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BEST POSSIBLE ESTIMATES OF SOLUTIONS TO THE TRANSMISSION PROBLEM FOR THE LAPLACE OPERATOR WITH N DIFFERENT MEDIA IN A CONICAL DOMAIN

Abstract. We investigate the behavior of weak solutions to the transmission problem for the Laplace operator with N different media in a neighborhood of a boundary conical point. We establish a precise exponent of the decreasing rate of the solution.

1. Introduction. Let $G \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near $\mathcal{O}$ it is a convex conical surface with vertex at $\mathcal{O}$ and aperture $\omega_0 \in (0, \pi)$. We assume that $G = \bigcup_{i=1}^{N} G_i$ is divided into N subdomains G_i, $i = 1, \ldots, N$, by $N-1$ hyperplanes Σ_k, $k = 1, \ldots, N-1$, where $\mathcal{O}$ belongs to every $\overline{\Sigma}_k$ and $G_i \cap G_j = \emptyset$, $i \neq j$. Let ω_i be the aperture of G_i at $\mathcal{O}$. Define $\theta_k = \omega_1 + \cdots + \omega_k$, thus $\omega_0 = \theta_N$. Let $\mathcal{D} \subseteq \partial G$ be the part of ∂G where we consider the Dirichlet boundary condition. We consider the transmission problem

$$(L) \quad \begin{cases} \mathcal{L}_i[u] \equiv a_i \Delta u_i - p_i u_i(x) = f_i(x), \quad x \in G_i, \ i = 1, \ldots, N; \\[2mm] [u]_{\Sigma_k} = 0, \quad \mathcal{S}_k[u] \equiv \left[a \dfrac{\partial u}{\partial n_k} \right]_{\Sigma_k} + \dfrac{1}{|x|} \beta_k(\phi) u(x) = h_k(x), \\[2mm] \hspace{3cm} x \in \Sigma_k, \ k = 1, \ldots, N-1; \\[2mm] \mathcal{B}[u] \equiv \alpha(x) \cdot a \dfrac{\partial u}{\partial \vec{n}} + \dfrac{1}{|x|} \gamma(\phi) u(x) = g(x), \quad x \in \partial G \setminus \mathcal{O}, \end{cases}$$

where $a_i > 0$, $p_i \geq 0$ $(i = 1, \ldots, N)$ are constants;

$$\alpha(x) = \begin{cases} 0 & \text{if } x \in \mathcal{D}, \\ 1 & \text{if } x \notin \mathcal{D}; \end{cases}$$

$\vec{n}$ denotes the unit outward normal to $\partial G \setminus \mathcal{O}$; moreover

- $u(x) = u_i(x)$, $x \in G_i$; $f(x) = f_i(x)$, $x \in G_i$; $a|_{\overline{G}_i} = a_i$, etc.;
- $[u]_{\Sigma_k}$ denotes the jump of u across Σ_k, i.e.

$$[u]_{\Sigma_k} = u_k(\overline{x})|_{\Sigma_k} - u_{k+1}(\overline{x})|_{\Sigma_k}, \quad u_k(\overline{x})|_{\Sigma_k} = \lim_{G_k \ni x \to \overline{x} \in \Sigma_k} u(x),$$

$$u_{k+1}(\overline{x})|_{\Sigma_k} = \lim_{G_{k+1} \ni x \to \overline{x} \in \Sigma_k} u(x);$$

- $\left[a \frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k}$ denotes the jump of the co-normal derivative of u across Σ_k, i.e.

$$\left[a \, \frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k} = a_k \, \frac{\partial u_k}{\partial \vec{n}_k}\bigg|_{\Sigma_k} - a_{k+1} \, \frac{\partial u_{k+1}}{\partial \vec{n}_k}\bigg|_{\Sigma_k},$$

where $\vec{n}_k$ denotes the unit outward normal to Σ_k.

The transmission problems often appear in different fields of physics and technique. For instance, an important problem in the electrodynamics of solid media is the electromagnetic processes in ferromagnetic media with different dielectric constants. These problems also appear in solid mechanics if a body consists of composite materials.

Here we obtain *best possible estimates* of weak solutions of problem (L) near a conical boundary point. Analogous results were established in [3] for the Dirichlet and Robin problems in a conical domain without interfaces. Many mathematicians have considered the transmission problems. First, V. A. Il'in [9], O. A. Ladyzhenskaya and N. N. Ural'tseva [10], Z. G. Sheftel [13] and M. V. Borsuk [2] studied general interface problems for second order elliptic operators in smooth domains. Later other mathematicians studied transmission problems in nonsmooth domains in some particular cases (see the references in [11, 12, 5]). General interface problems in polygonal and polyhedral domains were considered in [11, 12]. Regularity results in terms of weighted Sobolev spaces have been obtained in [5] for two- and three-dimensional transmission problems for the Laplace operator.

A principal new feature of this article is the derivation of sharp estimates for solutions of (L) in n-dimensional $(n \geq 2)$ conic domains with N different media $(N \geq 2)$. We show some examples. We also note that our methods are different from those in [11, 12].

We introduce the following notations:

- S^{n-1} : the unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- $(r, \phi), \phi = (\phi_1, \ldots, \phi_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$:

$$x_1 = r\cos\phi_1,$$
$$x_2 = r\cos\phi_2\sin\phi_1,$$
$$\vdots$$
$$x_{n-1} = r\cos\phi_{n-1}\sin\phi_{n-2}\ldots\sin\phi_1,$$
$$x_n = r\sin\phi_{n-1}\sin\phi_{n-2}\ldots\sin\phi_1.$$

- $\mathcal{C}$: the convex rotational cone $\{x_1 > r\cos(\omega_0/2)\}$ with vertex $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral surface of $\mathcal{C}$, $\{x_1 = r\cos(\omega_0/2)\}$;
- Ω : a domain with smooth boundary $\partial\Omega$ obtained by intersecting the cone lateral surface $\partial\mathcal{C}$ with S^{n-1}, $\partial\Omega = \partial\mathcal{C}\cap S^{n-1}$;
- Ω_i : intersection of G_i and S^{n-1} $(i = 1,\ldots,N)$; thus $\Omega = \bigcup_{i=1}^{N}\Omega_i$;
- $G_a^b = \{(r,\omega)\,|\,0 \le a < r < b;\ \omega\in\Omega\}\cap G$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \{(r,\phi)\,|\,0 \le a < r < b;\ \phi\in\partial\Omega\}\cap\partial G$: the lateral surface of G_a^b;
- $G_d = G\setminus G_0^d$, $\Gamma_d = \partial G\setminus\Gamma_0^d$, $d > 0$;
- $\Sigma = \sum_{k=1}^{N-1}\Sigma_k$, where $\Sigma_k = G\cap\{\phi_1 = \omega_0/2 - \theta_k\}$, $k = 1,\ldots,N-1$;
- $\sigma = \sum_{k=1}^{N-1}\sigma_k$, where $\sigma_k = \Sigma_k\cap\Omega$;
- $\Sigma_a^b = G_a^b\cap\Sigma$; $\Sigma_d = \Sigma\setminus\Sigma_0^d$, $d > 0$;
- $(G_i)_a^b = \{(r,\omega)\,|\,0 \le a < r < b;\ \omega\in\Omega\}\cap G_i$, $i = 1,\ldots,N$;
- $(\Sigma_k)_a^b = G_a^b\cap\Sigma_k$, $k = 1,\ldots,N-1$;
- $\Omega_\varrho = G_0^d\cap\{|x| = \varrho\}$, $0 < \varrho < d$.

We assume without loss of generality that there exists $d > 0$ such that G_0^d is *a convex rotational cone* with vertex $\mathcal{O}$ and aperture ω_0, thus

$$(1.1)\qquad \Gamma_0^d = \left\{(r,\phi)\,\middle|\,x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^{n}x_i^2;\ r\in(0,d),\ \phi_1 = \frac{\omega_0}{2},\ \omega_0\in(0,\pi)\right\}.$$
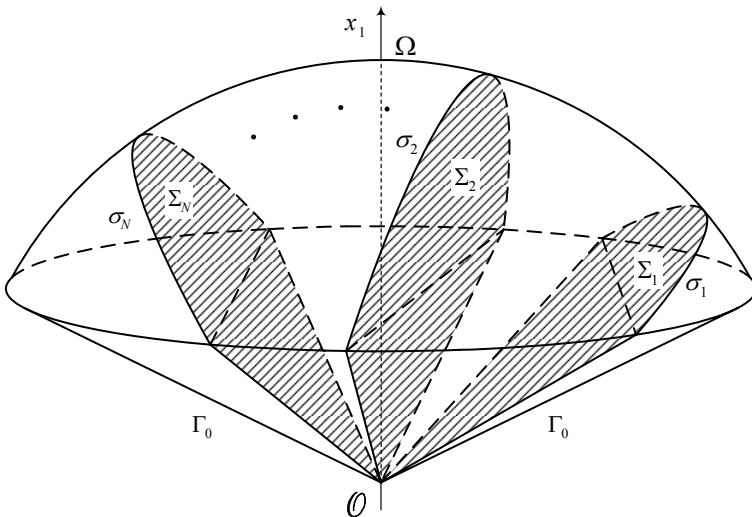

Fig. 1

We use the standard function spaces: $C^k(\overline{G}_i)$ with the norm $|u_i|_{k,G_i}$; the Lebesgue space $L_q(G_i)$, $q \geq 1$, with the norm $\|u_i\|_{q,G_i}$; the Sobolev space $W^{k,q}(G_i)$ with the norm $\|u_i\|_{k,q;G_i}$; the direct sums $\mathbf{C}^k(\overline{G}) = C^k(\overline{G}_1) \dot{+} \cdots \dot{+} C^k(\overline{G}_N)$ with the norm $|u|_{k,G} = \sum_{i=1}^{N} |u_i|_{k,G_i}$; $\mathbf{L}_q(G) = L_q(G_1) \dot{+} \cdots \dot{+} L_q(G_N)$ with the norm $\|u\|_{\mathbf{L}_q(G)} = \sum_{i=1}^{N} (\int_{G_i} |u_i|^q \, dx)^{1/q}$; $\mathbf{W}^{k,q}(G) = W^{k,q}(G_1) \dot{+} \cdots \dot{+} W^{k,q}(G_N)$ with the norm $\|u\|_{k,q;G} = \sum_{i=1}^{N} (\int_{G_i} \sum_{|\beta|=0}^{k} |D^\beta u_i|^q \, dx)^{1/q}$. We define the weighted Sobolev spaces $\mathbf{V}_{q,\alpha}^k(G) = V_{q,\alpha}^k(G_1) \dot{+} \cdots \dot{+} V_{q,\alpha}^k(G_N)$ for integer $k \geq 0$ and real α, where $V_{q,\alpha}^k(G_i)$ denotes the space of all distributions $u \in \mathcal{D}'(G_i)$ satisfying $r^{\alpha/q+|\beta|-k}|D^\beta u_i| \in L_q(G_i)$, $i = 1, \ldots, N$. $\mathbf{V}_{q,\alpha}^k(G)$ is a Banach space for the norm

$$\|u\|_{\mathbf{V}_{q,\alpha}^k(G)} = \sum_{i=1}^{N} \Big(\int_{G_i} \sum_{|\beta|=0}^{k} r^{\alpha+q(|\beta|-k)} |D^\beta u_i|^q \, dx \Big)^{1/q}.$$

$\mathbf{V}_{q,\alpha}^{k-1/q}(\partial G)$ is the space of functions φ on ∂G with the norm $\|\varphi\|_{\mathbf{V}_{q,\alpha}^{k-1/q}(\partial G)} = \inf \|\Phi\|_{\mathbf{V}_{q,\alpha}^k(G)}$, where the infimum is taken over all functions Φ such that $\Phi|_{\partial G} = \varphi$ in the sense of traces. We write

$$\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G), \quad \mathring{\mathbf{W}}_\alpha^k(G) \equiv \mathbf{V}_{2,\alpha}^k(G), \quad \mathring{\mathbf{W}}_\alpha^{k-1/2}(\partial G) \equiv \mathbf{V}_{2,\alpha}^{k-1/2}(\partial G).$$

DEFINITION 1. A function u is called a *weak solution* of problem (L) provided that $u \in \mathbf{C}^0(\overline{G}) \cap \mathring{\mathbf{W}}_0^1(G)$ and u satisfies the integral identity

$$(\text{II}) \quad \int_G a u_{x_j} \eta_{x_j} \, dx + \int_\Sigma \frac{1}{r} \beta(\phi) u(x) \eta(x) \, ds + \int_{\partial G} \alpha(x) \frac{1}{r} \gamma(\phi) u(x) \eta(x) \, ds$$

$$= \int_{\partial G} \alpha(x) g(x) \eta(x) \, ds + \int_\Sigma h(x) \eta(x) \, ds - \int_G (pu(x) + f(x)) \eta(x) \, dx$$

for all $\eta \in \mathbf{C}^0(\overline{G}) \cap \mathring{\mathbf{W}}_0^1(G)$ (summation over repeated indices from 1 to n is understood); here

$$\int_G f(x) \, dx = \sum_{i=1}^{N} \int_{G_i} f_i(x) \, dx, \quad \int_\Sigma h(x) \, ds = \sum_{k=1}^{N-1} \int_{\Sigma_k} h_k(x) \, ds, \quad \text{etc.}$$

REMARK 1. We assume that $u|_{\partial G \cap \mathcal{D}} = 0$.

We assume that $M_0 = \max_{x \in \overline{G}} |u(x)|$ is known.

LEMMA 1.1. *Let u be a weak solution of* (L). *For any $\eta \in \mathbf{C}^0(\overline{G}) \cap \mathring{\mathbf{W}}_0^1(G)$,*

$(\text{II})_{\text{loc}} \quad \int\limits_{G_0^\varrho} \{au_{x_i}\eta_{x_i} + (f(x) + pu(x))\eta(x)\}\, dx = \int\limits_{\Omega_\varrho} a\, \frac{\partial u}{\partial r}\, \eta(x)\, d\Omega_\varrho$

$+ \int\limits_{\Gamma_0^\varrho} \alpha(x)\left(g(x) - \frac{1}{r}\,\gamma(\phi)u(x)\right)\eta(x)\, ds + \int\limits_{\Sigma_0^\varrho} \left(h(x) - \frac{1}{r}\,\beta(\phi)u(x)\right)\eta(x)\, ds$

for a.e. $\varrho \in (0, d)$.

Proof. The proof is analogous to the proof of [3, Lemma 5.2, pp. 167–170]. ∎

Define numbers

$$(1.2) \quad \begin{cases} a_* = \min\{a_1, \ldots, a_N\} > 0, \quad a^* = \max\{a_1, \ldots, a_N\} > 0; \\ p^* = \max\{p_1, \ldots, p_N\} \geq 0; \\ [a]_{\Sigma_k} = a_k - a_{k+1},\ k = 1, \ldots, N-1; \\ a_0 = \max_{1 \leq k \leq N-1} |[a]_{\Sigma_k}|; \quad \widetilde{a} = \max(a^*, a_0). \end{cases}$$

We assume that:

(a) $f \in \mathbf{L}_{q/2}(G) \cap \mathbf{L}_2(G);\ q > n;$

(b) $\gamma(\phi) \geq \gamma_0 > \widetilde{a}\tan(\omega_0/2)$ on $\partial G;\ \beta_k(\phi) \geq \beta_0 > \widetilde{a}\tan(\omega_0/2)$ on $\Sigma_k,\ k = 1, \ldots, N-1;$

(c) there exist numbers $f_0, g_0, h_0 \geq 0,\ s > 1,\ \beta \geq s - 2$ such that

$$|f(x)| \leq f_0|x|^\beta, \quad |g(x)| \leq g_0|x|^{s-1}, \quad |h_k(x)| \leq h_0|x|^{s-1}, \quad k = 1, \ldots, N-1.$$

Our main result is the following theorem. Let

$$(1.3) \qquad \lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2},$$

where ϑ is the smallest positive eigenvalue of the problem (EVP) (see Subsection 2.2).

