APPLICATIONES MATHEMATICAE
34,2 (2007), pp. 143–167

W. M. Zajączkowski (Warszawa)

EXISTENCE OF SOLUTIONS TO THE NONSTATIONARY STOKES SYSTEM IN $H^{2,1}_{-\mu}$, $\mu \in (0,1)$, IN A DOMAIN WITH A DISTINGUISHED AXIS. PART 2. ESTIMATE IN THE 3d CASE

Abstract. We examine the regularity of solutions to the Stokes system in a neighbourhood of the distinguished axis under the assumptions that the initial velocity v_0 and the external force f belong to some weighted Sobolev spaces. It is assumed that the weight is the $(-\mu)$th power of the distance to the axis. Let $f \in L_{2,-\mu}$, $v_0 \in H^1_{-\mu}$, $\mu \in (0,1)$. We prove an estimate of the velocity in the $H^{2,1}_{-\mu}$ norm and of the gradient of the pressure in the norm of $L_{2,-\mu}$. We apply the Fourier transform with respect to the variable along the axis and the Laplace transform with respect to time. Then we obtain two-dimensional problems with parameters. Deriving an appropriate estimate with a constant independent of the parameters and using estimates in the two-dimensional case yields the result. The existence and regularity in a bounded domain will be shown in another paper.

1. Introduction. We consider the problem

$$
\begin{aligned}
&v_{,t} - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T = S\times(0,T),\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\
&v|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

in a bounded domain $\Omega \subset \mathbb{R}^3$, where v is the velocity of the fluid, p the

2000 *Mathematics Subject Classification*: 35Q30, 76D03, 76D99.
Key words and phrases: Stokes system, weighted Sobolev spaces, regularity in weighted Sobolev spaces, Fourier–Laplace transform, Parseval identity.
Research partially supported by MNiSW grant no 1 PO3A 021 30 and EC FP6 Marie Curie ToK programme SPADE2.

pressure, f the external force field, $\gamma > 0$ the constant slip coefficient, $\overline{n}$ the unit outward vector normal to the boundary S, and $\overline{\tau}_1, \overline{\tau}_2$ unit tangent vectors to S.

We denote by $\mathbb{T}(v,p)$ the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI, \tag{1.2}$$

where $\nu > 0$ is the constant viscosity coefficient, $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}, \tag{1.3}$$

and I is the unit matrix.

Let (x_1,x_2,x_3) denote the Cartesian coordinates in $\mathbb{R}^3$ and $t \in \mathbb{R}_+$. By dot we denote the scalar product in $\mathbb{R}^3$.

We assume that Ω contains a distinguished axis L.

To formulate the main result we introduce some weighted spaces. Let $\mu \in \mathbb{R}$ and $\varrho(x) = \operatorname{dist}\{x, L\}$ and set

$$H^{2,1}_\mu(\Omega^T) = \bigg\{u : \|u\|_{H^{2,1}_{\mu}(\Omega^T)} = \bigg[\int_{\Omega^T} [(u^2_{,xx} + u^2_{,t})\varrho^{2\mu}(x) + u^2_{,x}\varrho^{2\mu-2}(x) + u^2\varrho^{2\mu-4}(x)]\,dx\,dt\bigg]^{1/2} < \infty\bigg\},$$

$$L_{2,\mu}(\Omega^T) = \bigg\{u : \|u\|_{L_{2,\mu}(\Omega^T)} = \bigg(\int_{\Omega^T} u^2\varrho^{2\mu}(x)\,dx\,dt\bigg)^{1/2} < \infty\bigg\},$$

$$H^1_\mu(\Omega) = \bigg\{u : \|u\|_{H^1_\mu(\Omega)} = \bigg[\int_{\Omega}[u^2_{,x}\varrho^{2\mu}(x) + u^2\varrho^{2\mu-2}(x)]\,dx\bigg]^{1/2} < \infty\bigg\}.$$

Here and throughout, we do not distinguish vector and scalar valued functions. Let $u = (u_1,\ldots,u_n)$, $x = (x_1,\ldots,x_n)$. Then $u^2 = u_1^2 + \cdots + u_n^2$, $u^2_{,x} = \sum_{i,j=1}^n u^2_{i,x_j}$, $u_{,x} = \partial_x u$. The main result of [10, 11] and of this paper is the following

THEOREM A. *Assume that* $f \in L_{2,-\mu}(\Omega^T)$, $v_0 \in H^1_{-\mu}(\Omega)$, $\mu \in (0,1)$, $S \in C^3$. *Then there exists a solution to problem* (1.1) *such that* $v \in H^{2,1}_{-\mu}(\Omega^T)$, $\nabla p \in L_{2,-\mu}(\Omega^T)$, *and*

$$\|v\|_{H^{2,1}_{-\mu}(\Omega^T)} + \|\nabla p\|_{L_{2,-\mu}(\Omega^T)} \le c(\|f\|_{L_{2,-\mu}(\Omega^T)} + \|v_0\|_{H^1_{-\mu}(\Omega)}). \tag{1.4}$$

To prove Theorem A we shall use the localization technique (see [10]) and the existence in the 2d case (see [11]). To do this we need an estimate for solutions of (1.1) in a neighbourhood of the axis L in appropriate weighted Sobolev spaces. For this purpose we introduce a local Cartesian system $x = (x_1,x_2,x_3)$ with centre on L such that L is the x_3-axis. Next we define cylindrical coordinates (r, φ, z) by $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Let R, a be given positive numbers. We introduce the cylinder

$$C_R = \{x \in \mathbb{R}^3 : r < R,\ -a < z < a,\ \varphi \in [0, 2\pi]\}.$$

We assume that $C_R \cap S = \emptyset$.

To show the existence of solutions to problem (1.1) by the localization technique we examine the problem with vanishing initial data

$$\begin{aligned} &v_{,t} - \nu\Delta v + \nabla p = f && \text{in } C_R^T,\\ &\operatorname{div} v = 0 && \text{in } C_R^T,\\ &v = 0 && \text{on } \partial C_R^T,\\ &v|_{t=0} = 0 && \text{in } C_R. \end{aligned} \tag{1.5}$$

We underline that the v, p in (1.5) are different from those in (1.1).

The main result of this paper, crucial for the proof of Theorem A, is the following

Theorem 1. *Let $f \in L_{2,-\mu}(C_R^T)$, $\mu \in (0,1)$. Then solutions to problem (1.5) such that $v \in H^{1,2}_{-\mu}(C_R^T)$, $p \in L_2(0,T; H^1_{-\mu}(C_R))$ satisfy*

$$\|v\|_{H^{2,1}_{-\mu}(C_R^T)} + \|p\|_{L_2(0,T;H^1_{-\mu}(C_R^T))} \le c(T)\|f\|_{L_{2,-\mu}(C_R^T)}, \tag{1.6}$$

where $c(T)$ is an increasing positive function of T.

Theorem 1 is proved in Section 3. For this purpose (1.5) is extended to a problem on $\mathbb{R}^3 \times \mathbb{R}_+$ for solutions vanishing sufficiently fast at infinity. Applying the Laplace transform with respect to t and the Fourier transform with respect to x_3, because L is the x_3-axis, we transform problem (1.5) to a two-dimensional Stokes system and a two-dimensional Poisson equation (see (2.3) and (2.4)). Next, we need estimates for solutions to problems (2.3) and (2.4) (see Lemmas 2.1, 2.2 from [11]). Since problem (1.5) is three-dimensional and time dependent we need estimates for solutions to (1.5) with constants independent of the parameters of the Fourier and Laplace transforms (parameters ξ and s in (3.6)). This is done in Section 3 in Lemmas 3.1–3.6. Having such estimates we are able to prove Theorem 1.

Now, we introduce some notations. Let Ω be a bounded domain in $\mathbb{R}^3$. We denote by $V_2^0(\Omega^T)$ the Banach space of functions with the finite norm

$$\|u\|_{V_2^0(\Omega^T)} = \sup_{t\le T}\|u(t)\|_{L_2(\Omega)} + \Big(\int_0^T \|\nabla u(t)\|^2_{L_2(\Omega)}\,dt\Big)^{1/2}.$$

Let $W_2^{2,1}(\Omega^T)$ be the closure of $C^\infty(\overline{\Omega^T})$ in the norm

$$\|u\|_{W_2^{2,1}(\Omega^T)} = \Big[\int_{\Omega^T} (|u(x,t)|^2 + |u_{,xx}(x,t)|^2 + |u_{,t}(x,t)|^2)\,dx\,dt\Big]^{1/2},$$

and $W_2^1(\Omega)$ the closure of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|_{W_2^1(\Omega)} = \Big(\int\limits_\Omega (|u(x)|^2 + |u_{,x}(x)|^2)\,dx\Big)^{1/2}.$$

Consider the domain $\mathbb{D} = B_R(0)\times\mathbb{R}_{x_3}\times\mathbb{R}_t$, where $B_R(0) = \{x'\in\mathbb{R}^2 : |x'| < R\}$, $x' = (x_1,x_2)$, $|x'| = \sqrt{x_1^2+x_2^2}$. Let us introduce the Fourier–Laplace transform

$$\widetilde{u}(x',\xi,s) = \int\limits_{\mathbb{R}_+} dt \int\limits_{\mathbb{R}} dx_3\, e^{-st-ix_3\xi} u(x',x_3,t), \tag{1.7}$$

where $s = \gamma + i\xi_0$, $\gamma = \operatorname{Re} s > 0$, $\xi_0\in\mathbb{R}$.

Then we introduce the space $H^{k,k/2}_{\mu,\gamma}(\mathbb{D})$, $k\in\mathbb{Z}_+$ even, $\mu\in\mathbb{R}$, of functions with the finite norm

$$\|u\|_{H^{k,k/2}_{\mu,\gamma}(\mathbb{D})} = \Big[\sum_{|\alpha|+2\alpha_0\le k} \int ds \int\limits_{\mathbb{R}} d\xi \int\limits_{B_R(0)} |D^{\alpha'}_{x'}\widetilde{u}|^2 |\xi|^{2\alpha_3}|s|^{2\alpha_0}|x'|^{2(\mu+|\alpha|+2\alpha_0-k)}\,dx'\Big]^{1/2},$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha' = (\alpha_1,\alpha_2)$, $|\alpha'| = \alpha_1+\alpha_2$, $\alpha_1,\alpha_2,\alpha_3,\alpha_0$ are non-negative integers. Moreover,

$$L_{2,\mu,\gamma}(\mathbb{D}) = H^{0,0}_{\mu,\gamma}(\mathbb{D}).$$

In the stationary case, the spaces $H^k_\mu(B_R(0)\times\mathbb{R})$ appeared in [6]. The spaces $H^k_\gamma(B_R(0)\times\mathbb{R})$ were used in [3, 8, 9].

