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$, and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the *modular convergence* in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0 \quad \text{for all } |\alpha| \le 1.$$

This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If M satisfies the Δ_2-condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [15, 16]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. We recall some lemmas introduced in [5] which will be used in this paper.

LEMMA 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i}u & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\},\\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}.\end{cases}$$

LEMMA 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Suppose that the set of discontinuity points of F' is finite. Let M be an N-function. Then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

2.6. We now give the following lemma which concerns operators of the Nemytskiĭ type in Orlicz spaces (see [5]).

LEMMA 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,*

$$|f(x,s)| \le c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskiĭ operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), 1/k_2) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2\}$ into $E_Q(\Omega)$.

3. The main result. Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$, with the segment property.

Let M be an N-function satisfying the Δ_2-condition near infinity and let P be an N-function such that $P \ll M$. Let $A(u) = -\mathrm{div}(a(x, u, \nabla u))$ be a Leray–Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for a.e. $x \in \Omega$ and for all $\zeta, \zeta' \in \mathbb{R}^N$ ($\zeta \ne \zeta'$) and all $s \in \mathbb{R}$,

$$|a(x,s,\zeta)| \le h(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|), \tag{3.1}$$

$$(a(x,s,\zeta) - a(x,s,\zeta'))(\zeta - \zeta') > 0, \tag{3.2}$$

$$a(x,s,\zeta)\zeta \ge \alpha M(|\zeta|/\lambda), \tag{3.3}$$

with $\alpha, \lambda > 0,\ k_1, k_2, k_3, k_4 \ge 0,\ h \in E_{\overline{M}}(\Omega)$.

Furthermore let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$,

$$g(x,s,\zeta)s \ge 0, \tag{3.4}$$

$$|g(x,s,\zeta)| \le b(|s|)(c(x) + M(|\zeta|/\mu)), \tag{3.5}$$

where $b : \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function, c is a given nonnegative function in $L^1(\Omega)$, and $\mu > 0$. Finally, we assume that

$$f \in L^1(\Omega). \tag{3.6}$$

Consider the following Dirichlet problem:

$$A(u) + g(x, u, \nabla u) = f \quad \text{in } \Omega. \tag{3.7}$$

We define $T_0^{1,M}(\Omega)$ to be the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_M(\Omega)$, where

$$T_k(s) = \max(-k, \min(k, s)) \quad \text{for } s \in \mathbb{R} \text{ and } k \ge 0.$$

We shall prove the following existence theorem.

THEOREM 3.1. *Assume that* (3.1)–(3.6) *hold true. Then there exists at least one solution of* (3.7) *in the following sense*:

$$
\text{(P)} \quad \begin{cases} u \in T_0^{1,M}(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad \displaystyle\leq \int_\Omega fT_k(u-v)\,dx, \quad \forall v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega),\ \forall k>0. \end{cases}
$$

REMARK 3.1. Our result covers the critical case $M(t) = |t|\log(1+|t|)$ which satisfies the Δ_2-condition but $\overline{M} \notin \Delta_2$.

If $M(t) = |t|^p/p$, we obtain the result given in [21].

REMARK 3.2. If u is a solution of (P) such that $a(x,u,\nabla u) \in L^1(\Omega)$, then it is also a solution of (3.7) in the distributional sense. This is the case, for example, if we take $a(x,s,\zeta) = a(x,s)|\zeta|^{p-2}\zeta\log^\beta(1+|\zeta|)$ with $a(x,s)$ a Carathéodory function satisfying

$$
\alpha \leq a(x,s) \leq \gamma \quad \text{for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R}
$$

where $\alpha,\beta,\gamma>0$. Indeed, by choosing $0<\varepsilon<\frac{1}{\beta}\frac{p-1}{N-1}$, it is easy to see that there exists $C_\varepsilon>0$ such that

$$
\log^\beta(1+|\zeta|) \leq C_\varepsilon|\zeta|^{\beta\varepsilon} \quad \text{for } |\zeta| \text{ large enough,}
$$

and so that

$$
\int_\Omega |a(x,u,\nabla u)|\,dx \leq \gamma C_\varepsilon \int_\Omega |\nabla u|^{p-1+\beta\varepsilon}\,dx + C.
$$

In view of the fact that $p-1+\beta\varepsilon < N(p-1)/(N-1)$ one easily sees that $a(x,u,\nabla u) \in L^1(\Omega)$.