THEOREM 1.2. *Let u be a weak solution of problem (L), and assume that conditions (a)–(c) are satisfied. Let G and the parameters in (a)–(c) be such that $\lambda > 1$. Then there are $d \in (0,1)$ and constants $C_0, c > 0$ depending only on $n, a_*, a^*, p^*, \lambda, q, \omega_0, f_0, h_0, g_0, \beta_0, \gamma_0, s, M_0, \operatorname{meas} G, \operatorname{diam} G$ such that for all $x \in G_0^d$,*

$$(1.4) \qquad |u(x)| \leq C_0\left(\|u\|_{2,G} + f_0 + \frac{1}{\sqrt{\gamma_0}}\, g_0 + \frac{1}{\sqrt{\beta_0}}\, h_0\right)$$

$$\times \begin{cases} |x|^\lambda & \text{if } s > \lambda, \\ |x|^\lambda \ln^c(1/|x|) & \text{if } s = \lambda, \\ |x|^s & \text{if } s < \lambda. \end{cases}$$

Suppose, in addition, that

$$\gamma \in \mathbf{C}^1(\partial G), \quad f \in \mathbf{V}^0_{q,2q-n}(G), \quad h \in V^{1-1/q}_{q,2q-n}(\Sigma),$$

$$g \in \mathbf{V}^{1-1/q}_{q,2q-n}(\partial G), \quad q > n,$$

and there is a number

(1.5) $$\tau_s := \sup_{\varrho > 0} \varrho^{-s} (\|h\|_{\mathbf{V}^{1-1/q}_{q,2q-n}(\Sigma^\varrho_{\varrho/2})} + \|g\|_{V^{1-1/q}_{q,2q-n}(\Gamma^\varrho_{\varrho/2})}).$$

Then for all $x \in G^d_0$,

(1.6) $$|\nabla u(x)| \le C_1 \left(\|u\|_{2,G} + f_0 + \frac{1}{\sqrt{\gamma_0}} g_0 + \frac{1}{\sqrt{\beta_0}} h_0 + \tau_s \right)$$

$$\times \begin{cases} |x|^{\lambda-1} & \text{if } s > \lambda, \\ |x|^{\lambda-1} \ln^c(1/|x|) & \text{if } s = \lambda, \\ |x|^{s-1} & \text{if } s < \lambda. \end{cases}$$

Furthermore,

- $u \in \mathbf{V}^2_{q,2q-n}(G)$, $q > n$ *and*

(1.7) $$\|u\|_{\mathbf{V}^2_{q,2q-n}(G^\varrho_0)} \le C_2 \left(\|u\|_{2,G} + f_0 + \frac{1}{\sqrt{\gamma_0}} g_0 + \frac{1}{\sqrt{\beta_0}} h_0 + \tau_s \right)$$

$$\times \begin{cases} \varrho^\lambda & \text{if } s > \lambda, \\ \varrho^\lambda \ln^c(1/\varrho) & \text{if } s = \lambda, \\ \varrho^s & \text{if } s < \lambda; \end{cases}$$

- *if $f \in \mathring{\mathbf{W}}^0_\alpha(G)$, $\int_\Sigma r^{\alpha-1} h^2(x)\, ds < \infty$, $\int_{\partial G} r^{\alpha-1} g^2(x)\, ds < \infty$, where*

(1.8) $$4 - n - 2\lambda < \alpha \le 2,$$

  then $u \in \mathring{\mathbf{W}}^1_{\alpha-2}(G)$ and

(1.9) $$\int_G a(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\, dx + \int_\Sigma r^{\alpha-3}\beta(\phi)u^2(x)\, ds$$

$$+ \int_{\partial G} \alpha(x) r^{\alpha-3}\gamma(\phi)u^2(x)\, ds$$

$$\le C \Big\{ \int_G (u^2 + (1 + r^\alpha)f^2(x))\, dx + \int_\Sigma r^{\alpha-1}h^2(x)\, ds$$

$$+ \int_{\partial G} \alpha(x) r^{\alpha-1}g^2(x)\, ds \Big\},$$

  where the constant $C > 0$ depends only on q, n, a_*, a^*, α, λ and the domain G.

2. Preliminaries

2.1. *Auxiliary formulae.* Let us recall some well known formulae relating to spherical coordinates $(r, \omega_1, \ldots, \omega_{n-1})$ centered at the conical point $\mathcal{O}$:

- $dx = r^{n-1} dr d\Omega$,
- $d\Omega_\varrho = \varrho^{n-1} d\Omega$,
- $d\Omega = J(\phi) d\phi$ denotes the $(n-1)$-dimensional area element of the unit sphere,
- $J(\phi) = \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 \ldots \sin \phi_{n-2}$,
- $d\phi = d\phi_1 \ldots d\phi_{n-1}$,
- ds denotes the $(n-1)$-dimensional area element on ∂G;
- $d\sigma$ denotes the $(n-2)$-dimensional area element on $\partial \Omega$;
- $ds = r^{n-2} dr d\sigma$;
- $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} |\nabla_\phi u|^2$, where $|\nabla_\phi u|$ is the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ϕ,
- $|\nabla_\phi u|^2 = \sum_{i=1}^{n-1} \frac{1}{q_i} \left(\frac{\partial u}{\partial \phi_i}\right)^2$, where $q_1 = 1$ and $q_i = (\sin \phi_1 \cdots \sin \phi_{i-1})^2$ for $i \geq 2$,
- $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_\phi u$,
- $\Delta_\phi u = \frac{1}{J(\phi)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \phi_i} \left(\frac{J(\phi)}{q_i} \frac{\partial u}{\partial \phi_i}\right)$, the Laplace–Beltrami operator,
- $\operatorname{div}_\phi \vec{u} = \frac{1}{J(\phi)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \phi_i} \left(\frac{J(\phi)}{q_i} u_i\right)$.

$C = C(\ldots)$, $c = c(\ldots)$ denote constants depending only on the quantities appearing in parentheses. Hereafter, the same letters C, c will (generally) be used to denote different constants depending on the same set of arguments.

By a direct calculation we obtain

LEMMA 2.1.

$$(2.1) \qquad \begin{aligned} & x_i \cos(\vec{n}, x_i)|_{\Gamma_0^d} = 0, \qquad \cos(\vec{n}, x_1)|_{\Gamma_0^d} = -\sin \frac{\omega_0}{2}, \\ & x_i \cos(\vec{n}_k, x_i)|_{\Sigma_k} = 0, \qquad k = 1, \ldots, N-1 \end{aligned}$$

(*the latter is the equation of* Σ_k).

2.2. *Auxiliary inequalities.* We need some statements and inequalities.

The eigenvalue problem. Let $\Omega \subset S^{n-1}$ with smooth boundary $\partial \Omega$ be the intersection of the cone $\mathcal{C}$ with S^{n-1}. Let $\vec{\nu}$ be the exterior normal to $\partial \mathcal{C}$ at points of $\partial \Omega$ and $\vec{\tau}_k$ be the exterior normal to Σ_k with respect to Ω_k (lying in the tangent plane to Ω_k), $k = 1, \ldots, N-1$. Let $\gamma(\phi)$, $\phi \in \partial \Omega$, be a positive bounded piecewise smooth function, and $\beta_k(\phi)$ be a positive continuous function on Σ_k, $k = 1, \ldots, N-1$. We consider the eigenvalue

problem for the Laplace–Beltrami operator Δ_ϕ on the unit sphere

(EVP) $\begin{cases} a_i(\Delta_\phi\psi_i + \vartheta\psi_i) = 0, \quad \phi \in \Omega_i, \ a_i \text{ are positive constants; } i = 1, \ldots, N, \\[2mm] [\psi]_{\sigma_k} = 0, \quad \left[a\dfrac{\partial\psi}{\partial\vec{\tau}_k}\right]_{\sigma_k} + \beta_k(\phi)\psi|_{\sigma_k} = 0, \quad k = 1, \ldots, N-1, \\[2mm] \alpha(\phi)a\dfrac{\partial\psi}{\partial\vec{\nu}} + \gamma(\phi)\psi|_{\partial\Omega} = 0, \end{cases}$

which is to determine all values ϑ (eigenvalues) for which (EVP) has nonzero weak solutions (eigenfunctions).

From the variational principle we obtain the *Friedrichs–Wirtinger inequality* (the proof is analogous to one in §2.4 of [3]):

THEOREM 2.2. *Let ϑ be the smallest positive eigenvalue of problem* (EVP). *Let $\Omega \subset S^{n-1}$. Let $\psi \in \mathbf{W}^1(\Omega)$ satisfy the boundary and conjunction conditions from* (EVP) *in the weak sense. Let $\gamma(\phi)$ be a positive bounded piecewise smooth function on $\partial\Omega$, and $\beta_k(\phi)$ be a positive continuous function on σ_k, $k = 1, \ldots, N-1$. Then*

$$(2.2) \qquad \vartheta \int\limits_\Omega a\psi^2(\phi)\,d\Omega \le \int\limits_\Omega a|\nabla_\phi\psi|^2\,d\Omega + \sum_{k=1}^{N-1} \int\limits_{\sigma_k} \beta_k(\phi)\psi^2(\phi)\,d\sigma$$
$$+ \int\limits_{\partial\Omega} \alpha(\phi)\gamma(\phi)\psi^2(\phi)\,d\sigma.$$

Because of (1.3), the Friedrichs–Wirtinger inequality will be written in the following form:

$$(2.3) \qquad \lambda(\lambda + n - 2) \int\limits_\Omega a\psi^2(\phi)\,d\Omega \le \int\limits_\Omega a|\nabla_\phi\psi|^2\,d\Omega + \int\limits_\sigma \beta(\phi)\psi^2(\phi)\,d\sigma$$
$$+ \int\limits_{\partial\Omega} \alpha(\phi)\gamma(\phi)\psi^2(\phi)\,d\sigma$$

for all $\psi(\phi) \in \mathbf{W}^1(\Omega)$ satisfying the boundary and conjunction conditions from (EVP) in the weak sense. Multiplying (2.3) by $r^{n-5+\alpha}$ and integrating over $r \in (0, d)$ we obtain

COROLLARY 2.3.

$$(2.4) \qquad \int\limits_{G_0^d} ar^{\alpha-4}u^2\,dx \le \frac{1}{\lambda(\lambda+n-2)}\left\{ \int\limits_{G_0^d} ar^{\alpha-2}|\nabla u|^2\,dx + \int\limits_{\Sigma_0^d} r^{\alpha-3}\beta(\phi)u^2(x)\,ds \right.$$
$$\left. + \int\limits_{\Gamma_0^d} \alpha(x)r^{\alpha-3}\gamma(\phi)u^2(x)\,ds \right\}, \qquad \forall\alpha,$$

for all $u \in \mathring{\mathbf{W}}^1_{\alpha-2}(G_0^d)$ such that $u(\cdot, \omega)$ satisfies the boundary and conjunction conditions from (EVP) in the weak sense.

Now we use the Hardy inequality (see [8, Theorem 330]) to get:

PROPOSITION 2.4 (Hardy–Friedrichs–Wirtinger inequality). *Let* $u \in \mathbf{C}^0(\overline{G}) \cap \mathring{\mathbf{W}}_{\alpha-2}^1(G)$ *with* $u(0) = 0$ *and suppose* $u(\cdot, \omega)$ *satisfies the boundary and conjunction conditions from* (EVP) *in the weak sense. Let* $\beta_k(\phi), \gamma(\phi)$ *be positive bounded piecewise smooth functions. Then*

$$(2.5) \qquad \int\limits_{G_0^d} ar^{\alpha-4}u^2 \, dx \leq H(\lambda, n, \alpha) \Big\{ \int\limits_{G_0^d} ar^{\alpha-2}|\nabla u|^2 \, dx$$

$$+ \int\limits_{\Sigma_0^d} r^{\alpha-3}\beta(\phi)u^2(x) \, ds + \int\limits_{\Gamma_0^d} r^{\alpha-3}\alpha(x)\gamma(\phi)u^2(x) \, ds \Big\},$$

$$H(\lambda, n, \alpha) = \frac{1}{\lambda(\lambda + n - 2) + \frac{1}{4}(4 - n - \alpha)^2}, \qquad \alpha \leq 4 - n.$$

The proof is similar to that in [3, §2.5.2].

LEMMA 2.5. *Let* G_0^d *be a conical domain and* $\nabla u(\varrho, \cdot) \in \mathbf{L}_2(\Omega)$ *a.e.* $\varrho \in (0, d)$. *Assume that*

$$(2.6) \qquad U(\varrho) = \int\limits_{G_o^\varrho} ar^{2-n}|\nabla u|^2 \, dx + \int\limits_{\Sigma_0^\varrho} r^{1-n}\beta(\phi)u^2(x) \, ds$$

$$+ \int\limits_{\Gamma_0^\varrho} r^{1-n}\gamma(\phi)u^2(x) \, ds < \infty, \qquad \varrho \in (0, d).$$

Then

$$(2.7) \qquad \int\limits_{\Omega} a\left(\varrho u \, \frac{\partial u}{\partial r} + \frac{n-2}{2} \, u^2 \right) \bigg|_{r=\varrho} d\Omega \leq \frac{\varrho}{2\lambda} \, U'(\varrho).$$

Proof. Writing $U(\varrho)$ in spherical coordinates,

$$U(\varrho) = \int\limits_0^\varrho r^{2-n} \Big(\int\limits_\Omega a|\nabla u|^2 \, d\Omega \Big) r^{n-1} \, dr$$

$$+ \int\limits_0^\varrho r^{1-n} \Big(\int\limits_\Sigma \beta(\phi)|u|^2 \, d\sigma + \int\limits_{\partial\Omega} \gamma(\phi)|u|^2 \, d\sigma \Big) r^{n-2} \, dr$$

$$= \int\limits_0^\varrho r \int\limits_\Omega a\Big(u_r^2 + \frac{1}{r^2} \, |\nabla_\phi u|^2 \Big) d\Omega \, dr$$

$$+ \int\limits_0^\varrho \frac{1}{r} \Big(\int\limits_\Sigma \beta(\phi)|u|^2 \, d\sigma + \int\limits_{\partial\Omega} \gamma(\phi)|u|^2 \, d\sigma \Big) dr$$

and differentiating with respect to ϱ we obtain

$$(2.8) \qquad U'(\varrho) = \int\limits_{\Omega} a \left(\varrho \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{\varrho} |\nabla_{\phi} u|^2 \right) \Big|_{r=\varrho} d\Omega$$

$$+ \frac{1}{\varrho} \left(\int\limits_{\Sigma} \beta(\phi) u^2(\varrho, \phi) \, d\sigma + \int\limits_{\partial\Omega} \gamma(\phi) u^2(\varrho, \phi) \, d\sigma \right).$$

Moreover, by Cauchy's inequality, we have $\varrho u \frac{\partial u}{\partial r} \le \frac{\varepsilon}{2} u^2 + \frac{1}{2\varepsilon} \varrho^2 \left(\frac{\partial u}{\partial r} \right)^2$ for all $\varepsilon > 0$. Then

$$\int\limits_{\Omega} a \left(\varrho u \frac{\partial u}{\partial r} + \frac{n-2}{2} u^2 \right) \Big|_{r=\varrho} d\Omega \le \frac{\varepsilon + n - 2}{2} \int\limits_{\Omega} a u^2 \, d\Omega + \frac{\varrho^2}{2\varepsilon} \int\limits_{\Omega} a \left(\frac{\partial u}{\partial r} \right)^2 d\Omega.$$

