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A PARABOLIC SYSTEM IN A WEIGHTED SOBOLEV SPACE

Abstract. We examine the regularity of solutions of a certain parabolic system in the weighted Sobolev space $W^{2,1}_{2,\mu}$, where the weight is of the form r^μ, r is the distance from a distinguished axis and $\mu \in (0,1)$.

1. Introduction. In this paper we consider a certain parabolic linear system of two equations in a weighted Sobolev space. Here the weight is r^μ, where r denotes the distance from a distinguished axis and $\mu \in (0,1)$. The solutions of this system are conjugated by boundary conditions. Therefore we first analyze weak solutions of localized problems and next we prove that they are, together with their derivatives, square integrable with the above weight. Finally, we apply the idea of regularizer, i.e. we glue together the solutions of the localized problems and deduce some estimates which guarantee the unique solvability of the main problem. We want to stress that we have to be careful as regards constants. That is why we enumerate all parameters which the constants depend on.

Our main problem (see (1) in Sect. 3) can be obtained from the Stokes problem with the slip boundary conditions (see [Za04]), i.e.

$$\begin{aligned} &v_t - \operatorname{div} \mathbb{T}(v,p) = f,\\ &\operatorname{div} v = 0,\\ &v \cdot n_{|S} = 0,\\ &n \cdot \mathbb{T}(v,p) \cdot \tau_{i|S} = 0, \quad i = 1,2,\\ &v_{|t=0} = v(0), \end{aligned}$$

where $\mathbb{T}(v,p) = \{\nu(v^i_{x_j} + v^j_{x_i}) - p\delta_{i,j}\}^3_{i,j=1}$, n is the unit outward normal vector to the boundary and τ_i for $i = 1,2$ are tangent vectors on the boundary. Applying the rotation operator to the above problem we deduce problem (1).

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2. Notation and assumptions. For $x \in \mathbb{R}^k$ we denote by $r = r(x)$ the distance of x from the set $\{x \in \mathbb{R}^k;\ x_1 = x_2 = 0\}$, i.e. $r(x) = \sqrt{x_1^2 + x_2^2}$. If $\varepsilon > 0$, then we define $\chi_\varepsilon(t) := \chi(\varepsilon^{-1}t)$, where $\chi \in C^\infty([0,\infty))$ satisfies $0 \le \chi(t) \le 1$, $\chi(t) = 1$ for $t < 1$, $\chi(t) = 0$ for $t > 2$ and $|\chi^{(k)}(t)| \le 2^k$ for $k = 1, 2$. We recall the standard notation of function spaces. If $U \subseteq \mathbb{R}^n$, then $\|u\|_{L_{2,\mu}(U)} := \|u \cdot r^\mu\|_{L_2(U)}$. The space $H^m_\mu(U)$ is defined as the closure of the set of smooth functions with compact support in $U \setminus \{x;\ r(x) = 0\}$ with respect to the norm

$$\|u\|_{H^m_\mu(U)} := \Big\{ \sum_{|\alpha| \le m} \|D^\alpha u\|^2_{L_{2,\mu+m-|\alpha|}(U)} \Big\}^{1/2}.$$

We denote by $W^m_{2,\mu}(U)$ the space of functions $u \in L_{2,\mu}(U)$ such that $D^\alpha u \in L_{2,\mu}(U)$ for $|\alpha| \le m$. The norm is given by

$$\|u\|_{W^m_{2,\mu}(U)} := \Big\{ \sum_{|\alpha| \le m} \|D^\alpha u\|^2_{L_{2,\mu}(U)} \Big\}^{1/2}.$$

For $T > 0$, set $U^T := U \times (0,T)$; then for even m the space $W^{m,m/2}_\mu(U^T)$ is defined similarly and it is equipped with the norm

$$\|u\|_{W^{m,m/2}_{2,\mu}(U^T)} := \Big\{ \sum_{|\alpha|+2s \le m} \|D^s_t D^\alpha_x u\|^2_{L_{2,\mu}(U^T)} \Big\}^{1/2}.$$

We will also need the space $W^{1,0}_{2,\mu}(U^T) := \{u \in L_{2,\mu}(U^T);\ \nabla_x u \in L_{2,\mu}(U^T)\}$ with the norm $\|u\|_{W^{1,0}_{2,\mu}(U^T)} := \{\|u\|^2_{L_{2,\mu}(U^T)} + \|\nabla_x u\|^2_{L_{2,\mu}(U^T)}\}^{1/2}$. If $u \in W^{1,0}_2(U^T) := W^{1,0}_{2,0}(U^T)$, then we define

$$u \in V_2(U^T) \ \Leftrightarrow\ |u|_{U^T} := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(\cdot,t)\|_{L_2(U)} + \|\nabla_x u\|_{L_2(U^T)} < \infty.$$

The space $V^{1,0}_2(U^T)$ consists of $u \in V_2(U^T)$ such that the function $t \mapsto \|u(\cdot,t)\|_{L_2(U^T)}$ is continuous.

If X is a Banach space of functions which are defined on U (U^T resp.), then $\mathring{X}$ denotes the closure of the subset of X consisting of the smooth functions with compact support in U ($U \times [0,T]$ resp.). If X is a Banach space of functions defined on U^T, then $\underset{\circ}{X}$ denotes the closure of the subset of X consisting of the smooth functions vanishing for $t = 0$. If $S \subseteq \partial U$ is a part of the boundary of U, then $W^{3/2,3/4}_{2,\mu}(S^T)$ ($W^{1/2,1/4}_{2,\mu}(S^T)$ resp.) denotes the space of traces on S of functions u ($\partial u/\partial n$ resp.) for $u \in W^{2,1}_{2,\mu}(U^T)$. The spaces $\underset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^T)$ and $\underset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^T)$ are defined similarly. The norm of φ in $W^{3/2,3/4}_{2,\mu}(S^T)$ ($W^{1/2,1/4}_{2,\mu}(S^T)$ resp.) is the infimum of $\|u\|_{W^{2,1}_{2,\mu}(U^T)}$ taken over all $u \in W^{2,1}_{2,\mu}(U^T)$ such that $u_{|S} = \varphi$ ($\partial u/\partial n_{|S} = \varphi$ resp.). The norms in $\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^T)$ and $\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^T)$ are defined analogously. Finally, if m is even, then for $u \in W^{m,m/2}_{2,\mu}(U^T)$ we set

$$\|u\|_{L^m_{2,\mu}(U^T)} := \Big\{ \sum_{|\alpha|+2s=m} \|D^s_t D^\alpha_x u\|^2_{L_{2,\mu}(U^T)} \Big\}^{1/2}.$$

Our main problem will be considered in a domain Ω, which is an open and bounded subset of $\mathbb{R}^3$. We denote by L the axis $L := \{x \in \mathbb{R}^3;\, r(x) = 0\}$. We assume that the boundary $\partial\Omega$ is smooth and Ω is axially symmetric with respect to L and $\partial\Omega \cap L = \{p_1, p_2\}$. The boundary is described by the equality $\psi(r, x_3)=0$, where ψ is some smooth function. We assume that there exists a smooth vector-valued function $a = (a_1, a_2)$ defined in some neighborhood of $\partial\Omega$ such that $a_1 = \psi_r/|\nabla\psi|$, $a_2 = \psi_{x_3}/|\nabla\psi|$ on $\partial\Omega$ and $a_{|\partial\Omega}$ is the unit outward normal vector to $\partial\Omega$ denoted by n. Finally, we introduce the following notation: $S := \partial\Omega$, $\overline{\overline{a}} := (-a_2, a_1)$. Now we are able to formulate our main problem.

3. Main problem. We will consider the following system of parabolic equations:

$$\begin{cases} u_t - \nu\Delta u = F & \text{in } \Omega^T, \\ \overline{\overline{a}} u_{|S} = \phi_1 & \text{on } S^T, \\ \dfrac{\partial}{\partial n}(au)_{|S} = \phi_2 & \text{on } S^T, \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $u = (u^1, u^2)$ and $F = (F^1, F^2)$. We shall prove the following theorem.

THEOREM 1. *Assume that* $\mu \in (0, 1)$ *and* $T > 0$. *Then for each* $F = (F^1, F^2) \in L_{2,\mu}(\Omega^T)^2$, $u_0 \in W^1_{2,\mu}(\Omega)$, $\phi_1 \in W^{3/2,3/4}_{2,\mu}(S^T)$, $\phi_2 \in W^{1/2,1/4}_{2,\mu}(S^T)$, *if the compatibility conditions are fulfilled*, *i.e.*

$$\overline{\overline{a}} u_{0|S} = \phi_{1|t=0},$$

then there exists a unique solution $u = (u^1, u^2) \in W^{2,1}_{2,\mu}(\Omega^T)^2$ *of problem* (1), *and for some constant* $c = c(\Omega, T, \mu, \nu)$,

$$\|u\|_{W^{2,1}_{2,\mu}(\Omega^T)^2} \le c\{\|F\|_{L_{2,\mu}(\Omega^T)^2} + \|\phi_1\|_{W^{3/2,3/4}_{2,\mu}(S^T)} + \|\phi_2\|_{W^{1/2,1/4}_{2,\mu}(S^T)} + \|u_0\|_{W^1_{2,\mu}(\Omega)}\}.$$

REMARK 1. If we apply Lemma 2.11 in [KZ03], then it is enough to examine the existence of a solution of problem (1) with homogeneous initial condition. Thus by the compatibility conditions and the equality $W^{1/2,1/4}_{2,\mu}(S^T)$

$= \overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)$ we have to prove that for some $\tau > 0$ the problem

$$\begin{cases} u_t - \nu \Delta u = F, \\ \bar{\bar{a}} u_{|S} = \phi_1, \\ \dfrac{\partial}{\partial n}(au)_{|S} = \phi_2, \end{cases} \tag{2}$$

where $F \in L_{2,\mu}(\Omega^\tau)^2$, $\phi_1 \in \overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^\tau)$, $\phi_2 \in \overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)$, has a unique solution $u \in \overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^\tau)^2$. Then for some $c = c(\Omega, \mu, \nu)$,

$$\|u\|_{W^{2,1}_{2,\mu}(\Omega^\tau)^2} \le c\{\|F\|_{L_{2,\mu}(\Omega^\tau)^2} + \|\phi_1\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^\tau)} + \|\phi_2\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)}\}.$$

According to the above remark we only have to prove the existence of solutions of problem (2). For that purpose we apply the regularizer technique (see [La68, Chap. IV, §7]), which first requires the examination of localized problems.

4. The model problem. In this section we deal with a model problem, i.e. we consider a parabolic equation on $\mathbb{R}^3$ or on a half space $\mathbb{R}^3_+$ with homogeneous Dirichlet or Neumann boundary condition. Let us start with the following remark.