Proof of Theorem 3.1

STEP 1: *A priori estimates.* Consider the approximate problems:

$$
(3.8) \quad \begin{cases} u_n \in W_0^1L_M(\Omega),\ g(x,u_n,\nabla u_n) \in L^1(\Omega),\ g(x,u_n,\nabla u_n)u_n \in L^1(\Omega), \\ \displaystyle\langle A(u_n),v\rangle + \int_\Omega g(x,u_n,\nabla u_n)v\,dx = \int_\Omega f_n v\,dx, \\ \qquad\qquad\qquad\qquad \forall v\in W_0^1L_M(\Omega)\cap L^\infty(\Omega), \end{cases}
$$

where f_n is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$. By Theorem 3.1 of [5], there exists at least one solution u_n of (3.8). Taking $v = T_k(u_n)$ as test function in (3.8) gives

$$
\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n)\,dx = \int_\Omega f_nT_k(u_n)\,dx
$$

and by using the fact that $g(x,u_n,\nabla u_n)T_k(u_n)\geq 0$, we obtain

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \leq Ck.$$

Thanks to (3.3), one easily has

$$\alpha\int_{\Omega} M(|\nabla T_k(u_n)|/\lambda)\,dx \leq Ck. \tag{3.9}$$

On the other hand, thanks to Lemma 5.7 of [15], there exist two positive constants c_1 and c_2 such that

$$\int_{\Omega} M(T_k(u_n))\,dx \leq c_1\int_{\Omega} M(c_2|\nabla T_k(u_n)|)\,dx. \tag{3.10}$$

By the Δ_2-condition there exist another two positive constants c_1' and c_2' such that

$$M(c_2t)\leq c_1'+c_2'M(t/\lambda)\quad \text{for all } t\geq 0.$$

We then deduce, by using (3.9) and (3.10), that

$$\int_{\Omega} M(T_k(u_n))\,dx \leq c_3'+c_4'k,$$

which implies

$$M(k)\mathrm{meas}\{|u_n|>k\}\leq c_3'+c_4'k$$

and finally

$$\mathrm{meas}\{|u_n|>k\}\leq \frac{c_3'+c_4'k}{M(k)},\quad \forall n \text{ and } \forall k>0. \tag{3.11}$$

We have, for every $\delta>0$,

$$\begin{aligned}\mathrm{meas}\{|u_n-u_m|>\delta\}\leq{}& \mathrm{meas}\{|u_n|>k\}+\mathrm{meas}\{|u_m|>k\}\\ &+\mathrm{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.\end{aligned} \tag{3.12}$$

Since $T_k(u_n)$ is bounded in $W_0^1L_M(\Omega)$, there exists some $v_k\in W_0^1L_M(\Omega)$ such that

$$T_k(u_n)\rightharpoonup v_k \quad \begin{array}{l}\text{weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\prod L_M,\prod E_{\overline{M}}),\\ \text{strongly in } E_M(\Omega),\end{array}$$

and almost everywhere in Ω. Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon>0$. Then, by (3.11) and (3.12), there exists some $k(\varepsilon)>0$ such that

$$\mathrm{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon$$

for all $n,m\geq n_0(k(\varepsilon),\delta)$. This proves that (u_n) is a Cauchy sequence in measure, thus u_n converges almost everywhere to some measurable function u.