Thus choosing $\varepsilon = \lambda$ we obtain, by the Friedrichs–Wirtinger inequality (2.3),

$$\int\limits_{\Omega} a \left(\varrho u \frac{\partial u}{\partial r} + \frac{n-2}{2} u^2 \right) \Big|_{r=\varrho} d\Omega$$

$$\le \frac{\varepsilon + n - 2}{2\lambda(\lambda + n - 2)} \int\limits_{\Omega} a |\nabla_{\phi} u|^2 \, d\Omega + \frac{\varrho^2}{2\varepsilon} \int\limits_{\Omega} a \left(\frac{\partial u}{\partial r} \right)^2 d\Omega$$

$$+ \frac{\varepsilon + n - 2}{2\lambda(\lambda + n - 2)} \left(\int\limits_{\Sigma} \beta(\phi) u^2(\varrho, \phi) \, d\sigma + \int\limits_{\partial\Omega} \gamma(\phi) u^2(\varrho, \phi) \, d\sigma \right)$$

$$= \frac{\varrho}{2\lambda} U'(\varrho). \quad \blacksquare$$

We also need the well known inequalities

$$(2.9) \qquad \int\limits_{\Gamma} v \, ds \le C \int\limits_{G} (|v| + |\nabla v|) \, dx, \qquad \forall v \in \mathbf{W}^{1,1}(G), \, \forall \Gamma \subseteq \partial G,$$

$$(2.10) \qquad \int\limits_{\partial G} v^2 \, ds \le \int\limits_{G} \left(\delta |\nabla v|^2 + \frac{1}{\delta} c_0 v^2 \right) dx, \quad \forall v \in \mathbf{W}^{1,2}(G), \, \forall \delta > 0.$$

2.3. *Quasi-distance $r_{\varepsilon}(x)$.* Further, we define a function $r_{\varepsilon}(x)$ as follows. We fix $Q = (-1, 0, \dots, 0) \in S^{n-1} \setminus \overline{\Omega}$ and consider the unit radius-vector $\vec{l} = \mathcal{O}Q = \{-1, 0, \dots, 0\}$. We denote by $\vec{r}$ the radius-vector of the point $x \in \overline{G}$ and introduce the vector $\vec{r}_{\varepsilon} = \vec{r} - \varepsilon \vec{l}$ for $\varepsilon > 0$. Since $\varepsilon \vec{l} \notin G_0^d$ for all $\varepsilon \in (0, d)$, it follows that $r_{\varepsilon}(x) = |\vec{r} - \varepsilon \vec{l}| \ne 0$ for all $x \in \overline{G}$. It is easy to see that $r_{\varepsilon}(x)$ has the following properties:

1. *There exists $h > 0$ such that $r_{\varepsilon}(x) \ge hr$ and $r_{\varepsilon}(x) \ge h\varepsilon$ for all $x \in \overline{G}$, where*

$$h = \begin{cases} 1 & \text{if } x_1 \ge 0, \\ \sin(\omega_0/2) & \text{if } x_1 < 0. \end{cases}$$

Proof. From the definition of $r_\varepsilon(x)$ we know that

$$r_\varepsilon^2 = (x_1 + \varepsilon)^2 + \sum_{i=2}^{n} x_i^2 = (x_1 + \varepsilon)^2 + r^2 - x_1^2 = r^2 + 2\varepsilon x_1 + \varepsilon^2.$$

If $x_1 \geq 0$ we obtain either $r_\varepsilon^2 \geq r^2$, so $r_\varepsilon \geq r$, or $r_\varepsilon^2 \geq \varepsilon^2$, so $r_\varepsilon \geq \varepsilon$.

If $x_1 = r \cos \phi \leq 0$, $\phi \in [\pi/2, \omega_0/2]$, we obtain by the Cauchy inequality: either

$$|2\varepsilon r \cos \phi| \leq r^2 \cos^2 \phi + \varepsilon^2 \Rightarrow 2\varepsilon r \cos \phi \geq -r^2 \cos^2 \phi - \varepsilon^2 \Rightarrow r_\varepsilon \geq r \sin(\omega_0/2)$$

or

$$|2\varepsilon r \cos \phi| \leq \varepsilon^2 \cos^2 \phi + r^2 \Rightarrow 2\varepsilon r \cos \phi \geq -r^2 - \varepsilon^2 \cos^2 \phi \Rightarrow r_\varepsilon \geq \varepsilon \sin(\omega_0/2). \blacksquare$$

2. If $x \in G_d$, then $r_\varepsilon(x) \geq d/2$ for all $\varepsilon \in (0, d/2)$,

3. $\lim_{\varepsilon \to 0^+} r_\varepsilon(x) = r$ for all $x \in \overline{G}$.

4. $|\nabla r_\varepsilon|^2 = 1$ and $\Delta r_\varepsilon = (n-1)/r_\varepsilon$.

LEMMA 2.6. *Let $v \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$ with $v(0) = 0$ and suppose $v(\cdot, \omega)$ satisfies the boundary and conjunction conditions from* (EVP) *in the weak sense. Let $\beta(\phi), \gamma(\phi) > 0$. Then for any $\varepsilon > 0$,*

$$(2.11) \qquad \int_{G_0^d} a r_\varepsilon^{\alpha-4} v^2 \, dx \leq H(\lambda, n, \alpha) \left\{ \int_{G_0^d} a r_\varepsilon^{\alpha-2} |\nabla v|^2 \, dx \right.$$

$$\left. + \int_{\Sigma_0^d} r_\varepsilon^{\alpha-3} \beta(\phi) v^2(x) \, ds + \int_{\Gamma_0^d} r_\varepsilon^{\alpha-3} \alpha(x) \gamma(\phi) v^2(x) \, ds \right\},$$

where $H(\lambda, n, \alpha)$ is defined by (2.5).

LEMMA 2.7. *Let $v \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$ and suppose that $v(\cdot, \omega)$ satisfies the boundary and conjunction conditions from* (EVP) *in the weak sense. Let $\beta(\phi), \gamma(\phi) > 0$. Then for any $\varepsilon > 0$,*

$$(2.12) \qquad \int_{G_0^d} a r_\varepsilon^{\alpha-2} r^{-2} v^2 \, dx \leq \frac{1}{\lambda(\lambda+n-2)} \left\{ \int_{G_0^d} a r_\varepsilon^{\alpha-2} |\nabla v|^2 \, dx \right.$$

$$\left. + \int_{\Sigma_0^d} r^{-1} r_\varepsilon^{\alpha-2} \beta(\phi) v^2(x) \, ds + \int_{\Gamma_0^d} r^{-1} r_\varepsilon^{\alpha-2} \alpha(x) \gamma(\phi) v^2 \, ds \right\}.$$

For the proof of these lemmas see Lemmas 2.37 and 2.32 in [3].

2.4. *The Cauchy problem for a differential inequality*

THEOREM 2.8. *Let U be an increasing, nonnegative differentiable function defined on $[0, 2d]$ and satisfying*

$$(\text{CP}) \qquad \begin{cases} U'(\varrho) - \mathcal{P}(\varrho) U(\varrho) + \mathcal{N}(\varrho) U(2\varrho) + \mathcal{Q}(\varrho) \geq 0, & 0 < \varrho < d, \\ U(d) \leq U_0, \end{cases}$$

where $\mathcal{P}(\varrho)$, $\mathcal{N}(\varrho)$, $\mathcal{Q}(\varrho)$ *are nonnegative continuous functions defined on* $[0, 2d]$ *and* U_0 *is a constant. Then*

$$(2.13) \qquad U(\varrho) \leq \exp\Big(\int\limits_{\varrho}^{d} \mathcal{B}(\tau) \, d\tau\Big) \Big\{ U_0 \exp\Big(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau) \, d\tau\Big)$$

$$+ \int\limits_{\varrho}^{d} \mathcal{Q}(\tau) \exp\Big(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma) \, d\sigma\Big) \, d\tau \Big\}$$

with

$$(2.14) \qquad \mathcal{B}(\varrho) = \mathcal{N}(\varrho) \exp\Big(\int\limits_{\varrho}^{2\varrho} \mathcal{P}(\sigma) \, d\sigma\Big).$$

Proof. See [3, §1.10, Theorem 1.57]. ∎

2.5. *The comparison principle.* We consider the second order linear degenerate operator Q of the form

$$(2.15) \qquad Q(u, \eta) \equiv \int\limits_{G_0^d} \{a u_{x_i} \eta_{x_i} + (f(x) + pu(x))\eta(x)\} \, dx - \int\limits_{\Omega_d} a \frac{\partial u}{\partial r} \eta(x) \, d\Omega_d$$

$$- \int\limits_{\Gamma_0^d} \alpha(x)\Big(g(x) - \frac{1}{r}\gamma(\phi)u(x)\Big)\eta(x) \, ds - \int\limits_{\Sigma_0^d} \Big(h(x) - \frac{1}{r}\beta(\phi)u(x)\Big)\eta(x) \, ds$$

for $u \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G_0^d)$ and nonnegative $\eta \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G_0^d)$.

PROPOSITION 2.9. *Let* $\beta(\phi)$, $\gamma(\phi)$ *be positive piecewise smooth functions,* $0 < a_* \leq a \leq a^*$, $p \geq 0$ *and* $d \ll 1$. *Let* $u, w \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G_0^d)$ *satisfy*

$$(2.16) \qquad Q(u, \eta) \leq Q(w, \eta)$$

for all nonnegative $\eta \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G_0^d)$ *and also*

$$(2.17) \qquad u(x) \leq w(x), \qquad x \in \Omega_d \cup (\Gamma_0^d \cap \mathcal{D}),$$

in the weak sense. Then $u(x) \leq w(x)$ *in* G_0^d.

Proof. Define $z(x) = u(x) - w(x)$. Then

$$(2.18) \qquad 0 \geq Q(u, \eta) - Q(w, \eta) = \int\limits_{G_0^d} \{a z_{x_i} \eta_{x_i} + pz(x)\eta(x)\} \, dx$$

$$+ \int\limits_{\Gamma_0^d} \alpha(x) \frac{1}{r} \gamma(\phi) z(x) \eta(x) \, ds + \int\limits_{\Sigma_0^d} \frac{1}{r} \beta(\phi) z(x) \eta(x) \, ds$$

$$- \int\limits_{\Omega_d} a \frac{\partial z}{\partial r} \eta(x) \, d\Omega_d$$

for all nonnegative $\eta \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G_0^d)$. Define

$$(G_0^d)^+ := \{x \in G_0^d \mid u(x) > w(x)\} \subset G_0^d,$$

$$(\Sigma_0^d)^+ := \{x \in \Sigma_0^d \mid u(x) > w(x)\} \subset \Sigma_0^d,$$

$$(\Gamma_0^d)^+ := \{x \in \Gamma_0^d \mid u(x) > w(x)\} \subset \Gamma_0^d,$$

and assume that $(G_0^d)^+ \neq \emptyset$. As a test function in the integral inequality (2.18), we choose $\eta = \max\{(u - w), 0\}$. Then it follows from (2.17) and (2.18) that

$$\int\limits_{(G_0^d)^+} (a|\nabla z|^2 + pz^2(x)) \, dx + \int\limits_{(\Gamma_0^d)^+} \alpha(x) \frac{1}{r} \gamma(\phi) z^2(x) \, ds + \int\limits_{(\Sigma_0^d)^+} \frac{1}{r} \beta(\phi) z^2(x) \, ds \leq 0.$$

This implies that $z(x) = 0$ almost everywhere in $(G_0^d)^+$, contrary to our assumption that $(G_0^d)^+ \neq \emptyset$. Thus Proposition 2.9 is proved. ∎

3. The barrier function. A preliminary estimate of the modulus of the solution. Consider the linear operators $\mathcal{L}_i \equiv a_i \Delta$, $i = 1, \ldots, N$, with the numbers (1.2).

LEMMA 3.1 (Existence of the barrier function). *Fix numbers $\beta_0, \gamma_0 > \widetilde{a} \tan(\omega_0/2)$, $\delta > 0$, $g_0, h_0 \geq 0$ and let $\gamma(\phi) \geq \gamma_0$ on ∂G, $\beta_k(\phi) \geq \beta_0$ on Σ_k, $k = 1, \ldots, N - 1$. There exists $m > 0$, depending only on ω_0, a number $\varkappa_0 \in (0, (\delta_0/\widetilde{a}) \cot(\omega_0/2) - 1)$ (where $\delta_0 = \min(\beta_0, \gamma_0)$), numbers $B > 0$ and $d \in (0, 1)$ and a function $w \in \mathbf{C}^1(\overline{G}_0) \cap \mathbf{C}^2(G_0)$ that depend only on ω_0, the ellipticity constants a_*, a^* of the operators $\mathcal{L}_i$ and the quantities $\gamma_0, \beta_0, \delta_0, g_0, h_0, \omega_0$ such that for any $\varkappa \in (0, \min(\delta, \varkappa_0))$ the following hold:*

$$(3.1) \qquad \mathcal{L}[w(x)] \leq -a_* m^2 |x|^{\varkappa - 1}, \quad x \in G_0^d \cap G_\pm;$$

$$(3.2) \qquad \mathcal{B}[w(x)] \geq g_0 |x|^\delta, \quad x \in \Gamma_0^d;$$

$$(3.3) \qquad \mathcal{S}_k[w(x)] \geq h_0 |x|^\delta, \quad x \in \Sigma_k^d, \ k = 1, \ldots, N - 1;$$

$$(3.4) \qquad 0 \leq w(x) \leq c_0(\varkappa_0, B, \omega_0)|x|^{\varkappa + 1}, \quad x \in \overline{G_0^d};$$

$$(3.5) \qquad |\nabla w(x)| \leq c_1(\varkappa_0, B, \omega_0)|x|^\varkappa, \quad x \in \overline{G_0^d}.$$

Proof. Let $(x, y, x') \in \mathbb{R}^n$, where $x = x_1$, $y = x_2$, $x' = (x_3, \ldots, x_n)$. In $\{x_1 \geq 0\}$ we consider the cone K with vertex $\mathcal{O}$ such that $K \supset G_0^d$ (we recall that $G_0^d \subset \{x_1 \geq 0\}$). Let ∂K be the lateral surface of K and let $\partial K \cap y\mathcal{O}x = \Gamma_\pm$ be $x = \pm my$, where $m = \cot(\omega_0/2)$, $0 < \omega_0 < \pi$, such that $x > m|y|$ in the interior of K. We shall consider the function

$$(3.6) \quad w(x; y, x') \equiv x^{\varkappa - 1}(x^2 - m^2 y^2) + Bx^{\varkappa + 1}$$

with some $\varkappa \in (0, 1)$, $B > 0$.