Acknowledgements. The author thanks Ewa Zadrzyńska for important discussions during the preparation of this paper.

2. Auxiliary results. To formulate results from [11] for two-dimensional problems which follow from (1.5) after applying the Fourier-Laplace transform (1.7) we consider instead of (1.5) the more general problem

$$\begin{aligned} &v_{,t} - \operatorname{div}\mathbb{T}(v,p) = f_1 && \text{in } C_R^T,\\ &\operatorname{div} v = h_1 && \text{in } C_R^T,\\ &v = 0 && \text{on } \partial C_R^T,\\ &v|_{t=0} = 0 && \text{in } C_R,\\ &f_1|_{\partial C_R} = 0,\quad h_1|_{\partial C_R} = 0,\quad p|_{\partial C_R} = 0. \end{aligned} \tag{2.1}$$

To examine problem (2.1) in weighted Sobolev spaces we consider it in $\mathbb{R}^3\times\mathbb{R}_+$, so the last condition $(2.1)_5$ disappears. Moreover, to show the existence and estimates in weighted spaces we apply the methods of Kondrat'ev [6] for elliptic boundary value problems in cones. Since we consider elliptic two-dimensional problems in $\mathbb{R}^2$, our cone is the 2π angle. Therefore, instead of (2.1) we examine the problem in $\mathbb{R}^3$

$$
\begin{aligned}
&v_{,t} - \operatorname{div} \mathbb{T}(v,p) = f_1,\\
&\operatorname{div} v = h_1,\\
&v|_{\Gamma_0} = v|_{\Gamma_{2\pi}},\\
&\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0} = \overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}}.
\end{aligned}
\tag{2.2}
$$

In view of $(2.1)_5$ all functions in (2.1) are extended by zero to $\mathbb{R}^3$ and for $t<0$, where $\Gamma_0 = \Gamma_{2\pi} = \{x\in\mathbb{R}^3 : x_2 = 0\}$, $\overline{n}|_{\Gamma_0} = (0,-1,0)$, $\overline{n}|_{\Gamma_{2\pi}} = (0,1,0)$ and $(2.1)_5$ follows from localization of (1.1). Solvability of $(2.2)_2$ is considered in [2, 5].

Applying the Laplace–Fourier transform (1.7) to problem (2.2) we separate it into two problems

$$
\begin{aligned}
&-\nu\Delta'\widetilde{v}_j + \widetilde{p}_{,x_j} = \widetilde{f}_{1j} - q\widetilde{v}_j \equiv \widetilde{g}_j, \quad j=1,2, && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_{1,x_1} + \widetilde{v}_{2,x_2} = -i\xi\widetilde{v}_3 + \widetilde{h}_1 \equiv \widetilde{k} && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_j|_{\gamma_0} = \widetilde{v}_j|_{\gamma_{2\pi}}, \quad j=1,2, && x_2 = 0,\\
&\begin{pmatrix}\widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1}\\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p}\end{pmatrix}\bigg|_{\gamma_0} = \begin{pmatrix}\widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1}\\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p}\end{pmatrix}\bigg|_{\gamma_{2\pi}}, && x_2 = 0,
\end{aligned}
\tag{2.3}
$$

where $q = s+\nu\xi^2$, $\gamma_0 = \gamma_{2\pi} = \{x\in\mathbb{R}^2 : x_2 = 0\}$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$ and

$$
\begin{aligned}
&-\nu\Delta'\widetilde{v}_3 = \widetilde{f}_{13} - q\widetilde{v}_3 - i\xi\widetilde{p} \equiv \widetilde{g}_3 && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_3|_{\gamma_0} = \widetilde{v}_3|_{\gamma_{2\pi}}, && x_2 = 0,\\
&\widetilde{v}_{3,x_2}|_{\gamma_0} = \widetilde{v}_{3,x_2}|_{\gamma_{2\pi}}, && x_2 = 0.
\end{aligned}
\tag{2.4}
$$

Parabolic and elliptic equations in weighted spaces have been examined in [12, 13].

From [11] we have

LEMMA 2.1. *Let $\widetilde{g}\in L_{2,-\mu}(\mathbb{R}^2)$, $\widetilde{k}\in H^1_{-\mu}(\mathbb{R}^2)$, $\mu\in(0,1)$. Then there exists a solution to problem* (2.3) *such that $\widetilde{v}\in H^2_{-\mu}(\mathbb{R}^2)$, $\widetilde{p}\in H^1_{-\mu}(\mathbb{R}^2)$ and*

$$
\|\widetilde{v}\|_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|_{H^1_{-\mu}(\mathbb{R}^2)} \le c(\|\widetilde{g}\|_{L_{2,-\mu}(\mathbb{R}^2)} + \|\widetilde{k}\|_{H^1_{-\mu}(\mathbb{R}^2)}).
\tag{2.5}
$$

LEMMA 2.2 (see [11]). *Let $\widetilde{g}_3\in L_{2,-\mu}(\mathbb{R}^2)$, $\mu\in(0,1)$. Then there exists a solution to problem* (2.4) *such that $\widetilde{v}_3\in H^2_{-\mu}(\mathbb{R}^2)$ and*

$$
\|\widetilde{v}_3\|_{H^2_{-\mu}(\mathbb{R}^2)} \le c\|\widetilde{g}_3\|_{L_{2,-\mu}(\mathbb{R}^2)}.
\tag{2.6}
$$

Consider the problem

$$
\Delta\varphi = \operatorname{div} f, \qquad \varphi|_{\partial C_R} = 0.
\tag{2.7}
$$

Lemma 2.3. *Let $f \in L_{2,\mu}(C_R)$, $\mu \in (0,1)$. Then there exists a solution to* (2.7) *such that $\nabla\varphi \in L_{2,\mu}(C_R)$ and*

$$\|\nabla\varphi\|_{L_{2,\mu}(C_R)} \le c\|f\|_{L_{2,\mu}(C_R)}. \tag{2.8}$$

Proof. Let $G(x,y)$ be the Green function for problem (2.7). Then any solution to problem (2.7) can be represented in the form

$$\varphi(x) = \int_{C_R} G(x,y)\operatorname{div}_y f(y)\,dy = -\int_{C_R} \nabla_y G(x,y)\cdot f(y)\,dy. \tag{2.9}$$

Hence

$$\nabla_x\varphi(x) = -\int_{C_R} \nabla_x\nabla_y G(x,y) f(y)\,dy. \tag{2.10}$$

Then by [4] we obtain (2.8). This concludes the proof.

Finally, we consider the problem

$$\begin{aligned} & u_{,t} - \operatorname{div}\mathbb{T}(u,p) = g && \text{in } C_R^T,\\ & \operatorname{div} u = 0 && \text{in } C_R^T,\\ & u = 0 && \text{on } \partial C_R^T \times (0,T),\\ & u|_{t=0} = u_0 && \text{in } C_R. \end{aligned} \tag{2.11}$$

3. Estimates near L. In this section we find an estimate for solutions to problem (1.5) under the assumptions that $f \in L_{2,-\mu}(C_R^T)$, $\mu \in (0,1)$. For this purpose we assume that the problem is considered with f extended by zero for $x \in \mathbb{R}^3 \setminus C_R$ and for $t < 0$ and $t > T$.

Next, two-dimensional considerations require the form (2.2) with $f_1 = f$, $h_1 = 0$. Therefore, we consider the problem

$$\begin{aligned} & v_{,t} - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \mathbb{R}^3 \times \mathbb{R},\\ & \operatorname{div} v = 0 && \text{in } \mathbb{R}^3 \times \mathbb{R},\\ & v|_{\Gamma_0} = v|_{\Gamma_{2\pi}},\\ & \overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0} = \overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}}, \end{aligned} \tag{3.1}$$

with v, p vanishing outside C_R and for $t < 0$. The condition

$$v|_{\partial C_R} = 0 \tag{3.2}$$

is necessary for applying Lemma 2.3 and to prove the existence of solutions to problem (3.1), (3.2).

The property that v, p vanish for $t < 0$ is implied by the Paley–Wiener theorem for the inverse Laplace transform with respect to t (see [1, 7]).

To show that problem (2.11) implies (1.5) we extend initial data $u_0 \in H^1_{-\mu}(C_R)$, $\operatorname{div} u_0 = 0$, to a function $u_0' \in H^{2,1}_{-\mu}(C_R^T)$, $\operatorname{div} u_0' = 0$, such that

$$u_0'|_{t=0} = u_0 \tag{3.3}$$

and

$$\|u_0'\|_{H^{2,1}_{-\mu}(C_R^T)} \le c\|u_0\|_{H^1_{-\mu}(C_R)}. \tag{3.4}$$

Introducing the functions

$$v = u - u_0', \quad f = g - u_{0,t}' + \nu\Delta u_0',$$

where

$$\begin{aligned} \|f\|_{L_{2,-\mu}(C_R^T)} &\le c(\|g\|_{L_{2,-\mu}(C_R^T)} + \|u_0'\|_{H^{2,1}_{-\mu}(C_R^T)}) \\ &\le c(\|g\|_{L_{2,-\mu}(C_R^T)} + \|u_0\|_{H^1_{-\mu}(C_R)}), \end{aligned} \tag{3.5}$$

we see that v, p, f satisfy problem (1.5).

Let us introduce the Fourier–Laplace transforms

$$\begin{aligned} \widetilde{u}(x,s) &= \int_0^\infty e^{-st}u(x,t)\,dt, \qquad s = \gamma + i\xi_0,\ \gamma > 0,\ \xi_0 \in \mathbb{R}, \\ \widetilde{u}(x',\xi,t) &= \int_{\mathbb{R}} e^{-ix_3\xi}u(x',x_3,t)\,dx_3, \qquad \xi \in \mathbb{R}, \\ \widetilde{u}(x',\xi,s) &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}} dx_3\, e^{-st-ix_3\xi}u(x',x_3,t), \end{aligned} \tag{3.6}$$

where $x' = (x_1, x_2)$. We do not distinguish between these transforms in notations, because this will be clear from the context.

Applying the Laplace transform $(3.6)_1$ to $(3.1)_1$ yields

$$\begin{aligned} &s\widetilde{v} - \nu\Delta\widetilde{v} + \nabla\widetilde{p} = \widetilde{f} && \text{in } \mathbb{R}^3, \\ &\operatorname{div}\widetilde{v} = 0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{3.7}$$

Lemma 3.1. *Assume that* $\mu \in (0,1)$, $x' = (x_1, x_2)$ *and*

$$\int ds \int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx < \infty, \qquad \int ds \int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx < \infty.$$

Assume that v *vanishes outside the cylinder* $C_R = B_R(0)\times(-a,a)$, $B_R(0) \subset \mathbb{R}^2$ *and* $B_R(0)$ *is the ball of radius* R *and centre at* 0. *Then for any solution of* (3.7),

$$\begin{aligned} &\int ds \int_{\mathbb{R}^3} |s|^2|\widetilde{v}|^2|x'|^{-2\mu}\,dx + \nu\int ds \int_{\mathbb{R}^3} |s|\,|\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx \\ &\qquad \le c_1\int ds \int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + c_1\int ds \int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx, \end{aligned} \tag{3.8}$$

where the constant c_1 *may depend on* a, R, μ.