REMARK 2. If $\mu \in (0,1)$, $U = \mathbb{R}^3_+$ or $U = \mathbb{R}^3$ and $f \in L_{2,\mu}(U^T)$, then $f \in W^{1,0}_2(U^T)^*$, i.e. there exists a constant $c = c(\mu)$ such that for each $\eta \in W^{1,0}_2(U^T)$,

$$\Big|\int_{U^T} f\eta\,dx\,dt\Big| \le c\|f\|_{L_{2,\mu}(U^T)}\|\eta\|_{W^{1,0}_2(U^T)}.$$

Indeed, if we apply the Schwarz inequality twice, then we get (see Section 2 for the definition of χ_1)

$$\begin{aligned}\Big|\int_{U^T} f\eta\,dx\,dt\Big| &\le \Big|\int_{U^T} \chi_1 f\eta\,dx\,dt\Big| + \Big|\int_{U^T} (1-\chi_1) f\eta\,dx\,dt\Big| \\ &\le (\|\chi_1 r^{-\mu}\eta\|_{L^2(U^T)} + \|(1-\chi_1)r^{-\mu}\eta\|_{L^2(U^T)})\|f\|_{L_{2,\mu}(U^T)}.\end{aligned}$$

Clearly, we have $\|(1-\chi_1)r^{-\mu}\eta\|_{L^2(U^T)} \le \|\eta\|_{L^2(U^T)}$. Applying the Hardy inequality [Ha34, Th. 330] and the inclusion $\operatorname{supp}\chi_1 \subseteq B(0,2)$ we get the inequality $\|\chi_1 r^{-\mu}\eta\|_{L_2(U^T)} \le c(\mu)\|\eta\|_{W^{1,0}_2(U^T)}$.

In the next subsection we shall consider the existence and uniqueness of weak solutions of model problems.

4.1. *Weak solutions.* We will denote by B the boundary operator, which is of Dirichlet type $Bw = w_{|\partial U}$ or Neumann type $Bw = \frac{\partial w}{\partial x_3}{}_{|\partial U}$. We need the following lemmas:

LEMMA 1. *Assume that* $\mu \in (0,1)$, $U := \mathbb{R}^3_+$ *and* $T > 0$. *Then for each* $f \in L_{2,\mu}(U^T)$ *there exists a unique weak solution* $w \in V_2^{1,0}(U^T)$ *of the problem*

$$
\begin{cases} w_t - \nu \Delta w = f & \text{in } U^T, \\ Bw = 0 & \text{on } \partial U, \\ w_{|t=0} = 0. \end{cases} \tag{3}
$$

Furthermore, there exists a constant $c = c(T,\mu,\nu)$ *such that for each* τ *in* $(0,T)$,

$$
|w|_{U^\tau} \le c\|f\|_{L_{2,\mu}(U^\tau)}. \tag{4}
$$

LEMMA 2. *Assume that* $\mu \in (0,1)$, $U := \mathbb{R}^3$ *and* $T > 0$. *Then for each* $f \in L_{2,\mu}(U^T)$ *there exists a unique weak solution* $w \in V_2^{1,0}(U^T)$ *of the problem*

$$
\begin{cases} w_t - \nu \Delta w = f & \text{in } U^T, \\ w_{|t=0} = 0, \end{cases} \tag{5}
$$

Furthermore, there exists a constant $c = c(T,\mu,\nu)$ *such that for each* τ *in* $(0,T)$,

$$
|w|_{U^\tau} \le c\|f\|_{L_{2,\mu}(U^\tau)}.
$$

Proof of Lemma 1. We only consider the case of $Bw = w_{|\partial U}$, because in the other case we proceed similarly. Hence, suppose that $f \in L_{2,\mu}(U^T)$, where $U = \mathbb{R}^3_+$. Then with the help of Remark 2 and the Riesz theorem we get a unique $g \in \mathring{W}_2^{1,0}(U^T)$ such that

$$
\int_{U^T} f\overline{\eta}\,dx\,dt = \int_{U^T} g\overline{\eta}\,dx\,dt + \int_{U^T} \nabla g \nabla\overline{\eta}\,dx\,dt \quad \text{for all } \eta \in \mathring{W}_2^{1,0}(U^T). \tag{6}
$$

If we write $g_0 := g$, $g_i := \partial g/\partial x_i$, $i = 1,2,3$, then applying Theorem 4.1 and [La68, Chap. III, Lemma 4.1 ([1])] we get a unique weak solution $w \in \mathring{V}_2^{1,0}(U^T)$ of the problem

$$
w_t - \nu\Delta w = g_0 - \sum_{i=1}^{3} \frac{\partial g_i}{\partial x_i} \quad \text{in } U^T, \qquad w_{|\mathbb{R}^2} = 0, \qquad w_{|t=0} = 0,
$$

i.e. for all $\eta \in \mathring{W}_2^{1,1}(U^T)$ such that $\eta(\cdot,T) = 0$ we have the identity

$$
\int_0^T\!\!\int_U w \cdot \overline{\eta}_t\,dx\,dt + \nu \int_0^T\!\!\int_U \nabla w \cdot \nabla\overline{\eta}\,dx\,dt = \int_0^T\!\!\int_U \Big(g_0 \cdot \overline{\eta} + \sum_{i=1}^{3} g_i \cdot \overline{\eta}_{x_i}\Big)\,dx\,dt
$$

and for each $\tau \in (0,T)$ the estimate $|w|_{U^\tau} \le c\sum_{i=0}^{3} \|g_i\|_{L^2(U^\tau)}$ is satisfied, where $c = c(T,\nu)$. By the definition of a weak solution and the identity (6),

([1]) Theorem 5.1 in the Neumann case.

the above means that w is a weak solution of problem (3). In accordance with Remark 2 and the determination of g_i we have

$$\sum_{i=0}^{3}\|g_i\|^2_{L^2(U^T)} = \|g\|^2_{W_2^{1,0}(U^T)} = \int\limits_{U^T} f\overline{g}\,dx\,dt \le c(\mu)\|f\|_{L_{2,\mu}(U^T)}\|g\|_{W_2^{1,0}(U^T)},$$

hence $\sum_{i=0}^{3}\|g_i\|_{L^2(U^T)} \le 2c(\mu)\|f\|_{L_{2,\mu}(U^T)}$. Thus (4) holds and the proof is finished.

Clearly, Lemma 2 can be proved similarly and in this case we apply [La68, Chap. III, Theorem 5.2].

Now we shall obtain estimates in $W^{2,1}_{2,\mu}$ for the weak solutions given by Lemmas 1 and 2.

Remark 3. We notice that if w is a weak solution of (3) with Dirichlet (Neumann resp.) boundary condition, then if we extend w on $\mathbb{R}^3\times(0,T)$ by odd (even resp.) reflection with respect to $\{x;\, x_3 = 0\}$, we get a solution of (5) with the r.h.s. obtained by the same extension. Thus it is enough to deal with weak solutions of problem (5).

In the next two subsections we will deduce estimates in the weighted space $W^{2,1}_{2,\mu}$ for the weak solutions of (5).

4.2. *Estimate of the lower order terms.* We now show that weak solutions are integrable with weight if the data are. More precisely, we prove the following:

Lemma 3. *Assume that $U := \mathbb{R}^3$, $f \in L_{2,\mu}(U^T)$ and $w \in V_2^{1,0}(U^T)$ is a weak solution of the problem $w_t - \nu\Delta w = f$ in U^T and $w_{|t=0} = 0$. Then there exists a constant $c = c(\nu,\mu,T)$ such that for each $\tau\in(0,T)$,*

$$\|w\|_{W^{1,0}_{2,\mu}(U^\tau)} \le c\|f\|_{L_{2,\mu}(U^\tau)}. \tag{7}$$

Proof. From Lemma 2 we get the estimate

$$\|w\|_{W_2^{1,0}(U^\tau)} \le c\|f\|_{L_{2,\mu}(U^\tau)}, \tag{8}$$

where $c = c(\nu,\mu,T)$. Clearly,

$$\|w\|_{W^{1,0}_{2,\mu}(U^\tau)} \le \|\chi_1 w\|_{W^{1,0}_{2,\mu}(U^\tau)} + \|(1-\chi_1)w\|_{W^{1,0}_{2,\mu}(U^\tau)}$$

and

$$\|\chi_1 w\|_{W^{1,0}_{2,\mu}(U^\tau)} \le c(\mu)\|w\|_{W_2^{1,0}(U^\tau)}.$$

Thus we only have to estimate the expression $\|(1-\chi_1)w\|_{W^{1,0}_{2,\mu}(U^\tau)}$. Denote by $\{\eta_n\}_{n\in\mathbb{N}}$ a family of smooth functions $\eta_n = \eta_n(r)$ such that $\operatorname{supp}\eta_n \subset \{r;\, 2^{n-1} < r < 2^{n+1}\}$, $0\le\eta_n\le 1$ and $|\eta_n^{(k)}| \le 2^{-nk}$ for $k = 1,2$ and $n\in\mathbb{N}$ and $\sum_{n=0}^{\infty}\eta_n \equiv 1$ on $\operatorname{supp}(1-\chi_1)$. Setting $U_n = \{x\in\mathbb{R}^3;\, 2^{n-1} < r < 2^{n+1}\}$

we get

$$\|(1-\chi_1)w\|^2_{W^{1,0}_{2,\mu}(U^\tau)} \le 6\sum_{n=0}^{\infty} \|\eta_n w\|^2_{W^{1,0}_{2,\mu}(U^\tau)} + 2^{3+2\mu}\|w\|^2_{L_2(U^\tau)}.$$

The function $\eta_n w$ belongs to $V^{1,0}_2(U^T)$ and is the unique weak solution of the equation

$$(\eta_n w)_t - \nu\Delta(\eta_n w) = \eta_n f - 2\nu\nabla\eta_n\cdot\nabla w - \nu\Delta\eta_n\cdot w \quad \text{in } U^T,$$

where the r.h.s. is in $L_2(U^T)$. Hence there exists a constant $c = c(\nu, T)$ such that

$$\|\eta_n w\|_{W^{1,0}_2(U^\tau_n)} \le c\{\|f\|_{L_2(U^\tau_n)} + \|\nabla\eta_n\cdot\nabla w\|_{L_2(U^\tau_n)} + \|\Delta\eta_n\cdot w\|_{L_2(U^\tau_n)}\}.$$