Let us calculate the action of $\mathcal{L}_i$ on the function (3.6). For $t = y/x$, $|t| < 1/h$ we obtain $\mathcal{L}_i w = -m^2 x^{\varkappa-1}\varsigma_i(\varkappa)$, where $\varsigma_i(\varkappa) = a_i\{(\varkappa^2 - 3\varkappa + 2)t^2 + 2 - m^{-2}(1+B)(\varkappa^2 + \varkappa)\}$. Since $\varsigma_i(0) = 2a_i(t^2+1) \geq 2a_i \geq 2a_*$ and since $\varsigma_i(\varkappa)$ are quadratic polynomials there exists a number $\varkappa_0 > 0$ depending only on a_*, a^*, m such that $\varsigma_i(\varkappa) \geq a_*$ for $\varkappa \in [0, \varkappa_0]$. Therefore we obtain (3.1). Now, notice that

$$(3.7) \qquad \Gamma_\pm: \quad x = \pm my, \quad m = \cot(\omega_0/2), \ 0 < \omega_0 < \pi.$$

Then

$$(3.8) \qquad \text{on } \Gamma_+: \quad \begin{cases} x = r\cos(\omega_0/2), \\ y = r\sin(\omega_0/2), \end{cases} \quad \begin{cases} \angle(\vec{n}, x) = \pi/2 + \omega_0/2, \\ \angle(\vec{n}, y) = \omega_0/2, \end{cases}$$

$$(3.9) \qquad \text{on } \Gamma_-: \quad \begin{cases} x = r\cos(\omega_0/2), \\ y = -r\sin(\omega_0/2), \end{cases} \quad \begin{cases} \angle(\vec{n}, x) = \pi/2 + \omega_0/2, \\ \angle(\vec{n}, y) = \pi + \omega_0/2, \end{cases}$$

$$(3.10) \qquad \sin\frac{\omega_0}{2} = \frac{1}{\sqrt{1+m^2}}, \quad \cos\frac{\omega_0}{2} = \frac{m}{\sqrt{1+m^2}}.$$

Therefore

$$
\begin{aligned}
&w_x = (1+\varkappa)x^\varkappa(1+B) - (\varkappa - 1)m^2 y^2 x^{\varkappa-2} \\
(3.11) \qquad &\Rightarrow \ w_x\big|_{\Gamma_\pm} = [2 + B(1+\varkappa)]x^\varkappa, \\
&w_y = -2m^2 y x^{\varkappa-1} \ \Rightarrow \ w_y\big|_{\Gamma_\pm} = \mp 2mx^\varkappa.
\end{aligned}
$$

Because of $\frac{\partial w}{\partial \vec{n}}\big|_{\Gamma_\pm} = w_x \cos\angle(\vec{n}, x)\big|_{\Gamma_\pm} + w_y \cos\angle(\vec{n}, y)\big|_{\Gamma_\pm}$ and (3.8)–(3.11), we get

$$
\frac{\partial w}{\partial \vec{n}}\bigg|_{\Gamma_\pm} = -r^\varkappa \frac{m^\varkappa}{(1+m^2)^{(\varkappa+1)/2}}\{2(1+m^2) + B(1+\varkappa)\}.
$$

Hence, by (3.7) and $\gamma(\phi) \geq \gamma_0 > \widetilde{a}\tan(\omega_0/2)$ on ∂G, it follows that

$$
\begin{aligned}
\mathcal{B}[w]\big|_{\Gamma_\pm^d} &= \frac{m^\varkappa}{(1+m^2)^{(\varkappa+1)/2}} r^\varkappa\{Bm\gamma(\phi) - \alpha(x)a(B(1+\varkappa) + 2(1+m^2))\} \\
&\geq \frac{m^\varkappa}{(1+m^2)^{(\varkappa+1)/2}} r^\varkappa\{Bm\gamma_0 - Ba^*(1+\varkappa) - 2a^*(1+m^2)\}.
\end{aligned}
$$

Since $m > a^*/\gamma_0$ and $\varkappa_0 < m\gamma_0/a^* - 1$, for $\varkappa \leq \varkappa_0$ we obtain

$$(3.12) \quad \mathcal{B}[w]\big|_{\Gamma_\pm^d} \geq \frac{m^{\varkappa_0} r^\varkappa}{(1+m^2)^{(\varkappa_0+1)/2}}\{B(m\gamma_0 - a^* - a^*\varkappa_0) - 2a^*(1+m^2)\}$$

$$\geq g_0 r^\delta, \quad r \in (0, d),$$

if we choose $\varkappa \leq \delta$, so $r^\varkappa \geq r^\delta$, and

$$(3.13) \quad B \geq \left\{\frac{g_0(1+m^2)^{(\varkappa_0+1)/2}}{m^{\varkappa_0}} + 2a^*(1+m^2)\right\} \cdot \frac{1}{m\gamma_0 - a^*(1+\varkappa_0)}.$$

Thus, (3.2) is proved.

Now we prove (3.3). First, on Σ_k we have $x = r\cos(\omega_0/2 - \theta_k)$, $y = r\sin(\omega_0/2 - \theta_k)$, so $y = m_k x$, where $m_k = \tan(\omega_0/2 - \theta_k)$. Therefore

$$w|_{\Sigma_k} = (B + 1 - m^2 m_k^2)x^{1+\varkappa} = (B + 1 - m^2 m_k^2)r^{1+\varkappa}\cos^{1+\varkappa}(\omega_0/2 - \theta_k)$$

$$= \frac{B + 1 - m^2 m_k^2}{(1 + m_k^2)^{(1+\varkappa)/2}}r^{1+\varkappa}.$$

Further,

$$\cos(\vec{n}_k, x) = \cos(\pi/2 - \omega_0/2 + \theta_k) = \sin(\omega_0/2 - \theta_k) = \frac{m_k}{\sqrt{1 + m_k^2}},$$

$$\cos(\vec{n}_k, y) = \cos(\pi - \omega_0/2 + \theta_k) = -\cos(\omega_0/2 - \theta_k) = -\frac{1}{\sqrt{1 + m_k^2}}$$

on Σ_k, $k = 1, \ldots, N - 1$, and in virtue of

$$w_x|_{\Sigma_k} = \frac{(1+\varkappa)(1+B) + (1-\varkappa)m^2 m_k^2}{(1 + m_k^2)^{\varkappa/2}} \cdot r^\varkappa, \quad w_y|_{\Sigma_k} = -\frac{2m^2 m_k^2}{(1 + m_k^2)^{\varkappa/2}} \cdot r^\varkappa,$$

we obtain

$$\frac{\partial w}{\partial n_k}\bigg|_{\Sigma_k} = m_k \cdot \frac{(1+\varkappa)(1+B) + (1-\varkappa)m^2 m_k^2 + 2m^2}{(1 + m_k^2)^{(1+\varkappa)/2}} \cdot r^\varkappa.$$

Thus,

$$\mathcal{S}_k[w] =$$
$$\frac{\beta_k(\phi)(B+1-m^2 m_k^2) + m_k[a]_{\Sigma_k}\{(1+\varkappa)(1+B) + (1-\varkappa)m^2 m_k^2 + 2m^2\}}{(1+m_k^2)^{(1+\varkappa)/2}} \cdot r^\varkappa.$$

Since $0 < \omega_1 \le \theta_k < \omega_0$ for all $k = 1, \ldots, N-1$, we have $|m_k| < \tan(\omega_0/2) = 1/m$ for all $k = 1, \ldots, N - 1$. Now, as $\beta_k(\phi) \ge \beta_0 > \tilde{a}\tan(\omega_0/2)$ on Σ_k, $k = 1, \ldots, N - 1$, and $\varkappa_0 < m\beta_0/a_0 - 1$, for $\varkappa \le \varkappa_0$ we get

$$(3.14) \quad \mathcal{S}_k[w] \ge \{\beta_0 Bm - a_0((1+\varkappa_0)B + 2(1+m^2))\}\frac{m^{\varkappa_0}}{(1+m^2)^{(1+\varkappa_0)/2}} \cdot r^\varkappa$$

$$\ge r^\delta \cdot h_0,$$

if we choose $\varkappa \le \delta$, so $r^\varkappa \ge r^\delta$, and

$$(3.15) \quad B \ge \left\{\frac{h_0(1+m^2)^{(\varkappa_0+1)/2}}{m^{\varkappa_0}} + 2a_0(1+m^2)\right\} \cdot \frac{1}{m\beta_0 - a_0(1+\varkappa_0)}.$$

Thus, (3.3) is proved.

Now we show (3.4). Let us rewrite the function (3.6) in spherical coordinates. Recalling that $h = \cot(\omega_0/2)$, we obtain

$$w(x; y, x') = (1 + B)(r\cos\phi)^{1+\varkappa} - h^2 r^2 \sin^2\phi(r\cos\phi)^{\varkappa-1}$$

$$= r^{1+\varkappa}\cos^{\varkappa-1}\phi\left(B\cos^2\phi + \frac{\chi(\phi)}{\sin^2(\omega_0/2)}\right), \quad \forall\phi \in [-\omega_0/2, \omega_0/2],$$

where $\chi(\phi) = \sin(\omega_0/2 - \phi) \cdot \sin(\omega_0/2 + \phi)$. We find that $\chi'(\phi) = -\sin 2\phi$ and $\chi'(\phi) = 0$ for $\phi = 0$. Now we see that $\chi''(0) = -2\cos 0 = -2 < 0$. In this way we have

$$\max_{\phi\in[-\omega_0/2,\omega_0/2]}\chi(\phi) = \chi(0) = \sin^2(\omega_0/2)$$

and therefore

$$w(x; y, x') \le r^{1+\varkappa}\cos^{\varkappa-1}\phi\,(B\cos^2\phi + 1) \le r^{1+\varkappa}\cos^{\varkappa+1}\phi\left(B + \frac{1}{\cos^2\phi}\right)$$

$$\le r^{1+\varkappa_0}\left(B + \frac{1}{\cos^2(\omega_0/2)}\right).$$

Hence (3.4) follows. Finally, (3.5) follows in the same way, in virtue of (3.11). ∎

Now we can estimate $|u(x)|$ for u satisfying (L) in the neighborhood of the conical point.

THEOREM 3.2. *Let u be a weak solution of problem (L) and let u satisfy assumptions* (a)–(c). *Then there exist numbers $d \in (0,1)$ and $\varkappa > 0$ depending only on $a_*, n, \omega_0, f_0, h_0, g_0, \beta_0, \gamma_0, s$ and the domain G such that*

$$(3.16) \qquad\qquad |u(x) - u(0)| \le C_0|x|^{\varkappa+1}, \quad x \in G_0^d,$$

where the positive constant C_0 depends only on a_*, n, ω_0, f_0, h_0, g_0, β_0, γ_0, s, M_0 and the domain G, and does not depend on u.

Proof. Without loss of generality we may suppose that $u(0) \ge 0$. Consider the barrier function w defined by (3.6) with $\varkappa \in (0, \varkappa_0)$ and the function $v(x) = u(x) - u(0)$. For these, we shall verify Proposition 2.9. Let us calculate the action of Q on these functions. Because of the definition (2.15) and by integration by parts, we have

$$Q(Aw, \eta) \equiv \int\limits_{G_0^d} \{Aaw_{x_i}\eta_{x_i} + (f(x) + Apw(x))\eta(x)\}\,dx - \int\limits_{\Omega_d} Aa\,\frac{\partial w}{\partial r}\,\eta(x)\,d\Omega_d$$

$$- \int\limits_{\Gamma_0^d} \alpha(x)\left(g(x) - \frac{1}{r}\gamma(\phi)Aw(x)\right)\eta(x)\,ds$$

$$- \int\limits_{\Sigma_0^d} \left(h(x) - \frac{1}{r}\beta(\phi)Aw(x)\right)\eta(x)\,ds$$

$$= \int\limits_{G_0^d} (f(x) + Apw(x) - Aa\Delta w)\eta(x)\,dx$$

$$+ \int\limits_{\Gamma_0^d} (\mathcal{B}[Aw] - g(x))\eta(x)\,ds + \int\limits_{\Sigma_0^d} (\mathcal{S}[Aw] - h(x))\eta(x)\,ds$$

with any $A > 0$. Hence, by Lemma 3.1 and conditions (b), (c) we obtain

$$(3.17) \quad Q(Aw, \eta) \geq \int\limits_{G_0^d} \{f(x) + Aa_*m^2 r^{\varkappa-1}\}\eta(x)\,dx$$

$$+ \int\limits_{\Gamma_0^d} (Ag_0 r^\delta - g(x))\eta(x)\,ds + \int\limits_{\Sigma_0^d} (Ah_0 r^\delta - h(x))\eta(x)\,ds$$

$$\geq \int\limits_{G_0^d} r^{\varkappa-1}\{Aa_*m^2 - f_0 r^{\beta+1-\varkappa}\}\eta(x)\,dx$$

$$+ g_0\int\limits_{\Gamma_0^d} (Ar^\delta - r^{s-1})\eta(x)\,ds + h_0\int\limits_{\Sigma_0^d} (Ar^\delta - r^{s-1})\eta(x)\,ds$$

$$\geq 0$$

because of $0 < \varkappa < \varkappa_0$, if $\varkappa_0, \delta, A$ are chosen such that

$$(3.18) \qquad \varkappa_0 \leq \beta + 1, \quad \delta \leq s - 1, \quad A \geq \max\left\{1, \frac{f_0}{a_* m^2}\right\}.$$

On the other hand,

$$Q(v, \eta) \equiv -\int\limits_{G_0^d} pu(0)\eta(x)\,dx - \int\limits_{\Gamma_0^d} \frac{\gamma(\omega)}{r}\,u(0)\eta(x)\,ds - \int\limits_{\Sigma_0^d} \frac{\sigma(\omega)}{r}\,u(0)\eta(x)\,ds \leq 0$$

and thus

$$(3.19) \qquad\qquad Q(v, \eta) \leq Q(Aw, \eta)$$

for all nonnegative $\eta \in \mathbf{C}^0(\overline{G_0^d}) \cap \overset{\circ}{\mathbf{W}}{}^1_0(G_0^d)$.

Now we compare $v(x)$ and $w(x)$ on Ω_d. Since $x^2 \geq h^2 y^2$ in $\overline{K}$, from (3.6) we have

$$(3.20) \qquad\qquad w(x)|_{r=d} \geq Bd^{1+\varkappa}\cos^{\varkappa+1}(\omega_0/2).$$

On the other hand,

$$(3.21) \qquad\qquad v(x)|_{\Omega_d} = (u(x) - u(0))|_{\Omega_d} \leq M_0$$

and therefore from (3.20)–(3.21) we obtain

$$Aw(x)|_{\Omega_d} \geq ABd^{1+\varkappa}\cos^{\varkappa+1}(\omega_0/2) \geq ABd^{1+\varkappa}\frac{m^{1+\varkappa_0}}{(1+m^2)^{(1+\varkappa_0)/2}} \geq M_0$$

$$\geq v|_{\Omega_d},$$

if we choose A possibly greater:

$$(3.22) \qquad A \geq \frac{M_0(1+m^2)^{(1+\varkappa_0)/2}}{B(md)^{1+\varkappa_0}},$$

where B satisfies inequalities (3.13) and (3.15).

Finally, if $\Gamma_0^d \cap \mathcal{D} \neq \emptyset$ and $u(x) = g(x)$, $x \in \Gamma_0^d \cap \mathcal{D}$, where $|g(x)| \leq g_0|x|^s$, then

$$v(x) = u(x) - u(0) \leq u(x) = g(x) \leq g_0|x|^s, \quad x \in \Gamma_0^d \cap \mathcal{D};$$

$$Aw|_{\Gamma_0^d \cap \mathcal{D}} = ABr^{1+\varkappa}\cos^{\varkappa+1}\frac{\omega_0}{2} \geq ABr^{1+\varkappa}\frac{m^{1+\varkappa_0}}{(1+m^2)^{(1+\varkappa_0)/2}} \geq g_0 r^s$$

$$\geq v|_{\Gamma_0^d \cap \mathcal{D}},$$

if $\varkappa_0 \leq s - 1$ and if we increase A:

$$(3.23) \qquad A \geq \frac{(1+m^2)^{(1+\varkappa_0)/2}}{Bm^{1+\varkappa_0}}\, g_0,$$

where B satisfies inequalities (3.13) and (3.15). Thus, if we choose large numbers $B > 0$, $A \geq 1$ according to (3.13), (3.15), (3.18), (3.22), (3.23) we ensure the validity of Proposition 2.9.