Finally, by Lemma 4.4 of [15], we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \begin{array}{l}\text{weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}),\\ \text{strongly in } E_M(\Omega).\end{array}$$

Let Q be an N-function such that $M \ll Q$ and the continuous embedding $W_0^1L_M(\Omega) \subset E_Q(\Omega)$ holds (see [15]). Let $\varepsilon > 0$. Then there exists $C_\varepsilon > 0$, as in [6], such that

$$|a(x,s,\zeta)| \le h(x) + C_\varepsilon + k_1\overline{M}^{-1}Q(\varepsilon|s|) + k_3\overline{M}^{-1}M(\varepsilon|\zeta|) \tag{3.13}$$

for a.e. $x \in \Omega$ and for all $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$. From (3.9) and (3.13) we deduce that $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Step 2: *Almost everywhere convergence of the gradients.* Fix $r,k > 0$ and define $\Omega_r = \{x \in \Omega : |\nabla T_k(u(x))| \le r\}$. We denote by χ_r the characteristic function of Ω_r. Consider now, as in [9],

$$I_{n,r} = \int_{\Omega_r} \{[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\}^\theta\,dx$$

where $0 < \theta < 1$. Let A_n be the expression in braces above. Then for any $\eta > 0$,

$$I_{n,r} = \int_{\Omega_r\cap\{|T_k(u_n)-T_k(u)|\le\eta\}} A_n^\theta\,dx + \int_{\Omega_r\cap\{|T_k(u_n)-T_k(u)|>\eta\}} A_n^\theta\,dx.$$

From (3.13) we deduce that A_n is bounded in $L^1(\Omega)$ and by applying the Hölder inequality we obtain

$$I_{n,r} \le C_1\Big\{\int_{\Omega_r\cap\{|T_k(u_n)-T_k(u)|\le\eta\}} A_n\,dx\Big\}^\theta + C_2\,\mathrm{meas}\{x : |T_k(u_n) - T_k(u)| > \eta\}^{1-\theta}. \tag{3.14}$$

On the other hand, for any $s \ge r$ we have

$$\begin{aligned}
&\int_{\Omega_r\cap\{|T_k(u_n)-T_k(u)|\le\eta\}} A_n\,dx \\
&\le \int_{\{|T_k(u_n)-T_k(u)|\le\eta\}} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s)]\,dx \\
&\le \int_{\{|T_k(u_n)-T_k(u)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx \\
&\quad - \int_{\{|T_k(u_n)-T_k(u)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx
\end{aligned} \tag{3.15}$$

$$= \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_\eta(T_k(u_n) - T_k(u))\,dx$$
$$+ \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u) \chi_{\Omega\setminus\Omega_s} \chi_{\{|T_k(u_n)-T_k(u)|\le\eta\}}\,dx$$
$$- \int_{\{|T_k(u_n)-T_k(u)|\le\eta\}} a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx.$$

The use of the test function $T_\eta(u_n - T_k(u))$ in (3.8) gives

$$\langle A(u_n), T_\eta(u_n - T_k(u))\rangle + \int_\Omega g(x, u_n, \nabla u_n) T_\eta(u_n - T_k(u)\,dx \le \int_\Omega f_n T_\eta(u_n - T_k(u)\,dx, \tag{3.16}$$

which implies

$$\langle A(u_n), T_\eta(u_n - T_k(u))\rangle \le C\eta. \tag{3.17}$$

Note that

$$\begin{aligned}\langle A(u_n), T_\eta(u_n - T_k(u))\rangle &\ge \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_\eta(T_k(u_n) - T_k(u))\,dx \\ &\quad - \int_{|u_n|>k} |a(x, T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))|\,|\nabla T_k(u)|\,dx.\end{aligned} \tag{3.18}$$

We denote by $\varepsilon_i^\eta(n)$ $(i = 1, 2, \ldots)$ various sequences of real numbers which tend to 0 as $n \to \infty$ for η fixed.

The second term of the right hand side of (3.18) tends to zero since $a(x, T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$ while $\chi_{\{|u_n|>k\}}|\nabla T_k(u)| \to 0$ strongly in $(E_M(\Omega))^N$. Consequently, from (3.17) and (3.18), we have

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_\eta(T_k(u_n) - T_k(u))\,dx \le C\eta + \varepsilon_1^\eta(n). \tag{3.19}$$