Proof. Let $\widetilde{\varphi}$ be such that

$$\Delta\widetilde{\varphi} = \operatorname{div}((1 + i\operatorname{sign}\operatorname{Im} s)\widetilde{v}|x'|^{-2\mu}), \qquad \widetilde{\varphi}|_{\partial C_R} = 0. \tag{3.9}$$

Multiplying $(3.7)_1$ by $(1 - i\,\mathrm{sign}\,\mathrm{Im}\,s)\overline{\widetilde{v}}|x'|^{-2\mu} - \nabla\overline{\widetilde{\varphi}}$, where $\overline{u}$ means the complex conjugate to u, integrating with respect to $x \in \mathbb{R}^3$ and taking the real parts yields

$$
\begin{aligned}
(3.10)\qquad & \int_{\mathbb{R}^3} (|s|\,|\widetilde{v}|^2|x'|^{-2\mu} + \nu|\nabla\widetilde{v}|^2|x'|^{-2\mu})\,dx \\
&= \mathrm{Re}\int_{\mathbb{R}^3} s\widetilde{v}\cdot\nabla\overline{\widetilde{\varphi}}\,dx + 2\mu\nu\Big(\mathrm{Re}\int_{\mathbb{R}^3} \nabla\widetilde{v}\,\overline{\widetilde{v}}|x'|^{-2\mu-1}\nabla|x'|\,dx \\
&\qquad + \mathrm{sign}\,\mathrm{Im}\,s\,\mathrm{Im}\int_{\mathbb{R}^3} \nabla\widetilde{v}\,\overline{\widetilde{v}}|x'|^{-2\mu-1}\nabla|x'|\,dx\Big) \\
&\quad - \nu\,\mathrm{Re}\int_{\mathbb{R}^3} \Delta\widetilde{v}\cdot\nabla\overline{\widetilde{\varphi}}\,dx + \mathrm{Re}\int_{\mathbb{R}^3} \widetilde{f}\cdot\overline{\widetilde{v}}|x'|^{-2\mu}\,dx \\
&\quad + \mathrm{sign}\,\mathrm{Im}\,s\,\mathrm{Im}\int_{\mathbb{R}^3} \widetilde{f}\cdot\overline{\widetilde{v}}|x'|^{-2\mu}\,dx \\
&\quad - \mathrm{Re}\int_{\mathbb{R}^3} \widetilde{f}\cdot\nabla\overline{\widetilde{\varphi}}\,dx.
\end{aligned}
$$

Since $\widetilde{v}$ is divergence free, the first and third terms on the r.h.s. of (3.10) vanish after integration by parts.

By the Hölder and Young inequalities, the second term on the r.h.s. of (3.10) is estimated by

$$
\varepsilon_1\nu\int_{\mathbb{R}^3} |\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx + c(1/\varepsilon_1)\int_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx,
$$

the fourth by

$$
\varepsilon_2\int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu}\,dx + \frac{c(1/\varepsilon_2)}{|s|}\int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx,
$$

and the last by

$$
\varepsilon_3\int_{\mathbb{R}^3} |s|\,|\nabla\widetilde{\varphi}|^2|x'|^{2\mu}\,dx + \frac{c(1/\varepsilon_3)}{|s|}\int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx.
$$

Hence for sufficiently small ε_i, (3.10) implies

$$
\begin{aligned}
(3.11)\qquad & \int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu}\,dx + \nu\int_{\mathbb{R}^3} |\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx \\
&\le c\int_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + \varepsilon\int_{\mathbb{R}^3} |s|\,|\nabla\widetilde{\varphi}|^2|x'|^{2\mu}\,dx + \frac{c(1/\varepsilon)}{|s|}\int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx.
\end{aligned}
$$

Applying Lemma 2.3 to (3.9) yields

$$
(3.12)\qquad \|\nabla\widetilde{\varphi}\|_{L_{2,\mu}(C_R)} \le c\|\widetilde{v}|x'|^{-2\mu}\|_{L_{2,\mu}(C_R)} \le c\|\widetilde{v}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}.
$$

Using (3.12) in (3.11) gives

$$\begin{aligned}(3.13)\quad &\int\limits_{\mathbb{R}^3} (|\nabla\widetilde{v}|^2 + |s|\,|\widetilde{v}|^2)|x'|^{-2\mu}\,dx \\ &\le c\int\limits_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + c\varepsilon\int\limits_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu}\,dx + \frac{c}{|s|}\int\limits_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx.\end{aligned}$$

Assuming that ε is sufficiently small we obtain from (3.13) the inequality

$$\begin{aligned}(3.14)\quad &\int\limits_{\mathbb{R}^3} (|\nabla\widetilde{v}|^2 + |s|\,|\widetilde{v}|^2)|x'|^{-2\mu}\,dx \\ &\le c\int\limits_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + \frac{c}{|s|}\int\limits_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx.\end{aligned}$$

Multiplying both sides by $|s|$ and integrating with respect to s implies (3.8). This ends the proof.

To increase regularity with respect to x of solutions to (3.1), (3.2) we apply to (3.7) the Fourier transform with respect to x_3. Then we obtain the problems

$$(3.15)\qquad \xi^2\widetilde{v}' - \nu\Delta'\widetilde{v}' + \nabla'\widetilde{p} = \widetilde{f}' - s\widetilde{v}' \equiv \widetilde{g}', \qquad \operatorname{div}'\widetilde{v}' = i\xi\widetilde{v}_3,$$

where $v' = (v_1, v_2)$, $\nabla' = (\partial_{x_1}, \partial_{x_2})$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, and

$$(3.16)\qquad \xi^2\widetilde{v}_3 - \nu\Delta'\widetilde{v}_3 - i\xi\widetilde{p} = \widetilde{f}_3 - s\widetilde{v}_3 \equiv \widetilde{g}_3,$$

and $\widetilde{v}, \widetilde{p}$ vanish outside the ball $B_R(0)$. By Lemmas 2.1 and 2.2 we obtain for solutions of (3.15) the inequality

$$\begin{aligned}(3.17)\quad &\|\widetilde{v}'\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^2)} \\ &\le c\|\widetilde{g}'\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c|\xi|^4\|\widetilde{v}'\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c|\xi|^2\|\widetilde{v}_3\|^2_{H^1_{-\mu}(\mathbb{R}^2)},\end{aligned}$$

and for solutions of (3.16) the inequality

$$(3.18)\quad \|\widetilde{v}_3\|^2_{H^2_{-\mu}(\mathbb{R}^2)} \le c\|\widetilde{g}_3\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\xi^4\|\widetilde{v}_3\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\xi^2\|\widetilde{p}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}.$$

Integrating (3.17) and (3.18) with respect to ξ and s we get

$$\begin{aligned}(3.19)\quad &\int d\xi\,ds\,(\|\widetilde{v}'\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^2)}) \\ &\le c\int d\xi\,ds\,\|\widetilde{g}'\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\int d\xi\,ds\,(\xi^4\|\widetilde{v}'\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + \xi^2\|\widetilde{v}_3\|^2_{H^1_{-\mu}(\mathbb{R}^2)}),\end{aligned}$$

and

$$\begin{aligned}(3.20)\quad &\int d\xi\,ds\,\|\widetilde{v}_3\|^2_{H^2_{-\mu}(\mathbb{R}^2)} \le c\int d\xi\,ds\,\|\widetilde{g}_3\|^2_{L_{2,-\mu}(\mathbb{R}^2)} \\ &+ c\int d\xi\,ds\,\xi^4\|\widetilde{v}_3\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\int d\xi\,ds\,\xi^2\|\widetilde{p}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}.\end{aligned}$$

Therefore, we have to find estimates for the last two integrals in (3.19) and (3.20).

Lemma 3.2. *Assume that* $\mu \in (0,1)$, $\widetilde{g} = \widetilde{f} - s\widetilde{v}$, $\varepsilon \in (0,1)$,

$$\int ds\, d\xi \int_{\mathbb{R}^2} \xi^2 |\widetilde{p}|^2 |x'|^{-2\mu}\, dx' < \infty,$$

$$\int ds\, d\xi \int_{\mathbb{R}^2} \xi^2 |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx' < \infty,$$

$$\int ds\, d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2 |x'|^{-2\mu}\, dx' < \infty.$$

Then solutions to (3.1), (3.2) *satisfy*

$$\begin{aligned}(3.21)\quad \nu \int ds\, d\xi \int_{\mathbb{R}^2} \xi^2 (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\, dx' &\le \varepsilon \int ds\, d\xi \int_{\mathbb{R}^2} \xi^2 |\widetilde{p}|^2 |x'|^{-2\mu}\, dx' \\ &\quad + 4\left(1 + \frac{1}{\varepsilon}\right)\mu^2 \int ds\, d\xi \int_{\mathbb{R}^2} \xi^2 |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx' \\ &\quad + \frac{1}{\nu} \int ds\, d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2 |x'|^{-2\mu}\, dx'.\end{aligned}$$

Proof. Consider problem (3.7) in the form

$$(3.22)\qquad -\nu \Delta \widetilde{v} + \nabla \widetilde{p} = \widetilde{f} - s\widetilde{v} \equiv \widetilde{g}, \qquad \operatorname{div} \widetilde{v} = 0.$$

Applying the Fourier transform $(3.6)_2$, multiplying $(3.22)_1$ by $\overline{\widetilde{v}}|x'|^{-2\mu}$, integrating with respect to x' and integrating by parts we obtain

$$\begin{aligned}(3.23)\quad \nu \int_{\mathbb{R}^2} (|\nabla \widetilde{v}|^2 + \xi^2 |\widetilde{v}|^2)|x'|^{-2\mu}\, dx' &\le 2\nu\mu \int_{\mathbb{R}^2} \nabla \widetilde{v} \cdot \overline{\widetilde{v}} |x'|^{-2\mu-1} \nabla |x'|\, dx' - 2\mu \int_{\mathbb{R}^2} \widetilde{p}\, \overline{\widetilde{v}} |x'|^{-2\mu-1} \cdot \nabla |x'|\, dx' \\ &\quad + \int_{\mathbb{R}^2} \widetilde{g} \cdot \overline{\widetilde{v}} |x'|^{-2\mu}\, dx'.\end{aligned}$$

Applying the Hölder and Young inequalities we obtain

$$\begin{aligned}(3.24)\quad \frac{\nu}{2} \int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2 + \xi^2 |\widetilde{v}|^2)|x'|^{-2\mu}\, dx' &\le \frac{\varepsilon}{2} \int_{\mathbb{R}^2} |\widetilde{p}|^2 |x'|^{-2\mu}\, dx' + 2\left(1 + \frac{1}{\varepsilon}\right)\mu^2 \int_{\mathbb{R}^2} |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx' \\ &\quad + \frac{1}{2\nu} \frac{1}{\xi^2} \int_{\mathbb{R}^2} |\widetilde{g}|^2 |x'|^{-2\mu}\, dx'.\end{aligned}$$

Multiplying by $2\xi^2$ and integrating with respect to ξ and s implies (3.21). This concludes the proof.