Therefore, by that proposition,

$$(3.24) \qquad u(x) - u(0) \leq Aw(x), \quad x \in \overline{G_0^d}.$$

Similarly, we derive the estimate $u(x) - u(0) \geq -Aw(x)$, if we consider an auxiliary function $v(x) = u(0) - u(x)$. Our theorem is thus proved in virtue of (3.4). ∎

Now we will estimate the modulus of the gradient of the solution of problem (L) near a conical point.

THEOREM 3.3. *Let u be a weak solution of problem (L), and suppose that assumptions* (b)–(c) *are satisfied. Let $\varkappa > 0$ be defined by Lemma 3.1 with $\varkappa_0 \leq \beta + 1$. Then there exists a number $d > 0$ such that*

$$(3.25) \qquad |\nabla u(x)| < C_1|x|^{\varkappa}, \quad x \in G_0^d,$$

where the constant C_1 does not depend on u, but depends only on a_*, a^*, n, ω_0, f_0, h_0, g_0, β_0, γ_0, s, M_0 and the domain G.

Proof. Consider the set $G_{\varrho/2}^\varrho \subset G$, $0 < \varrho < d$. We make the transformation $x = \varrho x'$, $v(x') = \varrho^{-1-\varkappa}u(\varrho x')$. The function $v(x')$ satisfies

$$(L') \quad \begin{cases} a_i \Delta' v_i + p_i \varrho^2 v_i(x') = \varrho^{1-\varkappa} f_i(\varrho x'), & x' \in (G_i)^1_{1/2}, \ i = 1, \dots, N; \\[2mm] [v(x')]_{(\Sigma_k)^1_{1/2}} = 0, & k = 1, \dots, N-1; \\[2mm] \left[a \dfrac{\partial v}{\partial n'_k} \right]_{(\Sigma_k)^1_{1/2}} + \dfrac{1}{|x'|}\, \beta_k(\phi) v(x') = \varrho^{-\varkappa} h_k(\varrho x'), \\[2mm] \hspace{3cm} x' \in (\Sigma_k)^1_{1/2}, \ k = 1, \dots, N-1; \\[2mm] \alpha(x') \cdot a \dfrac{\partial v}{\partial n'} + \dfrac{1}{|x'|}\, \gamma(\phi) v(x') = \varrho^{-\varkappa} g(\varrho x'), & x' \in \Gamma^1_{1/2}. \end{cases}$$

Now we apply [10, Theorem 16.2, Chapter III] to get

$$(3.26) \qquad\qquad \max_{x' \in G^1_{1/2}} |\nabla' v(x')| \le M'_1.$$

Returning to the variable x and the function u we obtain, from (3.26),

$$|\nabla u(x)| \le M_1 \varrho^\varkappa, \quad x \in G^\varrho_{\varrho/2}, \ 0 < \varrho < d.$$

Putting now $|x| = \frac{2}{3}\varrho$ we obtain the desired estimate (3.25). ∎

COROLLARY 3.4. *Let u be a weak solution of problem (L) and suppose that the assumptions of Theorem 3.3 are satisfied. Then $u(0) = 0$ and therefore the inequality (3.16) takes the form*

$$(3.27) \qquad\qquad |u(x)| \le C_0 |x|^{\varkappa+1}, \quad x \in G^d_0.$$

Proof. From the boundary condition it follows that

$$\gamma(\phi) u(x) = |x| g(x) - \alpha(x) \cdot a |x| \frac{\partial u}{\partial n}, \quad x \in \partial G \setminus \mathcal{O}.$$

By the assumption (b)–(c) and the estimate (3.25), we obtain

$$\gamma_0 |u(x)| \le \gamma(\phi) |u(x)| \le |x|\,|g(x)| + a^* |x|\,|\nabla u| \le g_0 |x|^s + C_1 a^* |x|^{\varkappa+1}.$$

By letting $|x|$ tend to 0 we deduce, because of the continuity of u, that $\gamma_0 |u(0)| \le 0$ and taking into account $\gamma_0 > 0$, we find $u(0) = 0$. ∎

4. Local estimate at the boundary. We now derive the local boundedness (near the conical point) of a weak solution of (L) in case $\mathcal{D} = \emptyset$.

THEOREM 4.1. *Let u be a weak solution of (L). Let assumptions (a)–(c) be satisfied. Suppose, in addition, that $h \in \mathbf{L}_\infty(\Sigma_0)$ and $g \in L_\infty(\partial G)$. Then*

$$(4.1) \qquad \sup_{G^{\varkappa\varrho}_0} |u(x)| \le \frac{C}{(1-\varkappa)^{n/t}} \{ \varrho^{-n/t} \|u\|_{t, G^\varrho_0} + \varrho^{2(1-n/q)} \|f\|_{q/2, G^\varrho_0}$$

$$+ \varrho(\|g\|_{\infty, \Gamma^\varrho_0} + \|h\|_{\infty, \Sigma^\varrho_0}) \}$$

for any $t > 0$, $\varkappa \in (0,1)$ and $\varrho \in (0,d)$, where $C = C(n, a_*, a^*, t, q, G)$.

Proof. We can assume without loss of generality that $u \geq 0$ and $n \geq 3$ (for the general case see [7, §8.6] or [4, §1, Chapter 4]). We apply the Moser iteration method.

First we assume that $t \geq 2$. We consider the integral identity (II) and make the coordinate transformation $x = \varrho x'$. Let G' be the image of G, $\partial G'$ be the image of ∂G, and Σ'_k be the images of Σ_k, $k = 1, \ldots, N-1$. We have $dx = \varrho^n dx'$ and $ds = \varrho^{n-1} ds'$. In addition, we define

$$(4.2) \qquad \begin{aligned} v(x') &= u(\varrho x'), & \mathcal{F}(x') &= \varrho^2 f(\varrho x'), \\ \mathcal{G}(x') &= \varrho g(\varrho x'), & \mathcal{H}(x') &= \varrho h(\varrho x'). \end{aligned}$$

Then (II) means

$$(\mathrm{II})' \qquad \int\limits_{G'} \{a v_{x'_j} \eta_{x'_j} + p\varrho^2 v(x')\eta(x')\} \, dx' + \varrho \int\limits_{\Sigma'} \frac{1}{|x'|} \beta(\phi) v(x') \eta(x') \, ds'$$

$$+ \varrho \int\limits_{\partial G'} \frac{1}{|x'|} \gamma(\phi) v(x') \eta(x') \, ds'$$

$$= \int\limits_{\partial G'} \mathcal{G}(x')\eta(x') \, ds' + \int\limits_{\Sigma'_0} \mathcal{H}(x')\eta(x') \, ds' - \int\limits_{G'} \mathcal{F}(x')\eta(x') \, dx'$$

for all $\eta \in \mathbf{C}^0(\overline{G'}) \cap \mathring{\mathbf{W}}^1_0(G')$. Now define

$$(4.3) \qquad k = k(\varrho) = a_*^{-1}(\|\mathcal{F}\|_{q/2, G^1_0} + \|\mathcal{G}\|_{\infty, \Gamma^1_0} + \|\mathcal{H}\|_{\infty, \Sigma^1_0})$$

and set

$$(4.4) \qquad \overline{v}(x') = v(x') + k.$$

Observe that

$$(4.5) \qquad \begin{aligned} |\mathcal{F}|\overline{v} &= \frac{1}{k}\,|\mathcal{F}| \cdot k\overline{v} = \frac{1}{k}\,|\mathcal{F}|(\overline{v} - v) \cdot \overline{v} \\ &= \frac{1}{k}\,|\mathcal{F}| \cdot \overline{v}^2 - \frac{1}{k}\,|\mathcal{F}| \cdot v\overline{v} \leq \frac{1}{k}\,|\mathcal{F}| \cdot \overline{v}^2; \\ |\mathcal{H}|\overline{v} &\leq \frac{1}{k}\,|\mathcal{H}| \cdot \overline{v}^2; \quad |\mathcal{G}|\overline{v} \leq \frac{1}{k}\,|\mathcal{G}| \cdot \overline{v}^2 \end{aligned}$$

in the same way. As a test function in the integral identity $(\mathrm{II})'$ we choose

$$\eta(x') = \zeta^2(|x'|)\overline{v}^{t-1}(x'),$$

where $\zeta(|x'|) \in \mathbf{C}^\infty_0([0,1])$ is a nonnegative function to be further specified. By the chain and product rules, v is indeed a test function in $(\mathrm{II})'$ and also

$$\eta_{x'_i} = (t-1)\overline{v}^{t-2} v_{x'_i} \zeta^2(|x'|) + 2\zeta \zeta_{x'_i} \overline{v}^{t-1}(x'),$$

so that by substitution into (II)′, since $p \geq 0$ we obtain

$$(t-1)\int_{G_0^1} a|\nabla' v|^2 \overline{v}^{t-2}\zeta^2(|x'|)\,dx' + \varrho \int_{\Sigma_0^1} \frac{1}{|x'|}\,\beta(\phi)(\overline{v}(x')-k)\overline{v}^{t-1}(x')\zeta^2(|x'|)\,ds'$$

$$+ \varrho \int_{\Gamma_0^1} \frac{1}{|x'|}\,\gamma(\phi)(\overline{v}(x')-k)\overline{v}^{t-1}(x')\zeta^2(|x'|)\,ds'$$

$$\leq -2\int_{G_0^1} a(\varrho x')\zeta_{x_j'}v_{x_j'}\overline{v}^{t-1}\zeta(|x'|)\,dx' + \int_{\Gamma_0^1} \mathcal{G}(x')\overline{v}^{t-1}(x')\zeta^2(|x'|)\,ds'$$

$$+ \int_{\Sigma_0^1} \mathcal{H}(x')\overline{v}^{t-1}(x')\zeta^2(|x'|)\,ds' - \int_{G_0^1} \mathcal{F}(x')\overline{v}^{t-1}(x')\zeta^2(x')\,dx'.$$

By (c) and (4.5), it follows that

$$(4.6) \quad (t-1)\int_{G_0^1} a|\nabla' v|^2 \cdot \overline{v}^{t-2}\zeta^2(|x'|)\,dx'$$

$$\leq \int_{G_0^1} \left(2a^*|\nabla' v|\cdot|\nabla'\zeta|\cdot\overline{v}^{t-1}\zeta(|x'|) + \frac{1}{k}\,|\mathcal{F}(x')|\cdot\overline{v}^t\zeta^2(|x'|) \right)dx'$$

$$+ \frac{1}{k}\,\|\mathcal{G}\|_{\infty,\Gamma_0^1}\cdot\int_{\Gamma_0^1}\overline{v}^t(x')\zeta^2(|x'|)\,ds' + \frac{1}{k}\,\|\mathcal{H}\|_{\infty,\Sigma_0^1}\cdot\int_{\Sigma_0^1}\overline{v}^t(x')\zeta^2(|x'|)\,ds'.$$

By the Cauchy inequality, for any $\varepsilon > 0$,

$$2a^*|\nabla' v|\cdot|\nabla'\zeta|\cdot\overline{v}^{t-1}\zeta \leq 2(\sqrt{a}|\nabla' v|\cdot\overline{v}^{t/2-1}\zeta)\cdot\left(\frac{a^*}{\sqrt{a_*}}\,\overline{v}^{t/2}|\nabla'\zeta| \right)$$

$$\leq \varepsilon a|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2 + \frac{(a^*)^2}{\varepsilon a_*}\,\overline{v}^t|\nabla'\zeta|^2.$$

To estimate the boundary integrals we apply the inequality (2.10). Then from (4.6) it follows that

$$(4.7) \quad (t-1)\int_{G_0^1} a|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx' \leq \varepsilon \int_{G_0^1} a|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx'$$

$$+ \int_{G_0^1}\left(\frac{(a^*)^2}{\varepsilon a_*}\,\overline{v}^t\cdot|\nabla'\zeta|^2 + \frac{1}{k}\,|\mathcal{F}(x')|\cdot\overline{v}^t\zeta^2(|x'|) \right)dx'$$

$$+ \frac{1}{k}(\|\mathcal{G}\|_{\infty,\Gamma_0^1} + \|\mathcal{H}\|_{\infty,\Sigma_0^1})\cdot\int_{G_0^1}\left(\delta|\nabla'(\zeta\overline{v}^{t/2})|^2 + \frac{1}{\delta}\,c_0\overline{v}^t\zeta^2 \right)dx', \quad \forall\varepsilon,\delta > 0.$$

From

$$(4.8) \qquad |\nabla'(\zeta\overline{v}^{t/2})|^2 \le 2(\zeta^2|\nabla'(\overline{v}^{t/2})|^2 + \overline{v}^t|\nabla'\zeta|^2),$$
$$|\nabla'(\overline{v}^{t/2})|^2 = \frac{t^2}{4}\overline{v}^{t-2}|\nabla'v|^2$$

it follows that

$$(4.9) \qquad |\nabla'(\zeta\overline{v}^{t/2})|^2 \le \frac{t^2}{2}\overline{v}^{t-2}|\nabla'v|^2\zeta^2 + 2\overline{v}^t|\nabla'\zeta|^2.$$

Now, by (4.7), (4.9), choosing $\varepsilon = \frac{t-1}{2}$ we find that

$$(4.10) \qquad \frac{t-1}{2}\int\limits_{G_0^1} a|\nabla'v|^2 \cdot \overline{v}^{t-2}\zeta^2(|x'|)\, dx'$$

$$\le \frac{\delta t^2}{2a_*} \cdot \frac{\|\mathcal{G}\|_{\infty,\Gamma_0^1} + \|\mathcal{H}\|_{\infty,\Sigma_0^1}}{k} \cdot \int\limits_{G_0^1} a|\nabla'v|^2 \cdot \overline{v}^{t-2}\zeta^2(|x'|)\, dx'$$

$$+ \frac{1}{k}\int\limits_{G_0^1}\left\{|\mathcal{F}(x')| + \frac{c_0}{\delta}(\|\mathcal{G}\|_{\infty,\Gamma_0^1} + \|\mathcal{H}\|_{\infty,\Sigma_0^1})\right\} \cdot \overline{v}^t\zeta^2(|x'|)\, dx'$$

$$+ \int\limits_{G_0^1}\left\{\frac{2(a^*)^2}{(t-1)a_*} + \frac{2\delta}{k}(\|\mathcal{G}\|_{\infty,\Gamma_0^1} + \|\mathcal{H}\|_{\infty,\Sigma_0^1})\right\}\overline{v}^t \cdot |\nabla'\zeta|^2\, dx', \quad \forall\delta \in (0,1].$$

We now choose $\delta = \frac{t-1}{2t^2}$; by the definition of k in (4.3) the last inequality (4.10) may be rewritten as

$$\int\limits_{G_0^1} a|\nabla'v|^2 \cdot \overline{v}^{t-2}\zeta^2(|x'|)\, dx' \le \left(\frac{8(a^*)^2}{a_*(t-1)^2} + \frac{4a_*}{t^2}\right) \cdot \int\limits_{G_0^1} |\nabla'\zeta|^2\overline{v}^t(x')\, dx'$$

$$+ \frac{4}{t-1}\int\limits_{G_0^1}\left(\frac{2a_*c_0t^2}{t-1} + \frac{|\mathcal{F}(x')|}{k}\right) \cdot \overline{v}^t\zeta^2(|x'|)\, dx'.$$