Since $a(x, T_k(u_n), \nabla T_k(u_n))\chi_{\{|T_k(u_n)-T_k(u)|\le\eta\}}$ is bounded in $(L_{\overline{M}}(\Omega))^N$ it follows that $a(x, T_k(u_n), \nabla T_k(u_n))\chi_{\{|T_k(u_n)-T_k(u)|\le\eta\}}$ converges to h weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M(\Omega))$, for some $h \in (L_{\overline{M}}(\Omega))^N$. We deduce that the second term of the right hand side of (3.15) tends to

$$\int_{\Omega\setminus\Omega_s} h \nabla T_k(u)\,dx \quad \text{as } n \to \infty.$$

The third term of the right hand side of (3.15) tends to 0 since $a(x, T_k(u_n), \nabla T_k(u)\chi_s)\chi_{\{|T_k(u_n)-T_k(u)|\le\eta\}}$ converges strongly to $a(x, T_k(u), \nabla T_k(u)\chi_s)$ in $E_{\overline{M}}(\Omega)^N$ by Lemma 2.3 while $\nabla T_k(u_n)$ tends weakly to

$\nabla T_k(u)$ and

$$\int_\Omega a(x, T_k(u), \nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\,dx = 0.$$

Finally, from (3.15) and in view of (3.19) we have

$$I_{n,r} \le C_1\Big(C\eta + \varepsilon_2^\eta(n) + \int_\Omega h\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx\Big)^\theta + C_2 \operatorname{meas}\{x : |T_k(u_n) - T_k(u)| > \eta\}^{1-\theta},$$

which gives, by passing to the lim sup over n,

$$\limsup_{n\to\infty} I_{n,r} \le C\Big(\eta + \int_\Omega h\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx\Big)^\theta.$$

Then by, letting $s \to \infty$ and choosing η small enough, we obtain

$$\lim_{n\to\infty} I_{n,r} = 0$$

and so, as in [4],

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.20}$$

STEP 3: *Strong convergence of $M(|\nabla T_k(u_n)|/\mu)$ in $L^1(\Omega)$ (i.e. modular convergence of $\nabla T_k(u_n)$ in $(L_M(\Omega))^N$).* Now fix $k > 0$, and let

$$\gamma = \left(K\,\frac{b(k)}{2\alpha}\right)^2, \qquad \phi(s) = s\exp(\gamma s^2).$$

It is well known that

$$\phi'(s) - K\,\frac{b(k)}{\alpha}\,|\phi(s)| \ge \frac{1}{2}, \qquad \forall s \in \mathbb{R}, \tag{3.21}$$

where K is a constant which will be used later.

Consider now the function $h_m, m > 0$, defined by

$$h_m(t) = \begin{cases} 1 & \text{if } |t| \le m,\\ -(t/m)\operatorname{sgn}(t) + 2 & \text{if } m \le |t| \le 2m,\\ 0 & \text{if } |t| > 2m. \end{cases}$$

Let $v_{n,m} = h_m(u_n)\phi(z_n)$ with $z_n = T_k(u_n) - T_k(u)$. The use of $v_{n,m}$ as test function in (3.8) gives

$$\langle A(u_n), h_m(u_n)\phi(z_n)\rangle + \int_\Omega g(x, u_n, \nabla u_n)h_m(u_n)\phi(z_n)\,dx = \int_\Omega f_n h_m(u_n)\phi(z_n)\,dx.$$

It follows that

$$\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)\phi(z_n)\,dx + \int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n)\,dx \le \int_\Omega f_n h_m(u_n)\phi(z_n)\,dx. \tag{3.22}$$

Denote by $\varepsilon_m^1(n), \varepsilon_m^2(n), \ldots$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of m.

Since $g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n) \ge 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, we deduce from (3.22) that

$$\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)\phi(z_n)\,dx + \int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n)\,dx \le \int_\Omega f_n h_m(u_n)\phi(z_n)\,dx = \varepsilon_m^1(n). \tag{3.23}$$

The first term of the left hand side of (3.23) can be written as

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)&[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx \\ &= \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx \\ &\quad - \int_{\{|u_n|> k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\phi'(z_n)\,dx.\end{aligned} \tag{3.24}$$

For the second term of the right hand side of (3.24), we have

$$\Big|\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\phi'(z_n)\,dx\Big| \le C_k \int_\Omega |a(x,T_{2m}(u_n),\nabla T_{2m}(u_n))|\,|\nabla T_k(u)|\chi_{\{|u_n|>k\}}\,dx$$

where $C_k = \phi'(2k)$. The right hand side of the last inequality tends to 0 as n tends to infinity. Indeed, the sequence $(a(x,T_{2m}(u_n),\nabla T_{2m}(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ while $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(E_M(\Omega))^N$.