Now, we estimate the first integral on the r.h.s. of (3.21).

LEMMA 3.3. *Assume that*

$$\int d\xi\, ds \int_{\mathbb{R}^2} \xi^2|\widetilde{v}|^2|x'|^{-2\mu-2}\, dx' < \infty,$$
$$\int d\xi\, ds \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\, dx' < \infty.$$

Then solutions of (3.1), (3.2) *satisfy the inequality*

$$\begin{aligned}(3.25)\quad & \int d\xi\, ds\, (\|\widetilde{v}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^2)}) \\ &+ \int d\xi\, ds \int_{\mathbb{R}^2} \xi^2(|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\, dx' + \int d\xi\, ds \int_{\mathbb{R}^2} \xi^2|\widetilde{p}|^2|x'|^{-2\mu}\, dx' \\ &\le c\int d\xi\, ds \int_{\mathbb{R}^2} \xi^2|\widetilde{v}|^2|x'|^{-2\mu-2}\, dx' + c\int d\xi\, ds \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\, dx'.\end{aligned}$$

Proof. Consider the problem

$$(3.26)\qquad \begin{aligned} &-\Delta\psi' + \nabla\eta = 0,\\ &\operatorname{div}\psi' = p|x'|^{-\mu},\\ &\psi'|_{\partial C_R} = 0.\end{aligned}$$

Let us underline that problem (3.1) is obtained from extension outside some cylinder such that

$$\int_{\mathbb{R}^3} p|x'|^{-\mu}\, dx = 0.$$

The condition follows from extension with respect to x_3 such that

$$\int_{\mathbb{R}} p(x', x_3, t)\, dx_3 = 0 \quad \text{for any } x' \in \mathbb{R}^2,\ t \in \mathbb{R}_+.$$

For solutions of (3.26) we have the estimate

$$\|\psi'\|^2_{H^1(\mathbb{R}^3)} + \|\eta\|^2_{L_2(\mathbb{R}^3)} \le c\|p\|^2_{L_{2,-\mu}(\mathbb{R}^3)}.$$

Inserting $\psi' = \psi|x'|^{-\mu}$ yields

$$(3.27)\qquad \int_{\mathbb{R}^3} (|\nabla'(\psi|x'|^{-\mu})|^2 + |\partial_{x_3}\psi|^2|x'|^{-2\mu})\, dx \le c\int_{\mathbb{R}^3} |p|^2|x'|^{-2\mu}\, dx.$$

Passing in (3.27) to the Fourier transforms with respect to x_3 we have

$$(3.28)\qquad \int d\xi \int_{\mathbb{R}^2} [|\nabla'(\widetilde{\psi}|x'|^{-\mu})|^2 + \xi^2|\widetilde{\psi}|^2|x'|^{-2\mu}]\, dx' \le c\int d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\, dx'.$$

Differentiating (3.26) with respect to x_3 we obtain instead of (3.27) the inequality

$$\int_{\mathbb{R}^3}(|\nabla'(\psi_{,x_3}|x'|^{-\mu})|^2+|\partial^2_{x_3}\psi|^2|x'|^{-2\mu})\,dx\le c\int_{\mathbb{R}^3}|p_{,x_3}|^2|x'|^{-2\mu}\,dx. \tag{3.29}$$

Passing to the Fourier transforms with respect to x_3 yields

$$\int d\xi\int_{\mathbb{R}^2}[\xi^2|\nabla'(\widetilde{\psi}|x'|^{-\mu})|^2+\xi^4|\widetilde{\psi}|^2|x'|^{-2\mu}]\,dx' \le c\int d\xi\int_{\mathbb{R}^2}\xi^2|\widetilde{p}|^2|x'|^{-2\mu}\,dx'. \tag{3.30}$$

Multiplying $(3.15)_1$ and (3.16) by $\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta(|\xi|\,|x'|)$, where ζ is a smooth function such that $\zeta(t)=0$ for $t\le a_2/2$, $\zeta(t)=1$ for $t\ge a_2$, $|\dot{\zeta}|\le c/a_2$, and integrating with respect to x' we obtain

$$\begin{aligned}\nu\int_{\mathbb{R}^2}(-\Delta'\widetilde{v}+\xi^2\widetilde{v})\cdot\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta\,dx'&\\ +\int_{\mathbb{R}^2}\bigg(\frac{\partial p}{\partial x_1}\overline{\widetilde{\psi}}_1+\frac{\partial p}{\partial x_2}\overline{\widetilde{\psi}}_2-i\xi\widetilde{p}\overline{\widetilde{\psi}}_3\bigg)|x'|^{-2\mu}\zeta\,dx'&=\int_{\mathbb{R}^2}\widetilde{g}\cdot\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta\,dx'.\end{aligned} \tag{3.31}$$

Integrating by parts yields

$$\begin{aligned}&\int_{\mathbb{R}^2}[\widetilde{p}(\overline{\widetilde{\psi}}_1|x'|^{-\mu})_{,x_1}+\widetilde{p}(\overline{\widetilde{\psi}}_2|x'|^{-\mu})_{,x_2}+\widetilde{p}\,\overline{i\xi\widetilde{\psi}_3}|x'|^{-\mu}]|x'|^{-\mu}\zeta\,dx'\\ &\quad+\int_{\mathbb{R}^2}\widetilde{p}[\overline{\widetilde{\psi}}_1|x'|^{-\mu}(|x'|^{-\mu}\zeta)_{,x_1}+\overline{\widetilde{\psi}}_2|x'|^{-\mu}(|x'|^{-\mu}\zeta)_{,x_2}]\,dx'\\ &=\nu\int_{\mathbb{R}^2}[\nabla'\widetilde{v}\cdot\nabla'(\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta)+\xi^2\widetilde{v}\cdot\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta]\,dx'-\int_{\mathbb{R}^2}\widetilde{g}\cdot\overline{\widetilde{\psi}}|x'|^{-2\mu}\zeta\,dx'.\end{aligned} \tag{3.32}$$

From $(3.26)_2$ we have $\operatorname{div}(\psi|x'|^{-\mu})=p|x'|^{-\mu}$, which in the Fourier transforms with respect to x_3 equals

$$\operatorname{div}'(\widetilde{\psi}'|x'|^{-\mu})-i\xi\widetilde{\psi}_3|x'|^{-\mu}=\widetilde{p}|x'|^{-\mu}. \tag{3.33}$$

Hence, (3.32) implies

$$\begin{aligned}\int\xi^2\,d\xi\int_{\mathbb{R}^2}|\widetilde{p}|^2|x'|^{-2\mu}\zeta\,dx'&\\ \le c\int\xi^2\,d\xi\int_{\mathbb{R}^2}|\widetilde{p}|\,|\widetilde{\psi}|(|x'|^{-2\mu-1}\zeta+|x'|^{-2\mu}\dot{\zeta}|\xi|)\,dx'&\\ +c\int\xi^2\,d\xi\int_{\mathbb{R}^2}[|\nabla'\widetilde{v}|\,|\nabla'(\widetilde{\psi}|x'|^{-\mu})|\,|x'|^{-\mu}\zeta+|\nabla'\widetilde{v}|\,|\widetilde{\psi}|\,|x'|^{-2\mu-1}\zeta&\\ +|\nabla'\widetilde{v}|\,|\widetilde{\psi}|\,|x'|^{-2\mu}|\xi|\dot{\zeta}+\xi^2|\widetilde{v}|\,|\widetilde{\psi}|\,|x'|^{-2\mu}\zeta]\,dx'&\\ +c\int\xi^2\,d\xi\int_{\mathbb{R}^2}|\widetilde{g}|\,|\widetilde{\psi}|\,|x'|^{-2\mu}\zeta\,dx'.&\end{aligned} \tag{3.34}$$

By the Hölder and Young inequalities, we obtain

$$(3.35)\quad \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\zeta\,dx'$$
$$\le \varepsilon \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\zeta\,dx' + \frac{1}{4\varepsilon}\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{\psi}|^2|x'|^{-2\mu-2}\zeta\,dx'$$
$$+ \varepsilon_1 \int \xi^4\,d\xi \int_{\mathbb{R}^2} |\widetilde{\psi}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx' + \frac{1}{4\varepsilon_1}\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx'$$
$$+ \varepsilon_2 \int \xi^2\,d\xi \int_{\mathbb{R}^2} (|\nabla'(\widetilde{\psi}|x'|^{-\mu})|^2 + |\widetilde{\psi}|^2|x'|^{-2\mu-2} + \xi^2|\widetilde{\psi}|^2|x'|^{-2\mu})\zeta\,dx'$$
$$+ \frac{1}{4\varepsilon_2}\int \xi^2\,d\xi \int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\zeta\,dx'$$
$$+ \varepsilon_3 \int \xi^4\,d\xi \int_{\mathbb{R}^2} |\widetilde{\psi}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx' + \frac{1}{4\varepsilon_3}\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\nabla\widetilde{v}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx'$$
$$+ \varepsilon_4 \int \xi^4\,d\xi \int_{\mathbb{R}^2} |\widetilde{\psi}|^2|x'|^{-2\mu}\zeta\,dx' + \frac{1}{4\varepsilon_4}\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\zeta\,dx'.$$

Assuming that $\varepsilon = 1/2$, adding $\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}(1-\zeta)\,dx'$ to both sides and using

$$\operatorname{supp}\zeta = \{(\xi, x') : |x'|\,|\xi| \ge a_2/2 \text{ so } |x'|^{-1} \le (2/a_2)|\xi|\}$$

we obtain from (3.35) the inequality

$$(3.36)\quad \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\,dx'$$
$$\le \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}(1-\zeta)\,dx' + \frac{1}{2\varepsilon_1}\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx'$$
$$+ 2\left(\frac{4}{a_2^2} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4\right)\int \xi^4\,d\xi \int_{\mathbb{R}^2} |\widetilde{\psi}|^2|x'|^{-2\mu}\,dx'$$
$$+ 2\varepsilon_2 \int \xi^2\,d\xi \int_{\mathbb{R}^2} (|\nabla'(\widetilde{\psi}|x'|^{-\mu})|^2 + |\widetilde{\psi}|^2|x'|^{-2\mu-2})\zeta\,dx'$$
$$+ \frac{1}{2}\left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\int \xi^2\,d\xi \int_{\mathbb{R}^2 \cap \operatorname{supp}\zeta} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\,dx'$$
$$+ \frac{1}{2\varepsilon_4}\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx'.$$