But, by (4.8), this means that

$$(4.11) \qquad \int\limits_{G_0^1} a|\nabla'(\overline{v}^{t/2})|^2\zeta^2(|x'|)\, dx' \le \left(\frac{2t^2(a^*)^2}{a_*(t-1)^2} + a_*\right) \cdot \int\limits_{G_0^1} |\nabla'\zeta|^2\overline{v}^t(x')\, dx'$$

$$+ \frac{t^2}{t-1}\int\limits_{G_0^1}\left(\frac{2a_*c_0t^2}{t-1} + \frac{|\mathcal{F}(x')|}{k}\right) \cdot \overline{v}^t\zeta^2(|x'|)\, dx'.$$

Further, as $t \ge 2$ we have

$$2c_0 a_* \frac{t^2}{(t-1)^2} = 2c_0 a_* \left(1 + \frac{2}{t-1} + \frac{1}{(t-1)^2} \right) \le 8c_0 a_*;$$

$$\frac{2(a^*)^2 t^2}{a_*(t-1)^2} \le \frac{8(a^*)^2}{a_*}.$$

Therefore the inequality (4.11) can be rewritten as

$$(4.12) \quad \int\limits_{G_0^1} a|\nabla'(\overline{v}^{t/2})|^2 \zeta^2(|x'|) \, dx' \le C_1(1+t)^2 \int\limits_{G_0^1} a(|\nabla'\zeta|^2 + \zeta^2(|x'|))\overline{v}^t \, dx'$$

$$+ \, C_2(1+t)^2 \int\limits_{G_0^1} \frac{|\mathcal{F}(x')|}{k} \cdot a\overline{v}^t \zeta^2(|x'|) \, dx',$$

where the constants C_1, C_2 depend only on c_0, a_*, a^* and are independent of t. Setting

$$(4.13) \qquad\qquad w(x') = \sqrt{a} \cdot \overline{v}^{t/2}(x')$$

from (4.12) we obtain

$$(4.14) \quad \int\limits_{G_0^1} |\nabla'w|^2 \zeta^2(|x'|) \, dx' \le C_1(1+t)^2 \int\limits_{G_0^1} (|\nabla'\zeta|^2 + \zeta^2(|x'|))w^2(x') \, dx'$$

$$+ \, C_2(1+t)^2 \int\limits_{G_0^1} \frac{|\mathcal{F}(x')|}{k} \cdot w^2(x')\zeta^2(|x'|) \, dx'.$$

The desired iteration process can now be continued from (4.14). By the Sobolev imbedding theorem we have

$$(4.15) \quad \|\zeta w\|^2_{\frac{2n}{n-2}, G_0^1} \le C^* \int\limits_{G_0^1} (|\nabla'\zeta|^2 w^2(x') + \zeta^2(|x'|)|\nabla'w|^2) \, dx', \quad n \ge 3,$$

where C^* depends only on n and the domain G. Using the Hölder inequality

$$(4.16) \quad \int\limits_{G_0^1} |F(x')| \cdot w^2(x')\zeta^2(x') \, dx' \le \|F\|_{q/2, G_0^1} \cdot \|w\zeta\|^2_{\frac{2q}{q-2}, G_0^1}, \quad q > 2,$$

we get from (4.14)–(4.16)

$$(4.17) \quad \|\zeta w\|^2_{\frac{2n}{n-2}, G_0^1} \le C_3(1+t)^2 \int\limits_{G_0^1} (|\nabla'\zeta|^2 + \zeta^2(|x'|))w^2(x') \, dx'$$

$$+ \, C_4(1+t)^2 \left\| \frac{|\mathcal{F}(x')|}{k} \right\|_{q/2, G_0^1} \cdot \|w\zeta\|^2_{\frac{2q}{q-2}, G_0^1}, \quad q > n.$$

By the interpolation inequality for L_q norms,

$$(4.18) \quad \|\zeta w\|_{\frac{2q}{q-2}, G_0^1} \le \varepsilon \|\zeta w\|_{\frac{2n}{n-2}, G_0^1} + \varepsilon^{\frac{n}{n-q}} \|\zeta w\|_{2, G_0^1}, \quad q > n, \forall \varepsilon > 0,$$

from (4.17)–(4.18) it follows that

$$(4.19) \qquad \|\zeta w\|_{\frac{2n}{n-2},G_0^1} \leq (1+t)\sqrt{C_3} \cdot \|(\zeta + |\nabla'\zeta|)w\|_{2,G_0^1}$$
$$+ (1+t)\sqrt{C_4}\left\|\frac{|\mathcal{F}(x')|}{k}\right\|_{q/2,G_0^1}^{1/2}$$
$$\times (\varepsilon\|w\zeta\|_{\frac{2n}{n-2},G_0^1}^2 + \varepsilon^{\frac{n}{n-q}}\|\zeta w\|_{2,G_0^1}), \quad q > n, \forall \varepsilon > 0.$$

Choosing $\varepsilon = \frac{1}{2\sqrt{C_4}}(1+t)^{-1}\left\|\frac{|\mathcal{F}(x')|}{k}\right\|_{q/2,G_0^1}^{-1/2}$ we obtain

$$(4.20) \qquad \|\zeta w\|_{\frac{2n}{n-2},G_0^1} \leq C(1+t)^{\frac{q}{q-n}}\|(\zeta + |\nabla'\zeta|)w\|_{2,G_0^1}, \quad q > n \geq 3,$$

where C depends only on $c_0, n, a_*, a^*, q, \operatorname{diam} G$, and is independent of t. Recalling the definition (4.13) of w and taking into account that $t \geq 2$ from (4.20) we finally establish the inequality

$$(4.21) \qquad \|\zeta \cdot \overline{v}^{t/2}\|_{\frac{2n}{n-2},G_0^1} \leq Ct^{\frac{q}{q-n}}\|(\zeta + |\nabla'\zeta|) \cdot \overline{v}^{t/2}\|_{2,G_0^1}, \quad q > n \geq 3.$$

This inequality can now be iterated to yield the desired estimate.

For all $\varkappa \in (0,1)$ we define $G'_{(j)} \equiv G_0^{\varkappa+(1-\varkappa)2^{-j}}$, $j = 0, 1, 2, \ldots$. It is easy to verify that $G_0^{\varkappa} \equiv G'_{(\infty)} \subset \cdots \subset G'_{(j+1)} \subset G'_{(j)} \subset \cdots \subset G'_{(0)} \equiv G_0^1$. Now we consider a sequence of cut-off functions $\zeta_j \in \mathbf{C}^{\infty}(G'_{(j)})$ such that

$$0 \leq \zeta_j(x') \leq 1, \quad \zeta_j(x') \equiv 1 \text{ in } G'_{(j+1)}, \quad \zeta_j(x') \equiv 0 \text{ for } |x'| > \varkappa + 2^{-j}(1-\varkappa);$$
$$|\nabla'\zeta_j| \leq \frac{2^{j+1}}{1-\varkappa} \quad \text{for } \varkappa + 2^{-j-1}(1-\varkappa) < |x'| < \varkappa + 2^{-j}(1-\varkappa).$$

We also define the number sequence $t_j = t\left(\frac{n}{n-2}\right)^j$, $j = 0, 1, 2, \ldots$. Now we rewrite the inequality (4.21) replacing $\zeta(|x'|)$ by $\zeta_j(x')$ and t by t_j; then taking the t_jth roots, we obtain

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \left(\frac{C}{1-\varkappa}\right)^{2/t_j} \cdot 4^{j/t_j} \cdot (t_j)^{\frac{2q}{q-n}\cdot\frac{1}{t_j}}\|\overline{v}\|_{t_j,G'_{(j)}}.$$

After iteration, we find that

$$(4.22) \qquad \|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \left\{\frac{Ct^{\frac{1}{q-1}}}{1-\varkappa}\cdot\left(\frac{n}{n-2}\right)^{\frac{q}{q-n}}\right\}^{2\sum_{j=0}^{\infty}1/t_j}$$
$$\times 4^{\sum_{j=0}^{\infty}j/t_j} \cdot \|\overline{v}\|_{t,G_0^1}.$$

Notice that the series $\sum_{j=0}^{\infty}j/t_j$ is convergent by the d'Alembert ratio test, and $\sum_{j=0}^{\infty}1/t_j = n/2t$ is a geometric series. Therefore from (4.22) we get

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \frac{C}{(1-\varkappa)^{n/t}}\|\overline{v}\|_{t,G_0^1}.$$

Consequently, letting $j \to \infty$, we have

$$\sup_{x' \in G_0^\varkappa} |\overline{v}(x')| \leq \frac{C}{(1-\varkappa)^{n/t}} \|\overline{v}\|_{t,G_0^1}.$$

Hence, by the definition (4.4) of $\overline{v}(x')$ and (4.3) of k, we obtain

$$\sup_{x' \in G_0^\varkappa} v(x') \leq \frac{C}{(1-\varkappa)^{n/t}} (\|v\|_{t,G_0^1} + \|\mathcal{F}\|_{q/2,G_0^1} + \|\mathcal{G}\|_{\infty,\Gamma_0^1} + \|\mathcal{H}\|_{\infty,\Sigma_0^1}).$$

Returning to the variables x, u via (4.2) and considering the similar inequality with u replaced by $-u$, we obtain the required estimate (4.1) in the case $t \geq 2$.

Let now $0 < t < 2$. We consider (4.1) with $t = 2$:

$$(4.23) \qquad \sup_{x \in G_0^{\varkappa\varrho}} |u(x)| \leq \frac{C}{[(1-\varkappa)\varrho]^{n/2}} \|u\|_{2,G_0^\varrho} + K(\varrho),$$

where

$$K(\varrho) = \frac{C}{(1-\varkappa)^{n/2}} \{\varrho^{2(1-n/q)}\|f\|_{q/2,G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_0^\varrho} + \|h\|_{\infty,\Sigma_0^\varrho})\}.$$

Now, using the Young inequality with $s = \frac{2}{t}$ and $s' = \frac{2}{2-t}$ we can write

$$(4.24) \qquad \frac{C}{[(1-\varkappa)\varrho]^{n/2}}\|u\|_{2,G_0^\varrho} = \frac{C}{[(1-\varkappa)\varrho]^{n/2}} \Big(\int\limits_{G_0^\varrho} u^t \cdot u^{2-t} \, dx \Big)^{1/2}$$

$$\leq (\sup_{G_0^\varrho} |u(x)|)^{1-t/2} \cdot \frac{C}{[(1-\varkappa)\varrho]^{n/2}}\|u\|_{t,G_0^\varrho}^{t/2}$$

$$\leq \frac{1}{2} \sup_{G_0^\varrho} |u(x)| + \frac{C_1}{[(1-\varkappa)\varrho]^{n/t}} \|u\|_{t,G_0^\varrho}.$$

Define $\psi(s) = \sup_{x \in G_0^s} |u(x)|$. Then from (4.23)–(4.24) it follows that

$$(4.25) \qquad \psi(\varkappa\varrho) \leq \frac{1}{2} \psi(\varrho) + \frac{C_1}{[(1-\varkappa)\varrho]^{n/t}} \|u\|_{t,G_0^\varrho} + K(\varrho), \qquad \varkappa \in (0,1).$$

Further we apply the following statement:

PROPOSITION 4.2 (see [4, Lemma 4.1 in Chapter 2]). *Let ψ be a bounded nonnegative function defined on $[0, \varrho]$. Suppose that*

$$\psi(\sigma) \leq \delta\psi(s) + \frac{A}{(s-\sigma)^\alpha} + B$$

for any $0 \leq \sigma < s \leq \varrho$, where $\delta \in (0,1),$ and A, B and α are nonnegative constants. Then

$$(4.26) \qquad \psi(r) \leq C\Big[\frac{A}{(R-r)^\alpha} + B \Big], \qquad 0 \leq r < R \leq \varrho,$$

where C depends only on α, δ.

By this proposition, letting $r = \varkappa\varrho$, $R = \varrho$, $\delta = 1/2$, $\alpha = n/t$, $A = C_1\|u\|_{t,G_0^{\varrho}}$, $B = K(\varrho)$ from (4.25) we obtain the validity of the required estimate (4.1) in the case $0 < t < 2$.

The proof of Theorem 4.1 is complete. ∎

5. Global integral estimates. First we will obtain a global estimate for the Dirichlet integral.

THEOREM 5.1. *Let u be a weak solution of problem (L). Let assumptions* (a)–(c) *be satisfied. Suppose, in addition, that $\mathcal{D} = \emptyset$. Then*

$$(5.1) \quad \int\limits_G a|\nabla u|^2\, dx + \int\limits_\Sigma \frac{\beta(\phi)}{r}\, u^2(x)\, ds + \int\limits_{\partial G} \frac{\gamma(\phi)}{r}\, u^2(x)\, ds$$

$$\leq C\left\{\int\limits_G (u^2(x) + f^2(x))\, dx + \frac{1}{\beta_0} \int\limits_\Sigma rh^2(x)\, ds + \frac{1}{\gamma_0} \int\limits_{\partial G} rg^2(x)\, ds\right\},$$

where the constant C depends only on p^*.

Proof. Setting in (II) $\eta(x) = u(x)$ and using the classical Hölder inequality, by assumptions (a), (c), we get

$$(5.2) \quad \int\limits_G a|\nabla u|^2\, dx + \int\limits_\Sigma \frac{\beta(\phi)}{r}\, u^2(x)\, ds + \int\limits_{\partial G} \frac{\gamma(\phi)}{r}\, u^2(x)\, ds$$

$$\leq p^* \int\limits_G u^2(x)\, dx + \int\limits_\Sigma |u|\, |h(x)|\, ds + \int\limits_{\partial G} |u|\, |g(x)|\, ds + \int\limits_{\partial G} |u|\, |f(x)|\, dx.$$

By the Cauchy inequality, in view of (c), we obtain

$$\int\limits_\Sigma |u|\, |h(x)|\, ds = \int\limits_\Sigma \left(\sqrt{\frac{\beta(\phi)}{r}}|u|\right)\left(\sqrt{\frac{r}{\beta(\phi)}}|h(x)|\right)ds$$

$$\leq \frac{1}{2} \int\limits_\Sigma \frac{\beta(\phi)}{r}\, u^2(x)\, ds + \frac{1}{2\beta_0} \int\limits_\Sigma rh^2(x)\, ds;$$

$$\int\limits_{\partial G} |u|\, |g(x)|\, ds = \int\limits_{\partial G} \left(\sqrt{\frac{\gamma(\phi)}{r}}|u|\right)\left(\sqrt{\frac{r}{\gamma(\phi)}}|g(x)|\right)ds$$

$$\leq \frac{1}{2} \int\limits_{\partial G} \frac{\gamma(\phi)}{r}\, u^2(x)\, ds + \frac{1}{2\gamma_0} \int\limits_{\partial G} rg^2(x)\, ds;$$

$$\int\limits_{\partial G} |u|\, |f(x)|\, dx \leq \frac{1}{2} \int\limits_{\partial G} |u|^2\, dx + \frac{1}{2} \int\limits_{\partial G} |f|^2\, dx.$$

Hence we get the desired inequality (5.1). ∎

Further, we will obtain a global estimate for the weighted Dirichlet integral.

THEOREM 5.2. *Let u be a weak solution of problem (L) and let λ be as in (1.3), and $\varkappa_0$ be as in Lemma 3.1. Let assumptions (a)–(c) be satisfied. Suppose, in addition, that $\lambda > 1$ and $0 < \varkappa \leq \min(s-1, \varkappa_0, \lambda-1)$ as well as*

$$f \in \overset{\circ}{\mathbf{W}}{}^0_\alpha(G), \quad \int\limits_\Sigma r^{\alpha-1} h^2(x)\, ds < \infty, \quad \int\limits_{\partial G} r^{\alpha-1} g^2(x)\, ds < \infty,$$

where

(5.3) $$2 - n - 2\varkappa < \alpha \leq 2.$$

Then $u \in \overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$ and

(5.4) $$\int\limits_G a(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\, dx + \int\limits_\Sigma r^{\alpha-3}\beta(\phi)u^2(x)\, ds$$

$$+ \int\limits_{\partial G} \alpha(x) r^{\alpha-3}\gamma(\phi)u^2(x)\, ds$$

$$\leq C\Big\{\int\limits_G (u^2 + (1 + r^\alpha)f^2(x))\, dx + \int\limits_\Sigma r^{\alpha-1}h^2(x)\, ds + \int\limits_{\partial G} \alpha(x) r^{\alpha-1}g^2(x)\, ds\Big\},$$

where the constant $C > 0$ depends only on $q, n, a_*, a^*, \alpha, \lambda$ and the domain G.