For the first term of the right hand side of (3.24), we can write

$$\begin{aligned}\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)&[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx \\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)] \\ &\quad \times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\phi'(z_n)\,dx\end{aligned} \tag{3.25}$$

$$+\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\phi'(z_n)\,dx$$
$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u_n)\phi'(z_n)\,dx.$$

The second term of the right hand side of (3.25) tends to 0 since

$$a(x,T_k(u_n),\nabla T_k(u)\chi_s)h_m(u_n)\phi'(z_n)\to a(x,T_k(u),\nabla T_k(u)\chi_s)h_m(u) \quad \text{strongly in } (E_{\overline{M}}(\Omega))^N$$

by Lemma 2.3 and

$$\nabla T_k(u_n)\rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L_M(\Omega))^N \text{ for } \sigma\Big(\prod L_M(\Omega),\prod E_{\overline{M}}(\Omega)\Big).$$

The third term of the right hand side of (3.25) tends to the quantity $-\int_\Omega a(x,T_k(u),\,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx$ as $n\to\infty$ since

$$a(x,T_k(u_n),\nabla T_k(u)\chi_s)\rightharpoonup a(x,T_k(u),\nabla T_k(u)\chi_s) \quad \text{weakly for } \sigma\Big(\prod E_{\overline{M}}(\Omega),\prod L_M(\Omega)\Big).$$

Consequently, from (3.24) we have

$$\begin{aligned}(3.26)\qquad &\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\,dx\\ &\quad=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\phi'(z_n)\,dx\\ &\qquad-\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx+\varepsilon_m^2(n).\end{aligned}$$

On the other hand

$$\Big|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)\phi(z_n)\,dx\Big| \le \frac{2\phi(2k)}{m}\int_{\{m\le|u_n|\le 2m\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx$$

and by using $T_m(u_n-T_m(u_n))$ as test function in (3.8), we obtain

$$(3.27)\qquad \Big|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)\phi(z_n)\,dx\Big|\le 2\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|\,dx.$$

Thanks to the Δ_2-condition there exist two positive constants K and K' such that

$$(3.28)\qquad M(t/\mu)\le KM(t/\lambda)+K',\qquad \forall t\ge 0.$$

If we denote by $J_{n,m}$ the third term of the left hand side of (3.23), then (3.28) yields

$$(3.29)\quad |J_{n,m}| \le \int_{\{|u_n|\le k\}} b(k)(c(x)+K'+KM(|\nabla u_n|/\lambda))h_m(u_n)|\phi(z_n)|\,dx$$
$$\le b(k)\int_\Omega (c(x)+K')|\phi(z_n)|\,dx$$
$$+ K\,\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)|\phi(z_n)|\,dx$$
$$\le \varepsilon_m^3(n) + K\,\frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]$$
$$\times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)|\phi(z_n)|\,dx.$$

Indeed, we have

$$(3.30)\quad \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)|\phi(z_n)|\,dx$$
$$= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]$$
$$\times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)|\phi(z_n)|\,dx$$
$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s h_m(u_n)|\phi(z_n)|\,dx$$
$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)|\phi(z_n)|\,dx.$$

It is easy to see that the second term of the right hand side of (3.30) tends to 0 as n tends to infinity, since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, and

$$\nabla T_k(u)\chi_s h_m(u_n)|\phi(z_n)| \to 0 \quad \text{strongly in } (E_M(\Omega))^N$$

by Lebesgue's theorem.