It is clear that $\|\nabla\eta_n\cdot\nabla w\|_{L_2(U^\tau_n)} \le 2^{-n}\|\nabla w\|_{L_2(U^\tau_n)}$ and $\|\Delta\eta_n\cdot w\|_{L_2(U^\tau_n)} \le 2^{-2n}\|w\|_{L_2(U^\tau_n)}$. Therefore

$$\begin{aligned}
\|\eta_n w\|_{W^{1,0}_{2,\mu}(U^\tau_n)} &\le 2^{\mu n+\mu}\|\eta_n w\|_{W^{1,0}_2(U^\tau_n)} \\
&\le 2^{\mu n+\mu}c\{\|f\|_{L_2(U^\tau_n)} + 2^{-n}\|\nabla w\|_{L_2(U^\tau_n)} + 2^{-2n}\|w\|_{L_2(U^\tau_n)}\} \\
&\le 2^{2\mu}c\{\|f\|_{L_{2,\mu}(U^\tau_n)} + 2^{n(\mu-1)-\mu}\|\nabla w\|_{L_2(U^\tau_n)} + 2^{n(\mu-2)-\mu}\|w\|_{L_2(U^\tau_n)}\} \\
&\le 2^{2\mu+1}c\{\|f\|_{L_{2,\mu}(U^\tau_n)} + \|w\|_{W^{1,0}_2(U^\tau_n)}\},
\end{aligned}$$

where we have used the assumption $\mu < 1$. Thus we see that for some $c = c(\nu,\mu,T)$ the sum $\sum_{n=0}^{\infty}\|\eta_n w\|^2_{W^{1,0}_{2,\mu}(U^\tau_n)}$ is less than or equal to

$$c\Big\{\sum_{n=0}^{\infty}\|f\|^2_{L_{2,\mu}(U^\tau_n)} + \|w\|^2_{W^{1,0}_2(U^\tau_n)}\Big\} \le 2c\{\|f\|^2_{L_{2,\mu}(U^\tau)} + \|w\|^2_{W^{1,0}_2(U^\tau)}\}.$$

Thanks to the estimate (8) the proof is finished. ∎

4.3. *Estimate of the second derivatives.* As we will see later, estimating the second derivatives of a weak solution in $\mathbb{R}^3\times(0,T)$ can be reduced with the help of the partial Fourier transform to an appropriate estimate of solutions of a certain problem in $\mathbb{R}^2$ with parameter $s := \nu\xi_2^2 + i\xi_1$, where $\xi_1, \xi_2 \in \mathbb{R}$. That is why we need the following lemma.

LEMMA 4. *Assume that $\mu\in(0,1)$ and $h\in H^1(\mathbb{R}^2)$ is a weak solution of $-\Delta h + sh = p$, where $p\in L_{2,\mu}(\mathbb{R}^2)$. Then there exists a constant $c = c(\mu)$ such that*

$$\|D^2h\|_{L_{2,\mu}(\mathbb{R}^2)} + |s|^{1/2}\|Dh\|_{L_{2,\mu}(\mathbb{R}^2)} + |s|\,\|h\|_{L_{2,\mu}(\mathbb{R}^2)} \le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)}.$$

The proof will be divided into three steps, which we specify in Propositions 1–3. In the first and second steps we follow [SZ83].

PROPOSITION 1. *Assume that $\mu\in(0,1)$ and $h\in H^1(\mathbb{R}^2)$ is a weak solution of $-\Delta h + sh = p$, where $p\in L_{2,\mu}(\mathbb{R}^2)$. Then there exists a constant*

$c = c(\mu)$ *such that*

$$|s|^{1-\mu} \int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)\,dx \le c\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)}. \tag{9}$$

Proof. Recall the interpolation inequality

$$\int_{\mathbb{R}^2} |\eta|^2|x|^{-2\mu}\,dx \le \varepsilon^{2-2\mu} \int_{\mathbb{R}^2} |\nabla\eta|^2\,dx + c(\mu)\varepsilon^{-2\mu} \int_{\mathbb{R}^2} |\eta|^2\,dx, \tag{10}$$

which holds for all $\varepsilon > 0$ and $\eta \in H^1(\mathbb{R}^2)$. If we apply the Schwarz inequality and then the interpolation inequality with $\varepsilon := |s|^{-1/2}$, we get

$$\begin{aligned} |s|^{1-\mu}\Big|\int_{\mathbb{R}^2} p\overline{\eta}\,dx\Big| &\le |s|^{1-\mu}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} |\eta|^2|x|^{-2\mu}\,dx\Big)^{1/2} \\ &\le |s|^{1-\mu}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(|s|^{\mu-1}\int_{\mathbb{R}^2} |\nabla\eta|^2\,dx + c(\mu)|s|^{\mu}\int_{\mathbb{R}^2} |\eta|^2\,dx\Big)^{1/2} \\ &= |s|^{(1-\mu)/2}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} |\nabla\eta|^2\,dx + c(\mu)|s|\int_{\mathbb{R}^2} |\eta|^2\,dx\Big)^{1/2}. \end{aligned} \tag{11}$$

By the assumption h satisfies the identity

$$\int_{\mathbb{R}^2} \nabla h\cdot\nabla\overline{\eta}\,dx + s\int_{\mathbb{R}^2} h\cdot\overline{\eta}\,dx = \int_{\mathbb{R}^2} p\cdot\overline{\eta}\,dx \quad \text{for } \eta\in H^1(\mathbb{R}^2). \tag{12}$$

If we put $\eta := h(1 + i\,\mathrm{sign}\,\xi_1)|s|^{1-\mu} \in H^1(\mathbb{R}^2)$ in (12) and compare the real parts, then applying (11) we get

$$\begin{aligned} &|s|^{1-\mu}\int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)\,dx \\ &\qquad \le \sqrt{2}\,c(\mu)|s|^{(1-\mu)/2}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} (|\nabla h|^2 + c(\mu)|s|\,|h|^2)\,dx\Big)^{1/2}. \end{aligned}$$

Therefore we obtain (9) with the constant $2c(\mu)^3$. ∎

PROPOSITION 2. *Assume that $\mu \in (0,1)$ and $h \in H^1(\mathbb{R}^2)$ is a weak solution of $-\Delta h + sh = p$, where $p \in L_{2,\mu}(\mathbb{R}^2)$. Then there exists a constant $c = c(\mu)$ such that*

$$|s|\int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)|x|^{2\mu}\,dx \le c\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)}. \tag{13}$$

Proof. We fix s and for $\lambda > 1$ we define

$$\begin{aligned} d_1 &:= \{x\in\mathbb{R}^2 : 0 \le |x| < |s|^{-1/2}\}, \\ d_2 &:= \{x\in\mathbb{R}^2 : |s|^{-1/2} \le |x| < \lambda^{1/2\mu}|s|^{-1/2}\}, \\ d_3 &:= \{x\in\mathbb{R}^2 : \lambda^{1/2\mu}|s|^{-1/2} \le |x|\}, \end{aligned}$$

and

$$V_\lambda(x) := \begin{cases} |s|^{1-\mu} & \text{for } x \in d_1, \\ |s|\,|x|^{2\mu} & \text{for } x \in d_2, \\ \lambda|s|^{1-\mu} & \text{for } x \in d_3. \end{cases}$$

The function V_λ has the following properties:

$$|\nabla V_\lambda|^2 = \begin{cases} (2\mu)^2|s|^2|x|^{4\mu-2} & \text{for } x \in d_2, \\ 0 & \text{for } x \in d_1 \cup d_3, \end{cases} \tag{14}$$

$$|\nabla V_\lambda|^2 \cdot V_\lambda^{-1} = \begin{cases} (2\mu)^2|s|\,|x|^{\mu-2} & \text{for } x \in d_2, \\ 0 & \text{for } x \in d_1 \cup d_3, \end{cases} \tag{15}$$

$$V_\lambda|x|^{-2\mu} \le |s| \quad \text{for } x \in d_2 \cup d_3. \tag{16}$$

It is clear that $\eta := hV_\lambda(1 + i\operatorname{sign}\xi_1)$ belongs to $H^1(\mathbb{R}^2)$. Hence using (12) for such η and then applying the Schwarz inequality twice, we obtain

$$\begin{aligned} J_\lambda &:= \int_{\mathbb{R}^2} (|\nabla h|^2 + |s||h|^2)V_\lambda\, dx \le \sqrt{2}\Big|\int_{\mathbb{R}^2} p\bar{h}V_\lambda\, dx\Big| + \sqrt{2}\Big|\int_{\mathbb{R}^2} \nabla h\nabla V_\lambda \bar{h}\, dx\Big| \\ &\le \sqrt{2}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} |h|^2V_\lambda^2|x|^{-2\mu}\, dx\Big)^{1/2} \\ &\quad + \sqrt{2}\Big(\int_{\mathbb{R}^2} |\nabla h|^2V_\lambda\, dx\Big)^{1/2} \cdot \Big(\int_{\mathbb{R}^2} |h|^2|\nabla V_\lambda|^2V_\lambda^{-1}\, dx\Big)^{1/2}. \end{aligned}$$

Therefore, if we apply (14)–(16), then we get

$$\begin{aligned} J_\lambda &\le \sqrt{2}\|p\|_{L_{2,\mu}(\mathbb{R}^2)}\Big(|s|^{2-2\mu}\int_{d_1} |h|^2|x|^{-2\mu}\, dx + |s|\int_{d_2\cup d_3} |h|^2V_\lambda\, dx\Big)^{1/2} \\ &\quad + 2\sqrt{2}\mu\Big(\int_{\mathbb{R}^2} |\nabla h|^2V_\lambda\, dx\Big)^{1/2} \cdot \Big(|s|^{2-\mu}\int_{\mathbb{R}^2} |h|^2\, dx\Big)^{1/2}. \end{aligned} \tag{17}$$

If we apply (10) with $\eta := h$ and $\varepsilon := |s|^{-1/2}$, and next use Proposition 1, then we get

$$\begin{aligned} |s|^{2-2\mu}\int_{d_1} |h|^2|x|^{-2\mu}\, dx &\le c(\mu)|s|^{1-\mu}\int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)\, dx \\ &\le \widetilde{c}(\mu)\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)}. \end{aligned} \tag{18}$$

Applying Proposition 1 again we obtain

$$|s|^{2-\mu}\int_{\mathbb{R}^2} |h|^2\, dx \le |s|^{1-\mu}\int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)\, dx \le c(\mu)\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)}. \tag{19}$$

Hence from (17)–(19) we have $J_\lambda \le c(\mu)\|p\|_{L_{2,\mu}(\mathbb{R}^2)}(\|p\|_{L_{2,\mu}(\mathbb{R}^2)} + J_\lambda)^{1/2}$, thus

$$J_\lambda \le c\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)} \tag{20}$$

for some $c = c(\mu)$. Applying (20) and Proposition 1 we get

$$|s| \int_{d_1 \cup d_2} (|\nabla h|^2 + |s|\,|h|^2)|x|^{2\mu}\,dx$$
$$= |s| \int_{d_1} (|\nabla h|^2 + |s|\,|h|^2)|x|^{2\mu}\,dx + \int_{d_2} (|\nabla h|^2 + |s|\,|h|^2)V_\lambda\,dx$$
$$\le |s|^{1-\mu} \int_{d_1} (|\nabla h|^2 + |s|\,|h|^2)\,dx + \int_{\mathbb{R}^2} (|\nabla h|^2 + |s|\,|h|^2)V_\lambda\,dx \le c\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)},$$

where $c = c(\mu)$. The above estimate holds for $\lambda > 1$, thus if $\lambda \to \infty$, then we get (13). ■

Proposition 3. *Assume that $\mu \in (0,1)$ and $h \in H^1(\mathbb{R}^2)$ is a weak solution of $-\Delta h + sh = p$, where $p \in L_{2,\mu}(\mathbb{R}^2)$. Then there exists a constant $c = c(\mu)$ such that*

$$\|D^2 h\|_{L_{2,\mu}(\mathbb{R}^2)} \le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)}. \tag{21}$$

Proof. We shall multiply h by a suitable cut-off function and then we shall write the product as a sum (see (27)) whose components can be easily estimated.