Proof. Setting in (II) $\eta(x) = r_\varepsilon^{\alpha-2}u(x)$, since $\eta_{x_i} = r_\varepsilon^{\alpha-2}u_{x_i} + (\alpha-2) \cdot r_\varepsilon^{\alpha-3}\frac{x_i - \varepsilon l_i}{r_\varepsilon}u(x)$ we obtain

(5.5) $$\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla u|^2\, dx + \int\limits_\Sigma r^{-1}r_\varepsilon^{\alpha-2}\beta(\phi)u^2(x)\, ds + \int\limits_{\partial G} \alpha(x) r^{-1}r_\varepsilon^{\alpha-2}\gamma(\phi)u^2(x)\, ds$$

$$= \frac{2-\alpha}{2}\int\limits_G a r_\varepsilon^{\alpha-4}(x_j - \varepsilon l_j)(u^2)_{x_j}\, dx - \int\limits_G (pu(x) + f(x))r_\varepsilon^{\alpha-2}u(x)\, dx$$

$$+ \int\limits_\Sigma r_\varepsilon^{\alpha-2}u(x)h(x)\, ds + \int\limits_{\partial G} \alpha(x) r_\varepsilon^{\alpha-2}u(x)g(x)\, ds.$$

We transform the first integral on the right by integrating by parts:

(5.6) $$\int\limits_G a r_\varepsilon^{\alpha-4}(x_j - \varepsilon l_j)\frac{\partial u^2}{\partial x_j}\, dx = \sum_{i=1}^N \int\limits_{G_i} a_i r_\varepsilon^{\alpha-4}(x_j - \varepsilon l_j)\frac{\partial u_i^2}{\partial x_j}\, dx$$

$$= -\int\limits_G a u^2 \frac{\partial}{\partial x_i}(r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i))\, dx + \sum_{i=1}^N \int\limits_{\partial G_i} a_i u_i^2 r_\varepsilon^{\alpha-4}(x_j - \varepsilon l_j)\cos(\vec{n}, x_j)\, ds$$

$$= -\int\limits_{G} au^2 \frac{\partial}{\partial x_i}(r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i))\, dx + \int\limits_{\partial G} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\vec{n}, x_i)\, ds$$

$$+ \sum_{k=1}^{N-1}[a]_{\Sigma_k} \int\limits_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\vec{n}_k, x_i)\, ds,$$

because of $[u]_{\Sigma_k} = 0$, $k = 1, \ldots, N-1$. By elementary calculation we have:

(1) $\dfrac{\partial}{\partial x_i}(r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)) = nr_\varepsilon^{\alpha-4} + (\alpha - 4)(x_i - \varepsilon l_i)r_\varepsilon^{\alpha-5}\dfrac{x_i - \varepsilon l_i}{r_\varepsilon} = (n + \alpha - 4)r_\varepsilon^{\alpha-4}$;

(2) by (2.1), $(x_i - \varepsilon l_i)\cos(\vec{n}_k, x_i)|_{\Sigma_k} = \varepsilon\cos(\vec{n}_k, x_1)|_{\Sigma_k} = \varepsilon\sin(\omega_0/2 - \theta_k)$, $k = 1, \ldots, N-1$;

(3) from the representation $\partial G = \Gamma_0^d \cup \Gamma_d$ and by (2.1), we obtain $(x_i - \varepsilon l_i)\cos(\vec{n}, x_i)|_{\Gamma_0^d} = -\varepsilon\sin(\omega_0/2)$, so

$$\int\limits_{\partial G} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\vec{n}, x_i)\, ds$$

$$= \int\limits_{\Gamma_d} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\vec{n}, x_i)\, ds - \varepsilon\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} au^2 r_\varepsilon^{\alpha-4}\, ds.$$

Hence and from (5.6) it follows that

$$(5.7) \quad \frac{2-\alpha}{2}\int\limits_{G} ar_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\frac{\partial u^2}{\partial x_i}\, dx$$

$$= \frac{(2-\alpha)(4-n-\alpha)}{2}\int\limits_{G} ar_\varepsilon^{\alpha-4}u^2\, dx - \varepsilon\frac{2-\alpha}{2}\sin(\omega_0/2)\int\limits_{\Gamma_0^d} au^2 r_\varepsilon^{\alpha-4}ds$$

$$+ \frac{2-\alpha}{2}\int\limits_{\Gamma_d} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\vec{n}, x_i)\, ds$$

$$+ \varepsilon\sum_{k=1}^{N-1}[a]_{\Sigma_k}\int\limits_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4}\sin(\omega_0/2 - \theta_k)\, ds.$$

From (5.5), (5.7) we obtain

$$(5.8) \int\limits_{G} ar_\varepsilon^{\alpha-2}|\nabla u|^2\, dx + \int\limits_{\Sigma} r^{-1}r_\varepsilon^{\alpha-2}\beta(\phi)u^2(x)\, ds + \int\limits_{\partial G} \alpha(x)r^{-1}r_\varepsilon^{\alpha-2}\gamma(\phi)u^2(x)\, ds$$

$$+ \varepsilon\frac{2-\alpha}{2}\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} au^2 r_\varepsilon^{\alpha-4}\, ds = \frac{(2-\alpha)(4-n-\alpha)}{2}\int\limits_{G} ar_\varepsilon^{\alpha-4}u^2\, dx$$

$$+ \frac{2-\alpha}{2} \int\limits_{\Gamma_d} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) \cos(\vec{n}, x_i)\, ds$$

$$+ \varepsilon \sum_{k=1}^{N-1} [a]_{\Sigma_k} \int\limits_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4} \sin(\omega_0/2 - \theta_k)\, ds - \int\limits_G (pu(x) + f(x)) r_\varepsilon^{\alpha-2} u(x)\, dx$$

$$+ \int\limits_\Sigma r_\varepsilon^{\alpha-2} u(x) h(x)\, ds + \int\limits_{\partial G} \alpha(x) r_\varepsilon^{\alpha-2} u(x) g(x)\, ds.$$

Now we estimate the integral over Γ_d. Because on Γ_d, $r_\varepsilon \geq hr \geq hd$, so $(\alpha - 3) \ln r_\varepsilon \leq (\alpha - 3) \ln(hd)$, since $\alpha \leq 2$ we have $r_\varepsilon^{\alpha-3}|_{\Gamma_d} \leq (hd)^{\alpha-3}$ and therefore

$$(5.9) \qquad \frac{2-\alpha}{2} \int\limits_{\Gamma_d} au^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) \cos(\vec{n}, x_i)\, ds$$

$$\leq \frac{2-\alpha}{2} \int\limits_{\Gamma_d} a r_\varepsilon^{\alpha-3} u^2\, ds \leq \frac{2-\alpha}{2} (hd)^{\alpha-3} \int\limits_{\Gamma_d} au^2\, ds$$

$$\leq c_\delta \int\limits_{G_d} u^2\, dx + \delta \int\limits_{G_d} |\nabla u|^2\, dx, \qquad \forall \delta > 0,$$

by (2.10).

Further, by the Cauchy inequality and because of $\gamma(\phi) \geq \gamma_0 > 0$,

$$ug = \left(r^{1/2} \frac{1}{\sqrt{\gamma(\phi)}} |g| \right) (r^{-1/2} \sqrt{\gamma(\phi)} |u|)$$

$$\leq \frac{\delta_1}{2} r^{-1} \gamma(\phi) u^2 + \frac{1}{2\delta_1 \gamma_0} r g^2(x), \qquad \forall \delta_1 > 0;$$

taking into account property (1) of r_ε we obtain

$$(5.10) \qquad \int\limits_{\partial G} r_\varepsilon^{\alpha-2} |u|\, |g|\, ds \leq \frac{\delta_1}{2} \int\limits_{\partial G} r_\varepsilon^{\alpha-2} \frac{1}{r} \gamma(\phi) u^2\, ds$$

$$+ \frac{1}{2\delta_1 \gamma_0} \int\limits_{\partial G} r^{\alpha-1} g^2(x)\, ds, \qquad \forall \delta_1 > 0.$$

Similarly, because of $\beta(\phi) \geq \beta_0 > 0$,

$$(5.11) \qquad \int\limits_\Sigma r_\varepsilon^{\alpha-2} |u|\, |h(x)|\, ds \leq \frac{\delta_1}{2} \int\limits_\Sigma r_\varepsilon^{\alpha-2} \frac{1}{r} \beta(\phi) u^2\, ds$$

$$+ \frac{1}{2\delta_1 \beta_0} \int\limits_{\Sigma_0} r^{\alpha-1} h^2(x)\, ds, \qquad \forall \delta_1 > 0,$$

and

(5.12)
$$\int\limits_G r_\varepsilon^{\alpha-2} u f(x)\, dx \le \frac{\delta}{2} \int\limits_G a r^{-2} r_\varepsilon^{\alpha-2} u^2\, dx$$

$$+ \frac{1}{2a_* \delta} \int\limits_G r^\alpha f^2(x)\, dx, \quad \forall \delta > 0.$$

Now we use the representation $G = G_0^d \cup G_d$. Then

$$\int\limits_G pr_\varepsilon^{\alpha-2} u^2(x)\, dx = \int\limits_{G_0^d} pr_\varepsilon^{\alpha-2} u^2(x)\, dx + \int\limits_{G_d} pr_\varepsilon^{\alpha-2} u^2(x)\, dx.$$

Therefore

(5.13)
$$\int\limits_{G_0^d} pr_\varepsilon^{\alpha-2} u^2(x)\, dx = \int\limits_{G_0^d} a r^{-2} r_\varepsilon^{\alpha-2} u^2(x) \cdot \frac{pr^2}{a}\, dx$$

$$\le \frac{p^*}{a_*} d^2 \int\limits_{G_0^d} a r^{-2} r_\varepsilon^{\alpha-2} u^2(x)\, dx$$

as well as, in view of property (2) of $r_\varepsilon(x)$,

(5.14)
$$\int\limits_{G_d} pr_\varepsilon^{\alpha-2} u^2(x)\, dx \le p^* (d/2)^{\alpha-2} \int\limits_G u^2(x)\, dx.$$

Finally, we estimate the integral $\varepsilon \sum_{k=1}^{N-1} [a]_{\Sigma_k} \int_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4} \sin(\omega_0/2 - \theta_k)\, ds$. First, by (1.2), we get

(5.15)
$$\varepsilon \sum_{k=1}^{N-1} [a]_{\Sigma_k} \int\limits_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4} \sin(\omega_0/2 - \theta_k)\, ds \le a_0 \varepsilon \sum_{k=1}^{N-1} \int\limits_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4}\, ds$$

$$= a_0 \varepsilon \int\limits_\Sigma r_\varepsilon^{\alpha-4} u^2(x)\, ds.$$

Further, we use the representation $\Sigma = \Sigma_0^\varepsilon \cup \Sigma_\varepsilon$. Then, by property (1) of r_ε and in view of the inequality (3.27) of Corollary 3.4, we get

(5.16)
$$\varepsilon \int\limits_{\Sigma_0^\varepsilon} r_\varepsilon^{\alpha-4} u^2(x)\, ds \le \int\limits_{\Sigma_0^\varepsilon} r_\varepsilon^{\alpha-3} u^2(x)\, ds \le C_0^2 \operatorname{meas} \sigma \cdot \int\limits_0^\varepsilon r^{\alpha+2\varkappa+n-3}\, dr$$

$$= \frac{C_0^2 \operatorname{meas} \sigma}{\alpha + 2\varkappa + n - 2} \varepsilon^{\alpha+2\varkappa+n-2},$$

because of $\alpha + 2\varkappa + n - 2 > 0$, by our assumption (5.3). Similarly, for the integral over Σ_ε we obtain

$$(5.17) \quad \varepsilon \int_{\Sigma_\varepsilon} r_\varepsilon^{\alpha-4} u^2(x)\, ds \leq \varepsilon \cdot \operatorname{meas}\sigma \cdot \int_\varepsilon^R r^{\alpha+2\varkappa+n-4}\, dr$$

$$= \operatorname{meas}\sigma \cdot \begin{cases} \dfrac{\varepsilon R^{\alpha+2\varkappa+n-3} - \varepsilon^{\alpha+2\varkappa+n-2}}{\alpha+2\varkappa+n-3} & \text{if } \alpha+2\varkappa+n-3 \neq 0, \\[2mm] \varepsilon \ln \dfrac{R}{\varepsilon} & \text{if } \alpha+2\varkappa+n-3 = 0, \end{cases}$$

where $R = \max_{1 \leq k \leq N-1} \operatorname{diam}\Sigma_k$. Thus, from (5.15)–(5.17) we get

$$(5.18) \qquad \varepsilon \cdot \sum_{k=1}^{N-1} [a]_{\Sigma_k} \int_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4} \sin(\omega_0/2 - \theta_k)\, ds \leq J(\varepsilon),$$

where

$$(5.19) \quad J(\varepsilon) = \varepsilon \cdot a_0 \cdot \operatorname{meas}\sigma$$

$$\times \begin{cases} \dfrac{R^{\alpha+2\varkappa+n-3}}{\alpha+2\varkappa+n-3} + \left(\dfrac{C_0^2}{\alpha+2\varkappa+n-2} - \dfrac{1}{\alpha+2\varkappa+n-3} \right) \varepsilon^{\alpha+2\varkappa+n-3} \\ \hspace{6cm} \text{if } \alpha+2\varkappa+n-3 \neq 0, \\[2mm] C_0^2 + \ln \dfrac{R}{\varepsilon} \hspace{3.5cm} \text{if } \alpha+2\varkappa+n-3 = 0, \end{cases}$$

provided that $\alpha+2\varkappa+n-2 > 0$. Hence

COROLLARY 5.3.

$$\lim_{\varepsilon\to+0} J(\varepsilon) = 0, \quad so \quad \lim_{\varepsilon\to+0} \varepsilon \cdot \sum_{k=1}^{N-1} [a]_{\Sigma_k} \int_{\Sigma_k} u^2 r_\varepsilon^{\alpha-4} \sin(\omega_0/2 - \theta_k)\, ds = 0.$$