In view of (3.30) and for sufficiently small ε_i and sufficiently large a_2 we have

$$\begin{aligned}(3.37)\quad & \frac{1}{2}\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\,dx' \le \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}(1-\zeta)\,dx' \\ &+ c\int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}|\dot{\zeta}|\,dx' + c\int_{\mathrm{supp}\,\zeta} \xi^2\,d\xi \int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\,dx' \\ &+ c\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx'.\end{aligned}$$

Since $1-\zeta\ne 0$ for $|\xi|\,|x'|<a_2$ and $\dot{\zeta}\ne 0$ for $a_2/2<|\xi|\,|x'|<a_2$, we obtain $|\xi|^2<a_2^2|x'|^{-2}$. Then (3.37) takes the form

$$\begin{aligned}(3.38)\quad & \int \xi^2\,d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu}\,dx' \\ &\le c\int d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu-2}\,dx' + c\int_{\mathrm{supp}\,\zeta} d\xi \int_{\mathbb{R}^2} dx'\,\xi^2(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu} \\ &\quad + c\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx'.\end{aligned}$$

Using (3.38) in (3.21) and exploiting that ε is sufficiently small we have

$$\begin{aligned}(3.39)\quad & \int d\xi \int_{\mathbb{R}^2} \xi^2\,(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\,dx' \\ &\le \varepsilon c\int d\xi \int_{\mathbb{R}^2} |\widetilde{p}|^2|x'|^{-2\mu-2}\,dx' + c\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx' \\ &\quad + c\left(1+\frac{1}{\varepsilon}\right)\int d\xi \int_{\mathbb{R}^2} \xi^2|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx'.\end{aligned}$$

Using (3.39) in (3.19) and assuming that ε is sufficiently small we obtain

$$\begin{aligned}(3.40)\quad & \int d\xi\,(\|\widetilde{v}'\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^2)}) \\ &\le c\int d\xi \int_{\mathbb{R}^2} \xi^2|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx' + c\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx'.\end{aligned}$$

Employing (3.40) in (3.39), (3.38) and (3.20) we get

$$\begin{aligned}(3.41)\quad & \int d\xi\,(\|\widetilde{v}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^2)}) \\ &+ \int d\xi \int_{\mathbb{R}^2} \xi^2(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu}\,dx' + \int d\xi \int_{\mathbb{R}^2} \xi^2|\widetilde{p}|^2|x'|^{-2\mu}\,dx' \\ &\le c\int d\xi \int_{\mathbb{R}^2} \xi^2|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx' + c\int d\xi \int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\,dx'.\end{aligned}$$

This implies (3.25), and concludes the proof.

Next we estimate the first term on the r.h.s. of (3.25). Let a_1, a_2 be positive numbers. Then we introduce the sets (see [14])

$$\begin{aligned}Q_1 &= \{(\xi,x')\in\mathbb{R}^3 : |\xi|\,|x'|\le a_1\},\\ Q_2 &= \{(\xi,x')\in\mathbb{R}^3 : |\xi|\,|x'|\ge a_2\},\\ Q_3 &= \{(\xi,x')\in\mathbb{R}^3 : a_1\le|\xi|\,|x'|\le a_2\},\end{aligned}$$

LEMMA 3.4. *Assume that*

$$\begin{aligned}&\int d\xi\int_{\mathbb{R}^2}dx'\,|\widetilde v|^2|x'|^{-2\mu-4}<\infty,\\ &\int d\xi\int_{\mathbb{R}^2}dx'\,\xi^4|\widetilde v|^2|x'|^{-2\mu}<\infty,\\ &\int d\xi\int_{\mathbb{R}^2}dx'\,|\widetilde g|^2|x'|^{-2\mu}<\infty.\end{aligned}$$

Then solutions of (3.1), (3.2) *satisfy*

$$\begin{aligned}\int d\xi\int_{\mathbb{R}^2}dx'\,\xi^2|\widetilde v|^2|x'|^{-2\mu-2}&\le a_1^2\int d\xi\int_{\mathbb{R}^2}dx'|\widetilde v|^2|x'|^{-2\mu-4}\\ &+\frac{1}{a_2^2}\int d\xi\int_{\mathbb{R}^2}dx'\,\xi^4|\widetilde v|^2|x'|^{-2\mu}+\frac{4}{a_1^{2+2\mu}}\int d\xi\int_{\mathbb{R}^2}dx'\,|\widetilde g|^2|x'|^{-2\mu}.\end{aligned}\tag{3.42}$$

Proof. We write the first term on the r.h.s. of (3.25) in the form

$$\sum_{i=1}^3\int_{Q_i}d\xi\,dx'\,\xi^2|\widetilde v|^2|x'|^{-2\mu-2}\equiv\sum_{i=1}^3 I_i,$$

where

$$\begin{aligned}I_1&\le a_1^2\int_{Q_1}d\xi\,dx'\,|\widetilde v|^2|x'|^{-2\mu-4},\\ I_2&\le\frac{1}{a_2^2}\int_{Q_2}d\xi\,dx'\,\xi^4|\widetilde v|^2|x'|^{-2\mu},\\ I_3&\le\frac{1}{a_1^{2+2\mu}}\int_{Q_3}d\xi\,dx'\,\xi^{4+2\mu}|\widetilde v|^2\equiv I.\end{aligned}\tag{3.43}$$

To examine the integral I in (3.43) we introduce

$$\begin{aligned}d_1(\xi)&=\{x'\in\mathbb{R}^2:|\xi|\,|x'|\le a_1\},\\ d_2(\xi)&=\{x'\in\mathbb{R}^2:|\xi|\,|x'|\ge a_2\},\\ d_3(\xi)&=\{x'\in\mathbb{R}^2:a_1\le|\xi|\,|x'|\le a_2\}.\end{aligned}$$

For $\lambda>0$ we define

$$\Omega^\lambda=\{(x',\xi)\in\mathbb{R}^2\times\mathbb{R}:\lambda|\xi|\,|x'|\le1\},\qquad \omega^\lambda(\xi)=\{x'\in\mathbb{R}^2:\lambda|\xi|\,|x'|\le1\}.$$

We have $Q_3 \subset \Omega^\lambda$ for $\lambda \in (0, a_2^{-1}]$. Let $\chi = \chi(t)$ be a smooth function such that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$, $0 \le \chi(t) \le 1$, $\chi'(t) \le 2$.

Write $\chi_\lambda(x', \xi) = \chi(\lambda|\xi|\,|x'|)$. Then $\chi_\lambda(x', \xi) = 1$ for $|\xi|\,|x'| \le \lambda^{-1}$ and $\chi_\lambda(x', \xi) = 0$ for $|\xi|\,|x'| \ge 2\lambda^{-1}$. To estimate I we consider the system

$$
\begin{aligned}
&-\Delta'\widetilde{v}' + \xi^2\widetilde{v}' + \nabla'\widetilde{p} = \widetilde{g}',\\
&-\Delta'\widetilde{v}_3 + \xi^2\widetilde{v}_3 - i\xi\widetilde{p} = \widetilde{g}_3,\\
&\nabla'\cdot\widetilde{v}' - i\xi\widetilde{v}_3 = 0.
\end{aligned}
\tag{3.44}
$$

Let φ be such that

$$
\operatorname{div}'(\nabla'\widetilde{\varphi} - \widetilde{v}'\chi_\lambda^2) - i\xi(-i\xi\widetilde{\varphi} - \widetilde{v}_3\chi_\lambda^2) = 0, \qquad \widetilde{\varphi}|_{\partial B_R(0)} = 0.
\tag{3.45}
$$

Multiplying $(3.44)_1$ by $\overline{\widetilde{v}}'\chi_\lambda^2 - \nabla'\overline{\widetilde{\varphi}}$, $(3.44)_2$ by $\overline{\widetilde{v}}_3\chi_\lambda^2 + i\xi\overline{\widetilde{\varphi}}$, adding the results, integrating over $\mathbb{R}^2$, integrating by parts and using $(3.44)_3$ yields

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \nabla'\widetilde{v}'\cdot\nabla'(\overline{\widetilde{v}}'\chi_\lambda^2 - \nabla'\overline{\widetilde{\varphi}})\,dx' + \int_{\mathbb{R}^2} \xi^2\widetilde{v}(\overline{\widetilde{v}}\chi_\lambda^2 - \nabla'\overline{\widetilde{\varphi}})\,dx'\\
&\qquad + \int_{\mathbb{R}^2} \nabla'\widetilde{v}_3\cdot\nabla'(\overline{\widetilde{v}}_3\chi_\lambda^2 + \overline{i\xi\widetilde{\varphi}})\,dx' + \int_{\mathbb{R}^2} \xi^2\widetilde{v}_3(\overline{\widetilde{v}}_3\chi_\lambda^2 - \overline{i\xi\widetilde{\varphi}})\,dx'\\
&\qquad = \int_{\mathbb{R}^2} \widetilde{g}'(\overline{\widetilde{v}}'\chi_\lambda^2 - \nabla'\overline{\widetilde{\varphi}})\,dx' + \int_{\mathbb{R}^2} \widetilde{g}_3(\overline{\widetilde{v}}_3\chi_\lambda^2 + \overline{i\xi\widetilde{\varphi}})\,dx'.
\end{aligned}
$$

Since v is divergence free, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)\chi_\lambda^2\,dx' &= \int_{\mathbb{R}^2} \nabla'\widetilde{v}\overline{\widetilde{v}}2\nabla\chi_\lambda\chi_\lambda\,dx'\\
&\quad + \int_{\mathbb{R}^2} (\widetilde{g}'\cdot\overline{\widetilde{v}}' + \widetilde{g}_3\overline{\widetilde{v}}_3)\chi_\lambda^2\,dx' - \int_{\mathbb{R}^2} (\widetilde{g}'\cdot\nabla'\overline{\widetilde{\varphi}} + \widetilde{g}_3 i\xi\overline{\widetilde{\varphi}})\,dx'.
\end{aligned}
\tag{3.46}
$$