The third term of the right hand side of (3.30) also tends to 0 since

$$(3.31)\quad a(x,T_k(u_n),\nabla T_k(u)\chi_s)|\phi(z_n)| \to 0 \quad \text{strongly in } (E_{\overline{M}}(\Omega))^N$$

by Lemma 2.3 while

$$(3.32)\quad [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n) \quad \text{is bounded in } (L_M(\Omega))^N.$$

Combining (3.26) and (3.29) we obtain

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]$$
$$\times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\left(\phi'(z_n) - K\,\frac{b(k)}{\alpha}\,|\phi(z_n)|\right)dx$$
$$\le \varepsilon_m^4(n) - \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx$$
$$+ \phi(2k)\int_{\{|u_n|\ge m\}} |f_n|\,dx,$$

which implies, by (3.21),

$$\begin{aligned}\int_\Omega &[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\,dx\\ &\le 2\varepsilon_m^4(n) - 2\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx\\ &\qquad + 4\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega &a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)\,dx\\ &\le \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s\,dx\\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\,dx\\ &\quad+2\varepsilon_m^4(n) - 2\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx\\ &\quad+4\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|\,dx.\end{aligned}$$

By passing to the lim sup over n, one has

$$\begin{aligned}\limsup_{n\to\infty}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)\,dx\\ &\le \limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s h_m(u_n)\,dx\\ &\quad+\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n)\,dx\\ &\quad-2\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}h_m(u)\,dx\\ &\quad+4\phi(2k)\int_{\{|u|\ge m\}}|f|\,dx.\end{aligned}\tag{3.33}$$

The second term of the right hand side of (3.33) tends to 0, since $a(x,T_k(u_n),\nabla T_k(u)\chi_s)\to a(x,T_k(u),\nabla T_k(u)\chi_s)$ strongly in $E_{\overline{M}}(\Omega)$ while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$.

The first term of the right hand side of (3.33) tends to

$$\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s h_m(u)\,dx$$

since $a(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\rightharpoonup a(x,T_k(u),\nabla T_k(u))h_m(u)$ weakly in

$(L_{\overline{M}}(\Omega))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M)$ while $\nabla T_k(u)\chi_s \in E_M(\Omega)$. We deduce that

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) h_m(u)\,dx \\ \le \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\chi_s h_m(u)\,dx \\ - 2\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s} h_m(u)\,dx \\ + 4\phi(2k)\int_{\{|u|\ge m\}} |f|\,dx.\end{aligned}$$

Passing again to the lim sup but now over m, and using the fact that $a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \in L^1(\Omega)$, $f \in L^1(\Omega)$ and $h_m(u) \to 1$ as $m \to \infty$, one easily obtains by Lebesgue's theorem

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) h_m(u_n)\,dx \\ \le \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\chi_s\,dx \\ - 2\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx.\end{aligned}$$

Using again the fact that $a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$ we get, since $\operatorname{meas}(\Omega\setminus\Omega_s) \to 0$,

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) h_m(u_n)\,dx \\ \le \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\,dx.\end{aligned}$$

On the other hand, by Fatou's lemma,

$$\begin{aligned}\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\,dx \\ \le \limsup_{m\to\infty}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) h_m(u_n)\,dx,\end{aligned}$$

which implies finally

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) h_m(u_n)\,dx \\ = \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\,dx.\end{aligned}\tag{3.34}$$

Taking now $(1-h_m(u_n))T_k(u_n)$ as test function in (3.8) we obtain

$$\langle A(u_n),(1-h_m(u_n))T_k(u_n)\rangle+\int_\Omega g(x,u_n,\nabla u_n)(1-h_m(u_n))T_k(u_n))\,dx$$
$$=\int_\Omega f_n(1-h_m(u_n))T_k(u_n)\,dx$$

and thanks to the sign condition (3.4) we obtain

$$\langle A(u_n),(1-h_m(u_n))T_k(u_n)\rangle\le\int_\Omega f_n(1-h_m(u_n))T_k(u_n)\,dx,$$

and so

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))\,dx\\&\quad\le\int_\Omega f_n(1-h_m(u_n))T_k(u_n)\,dx+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)T_k(u_n)\,dx.\end{aligned}\tag{3.35}$$

