APPLICATIONES MATHEMATICAE
29,4 (2002), pp. 459–464

Ewa Sylwestrzak (Zielona Góra)

ON AN INVERSE PROBLEM IN THE THEORY OF THERMISTORS

Abstract. An inverse problem for a nonlocal problem describing the temperature of a conducting device is studied.

1. Introduction. In this paper we consider the following nonlocal problem for a nonlinear differential equation:

$$\theta'' + \lambda \frac{f(\theta)}{(\int_{-1}^{1} f(\theta(x))\,dx)^2} = 0, \qquad -1 < x < 1, \tag{1}$$

$$\theta(-1) = \theta(1) = 0, \tag{2}$$

where $\theta : [-1,1] \to [0,\infty)$ is an unknown function, $f : [0,\infty) \to (0,\infty)$ is a given function, and $\lambda > 0$ is a real parameter. This problem models the stationary temperature θ of a conducting device occupying the interval $-1 \le x \le 1$ when the electric current flows through the material with temperature-dependent electrical resistivity $f(\theta) > 0$, subject to a fixed potential difference V, and $\lambda = V^2$.

The temporal evolution of the temperature θ and the electric potential φ is described by the system

$$\theta_t = \theta_{xx} + \sigma\varphi_x^2, \tag{3}$$

$$(\sigma\varphi_x)_x = 0, \tag{4}$$

where $\sigma(\theta) = 1/f(\theta)$ is the electrical conductivity (see [4]). We assume that the thermal conductivity, the density and the specific heat of the one-dimensional conductor are equal to one.

We fix the electric potential at the ends of the conductor, i.e. $\varphi(-1) = V_1$, $\varphi(1) = V_2$, and we assume that $\theta = 0$ at $x = \pm 1$.

2000 *Mathematics Subject Classification*: 34K29, 34B15.
Key words and phrases: theory of thermistors, inverse problem, existence of solutions.
The preparation of this paper was supported by the KBN grant 2/P03A/011/19.

From (4) we have $\sigma\varphi_x = J(t)$, so

(5) $$\varphi_x = f(\theta)J(t).$$

Integrating (5) over $[-1,1]$ we get $J(t) = V/\int_{-1}^1 f(\theta)$, where $V = V_2 - V_1$. Then we have

$$\varphi_x = V\,\frac{f(\theta)}{\int_{-1}^1 f(\theta)}.$$

Hence we can write the equation (3) in the form

$$\theta_t = \theta_{xx} + \lambda\,\frac{f(\theta)}{(\int_{-1}^1 f(\theta))^2},$$

where $\lambda = V^2$.

Thus the stationary temperature θ satisfies the equation (1) with the boundary conditions (2).

2. Existence of solutions. Let θ_λ be a solution of the problem (1)–(2).

It is known that the solution of (1)–(2) is an even function and the maximum of θ_λ is attained at 0, $\theta_\lambda(0) \equiv M(\lambda)$ (see [2]).

Note that

(6) $$\lambda = 8\int_0^{M(\lambda)} f(s)\,ds.$$

Indeed, integrating (1) over $[-1,1]$ and using $\theta'_\lambda(1) = -\theta'_\lambda(-1)$ we obtain

(7) $$2\theta'_\lambda(1) + \frac{\lambda}{\int_{-1}^1 f(\theta_\lambda)} = 0.$$

On the other hand, multiplying (1) by $2\theta'_\lambda$ and then integrating over $[0,1]$ we get

$$(\theta'_\lambda(1))^2 + \frac{2\lambda}{(\int_{-1}^1 f(\theta_\lambda))^2}\int_0^1 f(\theta_\lambda)\theta'_\lambda = 0.$$

Hence

(8) $$(\theta'_\lambda(1))^2 - \frac{2\lambda}{(\int_{-1}^1 f(\theta_\lambda))^2}\int_0^{M(\lambda)} f(s)\,ds = 0.$$

Combining (7) and (8) we finally obtain the formula (6).