We fix s and $R > 1$. We set $\kappa := |s|^{-1/2}R$ and

$$h_\kappa := h \cdot \varsigma_\kappa, \quad \text{where } \varsigma_\kappa(x) := \chi_\kappa(|x|)\ (^2).$$

The function h_κ is in $H^1(\mathbb{R}^2)$ and it is a weak solution of $-\Delta h_\kappa + sh_\kappa = G$, where

$$G := p \cdot \varsigma_\kappa - 2\nabla h \cdot \nabla \varsigma_\kappa - h \cdot \Delta \varsigma_\kappa.$$

If we use the properties of functions χ_1, ς_κ and next apply Proposition 2 to h, then we get the estimate

$$\|G\|_{L_{2,\mu}(\mathbb{R}^2)} \le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)} \quad \text{and} \quad \operatorname{supp} G \subseteq B_{2\kappa}, \tag{22}$$

where $c = c(\mu)$ and B_ε denotes the ball with center at the origin and radius ε. Thus we can apply Proposition 2 to h_κ to obtain

$$|s| \int_{\mathbb{R}^2} (|\nabla h_\kappa|^2 + |s|\,|h_\kappa|^2)|x|^{2\mu}\,dx \le c(\mu)\|G\|^2_{L_{2,\mu}(\mathbb{R}^2)} \le \widetilde{c}(\mu)\|p\|^2_{L_{2,\mu}(\mathbb{R}^2)}. \tag{23}$$

In particular, h_κ is a weak solution of $-\Delta h_\kappa = q_\kappa$, where $q_\kappa := G - sh_\kappa$ and by (22) and (23) we have

$$\|q_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)} \le c(\mu)\|p\|_{L_{2,\mu}(\mathbb{R}^2)}. \tag{24}$$

On the other hand, there exists $W \in H^2_\mu(\mathbb{R}^2)$ such that (see Lemma 3.5 in

(2) See Section 2 for the definition of χ_ε.

[Za02])

$$-\Delta W = q_\kappa \quad \text{and} \quad \|W\|_{H^2_\mu(\mathbb{R}^2)} \le \widetilde{c}(\mu)\|q_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)}. \tag{25}$$

We set

$$V := W \cdot \varsigma_\kappa.$$

Then

$$-\Delta V = q_\kappa \varsigma_\kappa - 2\nabla W \cdot \nabla \varsigma_\kappa - W \cdot \Delta \varsigma_\kappa.$$

We define $U := h_\kappa - V$. Then $U \in H^1(\mathbb{R}^2)$, because $\operatorname{supp} V$ is compact. Furthermore,

$$-\Delta U = (1-\varsigma_\kappa)q_\kappa + 2\nabla W \cdot \nabla \varsigma_\kappa + W \cdot \Delta \varsigma_\kappa =: t_\kappa.$$

The support of t_κ is compact and does not contain the origin, hence t_κ is in $L_2(\mathbb{R}^2)$. Thus $U \in H^2(\mathbb{R}^2)$ and

$$\|D^2 U\|_{L_2(\mathbb{R}^2)} = \|t_\kappa\|_{L_2(\mathbb{R}^2)}. \tag{26}$$

Clearly, by the definition we have

$$h_\kappa = U + V. \tag{27}$$

Now we estimate each component in (27). Applying (26) we obtain

$$\begin{aligned}\|D^2U\|^2_{L_{2,\mu}(\mathbb{R}^2)} &= \int_{B_{2\kappa}} |D^2U|^2 |x|^{2\mu}\,dx \le (2\kappa)^{2\mu} \int_{B_{2\kappa}} |D^2U|^2\,dx \\ &= (2\kappa)^{2\mu}\|t_\kappa\|^2_{L_2(\mathbb{R}^2)}.\end{aligned}$$

We notice that t_κ has support in $B_{2\kappa} \setminus B_\kappa$, thus

$$\begin{aligned}\|t_\kappa\|^2_{L_2(\mathbb{R}^2)} &\le 6 \int_{B_{2\kappa}\setminus B_\kappa} (|q_\kappa|^2 + |\nabla W \cdot \nabla\varsigma_\kappa|^2 + |W\cdot \Delta\varsigma_\kappa|^2)\,dx \\ &\le 6\kappa^{-2\mu} \int_{B_{2\kappa}\setminus B_\kappa} (|q_\kappa|^2 + |\nabla W \cdot \nabla\varsigma_\kappa|^2 + |W\cdot \Delta\varsigma_\kappa|^2)|x|^{2\mu}\,dx \\ &\le 6\kappa^{-2\mu} \int_{B_{2\kappa}\setminus B_\kappa} (|q_\kappa|^2 + 4\kappa^{-2}|\nabla W|^2 + 16\kappa^{-4}|W|^2)|x|^{2\mu}\,dx \\ &\le 2^{11}\kappa^{-2\mu} \int_{B_{2\kappa}\setminus B_\kappa} (|q_\kappa|^2 + |x|^{-2}|\nabla W|^2 + |x|^{-4}|W|^2)|x|^{2\mu}\,dx \\ &\le 2^{11}\kappa^{-2\mu}(\|q_\kappa\|^2_{L_{2,\mu}(\mathbb{R}^2)} + \|W\|^2_{H^2_\mu(\mathbb{R}^2)}).\end{aligned}$$

Thus applying (24) and (25) we have

$$\|D^2U\|_{L_{2,\mu}(\mathbb{R}^2)} \le 2^{11}(\|q_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)} + \|W\|_{H^2_\mu(\mathbb{R}^2)}) \le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)}, \tag{28}$$

where c depends only on μ. We can similarly estimate the norm of D^2V. Indeed, using the definition of V we find that $\|D^2V\|^2_{L_{2,\mu}(\mathbb{R}^2)}$ is less than or equal to

$$3\int_{\mathbb{R}^2}(|W\cdot D^2\varsigma_\kappa|^2+2|\nabla W\cdot\nabla\varsigma_\kappa|^2+|D^2W\cdot\varsigma_\kappa|^2)|x|^{2\mu}\,dx$$
$$\le 3\int_{B_{2\kappa}\setminus B_\kappa}(16\kappa^{-4}|W|^2+4\kappa^{-2}|\nabla W|^2)|x|^{2\mu}\,dx+\int_{\mathbb{R}^2}|D^2W|^2|x|^{2\mu}\,dx$$
$$\le 2^{10}\int_{B_{2\kappa}\setminus B_\kappa}(|x|^{-4}|W|^2+|x|^{-2}|\nabla W|^2)|x|^{2\mu}\,dx+\int_{\mathbb{R}^2}|D^2W|^2|x|^{2\mu}\,dx$$
$$\le 2^{10}\|W\|^2_{H^2_\mu(\mathbb{R}^2)}.$$

Applying again (24) and (25) we get the estimate

$$\|D^2V\|_{L_{2,\mu}(\mathbb{R}^2)}\le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)},\tag{29}$$

where $c=c(\mu)$. Thus according to (27), if we apply (28) and (29), then we obtain

$$\|D^2h_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)}\le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)}.\tag{30}$$

By the definition of h_κ we have

$$\|\varsigma_\kappa\cdot D^2h\|_{L_{2,\mu}(\mathbb{R}^2)}\le 4\|D^2h_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)}+8\|\,|Dh|\cdot|D\varsigma_\kappa|\,\|_{L_{2,\mu}(\mathbb{R}^2)}+4\|h\cdot D^2\varsigma_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)}.\tag{31}$$

If we use the properties of ς_κ and next the assumption that $R>1$, and apply the estimate from Proposition 2 to h, then we get

$$\|\,|Dh|\cdot|D\varsigma_\kappa|\,\|_{L_{2,\mu}(\mathbb{R}^2)}+\|h\cdot D^2\varsigma_\kappa\|_{L_{2,\mu}(\mathbb{R}^2)}\le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)},$$

where $c=c(\mu)$. Hence the above estimate and estimates (30) and (31) give

$$\|\varsigma_\kappa\cdot D^2h\|_{L_{2,\mu}(\mathbb{R}^2)}\le c\|p\|_{L_{2,\mu}(\mathbb{R}^2)},$$

where c depends only on μ. Letting $R\to\infty$ yields (21). ■

Lemma 4 is a consequence of the estimates (13) and (21) from Propositions 2 and 3. Now we are able to deduce the crucial estimate for the weak solutions of problem (5).