We estimate the first term on the r.h.s. by

$$
\frac{\varepsilon_1}{2}\int_{\mathbb{R}^2} |\nabla'\widetilde{v}|^2\chi_\lambda^2\,dx' + \frac{4}{2\varepsilon_1}\int_{\mathbb{R}^2} |\widetilde{v}|^2|\nabla\chi_\lambda|^2\,dx',
$$

the second term by

$$
\frac{\varepsilon_2}{2}\int_{\mathbb{R}^2} |\widetilde{v}|^2|x'|^{2\mu}|\xi|^{2+2\mu}\chi_\lambda^2\,dx' + \frac{1}{2\varepsilon_2}\,\frac{1}{|\xi|^{2+2\mu}}\int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx',
$$

and the last term by

$$
\frac{\varepsilon_3}{2}\int_{\mathbb{R}^2} |\xi|^{2+2\mu}(|\nabla'\widetilde{\varphi}|^2 + |\xi|^2|\widetilde{\varphi}|^2)|x'|^{2\mu}\,dx' + \frac{1}{2\varepsilon_3}\,\frac{1}{|\xi|^{2+2\mu}}\int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx'.
$$

We assume that $\varepsilon_1 = 1$. On $\operatorname{supp}\chi_\lambda$ we have $|x'|^{2\mu}|\xi|^{2\mu} \le (2/\lambda)^{2\mu}$, so the term with ε_2 is estimated by

$$\frac{\varepsilon_2}{2}\left(\frac{2}{\lambda}\right)^{2\mu}\int\limits_{\mathbb{R}^2}|\widetilde{v}|^2|\xi|^2\chi_\lambda^2\,dx'.$$

We assume that $\varepsilon_2=(\lambda/2)^{2\mu}$. Using the above estimates in (3.46) yields

$$\begin{aligned}\frac{1}{2}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)\chi_\lambda^2\,dx' &\le 2\int\limits_{\mathbb{R}^2}|\widetilde{v}|^2|\nabla\chi_\lambda|^2\,dx'\\ &\quad+\frac{\varepsilon_3}{2}\int\limits_{\mathbb{R}^2}|\xi|^{2+2\mu}(|\nabla'\widetilde{\varphi}|^2+\xi^2|\widetilde{\varphi}|^2)|x'|^{2\mu}\,dx'\\ &\quad+\frac{1}{2}\left(\left(\frac{2}{\lambda}\right)^{2\mu}+\frac{1}{\varepsilon_3}\right)\frac{1}{|\xi|^{2+2\mu}}\int\limits_{\mathbb{R}^2}|\widetilde{g}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx'.\end{aligned}$$

Multiplying the above inequality by $|\xi|^{2+2\mu}$ and integrating with respect to ξ implies

$$\begin{aligned}\frac{1}{2}\int d\xi\,|\xi|^{2+2\mu}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)\chi_\lambda^2\,dx'\\ \le 2\int d\xi\,|\xi|^{2+2\mu}\int\limits_{\mathbb{R}^2}|\widetilde{v}|^2|\nabla\chi_\lambda|^2\,dx'\\ +\frac{\varepsilon_3}{2}\int d\xi\,|\xi|^{4+4\mu}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{\varphi}|^2+\xi^2|\widetilde{\varphi}|^2)|x'|^{2\mu}\,dx'\\ +\frac{1}{2}\left[\left(\frac{2}{\lambda}\right)^{2\mu}+\frac{1}{\varepsilon_3}\right]\int d\xi\int\limits_{\mathbb{R}^2}|\widetilde{g}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx'.\end{aligned}\tag{3.47}$$

Let F be the Fourier transform $(3.6)_2$. Then problem (3.45) can be written in the form

$$\varDelta\widetilde{\varphi}=\operatorname{div}(\widetilde{v}*F^{-1}\chi_\lambda^2),\qquad \widetilde{\varphi}|_{\partial C_R}=0,\tag{3.48}$$

where $*$ is convolution with respect to x_3 and $\sim$ is the Laplace transform with respect to t (see $(3.6)_1$). After applying the Parseval identity, the second term on the r.h.s. of (3.47) is

$$\int\limits_{\mathbb{R}^3}dx\,|\partial_{x_3}^{2+2\mu}\nabla\widetilde{\varphi}|^2|x'|^{2\mu}\equiv J,$$

where $\partial_{x_3}^{2+2\mu}$ is a fractional derivative. To estimate J we consider instead of (3.48) the problem

$$\varDelta\partial_{x_3}^{2+2\mu}\widetilde{\varphi}=\operatorname{div}\partial_{x_3}^{2+2\mu}(\widetilde{v}*F^{-1}\chi_\lambda^2),\qquad \partial_{x_3}^{2+2\mu}\widetilde{\varphi}|_{\partial C_R}=0.$$

In view of Lemma 2.3 we have

$$\int_{\mathbb{R}^3} dx\,|\partial_{x_3}^{2+2\mu}\nabla\widetilde{\varphi}|^2|x'|^{2\mu} \le c_1 \int_{\mathbb{R}^3} dx\,|\partial_{x_3}^{2+2\mu}\widetilde{v} * F^{-1}\chi_\lambda^2|^2|x'|^{2\mu}.$$

By the Parseval identity for the Fourier transform $(3.6)_2$ the above inequality implies

$$\int d\xi \int_{\mathbb{R}^2} dx'\,|\xi|^{4+4\mu}(|\nabla'\widetilde{\varphi}|^2 + \xi^2|\widetilde{\varphi}|^2)|x'|^{2\mu} \le c_1 \int d\xi \int_{\mathbb{R}^2} dx'\,|\xi|^{4+4\mu}|\widetilde{v}|^2\chi_\lambda^4|x'|^{2\mu}.$$

Since $|x'|^{2\mu}|\xi|^{2\mu} \le (2/\lambda)^{2\mu}$ on $\operatorname{supp}\chi_\lambda$ and $\chi_\lambda \le 1$ we obtain

$$J \le c_1\left(\frac{2}{\lambda}\right)^{2\mu} \int d\xi \int_{\mathbb{R}^2} dx'\,|\xi|^{4+2\mu}|\widetilde{v}|^2\chi_\lambda^2.$$

Assuming that $(\varepsilon_3/2)c_1(2/\lambda))^{2\mu} = 1/4$ we obtain from (3.47) the inequality

$$(3.49)\quad \frac{1}{4}\int d\xi\,|\xi|^{2+2\mu}\int_{\mathbb{R}^2}(|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)\chi_\lambda^2\,dx' \le 2\int d\xi\,|\xi|^{2+2\mu}\int_{\mathbb{R}^2}|\widetilde{v}|^2|\nabla\chi_\lambda|^2\,dx' + (1+c_1)\left(\frac{2}{\lambda}\right)^{2\mu}\int d\xi\int_{\mathbb{R}^2} dx'\,|\widetilde{g}|^2\chi_\lambda^2\,dx'.$$

Since $|\nabla\chi_\lambda| \le 2\lambda|\xi|$, assuming that $\lambda \le 2$ we obtain from (3.49) the inequality

$$(3.50)\quad \int d\xi\,|\xi|^{2+2\mu}\int_{\omega^\lambda(\xi)}\xi^2|\widetilde{v}|^2\,dx' \le 32\lambda^2\int d\xi\,|\xi|^{2+2\mu}\int_{\omega^{\lambda/2}(\xi)\setminus\omega^\lambda(\xi)}\xi^2|\widetilde{v}|^2\,dx' + c_2\left(\frac{2}{\lambda}\right)^{2\mu+2}\int d\xi\int_{\mathbb{R}^2}|\widetilde{g}|^2|x'|^{-2\mu}\,dx',$$

where $c_2 = 4(1+c_1)$. Multiplying by $(\lambda/2)^{2\mu+2}$ yields

$$(3.51)\quad \left(\frac{\lambda}{2}\right)^{2\mu+2}\int d\xi\,|\xi|^{2+2\mu}\int_{\omega^\lambda(\xi)}\xi^2|\widetilde{v}|^2\,dx' \le 2\cdot 4^{3+\mu}\lambda^2\left(\frac{\lambda/2}{2}\right)^{2\mu+2}\int d\xi\,|\xi|^{2+2\mu}\int_{\omega^{\lambda/2}(\xi)\setminus\omega^\lambda(\xi)}\xi^2|\widetilde{v}|^2\,dx' + c_2\int d\xi\int_{\mathbb{R}^2}|\widetilde{g}|^2|x'|^{-2\mu}\,dx'.$$

Let $2 \cdot 4^{3+\mu}\lambda^2 \le 1/2$. Then iterating (3.51) up to order k we obtain

$$\left(\frac{\lambda}{2}\right)^{2\mu+2} \int d\xi\, |\xi|^{2+2\mu} \int\limits_{\omega^\lambda(\xi)} \xi^2 |\widetilde{v}|^2\, dx' \tag{3.52}$$

$$\le \frac{1}{2^k}\left(\frac{\lambda/2^k}{2}\right)^{2\mu+2} \int d\xi\, |\xi|^{2+2\mu} \int\limits_{\omega^{\lambda/2^{k+1}}(\xi)\setminus\omega^{\lambda/2^k}(\xi)} \xi^2|\widetilde{v}|^2\, dx'$$

$$+ 2c_2 \int d\xi \int\limits_{\mathbb{R}^2} |\widetilde{g}|^2 |x'|^{-2\mu}\, dx',$$

where

$$\omega^{\lambda/2^{k+1}}(\xi) \setminus \omega^{\lambda/2^k}(\xi) = \{x' \in \mathbb{R}^2 : 2^k/\lambda \le |x'|\,|\xi| \le 2^{k+1}/\lambda\}. \tag{3.53}$$

On the set (3.53) we have

$$|\xi| \le \frac{2^{k+1}}{\lambda}\, |x'|^{-1}$$

so the first term on the r.h.s. of (3.52) is estimated by

$$\frac{1}{2^k}\left(\frac{\lambda}{2^{k+1}}\right)^{2\mu+2} \int d\xi\, \xi^2 \int\limits_{\omega^{\lambda/2^{k+1}}(\xi)\setminus\omega^{\lambda/2^k}(\xi)} |\widetilde{v}|^2 \left(\frac{2^{k+1}}{\lambda}\right)^{2\mu+2} \times |x'|^{-2\mu-2}\, dx' \tag{3.54}$$

$$= \frac{1}{2^k} \int d\xi\, \xi^2 \int\limits_{\omega^{\lambda/2^{k+1}}(\xi)\setminus\omega^{\lambda/2^k}(\xi)} |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx'$$

$$\le \frac{1}{2^k} \int d\xi\, \xi^2 \int\limits_{\mathbb{R}^2} |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx'.$$

In view of (3.43) and (3.52)–(3.54) we obtain

$$\int d\xi\, \xi^2 \int\limits_{\mathbb{R}^2} |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx' \tag{3.55}$$

$$\le a_1^2 \int\limits_{Q_1} d\xi\, dx'\, |\widetilde{v}|^2 |x'|^{-2\mu-4} + \frac{1}{a_2^2} \int\limits_{Q_2} d\xi\, dx'\, \xi^4 |\widetilde{v}|^2 |x'|^{-2\mu}$$

$$+ \frac{1}{2^k}\, \frac{1}{a_1^{2+2\mu}} \int d\xi\, \xi^2 \int\limits_{\mathbb{R}^2} |\widetilde{v}|^2 |x'|^{-2\mu-2}\, dx'$$

$$+ \frac{2c_2}{a_1^{2+2\mu}} \int d\xi \int\limits_{\mathbb{R}^2} |\widetilde{g}|^2 |x'|^{-2\mu}\, dx'.$$

This implies (3.42) for sufficiently large k and concludes the proof.