Hence the maximum $M(\lambda)$ of the solution of the problem (1)–(2) is an increasing function of λ. Moreover, in the case $\int_0^\infty f(s)\,ds = \infty$, we have $M(\lambda) \to \infty$ as $\lambda \to \infty$, and if $\int_0^\infty f(s)\,ds < \infty$, then $M(\lambda) \to \infty$ as $\lambda \to 8\int_0^\infty f(s)\,ds$ (see [5]).

Under the assumption that f is a positive decreasing function, the existence and uniqueness of solutions of (1)–(2) have been investigated in [2] and [5]. The following facts were proved:

(a) If $\int_0^\infty f(s)\,ds = \infty$, then there is a unique solution of the problem (1)–(2) for each $\lambda > 0$,

(b) If $\int_0^\infty f(s)\,ds < \infty$, then the problem (1)–(2) has

(i) a unique solution for each $\lambda < 8\int_0^\infty f(s)\,ds$,

(ii) no solution for $\lambda \ge 8\int_0^\infty f(s)\,ds$.

In [2] the following result was obtained. If $f(\theta)$ is a positive increasing function such that $\theta f'(\theta)/f(\theta) \to \infty$ as $\theta \to \infty$, then the problem (1)–(2) has a solution for each $\lambda > 0$.

Here we generalize the above result proving the existence of a solution of the problem (1)–(2) for an arbitrary positive increasing function f.

THEOREM 1. *Let f be a continuous positive increasing function. Then there is a solution θ in $C^0[-1,1] \cap C^2(-1,1)$ of the problem* (1)–(2) *for each $\lambda > 0$.*

Proof. We introduce the following operator T on the space $C^0[-1,1]$:

$$T\theta(x) = \frac{\lambda}{(\int_{-1}^1 f(|\theta|))^2} \int_{-1}^{1} G(x,y) f(|\theta(y)|)\,dy,$$

where

$$G(x,y) = \begin{cases} (1-x)(y+1)/2 & \text{for } -1 \le y \le x \le 1, \\ (1-y)(x+1)/2 & \text{for } -1 \le x \le y \le 1, \end{cases}$$

is the Green function for the operator $-\theta''$ (see [1]). The solution of the problem (1)–(2) is a fixed point of the map T.

The proof of the existence of a fixed point will be based on the Leray–Schauder Theorem applied in the space $C^0[-1,1]$ with the supremum norm $\|\theta\|_\infty$.

The operator T is continuous on $C^0[-1,1]$. Moreover $(T\theta(x))'$ is uniformly bounded if θ belongs to a bounded subset of $C^0[-1,1]$. Indeed, if $\|\theta\|_\infty \le C$, then $\|(T\theta)'\|_\infty \le \lambda f(C)/(4f^2(0))$. Hence T is a compact operator.

To apply the Leray–Schauder Theorem, it is sufficient to prove a uniform *a priori* estimate of solutions θ^α of the equation $\theta = \alpha T\theta$ for $\alpha \in [0,1]$.

Obviously we have $\|\theta^\alpha\|_\infty \le \lambda/(4f(0)) \equiv C$ for $\alpha \in [0,1]$.

3. The inverse problem. Here we study an inverse problem related to (1)–(2), consisting in the unique identification of the function f in the case when we have some information about the solutions of the problem (1)–(2).

First we investigate the problem of the determination of f by the values of the solutions of (1)–(2) on $[0,1]$.

THEOREM 2. *The solution* θ_λ *of the problem* (1)–(2) *uniquely determines the function* f *on the interval* $[0, M(\lambda)]$.