Since

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)T_k(u_n)\,dx\le\frac{k}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx$$

inequality (3.35) becomes

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))\,dx\\&\quad\le\int_\Omega f_n(1-h_m(u_n))T_k(u_n)\,dx+\frac{k}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx.\end{aligned}\tag{3.36}$$

By passing to the lim sup over n, one has

$$\begin{aligned}&\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))\,dx\\&\quad\le\int_\Omega f(1-h_m(u))T_k(u)\,dx+\limsup_{n\to\infty}\frac{k}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx.\end{aligned}$$

Passing again to the lim sup, but now over m, we obtain

$$\begin{aligned}&\limsup_{m\to\infty}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))\,dx\\&\quad\le\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{k}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx.\end{aligned}\tag{3.37}$$

On the other hand, since

$$\frac{1}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx\le\int_{\{|u_n|\ge m\}}|f_n|\,dx$$

one easily has

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{m}\int_{\{m\le|u_n|\le 2m\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0.$$

Hence, from (3.37) we deduce that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))\,dx = 0. \tag{3.38}$$

Now, write

$$\begin{aligned}a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)&\\ &= a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)\\ &\quad + a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n)),\end{aligned}$$

which gives, by (3.34) and (3.38),

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx&\\ &\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\end{aligned}$$

and so

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx&\\ &\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx.\end{aligned} \tag{3.39}$$

On the other hand, thanks to Fatou's lemma, we have

$$\begin{aligned}\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx&\\ &\le \liminf_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx.\end{aligned}$$

Consequently, in view of (3.39), we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx&\\ &= \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx.\end{aligned} \tag{3.40}$$

Thanks to (3.28) we have

$$M(|\nabla T_k(u_n)|/\mu)\le K' + KM(|\nabla T_k(u_n)|/\lambda)$$

and then by using (3.40), one obtains, by Vitali's theorem,

$$M(|\nabla T_k(u_n)|/\mu)\to M(|\nabla T_k(u)|/\mu)\quad \text{in } L^1(\Omega). \tag{3.41}$$

STEP 4: *Passage to the limit.* By using $T_k(u_n - v)$ as test function in (3.8), with $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, we get

$$\begin{aligned}(3.42)\qquad &\int_\Omega a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx \\ &\qquad + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n - v)\,dx = \int_\Omega f_n T_k(u_n - v)\,dx.\end{aligned}$$

By Fatou's lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M)$ one easily sees that

$$\begin{aligned}(3.43)\qquad &\int_\Omega a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u - v)\,dx \\ &\qquad \le \liminf_{n\to\infty} \int_\Omega a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx\end{aligned}$$

Our next purpose is to prove that

$$g(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega).$$

In virtue of Vitali's theorem, it is sufficient to prove that $g(x, u_n, \nabla u_n)$ is equiintegrable in $L^1(\Omega)$. On the one hand, by taking $T_1(u_n - T_l(u_n))$ as test function in (3.8), we obtain

$$\int_{\{|u_n|>l+1\}} |g(x, u_n, \nabla u_n)|\,dx \le \int_{\{|u_n|>l\}} |f_n|\,dx.$$

Let $\varepsilon > 0$. Then there exists $l(\varepsilon) \ge 1$ such that

$$(3.44)\qquad \int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\,dx < \varepsilon/2.$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |g(x, u_n, \nabla u_n)|\,dx \le &\int_E b(l(\varepsilon))(c(x) + M(|\nabla T_{l(\varepsilon)}(u_n)|/\mu))\,dx \\ &+ \int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\,dx.\end{aligned}$$

In view of (3.41) there exists $\eta(\varepsilon) > 0$ such that

$$\begin{aligned}(3.45)\qquad &\int_E b(l(\varepsilon))(c(x) + M(|\nabla T_{l(\varepsilon)}(u_n)|/\mu))\,dx < \varepsilon/2 \\ &\qquad\qquad \text{for all } E \text{ such that } |E| < \eta(\varepsilon).\end{aligned}$$

Finally, by combining (3.44) and (3.45) one easily has

$$\int_E |g(x, u_n, \nabla u_n)|\,dx < \varepsilon \quad \text{for all } E \text{ such that } |E| < \eta(\varepsilon),$$

which allows us, by using (3.43), to pass to the limit in (3.42).