LEMMA 5. *Assume that $\mu\in(0,1)$, $U:=\mathbb{R}^3$ and $T>0$. Then there exists a constant $c=c(\mu,\nu,T)$ with the following property. If $w\in V_2^{1,0}(U^T)$ is a weak solution of (5) with $f\in L_{2,\mu}(U^T)$, then for each $\tau\in(0,T)$,*

$$\|w\|_{W^{2,1}_{2,\mu}(U^\tau)}\le c\|f\|_{L_{2,\mu}(U^\tau)}.\tag{32}$$

Proof. By Lemma 3 it is enough to estimate the norms $\|w_t\|_{L_{2,\mu}(U^\tau)}$ and $\|D^2_xw\|_{L_{2,\mu}(U^\tau)}$. To do it we shall apply partial Fourier transform with respect to x_3 and t. However, w is defined only for $t\in(0,T)$, so we have to extend it. Moreover, we want to obtain estimates of the norm on U^τ for each $\tau\in(0,T)$

with a constant independent of τ. Therefore suppose that $\tau \in (0, T)$; we may assume that $f(x,t)$ is zero for negative t. Then we define $f^*(x,t) = f(x,t)$ for $t < \tau$ and $f^*(x,t) = f(x, 2\tau - t)$ for $t > \tau$. Let $\omega = \omega(t)$ be a smooth function such that $\omega(t) = 1$ for $t < \tau$, $\omega(t) = 0$ for $t > \tau + 1 =: \tau'$ and $|\omega^{(k)}(t)| \le 2^k$ for $k = 0, 1$. Clearly, $\|f^*\|_{L_{2,\mu}(U^{\tau'})} \le 2\|f\|_{L_{2,\mu}(U^\tau)}$, hence from Lemma 2 we get a unique weak solution $w^* \in V_2^{1,0}(U^{\tau'})$ of the problem $w_t^* - \nu\Delta w^* = f^*$ in $U^{\tau'}$ and $w^*{}_{|t=0} = 0$. We extend w^* onto $\mathbb{R}^3 \times (-\infty, \tau')$ by putting zero for negative t. We define

$$v := \omega w^*. \tag{33}$$

Then $v_t - \nu\Delta v = \omega f^* + \omega_t w^* =: g$. It is clear that $\|\omega f^*\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})} \le 2\|f\|_{L_{2,\mu}(U^\tau)}$, and applying Lemma 3 we get

$$\|\omega_t w^*\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})} \le 2\|w^*\|_{L_{2,\mu}(U^{\tau'})} \le 2c\|f^*\|_{L_{2,\mu}(U^{\tau'})} \le 4c\|f\|_{L_{2,\mu}(U^\tau)},$$

where c comes from Lemma 3. Thus we showed that for some $c = c(\nu, \mu, T)$,

$$\|g\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})} \le c\|f\|_{L_{2,\mu}(U^\tau)}. \tag{34}$$

Therefore $v \in V_2(\mathbb{R}^3 \times \mathbb{R}) \cap L_2(\mathbb{R}^3 \times \mathbb{R})$ is a weak solution of $v_t - \nu\Delta v = g$ with $g \in L_{2,\mu}(\mathbb{R}^3 \times \mathbb{R})$, i.e. for each $\eta \in W_2^{1,1}(\mathbb{R}^3 \times \mathbb{R})$,

$$-\int_{\mathbb{R}^3\times\mathbb{R}} v \cdot \overline{\eta}_t \, dx\, dt + \nu \int_{\mathbb{R}^3\times\mathbb{R}} \nabla v \cdot \nabla\overline{\eta} \, dx\, dt = \int_{\mathbb{R}^3\times\mathbb{R}} g \cdot \overline{\eta} \, dx\, dt. \tag{35}$$

We shall show that there exists a constant $c = c(\nu, \mu)$ such that

$$\|v_t\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})} + \|D_x^2 v\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})} \le c\|g\|_{L_{2,\mu}(\mathbb{R}^3\times\mathbb{R})}. \tag{36}$$

For this purpose we denote by $\widetilde{v}$ the partial Fourier transform of v with respect to x_3 and t, i.e.

$$\widetilde{v}(x', \xi_2, \xi_1) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(x', x_3, t) \cdot e^{-i(x_3\xi_2 + t\xi_1)} \, dx_3\, dt,$$

where $x' = (x_1, x_2)$. The identity (35) shows that for a.e. $\xi_1, \xi_2 \in \mathbb{R}$,

$$\nu \int_{\mathbb{R}^2} \nabla'\widetilde{v} \cdot \nabla'\overline{\eta} \, dx' + s \int_{\mathbb{R}^2} \widetilde{v} \cdot \overline{\eta} \, dx' = \int_{\mathbb{R}^2} \widetilde{g} \cdot \overline{\eta} \, dx' \quad \text{for each } \eta \in H^1(\mathbb{R}^2),$$

where $\widetilde{g}$ is the partial Fourier transform of g, $s := \nu\xi_2^2 + i\xi_1$ and the operator ∇' acts on the x' variable. Thus for a.e. $\xi_1, \xi_2 \in \mathbb{R}$ the function $\widetilde{v}(\cdot, \xi_2, \xi_1) \in H^1(\mathbb{R}^2)$ is a weak solution of $-\nu\Delta'\widetilde{v} + s\widetilde{v} = \widetilde{g}$, where $\widetilde{g}(\cdot, \xi_2, \xi_1) \in L_{2,\mu}(\mathbb{R}^2)$. Lemma 4 yields the estimate

$$\int_{\mathbb{R}^2} (|D^2\widetilde{v}(x', \xi_2, \xi_1)|^2 + |\xi_2 D\widetilde{v}(x', \xi_2, \xi_1)|^2 + |\xi_2^2\widetilde{v}(x', \xi_2, \xi_1)|^2)|x'|^{2\mu} \, dx'$$
$$+ \int_{\mathbb{R}^2} |\xi_1\widetilde{v}(x', \xi_2, \xi_1)|^2|x'|^{2\mu} \, dx' \le c(\nu, \mu) \int_{\mathbb{R}^2} \widetilde{g}(x', \xi_2, \xi_1)|x'|^{2\mu} \, dx'$$

for a.e. $\xi_1, \xi_2 \in \mathbb{R}$. If we integrate it with respect to ξ_1, $\xi_2 \in \mathbb{R}$ and next apply the Parseval identity, then we get (36). By the uniqueness of solutions of (5) in $V_2^{1,0}(U^\tau)$ and the definition (33) we have $w = w^* = v$ in U^τ. Hence applying Lemma 3 to $v \in V_2^{1,0}(U^T)$, next (36) and finally (34) we deduce the estimate

$$\|w\|_{W^{2,1}_{2,\mu}(U^\tau)} = \|v\|_{W^{2,1}_{2,\mu}(U^\tau)} \le c\|g\|_{L_{2,\mu}(U^\tau)} \le \widetilde{c}\|f\|_{L_{2,\mu}(U^\tau)},$$

where $\widetilde{c}$ depends only on ν, μ and T. ■

REMARK 4. Evidently, Lemma 4 is valid if we replace $s = \nu\xi_2^2 + i\xi_1$ by $\widetilde{s} := \xi^2$, where $\xi \in \mathbb{R}$. Hence, if $u \in H^1(\mathbb{R}^3)$ is a weak solution of $-\Delta u = f \in L_{2,\mu}(\mathbb{R}^3)$, then $\|D^2u\|_{L_{2,\mu}(\mathbb{R}^3)} \le c(\mu)\|f\|_{L_{2,\mu}(\mathbb{R}^3)}$. The lower order terms can be estimated as in the proof of Lemma 3, provided we already have the estimate $\|u\|_{H^1(\mathbb{R}^3)} \le c\|\Delta u\|_{L^2(\mathbb{R}^3)}$.

To apply the regularizer method for problem (2) we need the following lemma.

LEMMA 6. *Assume that* $\mu \in (0,1)$, $U := \mathbb{R}^3_+$, $S := \partial U$ *and* $T > 0$. *Then there exists a constant* $c = c(\nu, \mu, T)$ *with the following property. If* $d_1, d_2 \in \mathbb{R}$ *satisfy* $d_1^2 + d_2^2 = 1$, *then for each* $f = (f_1, f_2) \in L_{2,\mu}(U^T)^2$, *and any* $\psi_1 \in \overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^T)$, $\psi_2 \in \overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^T)$ *there exists a unique weak solution* $w = (w_1, w_2) \in V_2^{1,0}(U^T)^2$ *of the problem*

$$(37)\qquad \begin{cases} w_t - \nu\Delta w = f & \text{in } U^T,\\ \bar{\bar{d}}w = \psi_1 & \text{on } S^T,\\ \dfrac{\partial}{\partial x_3}(dw) = \psi_2 & \text{on } S^T,\\ w_{|t=0} = 0 & \text{on } U,\end{cases}$$

where $d = (d_1, d_2)$, $\bar{\bar{d}} = (-d_2, d_1)$. *Furthermore, for each* $\tau \in (0,T)$,

$$(38)\quad \|w\|_{W^{2,1}_{2,\mu}(U^\tau)^2} \le c\{\|f\|_{L_{2,\mu}(U^\tau)^2} + \|\psi_1\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^\tau)} + \|\psi_2\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)}\}.$$

Proof. We define $g := (\bar{\bar{d}}f, df)$ and consider the system

$$(39)\qquad \begin{cases} u_t - \nu\Delta u = g & \text{in } U^T,\\ u_1 = \psi_1 & \text{on } S^T,\\ \dfrac{\partial}{\partial x_3}u_2 = \psi_2 & \text{on } S^T,\\ u_{|t=0} = 0 & \text{on } U,\end{cases}$$

where $u = (u_1, u_2)$. The existence of weak solutions of (39) with homogeneous boundary conditions follows from Lemma 1. The estimate can be deduced with the help of Remark 3 and Lemma 5. Finally, system (39) with nonhomogeneous boundary conditions can be reduced to the homogeneous one with the help of the trace theorem. Thus system (39) has a unique weak solution $u=(u_1,u_2)\in V_2^{1,0}(U^T)^2$, which satisfies the estimate

$$\|u\|_{W^{2,1}_{2,\mu}(U^\tau)^2}\le c\{\|g\|_{L_{2,\mu}(\Omega^\tau)^2}+\|\psi_1\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^\tau)}+\|\psi_2\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)}\},$$

where $c=c(\nu,\mu,T)$. We define $w_1:=\bar{\bar{d}}u$, $w_2:=du$. Then simple calculations show that $w=(w_1,w_2)$ is a weak solution of (37) and $\|w_1\|^2+\|w_2\|^2=\|u_1\|^2+\|u_2\|^2$, where $\|\cdot\|:=\|\cdot\|_{W^{2,1}_{2,\mu}(U^\tau)}$. Clearly, we have $\|g\|_{L_{2,\mu}(\Omega^\tau)^2}=\|f\|_{L_{2,\mu}(\Omega^\tau)^2}$, thus the above estimate gives (38). Finally, system (37) is uniquely solvable, because if $w\in V_2^{1,0}(U^T)^2$ is a weak solution of the homogeneous problem, then $u=(\bar{\bar{d}}w,dw)$ is a weak solution of the homogeneous system (39). By the uniqueness of solutions of the latter problem we have $u\equiv 0$. Thus $w\equiv 0$, since $w=(\bar{\bar{d}}u,du)$. ∎

5. Problem in a bounded domain. In this section we prove Theorem 1. By Remark 1 we have to show that there exists $\tau>0$ such that problem (2) is solvable. We apply the regularizer technique (see [La68, Chap. IV, §7]). Thus we have to define a continuous linear operator R such that if A denotes the operator associated with problem (2), then for $\tau>0$ small enough, $\|AR-\mathrm{Id}\|<1$ and $\|RA-\mathrm{Id}\|<1$. Clearly, this means that A is invertible, hence problem (2) is uniquely solvable.