Proof. From (1) we have

$$f(\theta_\lambda(x)) = -\frac{4(\int_0^1 f(\theta_\lambda))^2}{\lambda}\,\theta_\lambda''(x).$$

Integrating (1) over $[0,1]$ we get $\int_0^1 f(\theta_\lambda) = -\lambda/(4\theta_\lambda'(1))$. Then

$$f(\theta_\lambda(x)) = -\frac{\lambda\theta_\lambda''(x)}{4(\theta_\lambda'(1))^2}.$$

Note that the solution of (1)–(2) is one-to-one on $[0,1]$. Hence

$$f(s) = -\frac{\lambda\theta_\lambda''(\theta_\lambda^{-1}(s))}{4(\theta_\lambda'(1))^2} \quad \text{for } s \in [0, M(\lambda)]. \tag{9}$$

COROLLARY 1. *Let* $\int_0^\infty f(s)\,ds = \int_0^\infty \bar{f}(s)\,ds = \infty$. *If* $\theta_{\lambda_n} = \overline{\theta}_{\lambda_n}$ *for some increasing sequence* λ_n *tending to* ∞, *then* $f = \bar{f}$ *on the whole half-line* $[0,\infty)$.

COROLLARY 2. *Let* $\int_0^\infty f(s)\,ds = \int_0^\infty \bar{f}(s)\,ds \equiv \lambda^* < \infty$. *If* $\theta_{\lambda_n} = \overline{\theta}_{\lambda_n}$ *for some increasing sequence* λ_n *tending to* $8\lambda^*$, *then* $f = \bar{f}$ *on the whole half-line* $[0,\infty)$.

The formula (9) involves the first and second derivatives of θ_λ. Therefore small C^0-changes of θ_λ may give big changes in f. This kind of instability is typical for inverse problems. Our aim is to describe a procedure for finding f which is stable under C^0-perturbations of θ_λ.

We start with the following example. In [3] the authors assumed that the function f has the form $f(\theta) = ae^{b\theta}$, where $a > 0$, $b > 0$ are real parameters. In that case the problem (1)–(2) has a unique solution for all $\lambda > 0$ (see [2]). Our aim is to determine a and b if we know the value of the temperature $\theta_\lambda(0) = M(\lambda)$ for some $\lambda = \lambda_j$, $j = 1,2$. The formula (6) takes the form $\lambda/8 = a(e^{bM(\lambda)} - 1)/b$, and we obtain $a = \lambda_1 b/(8(e^{bM(\lambda_1)} - 1))$ and $\lambda_1/\lambda_2 = (e^{bM(\lambda_1)} - 1)/(e^{bM(\lambda_2)} - 1)$. Let $\lambda_1 > \lambda_2$. Then $M(\lambda_1) > M(\lambda_2)$ and we note that the right hand side of the last equation is an increasing function of b. Hence we can determine uniquely the parameters a, b, provided $\lambda_1/\lambda_2 > M(\lambda_1)/M(\lambda_2)$.

On the other hand, if $\lambda_1/\lambda_2 \le M(\lambda_1)/M(\lambda_2)$ for some λ_1, λ_2, we can conclude that the dependence of the resistivity of the thermistor on the temperature is not given by a function of the form $f(\theta) = ae^{b\theta}$.

We prove that $M(\lambda)$ determines f not only for f of the special form $f(\theta) = ae^{b\theta}$ but also for arbitrary f. Assume that $M(\lambda)$ is a differentiable function of λ. Then using the formula (6) we get the relation $f(M(\lambda)) = (8M'(\lambda))^{-1}$. Hence f is uniquely determined by M. We see that small changes of $M(\lambda)$ in C^0-norm may give big changes of f. In practice the values $M(\lambda)$ are obtained experimentally, hence it is reasonable to ask about the influence of inaccuracy in measuring $M(\lambda)$ on the function f. Below we show that under some assumptions on the dependence of the resistivity f on the temperature θ, small C^0-perturbations of M lead to small changes of f.

THEOREM 3. *Let f_n, f be differentiable positive functions such that the sequences f_n, $f' - f_n'$ are uniformly bounded on $[0,\infty)$, and suppose $\int_0^\infty f(s)\,ds = \int_0^\infty f_n(s)\,ds = \infty$. Assume that the sequence $M_n(\lambda)$ converges uniformly to $M(\lambda)$ on $[0,\overline{\lambda}]$. Then the sequence $f_n(s)$ tends uniformly to $f(s)$ on $[0, M(\overline{\lambda})]$.*

Proof. Let $f_n \le c$ for some constant $c > 0$.