Using (3.55) in (3.25) and assuming that a_1 is sufficiently small and a_2, k are sufficiently large we obtain

$$\int ds\,(\|\widetilde{v}\|^2_{H^2_{-\mu}(\mathbb{R}^3)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^3)}) \le c\int ds\,\|\widetilde{g}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}. \tag{3.56}$$

Finally, we have to estimate the first term on the r.h.s. of (3.8).

LEMMA 3.5. *Assume that*

$$\int ds \int_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-4}\,dx < \infty,$$

$$\int ds \int_{\mathbb{R}^3} |s|^2|\widetilde{v}|^2|x'|^{-2\mu}\,dx < \infty,$$

$$\int ds \int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx < \infty.$$

Then solutions to problem (3.1), (3.2) *satisfy*

$$\int ds \int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx \le 2\overline{a}_1\int ds \int_{\mathbb{R}^3} |\widetilde{v}|^2|x'|^{-2\mu-4}\,dx + \frac{2}{\overline{a}_2}\int ds \int_{\mathbb{R}^3} dx\,|s|^2|\widetilde{v}|^2|x'|^{-2\mu} + 8\left(\frac{2}{\lambda}\right)^{1+\mu}\int ds \int_{\mathbb{R}^3} |\widetilde{f}|^2|x'|^{-2\mu}\,dx, \tag{3.57}$$

where $\overline{a}_1, \overline{a}_2$ *are positive constants defined in* (3.58).

Proof. First we introduce the sets (see [14])

$$\begin{aligned} \overline{Q}_1 &= \{(s,x') : |s|\,|x'|^2 \le \overline{a}_1\},\\ \overline{Q}_2 &= \{(s,x') : |s|\,|x'|^2 \ge \overline{a}_2\},\\ \overline{Q}_3 &= \{(s,x') : \overline{a}_1 \le |s|\,|x'|^2 \le \overline{a}_2\}, \end{aligned} \tag{3.58}$$

where the numbers $\overline{a}_i$, $i = 1,2,3$, will be chosen later.

In view of (3.58) we express the first integral on the r.h.s. of (3.8) in the form

$$\int ds \int_{\mathbb{R}^3} |s|\,|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx = \sum_{i=1}^{3} \int_{\overline{Q}_i} ds\,dx\,|s|\,|\widetilde{v}|^2|x'|^{-2\mu-2} \equiv \sum_{i=1}^{3} \bar{I}_i. \tag{3.59}$$

From the properties of the sets $\overline{Q}_i$, $i = 1,2,3$, we have

$$\begin{aligned} \bar{I}_1 &\le \overline{a}_1\int ds \int_{\mathbb{R}^3} dx\,|\widetilde{v}|^2|x'|^{-2\mu-4},\\ \bar{I}_2 &\le \frac{1}{\overline{a}_2}\int ds \int_{\mathbb{R}^3} dx\,|s|^2|\widetilde{v}|^2|x'|^{-2\mu}\\ \bar{I}_3 &\le \frac{1}{\overline{a}_1^{\mu+1}}\int ds \int_{\mathbb{R}^3} dx\,|s|^{2+\mu}|\widetilde{v}|^2 \equiv \bar{I}. \end{aligned} \tag{3.60}$$

To estimate $\bar{I}$ we introduce the sets

$$\begin{aligned}\bar{d}_1(s) &= \{x' \in \mathbb{R}^2 : |s|\,|x'|^2 \le \bar{a}_1\},\\ \bar{d}_2(s) &= \{x' \in \mathbb{R}^2 : |s|\,|x'|^2 \ge \bar{a}_2\},\\ \bar{d}_3(s) &= \{x' \in \mathbb{R}^2 : \bar{a}_1 \le |s|\,|x'|^2 \le \bar{a}_2\}.\end{aligned}$$

Moreover, for $\lambda > 0$ we define

$$\begin{aligned}\overline{\Omega}^\lambda &= \{(x', |s|) \in \mathbb{R}^2 \times \mathbb{R}_+ : \lambda|s|\,|x'|^2 \le 1\},\\ \overline{\omega}^\lambda(s) &= \{x' \in \mathbb{R}^2 : \lambda|s|\,|x'|^2 \le 1\}.\end{aligned}$$

We have $\overline{Q}_3 \subset \Omega^\lambda$ for $\lambda \in (0, \overline{a}_2^{-1}]$. Let $\chi(t)$ be the function introduced in the proof of Lemma 3.4. Then $\overline{\chi}_\lambda(x', s) = \chi(\lambda|s|\,|x'|^2)$. Hence $\overline{\chi}_\lambda(x', s) = 1$ for $|s|\,|x'|^2 \le \lambda^{-1}$ and $\overline{\chi}_\lambda(x', s) = 0$ for $|s|\,|x'|^2 \ge 2\lambda^{-1}$.

Let $\widetilde{\varphi}$ be such that

$$\operatorname{div}(\nabla\widetilde{\varphi} - (1 + i \operatorname{sign} \operatorname{Im} s)\widetilde{v}\overline{\chi}_\lambda^2) = 0, \qquad \widetilde{\varphi}|_{\partial C_R} = 0. \tag{3.61}$$

Since v is divergence free, (3.61) assumes the form

$$\Delta\widetilde{\varphi} = 2(1 + i \operatorname{sign} \operatorname{Im} s)\widetilde{v}\nabla\overline{\chi}_\lambda\,\overline{\chi}_\lambda, \qquad \widetilde{\varphi}|_{\partial C_R} = 0. \tag{3.62}$$

Multiplying $(3.7)_1$ by $\overline{\widetilde{v}}\overline{\chi}_\lambda^2 - \nabla\overline{\widetilde{\varphi}}$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned}\int_{\mathbb{R}^3} s\widetilde{v} \cdot ((1 - i \operatorname{sign} \operatorname{Im} s)\overline{\widetilde{v}}\overline{\chi}_\lambda^2 - \nabla\overline{\widetilde{\varphi}})\,dx \\ + \nu \int_{\mathbb{R}^3} \nabla\widetilde{v} \cdot \nabla((1 - i \operatorname{sign} \operatorname{Im} s)\overline{\widetilde{v}}\overline{\chi}_\lambda^2 - \nabla\overline{\widetilde{\varphi}})\,dx \\ = \int_{\mathbb{R}^3} \widetilde{f}((1 - i \operatorname{sign} \operatorname{Im} s)\overline{\widetilde{v}}\overline{\chi}_\lambda^2 - \nabla\overline{\widetilde{\varphi}})\,dx.\end{aligned} \tag{3.63}$$

Since $\operatorname{div}\widetilde{v} = 0$, taking the real part in (3.63) gives

$$\begin{aligned}&\int_{\mathbb{R}^3} (|s|\,|\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)\overline{\chi}_\lambda^2\,dx \\ &= -2\Big(\operatorname{Re}\int_{\mathbb{R}^3} \nabla\widetilde{v}\,\overline{\widetilde{v}}\nabla\overline{\chi}_\lambda\,\overline{\chi}_\lambda\,dx + \operatorname{sign} \operatorname{Im} s \operatorname{Im}\int_{\mathbb{R}^3} \nabla\widetilde{v}\,\overline{\widetilde{v}}\nabla\overline{\chi}_\lambda\,\overline{\chi}_\lambda\,dx\Big) \\ &\quad + \Big(\operatorname{Re}\int_{\mathbb{R}^3} \widetilde{f} \cdot \overline{\widetilde{v}}\overline{\chi}_\lambda^2\,dx + \operatorname{sign} \operatorname{Im} s \operatorname{Im}\int_{\mathbb{R}^3} \widetilde{f} \cdot \overline{\widetilde{v}}\overline{\chi}_\lambda^2\,dx\Big) - \operatorname{Re}\int_{\mathbb{R}^3} \widetilde{f} \cdot \nabla\overline{\widetilde{\varphi}}\,dx.\end{aligned} \tag{3.64}$$

Applying the Hölder and Young inequalities yields

$$\begin{aligned}\int_{\mathbb{R}^3} (|s|\,|\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)\overline{\chi}_\lambda^2\,dx \le \frac{\nu}{2}\int_{\mathbb{R}^3} |\nabla\widetilde{v}|^2\overline{\chi}_\lambda^2\,dx \\ + 2\nu\int_{\mathbb{R}^3} |\widetilde{v}|^2|\nabla\overline{\chi}_\lambda|^2\,dx + \frac{\varepsilon_1}{2}\int_{\mathbb{R}^3} |s|^{1+\mu}|\widetilde{v}|^2|x'|^{2\mu}\overline{\chi}_\lambda^2\,dx\end{aligned} \tag{3.65}$$

$$+\frac{1}{2\varepsilon_1}\frac{1}{|s|^{1+\mu}}\int|\widetilde{f}|^2|x'|^{-2\mu}\overline{\chi}_\lambda^2\,dx$$

$$+\frac{\varepsilon_2}{2}\int_{\mathbb{R}^3}|\nabla\widetilde{\varphi}|^2|s|^{1+\mu}|x'|^{2\mu}\,dx$$

$$+\frac{1}{2\varepsilon_2}\frac{1}{|s|^{1+\mu}}\int_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.$$

Since $(|s|\,|x'|^2)^\mu\le(2/\lambda)^\mu$ on $\operatorname{supp}\overline{\chi}_\lambda$, the third term on the r.h.s. of (3.65) is estimated by

$$\frac{\varepsilon_1}{2}\left(\frac{2}{\lambda}\right)^\mu\int_{\mathbb{R}^3}|s|\,|\widetilde{v}|^2\overline{\chi}_\lambda^2\,dx.$$

We choose $\varepsilon_1=\frac{1}{2}(\lambda/2)^\mu$.