5.1. *The regularizer.* By the assumption the boundary of Ω is smooth. Thus there exists $\lambda_0\in(0,1)$ such that for each $\lambda\in(0,\lambda_0)$ there are families of subsets of Ω denoted by $\{\omega^{(k)}\}_{k=1,\dots,K_\lambda}$ and $\{\Omega^{(k)}\}_{k=1,\dots,K_\lambda}$ which have the following properties:

- (P_1) $\omega^{(k)}\subseteq\Omega^{(k)}$ for each k, $\bigcup_{k=1}^{K_\lambda}\omega^{(k)}=\bigcup_{k=1}^{K_\lambda}\Omega^{(k)}=\Omega$,
- (P_2) there exists N_0, independent of λ, such that any intersection of N_0+1 sets from the family $\{\Omega^{(k)}\}_{k=1,\dots,K_\lambda}$ is empty,
- (P_3) $\{1,\dots,K_\lambda\}$ is a disjoint union of subsets $\mathcal{D}_\lambda$, $\mathcal{O}_\lambda$, $\mathcal{N}_\lambda$, $\mathcal{M}_\lambda$ and there exists $\beta\in(1,3/2)$, independent of λ, such that if $k\in\mathcal{O}_\lambda\cup\mathcal{M}_\lambda$, then $\omega^{(k)}$ and $\Omega^{(k)}$ are cubes with sides of length λ and $\beta\lambda$ respectively and center $q^{(k)}\in\Omega$. If $k\in\mathcal{N}_\lambda\cup\mathcal{D}_\lambda$, then $\omega^{(k)}=\Omega\cap K_1$ and $\Omega^{(k)}=\Omega\cap K_2$, where K_1 and K_2 are cubes with sides of length λ and $\beta\lambda$ respectively and center $q^{(k)}\in\partial\Omega\cap\overline{\omega^{(k)}}$. If $k\in\mathcal{O}_\lambda$, then $q^{(k)}\in L$ and $\mathrm{dist}(\Omega^{(k)},\partial\Omega)\ge\lambda\beta/2$. If $k\in\mathcal{N}_\lambda$, then $\overline{\omega^{(k)}}\cap\partial\Omega\ne\emptyset$ and $\Omega^{(k)}\cap L=\emptyset$. If $k\in\mathcal{D}_\lambda$, then $\overline{\omega^{(k)}}\cap\partial\Omega\ne\emptyset$ and $q^{(k)}=p_1$ or $q^{(k)}=p_2$, where $\{p_1,p_2\}=\partial\Omega\cap L$,
- (P_4) there exists $\gamma\in(\sqrt{3}/2,1)$, independent of λ, such that if $d^{(k)}:=\mathrm{dist}(q^{(k)},L)$, then $\min_{k\in\mathcal{M}_\lambda\cup\mathcal{N}_\lambda}d^{(k)}\ge\gamma\lambda$.

Remark 5. In order to apply the regularizer we have to consider the local problems in four cases: in the neighborhoods of the axis L disjoint ($k \in \mathcal{O}_\lambda$) and not disjoint ($k \in \mathcal{D}_\lambda$) from $\partial\Omega$, and away from the axis L in the neighborhoods disjoint ($k \in \mathcal{M}_\lambda$) and not disjoint ($k \in \mathcal{N}_\lambda$) from $\partial\Omega$. The first two cases will be considered in a weighted space. The remaining ones will be considered in a Sobolev space without weight. Then the desired estimates in weighted spaces are guaranteed by the condition (P_4).

The smooth boundary of Ω is locally the graph of a smooth function. Hence we introduce a local coordinate system $y = (y_1, y_2, y_3)$ with center at $q^{(k)}$ such that if $k \in \mathcal{D}_\lambda \cup \mathcal{N}_\lambda$, then $S^{(k)} := \partial\Omega \cap \overline{\Omega^{(k)}}$ is described by $y_3 = f^{(k)}(y_1, y_2)$, where $f^{(k)}$ is smooth and $\max(|y_1|, |y_2|) < \lambda\beta$. We also introduce the coordinates $Z_k = (Z_{k,1}, Z_{k,2}, Z_{k,3})$, where $Z_{k,i} := y_i$ for $i = 1, 2$ and $Z_{k,3} := y_3 - f^{(k)}(y_1, y_2)$. If $k \in \mathcal{O}_\lambda \cup \mathcal{M}_\lambda$, then $Z_k = (Z_{k,1}, Z_{k,2}, Z_{k,3})$ is the cartesian coordinate system with center at $q^{(k)}$. The smoothness of $\partial\Omega$ guarantees that the functions

$$\eta_1(\lambda) := \max_{k\in\mathcal{D}_\lambda\cup\mathcal{N}_\lambda} \sup_{x\in\Omega^{(k)}} \Big| \sum_{l,m=1}^{3} \nabla Z_{k,m} \cdot \nabla Z_{k,l}(x) - \nabla Z_{k,m} \cdot \nabla Z_{k,l}(q^{(k)}) \Big|,$$

$$\eta_2(\lambda) := \max_{k\in\mathcal{D}_\lambda\cup\mathcal{N}_\lambda} \sup_{x\in\Omega^{(k)}} |a - a(q^{(k)})|$$

have the following property:

$$\eta_1(\lambda) \to 0 \quad \text{and} \quad \eta_2(\lambda) \to 0 \quad \text{as } \lambda \to 0. \tag{40}$$

Let c_a denote the norm of a in $W^{1,\infty}$ and

$$c_\Omega = \max\{\|D^l Z_k\|_{L^\infty(\Omega^{(k)})}, \|D^l Z_k^{-1}\|_{L^\infty(\Omega^{(k)})};\ k \in \mathcal{D}_\lambda \cup \mathcal{N}_\lambda,\ l = 1, 2\}.$$

We set

$$\widehat{\Omega^{(k)}} := Z_k(\Omega^{(k)}), \quad \widehat{\omega^{(k)}} := Z_k(\omega^{(k)}), \quad \widehat{S^{(k)}} := Z_k(S^{(k)}).$$

Let $\{\xi^{(k)};\ k = 1, \ldots, K_\lambda\}$ be a family of smooth functions such that $0 \le \xi^{(k)}(x) \le 1$, $\xi^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\xi^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$ and $\frac{\partial \xi^{(k)}}{\partial n}{}_{|\partial U} = 0$. It is clear that ($\mathrm{P}_3$) gives $|D_x^m \xi^{(k)}(x)| \le c\lambda^{-|m|}$ for $|m| \le 2$, where $c = c(\beta)$. From (P_1) and (P_2) we deduce that $1 \le \sum_{k=1}^{K_\lambda} (\xi^{(k)}(x))^2 \le N_0$. We define

$$\eta^{(k)}(x) := \frac{\xi^{(k)}}{\sum_{l=1}^{K_\lambda} (\xi^{(l)}(x))^2}. \tag{41}$$

Then $\eta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$, $\frac{\partial \eta^{(k)}}{\partial n}{}_{|\partial U} = 0$ and

$$\sum_{k=1}^{K_\lambda} \eta^{(k)} \cdot \xi^{(k)} \equiv 1 \quad \text{on } \Omega, \tag{42}$$

$$|D_x^m \eta^{(k)}(x)| \le c\lambda^{-|m|} \quad \text{for } |m| \le 2, \tag{43}$$

where $c = c(\beta, N_0)$. We introduce the following notation:

$$Y^\tau(\Omega^{(k)}) := L_{2,\mu}(\Omega^{(k),\tau})^2 \times \overset{\circ}{W}{}_{2,\mu}^{3/2,3/4}(S^{(k),\tau}) \times \overset{\circ}{W}{}_{2,\mu}^{1/2,1/4}(S^{(k),\tau}),$$
$$X^\tau(\Omega^{(k)}) := \overset{\circ}{W}{}_{2,\mu}^{2,1}(\Omega^{(k),\tau})^2.$$

The spaces $Y^\tau(\Omega)$ and $X^\tau(\Omega)$ are defined similarly, by dropping the superscript (k). We denote by A the operator associated with problem (2), i.e. $A : X^\tau(\Omega) \to Y^\tau(\Omega)$, and for $u = (u^1, u^2) \in X^\tau(\Omega)$ we define

$$Au = \left[\left(\frac{\partial}{\partial t} - \nu\Delta\right)u,\ \overline{\overline{a}}u_{|S},\ \frac{\partial}{\partial n}(au)_{|S}\right].$$

We shall show that for some $\tau \in (0,1)$ small enough, the operator A is invertible. In order to do it we shall define a continuous linear operator $R : Y^\tau(\Omega) \to X^\tau(\Omega)$ such that

$$\|AR - \mathrm{Id}\|_{Y^\tau(\Omega)} < 1 \quad \text{and} \quad \|RA - \mathrm{Id}\|_{X^\tau(\Omega)} < 1 \tag{44}$$

for some positive τ. Clearly, (44) guarantees that A^{-1} exists. We shall define R to be the sum of $\eta^{(k)}R^{(k)}$, where $R^{(k)} : Y^\tau(\Omega) \to X^\tau(\Omega^{(k)})$ will be linear, continuous and for each $k \in \{1, \ldots, K_\lambda\}$ and $h \in Y^\tau(\Omega)$,

$$\|R^{(k)}h\|_{X^\tau(\Omega^{(k)})} \le c\|h\|_{Y^\tau(\Omega^{(k)})}, \tag{45}$$

where $c = c(c_\Omega, \mu, \nu)$. In further considerations we shall assume that the numbers λ and τ satisfy

$$\tau/\lambda^2 \le \kappa, \quad \text{where} \quad \kappa \le 1. \tag{46}$$

Assume that $h = (F^1, F^2, \phi_1, \phi_2) \in Y^\tau(\Omega)$. We have to consider each case of k. We define $R^{(k)}h$ as follows.