From (6) we get $\int_0^{M(\lambda)} f(s)\,ds = \int_0^{M_n(\lambda)} f_n(s)\,ds$. Thus

$$\left|\int_0^{M(\lambda)} (f(s) - f_n(s))\,ds\right| = \left|\int_{M(\lambda)}^{M_n(\lambda)} f_n(s)\,ds\right| \le c|M_n(\lambda) - M(\lambda)|. \tag{10}$$

Let $g_n(x) = \int_0^x h_n(s)\,ds$, where $x \equiv M(\lambda)$ and $h_n \equiv f - f_n$. Then by the assumption of the theorem it follows from (10) that the sequence g_n tends uniformly to 0 on $[0, M(\overline{\lambda})]$.

The sequence h_n' is uniformly bounded, so by the Arzelà–Ascoli theorem from each subsequence of h_n we can choose some subsequence h_{n_k} uniformly convergent to some function h. We have $g_{n_k}(x) = \int_0^x h_{n_k}(s)\,ds$ and $g_{n_k} \to 0$ uniformly on $[0, M(\overline{\lambda})]$. Hence $\int_0^x h(s)\,ds = 0$ for each $x > 0$, which implies $h = 0$, and thus the sequence $h_n = f - f_n$ tends uniformly to 0 on $[0, M(\overline{\lambda})]$.

REMARK. *The assertion of Theorem 3 holds if we consider the differentiable positive functions f_n, f such that f_n are decreasing, the sequences f_n', $f' - f_n'$ are uniformly bounded on $[0,\infty)$ and $\int_0^\infty f(s)\,ds = \int_0^\infty f_n(s)\,ds = \infty$.*

It is sufficient to prove that the sequence $f_n(0)$ is bounded. Suppose that $f_n(0) \to \infty$. Then from the relation $\lambda = 8\int_0^{M_n(\lambda)} f_n(s)\,ds$ and uniform boundedness of f_n' it follows that $M_n(\lambda)$ tends to 0 for each $\lambda > 0$. Moreover the $M_n(\lambda)$ are increasing functions of λ. Thus we get uniform convergence of the sequence $M_n(\lambda)$ to $M(\lambda) \equiv 0$, a contradiction.

Immediately from Theorem 3 we get

COROLLARY 3. *Let* $\int_0^\infty f_j(s)\,ds = \infty$, $j = 1, 2$. *If* $M_1(\lambda) = M_2(\lambda)$ *for each* $\lambda > 0$, *then* $f_1 = f_2$ *on the whole half-line* $[0, \infty)$.

Acknowledgments. The author would like to thank P. Biler and T. Nadzieja for interesting conversations during the preparation of this paper.

References

[1] P. B. Bailey, L. F. Shampine and P. E. Waltman, *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York, 1968.

[2] J. W. Bebernes and A. A. Lacey, *Global existence and finite-time blow-up for a class of nonlocal parabolic problems*, Adv. Differ. Equ. 2 (1997), 927–953.

[3] E. Kuźma and B. Schmidt, *Thermistors*, Wydawnictwa Naukowo-Techniczne, Warszawa, 1972 (in Polish).

[4] A. A. Lacey, *Thermal runaway in a nonlocal problem modelling Ohmic heating. Part I: Model derivation and some special cases*, Euro. J. Appl. Math. 6 (1995), 127–144.

[5] —, *Thermal runaway in a nonlocal problem modelling Ohmic heating. Part II: General proof of blow-up and asymptotics of runaway*, ibid., 201–224.

Institute of Mathematics
University of Zielona Góra
Podgórna 50
65-246 Zielona Góra, Poland
E-mail: e.sylwestrzak@im.uz.zgora.pl

Received on 13.5.2002;
revised version on 30.8.2002 (1629)