This completes the proof. ■

REMARK 3.3. We obtain the same result of our theorem if we replace (3.1) by the general growth condition

$$|a(x,s,\zeta)| \le \overline{b}(s)(\overline{h}(x) + \overline{M}^{-1}M(k|\zeta|)) \tag{3.46}$$

where $k \ge 0$, $\overline{h} \in E_{\overline{M}}(\Omega)$ and $\overline{b} : \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function. Indeed, we consider the following approximate problems:

$$\begin{cases} -\operatorname{div}(a(x,T_n(u_n),\nabla u_n)) + g(x,u_n,\nabla u_n) = f_n \quad \text{in } \mathcal{D}'(\Omega), \\ u_n \in W_0^1 L_M(\Omega), \quad g(x,u_n,\nabla u_n) \in L^1(\Omega), \quad g(x,u_n,\nabla u_n)u_n \in L^1(\Omega), \end{cases}$$

and we end the proof by using the same last steps.

For some results obtained in the L^p case under the assumption (3.46), we refer to [19] and [20].

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vázquez, *An L^1-theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa 22 (1995), 240–273.

[3] A. Benkirane, *Approximation de type Hedberg dans les espaces $W^m L \log L(\Omega)$ et applications*, Ann. Fac. Sci. Toulouse 11 (1990), 67–78.

[4] A. Benkirane and A. Elmahi, *Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Anal. 28 (1997), 1769–1784.

[5] —, —, *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*, ibid. 36 (1999), 11–24.

[6] —, —, *A strongly nonlinear elliptic equation having natural growth terms and L^1 data*, ibid. 39 (2000), 403–411.

[7] A. Benkirane and J.-P. Gossez, *An approximation theorem for higher order Orlicz–Sobolev spaces*, Studia Math. 92 (1989), 231–255.

[8] A. Bensoussan, L. Boccardo and F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Ann. Inst. H. Poincaré 5 (1988), 347–364.

[9] L. Boccardo, *Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form*, in: Progress in Elliptic and Parabolic Partial Differential Equations (Capri, 1994), Pitman Res. Notes Math. Ser. 350, Longman, Harlow, 1996, 43–57.

[10] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87 (1989), 149–169.

[11] —, —, *Strongly nonlinear elliptic equations having natural growth and L^1 data*, Nonlinear Anal. 19 (1992), 573–578.

[12] L. Boccardo, T. Gallouët and F. Murat, *A unified presentation of two existence results for problems with natural growth*, in: Progress in Partial Differential Equations: the Metz Surveys 2, M. Chipot (ed.), Pitman Res. Notes Math. Ser. 296, Longman, 1993, 127–137.

[13] L. Boccardo, T. Gallouët and L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. 73 (1997), 203–223.

[14] L. Boccardo, F. Murat and J.-P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. 152 (1988), 183–196.

[15] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly* (*or slowly*) *increasing coefficients*, Trans. Amer. Math. Soc. 190 (1974), 163–205.

[16] —, *Some approximation properties in Orlicz–Sobolev spaces*, Studia Math. 74 (1982), 17–24.

[17] —, *A strongly nonlinear elliptic problem in Orlicz–Sobolev spaces*, in: Proc. Sympos. Pure Math. 45, Amer. Math. Soc., 1986, 455–462.

[18] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz–Sobolev spaces*, Nonlinear Anal. 11 (1987), 379–392.

[19] C. Leone and A. Porretta, *Entropy solutions for nonlinear elliptic equations in L^1*, Nonlinear Anal. 32 (1998), 325–334.

[20] A. Porretta, *Some remarks on the regularity of solutions for a class of elliptic equations with measure data*, Houston J. Math. 26 (2000), 183–213.

[21] —, *Existence for elliptic equations in L^1 having lower order terms with natural growth*, Portugal. Math. 57 (2000), 179–190.

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P. 1796 Atlas Fès, Maroc
E-mail: abdelmoujib@iam.net.ma
meskinedriss@hotmail.com

C.P.R. de Fès
B.P. 49, Fès, Maroc

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