Case of $k \in \mathcal{D}_\lambda$. We set

$$\begin{aligned} f_i(x,t) &:= \xi^{(k)}F^i(Z_k^{-1}(x),t),\\ \psi_i(x,t) &:= \xi^{(k)}\phi_i(Z_k^{-1}(x),t) \quad \text{for } i = 1,2. \end{aligned} \tag{47}$$

Clearly, f_i, ψ_i satisfy the assumptions of Lemma 6, thus we get a weak solution $w^{(k)} = (w_1^{(k)}, w_2^{(k)})$ of problem (37), where we set $d := a(q^{(k)})$. We define

$$R^{(k)}h(x,t) := w^{(k)}(Z_k(x),t). \tag{48}$$

Hence using the estimate (38) we get

$$\begin{aligned} \|R^{(k)}h\|_{X^\tau(\Omega^{(k)})} &\le c_\Omega\|w^{(k)}\|_{W_{2,\mu}^{2,1}(\widehat{\Omega^{(k)}},\tau)^2}\\ &\le c\{\|f\|_{L_{2,\mu}(\widehat{\Omega^{(k)}},\tau)^2} + \|\psi_1\|_{\overset{\circ}{W}{}_{2,\mu}^{3/2,3/4}(\widehat{S^{(k)}},\tau)} + \|\psi_2\|_{\overset{\circ}{W}{}_{2,\mu}^{1/2,1/4}(\widehat{S^{(k)}},\tau)}\}, \end{aligned} \tag{49}$$

where $c = c(\nu, \mu, c_\Omega)$. Clearly, $\|f_i\|_{L_{2,\mu}(\widehat{\Omega^{(k)}},\tau)} \le c_\Omega\|F^i\|_{L_{2,\mu}(\Omega^{(k),\tau})}$. Before we estimate the remaining terms in (49) we notice the following

REMARK 6. There exists a constant $c = c(\mu, c_\Omega)$ such that for each $w \in \overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^{(k),\tau})$ and for $|m| \le 1$, $\|D^m_x w\|_{L_{2,\mu}(\Omega^{(k)})} \le c\tau^{1-|m|/2}\|w\|_{L^2_{2,\mu}(\Omega^{(k),\tau})}$. Indeed, this is a consequence of vanishing of w for $t = 0$ and for $|m| = 1$ it may be deduced with the help of an interpolation inequality.

REMARK 7. The trace operator $v \mapsto v_{|S^{(k)}}$ ($v \mapsto \frac{\partial v}{\partial n}{}_{|S^{(k)}}$ resp.) is linear and continuous from $\overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^{(k),\tau})$ onto $\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^{(k),\tau})$ ($\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^{(k),\tau})$ resp.), therefore it has a right inverse, which is also linear and continuous (3). We chose the norm in the space of traces so as to get independence of the norm of the trace operator and of its right inverse from λ and τ.

Utilizing the right inverse of the trace operator we get a continuation $\Phi_1 \in \overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^{(k),\tau})$ of $\phi_1 \in \overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^{(k),\tau})$. Then

$$\begin{aligned}
\|\psi_1\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(\widehat{S^{(k)}},\tau)} &\le \|\xi^{(k)}\Phi_1(Z_k^{-1}(\cdot))\|_{\overset{\circ}{W}{}^{2,1}_{2,\mu}(\widehat{\Omega^{(k)}},\tau)} \\
&\le c_\Omega \|\xi^{(k)}\Phi_1\|_{\overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^{(k),\tau})} \le c(c_\Omega)(1+\lambda^{-1}+\lambda^{-2})\|\Phi_1\|_{L_{2,\mu}(\Omega^{(k),\tau})} \\
&\quad + c(c_\Omega)[(1+\lambda^{-1})\|\nabla\Phi_1\|_{L_{2,\mu}(\Omega^{(k),\tau})} + \|\Phi_1\|_{L^2_{2,\mu}(\Omega^{(k),\tau})}] \\
&\le c(c_\Omega,\mu)\{\tau^{1/2}(\kappa^{1/2}+\tau^{1/2}+1)+\kappa^{1/2}+\kappa+1\}\|\Phi_1\|_{L^2_{2,\mu}(\Omega^{(k),\tau})} \\
&\le c(c_\Omega,\mu)\|\phi_1\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^{(k),\tau})},
\end{aligned}$$

where we applied property (P_3), condition (46) and Remark 6.

Similarly we prove the estimate

$$\|\psi_2\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(\widehat{S^{(k)}},\tau)} \le c(c_\Omega,\mu)\|\phi_2\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^{(k),\tau})}.$$

Thus from (49) we deduce (45) for $k \in \mathcal{D}_\lambda$, where $c = c(c_\Omega, \mu, \nu)$, provided (46) holds.

CASE OF $k \in \mathcal{O}_\lambda$. We define f_i as in (47). With the help of Lemma 2 we get $w^{(k)} = (w^{(k)}_1, w^{(k)}_2)$ such that $w^{(k)}_i$ is a weak solution of (5) with r.h.s. equal to f_i, $i = 1, 2$. We define $R^{(k)}h$ by (48). Then the estimate (32) gives (45) with $c = c(\mu, \nu)$.

REMARK 8. In the remaining cases the sets $\overline{\Omega^{(k)}}$ are disjoint from the axis L. Therefore we may consider the local solutions in the space without weight. However, we have to proceed carefully, because we need estimates in weighted spaces.

CASE OF $k \in \mathcal{M}_\lambda$. Let f_i be as in (47). Then

$$\|f_i\|_{L^2(\widehat{\Omega^{(k)}},\tau)} \le \sup_{\Omega^{(k)}} r^{-\mu} \cdot \|F^i\|_{L_{2,\mu}(\Omega^{(k),\tau})}.$$

(3) Some details may be found in §6 of [Ku05].

By (P_3) and (P_4) we have $\sup_{\Omega^{(k)}} r^{-\mu} \le \lambda^\mu(\gamma - \beta/2)^\mu < \infty$, thus f_i are square integrable. Let $w_i^{(k)}$ be a weak solution of (5) with r.h.s. f_i, and set $w^{(k)} = (w_1^{(k)}, w_2^{(k)})$. We define $R^{(k)}h$ by (48) and apply the Hölder inequality to get

$$\begin{aligned}\|R^{(k)}h\|_{W^{2,1}_{2,\mu}(\Omega^{(k)},\tau)^2} &\le \sup_{\Omega^{(k)}} r^\mu \cdot \|R^{(k)}h\|_{W^{2,1}_{2}(\Omega^{(k)},\tau)^2} \le c \sup_{\Omega^{(k)}} r^\mu \cdot \|f\|_{L^2(\widehat{\Omega^{(k)}},\tau)^2} \\ &\le c \sup_{\Omega^{(k)}} r^\mu \cdot \|F\|_{L^2(\Omega^{(k)},\tau)^2} \le c \sup_{\Omega^{(k)}} r^\mu \cdot \sup_{\Omega^{(k)}} r^{-\mu} \cdot \|F\|_{L_{2,\mu}(\Omega^{(k)},\tau)^2},\end{aligned}$$

where $c = c(\mu,\nu)$. The sets $\Omega^{(k)}$ satisfy (P_3) and (P_4), which guarantee the estimate

$$\sup_{\Omega^{(k)}} r^\mu \cdot \sup_{\Omega^{(k)}} r^{-\mu} \le \left(\frac{\gamma+\beta/2}{\gamma-\beta/2}\right)^\mu$$

independently of λ. Thus we get (45) with $c = c(\mu,\nu)$.

Case of $k \in \mathcal{N}_\lambda$. We define f_i, ψ_i as in (47) and let $w^{(k)} = (w_1^{(k)}, w_2^{(k)})$ be a weak solution of (37). We define $R^{(k)}h$ by (48). We use properties (P_3) and (P_4) as in the previous case and the estimates used in the first case. Thus we get (45) with $c = c(c_\Omega,\mu,\nu)$.

In this way we have shown the estimate (45) for each $k \in \{1,\dots,K_\lambda\}$, where $c = c(c_\Omega,\mu,\nu)$, provided (46) holds. Clearly, if $h \in Y^\tau(\Omega)$, then $\eta^{(k)}R^{(k)}h \in X^\tau(\Omega)$. We define the operator $R : Y^\tau(\Omega) \to X^\tau(\Omega)$ by

$$Rh := \sum_{k=1}^{K_\lambda} \eta^{(k)}R^{(k)}h \quad \text{for } h \in Y^\tau(\Omega). \tag{50}$$

It is clear that R is linear. From the estimate (45) we get

$$\begin{aligned}\|Rh\|^2_{X^\tau(\Omega)} &\le c(\mu,c_\Omega)\|Rh\|^2_{L^2_{2,\mu}(\Omega^\tau)^2} \\ &\le c(\mu,c_\Omega,N_0)\sum_{k=1}^{K_\lambda} \|\eta^{(k)}R^{(k)}h\|^2_{L^2_{2,\mu}(\Omega^{(k)},\tau)^2} \\ &\le c(\mu,c_\Omega,\kappa,N_0)\sum_{k=1}^{K_\lambda} \|R^{(k)}h\|^2_{L^2_{2,\mu}(\Omega^{(k)},\tau)^2} \le c\sum_{k=1}^{K_\lambda} \|h\|^2_{Y^\tau(\Omega^{(k)})},\end{aligned}$$

where $c = c(\mu,\nu,c_\Omega,N_0)$, provided (46) holds. Hence R is continuous, because $\sum_{k=1}^{K_\lambda} \|h\|^2_{Y^\tau(\Omega^{(k)})} \le N_0\|h\|^2_{Y^\tau(\Omega)}$.

5.2. *Estimate of the regularizer.* Now we prove the estimates (44). Let us start with the first one. If $h = (F^1, F^2, \phi_1, \phi_2) \in Y^\tau(\Omega)$, then by (42) we may write

$$ARh - h = A\sum_{k=1}^{K_\lambda} \eta^{(k)} R^{(k)} h - \sum_{k=1}^{K_\lambda} \eta^{(k)} \cdot \xi^{(k)} h. \tag{51}$$

REMARK 9. In this subsection all constants depend only on μ, ν, c_Ω and N_0 and may vary from line to line. The estimates below hold for all λ and τ that satisfy (46).