By Lemma 2.3 solutions to problem (3.61) satisfy

$$\|\nabla\widetilde{\varphi}\|_{L_{2,\mu}(\mathbb{R}^3)}\le c\|\widetilde{v}\overline{\chi}_\lambda^2\|_{L_{2,\mu}(\mathbb{R}^3)}\tag{3.66}$$

so the fifth term on the r.h.s. of (3.65) is bounded by

$$\frac{\varepsilon_2}{2}\int_{\mathbb{R}^3}|\widetilde{v}|^2|s|^{1+\mu}\overline{\chi}_\lambda^2|x'|^{2\mu}\,dx\le\frac{\varepsilon_2}{2}\left(\frac{2}{\lambda}\right)^\mu\int_{\mathbb{R}^3}|\widetilde{v}|^2\overline{\chi}_\lambda^2\,dx,$$

where we have used the fact that $|s|^\mu|x'|^{2\mu}\le(2/\lambda)^\mu$ on $\operatorname{supp}\overline{\chi}_\lambda$. We choose $\varepsilon_2=\varepsilon_1$.

In view of the above estimates we obtain from (3.65) the inequality

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}(|s|\,|\widetilde{v}|^2+\nu|\nabla\widetilde{v}|^2)\overline{\chi}_\lambda^2\,dx&\\ \le 2\nu\int_{\mathbb{R}^3}|\widetilde{v}|^2|\nabla\overline{\chi}_\lambda|^2\,dx+2\left(\frac{2}{\lambda}\right)^\mu\frac{1}{|s|^{1+\mu}}\int_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\overline{\chi}_\lambda^2\,dx.&\end{aligned}\tag{3.67}$$

Since

$$|\nabla\overline{\chi}_\lambda|\le 2\lambda|s|\,|\nabla|x'|\,|\,|x'|\le 2\lambda|s|\,|x'|,$$

multiplying (3.67) by 2 we obtain from (3.67) the relation

$$\begin{aligned}\int_{\omega^\lambda(s)}|s|\,|\widetilde{v}|^2\,dx\le 16\nu\lambda^2\int_{\omega^{\lambda/2}(s)\setminus\omega^\lambda(s)}|s|^2|\widetilde{v}|^2|x'|^2\,dx&\\ +4\left(\frac{2}{\lambda}\right)^{\mu+1}\frac{1}{|s|^{1+\mu}}\int_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.&\end{aligned}\tag{3.68}$$

Multiplying (3.68) by $(\lambda/2)^{\mu+1}|s|^{1+\mu}$, using the estimate $\lambda|s|\,|x'|^2\le 2$ on $\omega^{\lambda/2}(s)$ and integrating with respect to s we have

$$(3.69)\quad \Big(\frac{\lambda}{2}\Big)^{\mu+1}\int ds \int\limits_{\omega^\lambda(s)} |s|^{2+\mu}|\widetilde{v}|^2\,dx \le 64\cdot 2^\mu\nu\lambda\Big(\frac{\lambda/2}{2}\Big)^{\mu+1}\int ds \int\limits_{\omega^{\lambda/2}(s)\setminus\omega^\lambda(s)} |s|^{2+\mu}|\widetilde{v}|^2\,dx + 4\int ds\int\limits_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.$$

Let $64\cdot 2^\mu\nu\lambda\le 1/2$. Iterating (3.69) up to order k we obtain

$$(3.70)\quad \Big(\frac{\lambda}{2}\Big)^{\mu+1}\int ds \int\limits_{\omega^\lambda(s)} |s|^{2+\mu}|\widetilde{v}|^2\,dx \le \frac{1}{2^k}\Big(\frac{\lambda/2^k}{2}\Big)^{\mu+1}\int ds \int\limits_{\omega^{\lambda/2^{k+1}}(s)\setminus\omega^{\lambda/2^k}(s)} |s|^{2+\mu}|\widetilde{v}|^2\,dx + 8\int ds\int\limits_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx,$$

where

$$(3.71)\quad \omega^{\lambda/2^{k+1}}(s)\setminus\omega^{\lambda/2^k}(s)=\{x'\in\mathbb{R}^2: 2^k/\lambda\le|s|\,|x'|^2\le 2^{k+1}/\lambda\}.$$

On the set (3.71) we have

$$|s|\le\frac{2^{k+1}}{\lambda}|x'|^{-2}.$$

Hence the first term on the r.h.s. of (3.70) is bounded by

$$\frac{1}{2^k}\int ds\int\limits_{\mathbb{R}^3}|s|\,|\widetilde{v}|^2|x'|^{-2-2\mu}\,dx.$$

Inserting this estimate in (3.70) and using (3.60) we obtain from (3.59) the inequality

$$(3.72)\quad \int ds\int\limits_{\mathbb{R}^3}|s|\,|\widetilde{v}|^2|x'|^{-2-2\mu}\,dx \le \overline{a}_1\int ds\int\limits_{\mathbb{R}^3}dx\,|\widetilde{v}|^2|x'|^{-2-4\mu} + \frac{1}{\overline{a}_2}\int ds\int\limits_{\mathbb{R}^3}dx\,|s|^2|\widetilde{v}|^2|x'|^{-2\mu} + \frac{1}{\overline{a}_1^{\mu+1}}\Big(\frac{\lambda}{2}\Big)^{\mu+1}\frac{1}{2^k}\int ds\int\limits_{\mathbb{R}^3}|s|\,|\widetilde{v}|^2|x'|^{-2-2\mu}\,dx + 8\Big(\frac{2}{\lambda}\Big)^{\mu+1}\int ds\int\limits_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.$$

Assuming that k is so large that

$$\frac{1}{\overline{a}_1^{\mu+1}}\left(\frac{\lambda}{2}\right)^{\mu+1}\frac{1}{2^k}\le\frac{1}{2}$$

we obtain from (3.72) the inequality (3.57). This concludes the proof.

Finally, we have

LEMMA 3.6 *Assume that* $f\in L_{2,-\mu,\gamma}(C_R\times\mathbb{R}_+)$, $\mu\in(0,1)$, $\gamma>0$. *Then for sufficiently smooth solutions to problem* (3.1), (3.2) *vanishing sufficiently quickly near L we have*

$$\|v\|_{H^{2,1}_{-\mu,\gamma}(C_R\times\mathbb{R}_+)}+\|p\|_{L_{2,\gamma}(\mathbb{R}_+;H^1_{-\mu}(C_R))}\le c\|f\|_{L_{2,-\mu,\gamma}(C_R\times\mathbb{R}_+)}. \tag{3.73}$$

Proof. Inequalities (3.8) and (3.57) imply for sufficiently large $\overline{a}_2$ the inequality

$$\begin{aligned}\int ds\int_{\mathbb{R}^3}|s|^2|\widetilde{v}|^2|x'|^{-2\mu}\,dx+\nu\int ds\int_{\mathbb{R}^3}|s|\,|\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx\\ \le c_1\overline{a}_1\int ds\int_{\mathbb{R}^3}|\widetilde{v}|^2|x'|^{-2\mu-4}\,dx+c\int ds\int_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx,\end{aligned} \tag{3.74}$$

where the Laplace transforms with respect to t appear only.

Inequalities (3.56) and (3.74) imply that for sufficiently small $\overline{a}_1$,

$$\begin{aligned}\int ds\int_{\mathbb{R}^3}|s|^2|\widetilde{v}|^2|x'|^{-2\mu}\,dx+\int ds\int_{\mathbb{R}^3}|s|\,|\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx\\ +\int ds\,(\|\widetilde{v}\|^2_{H^2_{-\mu}(\mathbb{R}^3)}+\|\widetilde{p}\|^2_{H^1_{-\mu}(\mathbb{R}^3)})\le c\int ds\int_{\mathbb{R}^3}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.\end{aligned} \tag{3.75}$$

Hence by the Parseval identity (see [1]) we have

$$\begin{aligned}\int_0^\infty dt\,e^{-\gamma t}(\|v_t\|^2_{L_{2,-\mu}(\mathbb{R}^3)}+\|v_{xt}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}\\ +\|v\|^2_{H^2_{-\mu}(\mathbb{R}^3)}+\|p\|^2_{H^1_{-\mu}(\mathbb{R}^3)})\le c\int_0^\infty dt\,e^{-\gamma t}\|f\|^2_{L_{2,-\mu}(\mathbb{R}^3)},\end{aligned} \tag{3.76}$$

where v,p and f have compact supports in C_R. From (3.76) we obtain (3.73). This concludes the proof.

Lemma 3.6 implies Theorem 1.

References

[1] M. S. Agranovich and M. I. Vishik, *Elliptic problems with parameters and parabolic problems of general type*, Uspekhi Mat. Nauk 19 (117) (1964), no. 3, 53–161 (in Russian).

[2] M. E. Bogovskiĭ, *Solutions of some vector analysis problems connected with operators div and grad*, Trudy Sem. Soboleva 1 (1980), 5–40 (in Russian).
[3] M. Burnat and W. M. Zajączkowski, *On local motion of a compressible barotropic viscous fluid with the boundary slip conditions*, Topol. Methods Nonlinear Anal. 10 (1997), 195–223.
[4] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 60 (1974), 241–250.
[5] L. Kapitanski and K. Pileckas, *On some problems of vector analysis*, Zap. Nauchn. Sem. LOMI 138 (1984), 65–84 (in Russian).
[6] V. A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical and angular points*, Trudy Moskov. Mat. Obshch. 16 (1967), 209–292 (in Russian).
[7] R. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc., New York, 1934.
[8] V. A. Solonnikov, *Estimates of solutions to the nonstationary linearized Navier–Stokes system*, Trudy Mat. Inst. Steklov. 70 (1964), 213–217 (in Russian).
[9] —, *On an initial-boundary value problem for the Stokes system which appears in free boundary problems*, ibid. 188 (1990), 150–188 (in Russian).
[10] E. Zadrzyńska and W. M. Zajączkowski, *Existence of solutions to the nonstationary Stokes system in $H^{2,1}_{-\mu}$, $\mu \in (0,1)$, in a domain with a distinguished axis, Part 3. Existence in a bounded domain*, to appear.
[11] W. M. Zajączkowski, *Existence of solutions to the nonstationary Stokes system in $H^{2,1}_{-\mu}$, $\mu \in (0,1)$, in a domain with a distinguished axis. Part 1. Existence near the axis in 2d*, Appl. Math. (Warsaw) 34 (2007), 121–141.
[12] —, *Existence of solutions of initial-boundary value problems for the heat equation in Sobolev spaces with the weight as a power of the distance to some axis*, in: Lecture Notes in Nonlinear Anal. 3 (2002), 185–192.
[13] —, *Existence of solutions to some elliptic system in Sobolev spaces with the weight as a power of the distance to some axis*, Topol. Methods Nonlinear Anal. 19 (2002), 91–108.
[14] —, *Existence of solutions vanishing near some axis for the nonstationary Stokes system with boundary slip conditions*, Dissertationes Math. 400 (2002).

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: wz@impan.gov.pl

Institute of Mathematics and Cryptology
Military University of Technology
Kaliskiego 2
00-908 Warszawa, Poland

Received on 20.3.2007;
revised version on 28.5.2007 (1864)