If $i = 1, 2$, then we denote by $R^{(k),i}$ the ith coordinate of $R^{(k)}h$, hence the first two coordinates in (51) are equal to

$$\sum_{k=1}^{K_\lambda} \eta^{(k)} \Big[\Big(\frac{\partial}{\partial t} - \nu \Delta\Big) R^{(k),i} h - \xi^{(k)} F^i\Big] - 2\nu \sum_{k=1}^{K_\lambda} \nabla \eta^{(k)} \cdot \nabla R^{(k),i} h - \nu \sum_{k=1}^{K_\lambda} \Delta \eta^{(k)} \cdot R^{(k),i} h. \tag{52}$$

The function $R^{(k),i}$ was defined by a smooth change of variables from w^i which satisfies the equation $(\partial/\partial t - \nu \nabla) w^i(x,t) = \xi^{(k)} F^i(Z_k^{-1}(x), t)$. Hence, with the help of (45), after straightforward calculations, the first sum in (52) is estimated by $c(\eta_1(\lambda) + \tau^{1/2}) \|h\|_{Y^\tau(\Omega)}$. The norm of the second sum in (52) can be estimated by $c\kappa^{1/2} (\sum_{k=1}^{K_\lambda} \|R^{(k),i} h\|^2_{L^2_{2,\mu}(\Omega^{(k),\tau})})^{1/2}$. Thus by (45) it is at most $c\kappa^{1/2} \|h\|_{Y^\tau(\Omega)}$. Similarly the norm of the last sum in (52) can be estimated by $c\kappa \|h\|_{Y^\tau(\Omega)}$. Summarizing, the norm in $L_{2,\mu}(\Omega^\tau)$ of the first two coordinates of (51) is estimated by

$$c(\eta_1(\lambda) + \tau^{1/2} + \kappa^{1/2} + \kappa) \|h\|_{Y^\tau(\Omega)}. \tag{53}$$

The third coordinate of (51) equals

$$(\overline{\overline{a}} Rh)_{|S} - \phi_1 = \sum_{k \in \mathcal{D}_\lambda \cup \mathcal{N}_\lambda} \eta^{(k)} [\overline{\overline{a}} R^{(k)} h_{|S} - \xi^{(k)} \phi_1] = \sum_{k \in \mathcal{D}_\lambda \cup \mathcal{N}_\lambda} \eta^{(k)} [(\overline{\overline{a}} - \overline{\overline{a}}(q^{(k)})) R^{(k)} h_{|S}],$$

because $R^{(k)}h$ satisfies the appropriate boundary condition. Hence, if we apply the trace theorem and the estimate (45), then the norm of this function in $\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(S^\tau)$ is at most

$$c\eta_2(\lambda) \|h\|_{Y^\tau(\Omega)}. \tag{54}$$

Finally, the last coordinate of (51) is equal to

$$\Big[\frac{\partial}{\partial n}(aRh)\Big]_{|S} - \phi_2 = \Big[\frac{\partial}{\partial n}\Big(a \sum_{k=1}^{K_\lambda} \eta^{(k)} R^{(k)} h\Big)\Big]_{|S} - \phi_2 = \Big[\frac{\partial a}{\partial n} \sum_{k=1}^{K_\lambda} \eta^{(k)} R^{(k)} h\Big]_{|S} + \Big[a \sum_{k=1}^{K_\lambda} \frac{\partial \eta^{(k)}}{\partial n} R^{(k)} h\Big]_{|S}$$

$$+\Big[\sum_{k=1}^{K_\lambda}\eta^{(k)}(a-a(q^{(k)}))\frac{\partial R^{(k)}h}{\partial n}\Big]_{|S}+\Big[\sum_{k=1}^{K_\lambda}\eta^{(k)}a(q^{(k)})\frac{\partial R^{(k)}h}{\partial n}\Big]_{|S}-\phi_2.$$

Clearly, the last two terms vanish, because $R^{(k)}h$ satisfies the appropriate boundary condition. We recall that the normal derivative of $\eta^{(k)}$ vanishes on the boundary, therefore the above expression is equal to

$$\Big\{\frac{\partial}{\partial n}\Big[\sum_{k=1}^{K_\lambda}\eta^{(k)}(a-a(q^{(k)}))R^{(k)}h\Big]\Big\}_{|S}.$$

Hence, the trace theorem, Remark 6 and the estimate (45) show that the norm in $\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(S^\tau)$ is less than or equal to

$$c\eta_2(\lambda)\|h\|_{Y^\tau(\Omega)}. \tag{55}$$

To sum up, from (53)–(55) we get

$$\|ARh-h\|_{Y^\tau(\Omega)}\le c_1(\eta_1(\lambda)+\eta_2(\lambda)+\tau^{1/2}+\kappa^{1/2}+\kappa)\|h\|_{Y^\tau(\Omega)}. \tag{56}$$

Now we turn our attention to the second estimate in (44). Suppose that $u\in X^\tau(\Omega)$. Then we get

$$\begin{aligned}\|RAu-u\|^2_{X^\tau(\Omega)}&\le c\sum_{k=1}^{K_\lambda}\|\eta^{(k)}(R^{(k)}Au-\xi^{(k)}u)\|^2_{X^\tau(\Omega^{(k)})}\\&\le c\sum_{k=1}^{K_\lambda}\|\eta^{(k)}R^{(k)}(A-A^{(k)})u\|^2_{X^\tau(\Omega^{(k)})}\\&\quad+c\sum_{k=1}^{K_\lambda}\|\eta^{(k)}\big(R^{(k)}A^{(k)}u-\xi^{(k)}u\big)\|^2_{X^\tau(\Omega^{(k)})},\end{aligned} \tag{57}$$

where $A^{(k)}$ denotes the operator A with coefficients "frozen" at $q^{(k)}$, i.e.

$$A^{(k)}u:=\Big[\Big(\frac{\partial}{\partial t}-\nu\Delta\Big)u,\ \overline{\overline{a}}(q^{(k)})u_{|S},\ \frac{\partial}{\partial n}(a(q^{(k)})u)_{|S}\Big].$$

For the first sum in (57), according to Remark 6, the estimate (45) and the trace theorem we obtain

$$\begin{aligned}\|\eta^{(k)}R^{(k)}(A-A^{(k)})u\|_{X^\tau(\Omega^{(k)})}&\le c\|R^{(k)}(A-A^{(k)})u\|_{X^\tau(\Omega^{(k)})}\\&\le c\|(A-A^{(k)})u\|_{Y^\tau(\Omega^{(k)})}\\&\le c\eta_2(\lambda)\|u\|_{X^\tau(\Omega^{(k)})}.\end{aligned} \tag{58}$$

The second sum in (57) may be estimated in the following manner. Let $y:=Z_k(x)$ be a local coordinate with origin at $q^{(k)}$. Then $R^{(k)}A^{(k)}u(x,t)=w^{(k)}(y,t)$, where $w^{(k)}$ is a solution of the appropriate problem. Hence, the estimates (32) and (38) yield

$$
\begin{aligned}
&\|\eta^{(k)}(R^{(k)}A^{(k)}u-\xi^{(k)}u)\|_{\overset{\circ}{W}{}^{2,1}_{2,\mu}(\Omega^{(k)},\tau)^2}\\
&\quad\le c\|\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u\|_{\overset{\circ}{W}{}^{2,1}_{2,\mu}(\widehat{\Omega^{(k)}},\tau)^2}\\
&\quad\le c\bigg\{\bigg\|\bigg(\frac{\partial}{\partial t}-\nu\Delta_y\bigg)[\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u]\bigg\|_{L_{2,\mu}(\widehat{\Omega^{(k)}},\tau)^2}\\
&\qquad+\delta_k\|\overline{\overline{a}}(q^{(k)})[\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u]_{|\widehat{S^{(k)}}}\|_{\overset{\circ}{W}{}^{3/2,3/4}_{2,\mu}(\widehat{S^{(k)}},\tau)}\\
&\qquad+\delta_k\bigg\|\frac{\partial}{\partial y_3}[a(q^{(k)})(\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u)]_{|\widehat{S^{(k)}}}\bigg\|_{\overset{\circ}{W}{}^{1/2,1/4}_{2,\mu}(\widehat{S^{(k)}},\tau)}\bigg\},
\end{aligned}
$$

where the hat $\widehat{\ }$ over a function denotes this function in the y coordinates, and $\delta_k=0$ for $k\in\mathcal{M}_\lambda\cup\mathcal{O}_\lambda$ and $\delta_k=1$ for $k\in\mathcal{D}_\lambda\cup\mathcal{N}_\lambda$. A straightforward calculation gives

$$
\begin{aligned}
&\bigg(\frac{\partial}{\partial t}-\nu\Delta_y\bigg)[w^{(k)}(y,t)-\xi^{(k)}u(Z_k^{-1}(y),t)]=\nu\Delta_y\xi^{(k)}\cdot u(Z_k^{-1}(y),t)\\
&\qquad+2\nu\nabla_y\xi^{(k)}\cdot\nabla_y[u(Z_k^{-1}(y),t)]+\nu\xi^{(k)}\sum_{m=1}^{3}\frac{\partial u}{\partial x_m}(Z_k^{-1}(y),t)\cdot\Delta_y Z_{k,m}^{-1}\\
&\qquad+\nu\xi^{(k)}\sum_{n,m=1}^{3}\frac{\partial^2 u}{\partial x_n\partial x_m}(Z_k^{-1}(y),t)\\
&\qquad\cdot[\nabla_y Z_{k,m}^{-1}\cdot\nabla_y Z_{k,n}^{-1}(y)-\nabla_y Z_{k,m}^{-1}\cdot\nabla_y Z_{k,n}^{-1}(0)].
\end{aligned}
$$

Thus, as before, the above norm in $L_{2,\mu}(\widehat{\Omega^{(k)},\tau})^2$ can be estimated by

$$
c(\kappa^{1/2}+\kappa+\tau^{1/2}+\eta_1(\lambda))\|u\|_{L^2_{2,\mu}(\Omega^{(k)},\tau)^2}.
$$

If $k\in\mathcal{D}_\lambda\cup\mathcal{N}_\lambda$, then according to the boundary conditions for $w^{(k)}$ on $\widehat{S^{(k)},\tau}$ we have

$$
\begin{aligned}
\overline{\overline{a}}(q^{(k)})[\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u]&=0,\\
\frac{\partial}{\partial y_3}[a(q^{(k)})(\widehat{\eta^{(k)}}w^{(k)}-\widehat{\eta^{(k)}\xi^{(k)}}u)]&=0.
\end{aligned}
$$

Hence

$$
\|\eta^{(k)}(R^{(k)}A^{(k)}u-\xi^{(k)}u)\|_{X^\tau(\Omega^{(k)})}\le c(\kappa^{1/2}+\kappa+\tau^{1/2}+\eta_1(\lambda))\|u\|_{L^2_{2,\mu}(\Omega^{(k)},\tau)^2}.
$$

Thus the above inequality, (57) and (58) give

$$
(59)\qquad \|RAu-u\|_{X^\tau(\Omega)}\le c_2(\kappa^{1/2}+\kappa+\tau^{1/2}+\eta_1(\lambda)+\eta_2(\lambda))\|u\|_{X^\tau(\Omega)}.
$$

Now we fix $\lambda \in (0, \lambda_0)$ and $\tau \in (0,1)$ such that

$$c_i(\kappa^{1/2} + \kappa + \tau^{1/2} + \eta_1(\lambda) + \eta_2(\lambda)) < 1 \quad \text{for } i = 1, 2,$$

where c_i for $i = 1, 2$ are the constants from (56) and (59) respectively. Hence, for such λ and τ the estimates (44) hold and so the operator $A : X^\tau(\Omega) \to Y^\tau(\Omega)$ is invertible. Thus we have shown that problem (2) is uniquely solvable for some $\tau > 0$, and according to Remark 1 the proof of Theorem 1 is complete.

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