As a result, from (5.8)–(5.18) we obtain

$$(5.20) \quad \int_G a r_\varepsilon^{\alpha-2} |\nabla u|^2\, dx + \int_\Sigma r^{-1} r_\varepsilon^{\alpha-2} \beta(\phi) u^2(x)\, ds$$

$$+ \int_{\partial G} \alpha(x) r^{-1} r_\varepsilon^{\alpha-2} \gamma(\phi) u^2(x)\, ds + \varepsilon \frac{2-\alpha}{2} \sin\frac{\omega_0}{2} \int_{\Gamma_0^d} a u^2 r_\varepsilon^{\alpha-4}\, ds$$

$$\leq J(\varepsilon) + \frac{(2-\alpha)(4-n-\alpha)}{2} \int_G a r_\varepsilon^{\alpha-4} u^2\, dx$$

$$+ \left(\frac{\delta}{2} + \frac{p^*}{a_*} d^2 \right) \int_{G_0^d} a r^{-2} r_\varepsilon^{\alpha-2} u^2\, dx + c(p^*, a_*, \alpha, d) \int_G (a|\nabla u|^2 + u^2)\, dx$$

$$+ \frac{\delta_1}{2} \left\{ \int_\Sigma r^{-1} r_\varepsilon^{\alpha-2} \beta(\phi) u^2(x)\, ds + \int_{\partial G} \alpha(x) r^{-1} r_\varepsilon^{\alpha-2} \gamma(\phi) u^2(x)\, ds \right\}$$

$$+ \frac{1}{2\delta_1\gamma_0} \int\limits_{\partial G} r^{\alpha-1} g^2(x)\, ds + \frac{1}{2\delta_1\beta_0} \int\limits_{\Sigma} r^{\alpha-1} h^2(x)\, ds$$

$$+ \frac{1}{2a_*\delta} \int\limits_{G} r^{\alpha} f^2(x)\, dx, \quad \forall \delta, \delta_1 > 0.$$

Now it is clear that it is sufficient to consider the case $\alpha < 4 - n$. First we apply Lemma 2.6. Then from (5.20) we get

$$(5.21) \quad \left(1 - \frac{2(2-\alpha)(4-\alpha-n)}{(4-n-\alpha)^2 + 4\lambda(\lambda+n-2)}\right)\left\{\int\limits_{G} a r_\varepsilon^{\alpha-2} |\nabla u|^2\, dx \right.$$

$$+ \int\limits_{\Sigma} \frac{1}{r} r_\varepsilon^{\alpha-2} \beta(\phi) u^2(x)\, ds + \int\limits_{\partial G} \alpha(x) \frac{1}{r} r_\varepsilon^{\alpha-2} \gamma(\phi) u^2(x)\, ds \right\}$$

$$\leq J(\varepsilon) + \left(\frac{\delta}{2} + \frac{p^*}{a_*} d^2\right) \int\limits_{G_0^d} a r^{-2} r_\varepsilon^{\alpha-2} u^2\, dx$$

$$+ c(p^*, a_*, \alpha, d) \int\limits_{G} (a|\nabla u|^2 + u^2)\, dx + \frac{1}{2\delta_1\gamma_0} \int\limits_{\partial G} r^{\alpha-1} g^2(x)\, ds$$

$$+ \frac{\delta_1}{2}\left\{\int\limits_{\Sigma} r^{-1} r_\varepsilon^{\alpha-2} \beta(\phi) u^2(x)\, ds + \int\limits_{\partial G} \alpha(x) r^{-1} r_\varepsilon^{\alpha-2} \gamma(\phi) u^2(x)\, ds\right\}$$

$$+ \frac{1}{2\delta_1\beta_0} \int\limits_{\Sigma} r^{\alpha-1} h^2(x)\, ds + \frac{1}{2a_*\delta} \int\limits_{G} r^{\alpha} f^2(x)\, dx, \quad \forall \delta, \delta_1 > 0, \forall \varepsilon > 0.$$

But we have

$$(5.22) \qquad \frac{2(2-\alpha)(4-\alpha-n)}{(4-n-\alpha)^2 + 4\lambda(\lambda+n-2)} < 1,$$

provided $\alpha > 4 - n - 2\lambda$; this is indeed satisfied, in view of $\alpha > 2 - n - 2\varkappa$ and $0 < \varkappa \leq \lambda - 1$. Therefore (5.22) is true. Now we apply Lemma 2.7 and choose $\delta_1 = \frac{2}{\lambda(\lambda+n-2)}\left(\frac{\delta}{2} + \frac{p^*}{a_*} d^2\right)$. As a result we obtain

$$(5.23) \quad \left(1 - \frac{2(2-\alpha)(4-\alpha-n)}{(4-n-\alpha)^2 + 4\lambda(\lambda+n-2)}\right)\left\{\int\limits_{G} a r_\varepsilon^{\alpha-2} |\nabla u|^2\, dx \right.$$

$$+ \int\limits_{\Sigma} \frac{1}{r} r_\varepsilon^{\alpha-2} \beta(\phi) u^2(x)\, ds + \int\limits_{\partial G} \alpha(x) \frac{1}{r} r_\varepsilon^{\alpha-2} \gamma(\phi) u^2(x)\, ds \right\}$$

$$\leq J(\varepsilon) + c(p^*, a_*, \alpha, d) \int\limits_{G} (a|\nabla u|^2 + u^2)\, dx$$

$$+ \frac{1}{2\delta_1\gamma_0} \int\limits_{\partial G} r^{\alpha-1} g^2(x)\, ds + \frac{2}{\lambda(\lambda+n-2)}\left(\frac{\delta}{2} + \frac{p^*}{a_*} d^2\right)\left\{\int\limits_{G} a r_\varepsilon^{\alpha-2} |\nabla u|^2\, dx\right.$$

$$+ \int\limits_{\Sigma} r^{-1} r_{\varepsilon}^{\alpha-2} \beta(\phi) u^2(x) \, ds + \int\limits_{\partial G} \alpha(x) r^{-1} r_{\varepsilon}^{\alpha-2} \gamma(\phi) u^2(x) \, ds \Big\}$$

$$+ \frac{1}{2\delta_1 \beta_0} \int\limits_{\Sigma} r^{\alpha-1} h^2(x) \, ds + \frac{1}{2a_* \delta} \int\limits_{G} r^{\alpha} f^2(x) dx, \quad \forall \delta, \varepsilon > 0.$$

Now we can choose

$$d = \sqrt{\frac{\delta a_*}{2p^*}}, \quad \delta = \frac{\lambda(\lambda+n-2)}{4} \left(1 - \frac{2(2-\alpha)(4-\alpha-n)}{(4-n-\alpha)^2 + 4\lambda(\lambda+n-2)} \right).$$

Thus we get

$$(5.24) \quad \int\limits_{G} ar_{\varepsilon}^{\alpha-2} |\nabla u|^2 \, dx + \int\limits_{\Sigma} \frac{1}{r} r_{\varepsilon}^{\alpha-2} \beta(\phi) u^2(x) \, ds + \int\limits_{\partial G} \alpha(x) \frac{1}{r} r_{\varepsilon}^{\alpha-2} \gamma(\phi) u^2(x) \, ds$$

$$\leq C(p^*, n, a_*, \alpha, \lambda) \Big\{ J(\varepsilon) + \int\limits_{G} (a|\nabla u|^2 + u^2 + r^{\alpha} f^2(x)) \, dx$$

$$+ \frac{1}{\gamma_0} \int\limits_{\partial G} \alpha(x) r^{\alpha-1} g^2(x) \, ds + \frac{1}{\beta_0} \int\limits_{\Sigma} r^{\alpha-1} h^2(x) \, ds \Big\}, \quad \forall \varepsilon > 0.$$

Now we can let $\varepsilon \to +0$ by the Fatou theorem. From Corollary 5.3 it follows that

$$(5.25) \quad \int\limits_{G} ar^{\alpha-2} |\nabla u|^2 \, dx + \int\limits_{\Sigma} r^{\alpha-3} \beta(\phi) u^2(x) \, ds + \int\limits_{\partial G} \alpha(x) r^{\alpha-3} \gamma(\phi) u^2(x) \, ds$$

$$\leq C(p^*, n, a_*, \alpha, \lambda) \Big\{ \int\limits_{G} (a|\nabla u|^2 + u^2 + r^{\alpha} f^2(x)) \, dx$$

$$+ \frac{1}{\gamma_0} \int\limits_{\partial G} \alpha(x) r^{\alpha-1} g^2(x) \, ds + \frac{1}{\beta_0} \int\limits_{\Sigma} r^{\alpha-1} h^2(x) \, ds \Big\}.$$

Now applying Theorem 5.1, by the Hardy–Friedrichs–Wirtinger inequality (2.5), from (5.25) we get the desired estimate (5.4). ∎

6. Local integral weighted estimates

THEOREM 6.1. *Let u be a weak solution of problem (L) and $\lambda > 1$ be as in (1.3). Let assumptions* (a)–(c) *be satisfied and $\mathcal{D} = \emptyset$. Then $u \in \overset{\circ}{\mathbf{W}}{}^1_{2-n}(G)$ and there are $d \in (0,1)$, a constant $C > 0$ depending only on $n, s, \lambda, a_*, G, \Sigma$ and a $c = \mathrm{const}(n, \lambda, a_*, p^*)$ such that for all $\varrho \in (0,d)$,*

$$(6.1) \quad \int\limits_{G_0^{\varrho}} (ar^{2-n} |\nabla u|^2 + r^{-n} u^2(x)) \, dx + \int\limits_{\Sigma_0^{\varrho}} r^{1-n} \beta(\phi) u^2(x) \, ds$$

$$+ \int\limits_{\Gamma_0^{\varrho}} r^{1-n} \gamma(\phi) u^2(x) \, ds$$

$$\leq C\left(\|u\|_{2,G}^2 + f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right) \cdot \begin{cases} \varrho^{2\lambda} & \text{if } s > \lambda, \\ \varrho^{2\lambda}\ln^c(1/\varrho) & \text{if } s = \lambda, \\ \varrho^{2s} & \text{if } s < \lambda. \end{cases}$$

Proof. From Theorem 5.2 it follows that $u \in \mathring{\mathbf{W}}_{2-n}^1(G)$, so it is enough to prove the estimate (6.1). Setting $\eta(x) = r^{2-n}u(x)$ in $(\mathrm{II})_{\mathrm{loc}}$, by the definition (2.6) we obtain

$$(6.2) \quad U(\varrho) = \varrho \int\limits_\Omega au(x)\left.\frac{\partial u}{\partial r}\right|_{r=\varrho} d\Omega + \int\limits_{\Gamma_0^\varrho} \alpha(x)r^{2-n}u(x)g(x)\,ds$$

$$+ \int\limits_{\Sigma_0^\varrho} r^{2-n}u(x)h(x)\,ds$$

$$+ \int\limits_{G_0^\varrho} \{(n-2)ar^{-n}u(x)x_iu_{x_i} - pr^{2-n}u^2(x) - r^{2-n}u(x)f(x)\}\,dx.$$

By the Gauss–Ostrogradskiĭ divergence theorem, we find that

$$(6.3) \quad (n-2)\int\limits_{G_0^\varrho} ar^{-n}u(x)x_iu_{x_i}\,dx = \frac{n-2}{2}\int\limits_{G_0^\varrho} ar^{-n}x_i\frac{\partial u^2}{\partial x_i}\,dx$$

$$= \frac{n-2}{2}\Big\{-\int\limits_{G_0^\varrho} au^2(x)(nr^{-n} - nr^{-n})\,dx + \varrho^{-n}\int\limits_{\Omega_\varrho} au^2(x)x_i\cos(r,x_i)\,d\Omega_\varrho$$

$$+ \sum_{k=1}^{N-1}[a]_{\Sigma_k}\int\limits_{(\Sigma_k)_0^\varrho} r^{-n}u^2(x)x_i\cos(n_k,x_i)\,ds + \int\limits_{\Gamma_0^\varrho} \alpha(x)ar^{-n}u^2(x)x_i\cos(n,x_i)\,ds\Big\}.$$

By Lemma 2.1,

$$x_i\cos(n_k,x_i)|_{\Sigma_k} = 0, \quad k = 1,\ldots,N-1;$$
$$x_i\cos(n,x_i)|_{\Gamma_0^\varrho} = 0, \quad x_i\cos(r,x_i)|_{\Omega_\varrho} = \varrho,$$

we have

$$(6.4) \quad (n-2)\int\limits_{G_0^\varrho} ar^{-n}u(x)x_iu_{x_i}\,dx = \frac{n-2}{2}\int\limits_\Omega au^2(x)\,d\Omega.$$

Because of Lemma 2.5, from (6.2)–(6.4) it follows that

$$(6.5) \quad U(\varrho) \leq \frac{\varrho}{2\lambda}U'(\varrho) + \int\limits_{\Gamma_0^\varrho} r^{2-n}|u(x)| \cdot |g(x)|\,ds + \int\limits_{\Sigma_0^\varrho} r^{2-n}|u(x)| \cdot |h(x)|\,ds$$

$$+ p^* \int\limits_{G_0^\varrho} r^{2-n}u^2\,dx + \int\limits_{G_0^\varrho} r^{2-n}|u(x)| \cdot |f(x)|\,dx.$$

We shall obtain an upper bound for each integral on the right. First, applying the Cauchy and Hardy–Friedrichs–Wirtinger inequality (2.5), we have, for any $\delta > 0$,

$$(6.6) \qquad \int\limits_{\Gamma_0^\varrho} r^{2-n}|u|\,|g|\,ds = \int\limits_{\Gamma_0^\varrho} (r^{(1-n)/2}\sqrt{\gamma(\phi)}|u|)\left(r^{(3-n)/2}\frac{1}{\sqrt{\gamma(\phi)}}|g|\right)ds$$

$$\le \frac{\delta}{2}\int\limits_{\Gamma_0^\varrho} r^{1-n}\gamma(\phi)|u|^2 ds + \frac{1}{2\delta\gamma_0}\int\limits_{\Gamma_0^\varrho} r^{3-n}|g|^2\,ds;$$

$$(6.7) \qquad \int\limits_{\Sigma_0^\varrho} r^{2-n}|u|\,|h|\,ds = \int\limits_{\Sigma_0^\varrho} (r^{(1-n)/2}\sqrt{\beta(\phi)}|u|)\left(r^{(3-n)/2}\frac{1}{\sqrt{\beta(\phi)}}|g|\right)ds$$

$$\le \frac{\delta}{2}\int\limits_{\Sigma_0^\varrho} r^{1-n}\beta(\phi)|u|^2\,ds + \frac{1}{2\delta\beta_0}\int\limits_{\Sigma_0^\varrho} r^{3-n}|h|^2\,ds;$$

$$(6.8) \qquad \int\limits_{G_0^\varrho} r^{2-n}|u(x)|\,|f(x)|\,dx \le \frac{\delta}{2a_*}\int\limits_{G_0^\varrho} ar^{-n}|u|^2\,dx + \frac{1}{2\delta}\int\limits_{G_0^\varrho} r^{4-n}|f|^2\,dx$$

$$\le \frac{\delta}{2a_*\lambda(\lambda+n-2)}U(\varrho) + \frac{1}{2\delta}\int\limits_{G_0^\varrho} r^{4-n}|f|^2\,dx;$$

$$(6.9) \qquad p^*\int\limits_{G_0^\varrho} r^{2-n}u^2\,dx \le p^*\varrho^2\int\limits_{G_0^\varrho} r^{-n}u^2\,dx \le \frac{p^*}{a_*\lambda(\lambda+n-2)}\varrho^2 U(\varrho).$$

Thus from (6.5)–(6.9) we get

$$(6.10) \qquad \{1 - c_1(n,\lambda,a_*,p^*)(\delta+\varrho^2)\}U(\varrho) \le \frac{\varrho}{2\lambda}U'(\varrho)$$

$$+ \frac{1}{2\delta}\left\{\int\limits_{G_0^\varrho} r^{4-n}|f|^2\,dx + \frac{1}{\gamma_0}\int\limits_{\Gamma_0^\varrho} r^{3-n}|g|^2\,ds + \frac{1}{\beta_0}\int\limits_{\Sigma_0^\varrho} r^{3-n}|h|^2\,ds\right\}, \qquad \forall\delta>0.$$

But, by condition (c),

$$\int\limits_{G_0^\varrho} r^{4-n}|f|^2\,dx + \frac{1}{\gamma_0}\int\limits_{\Gamma_0^\varrho} r^{3-n}|g|^2\,ds + \frac{1}{\beta_0}\int\limits_{\Sigma_0^\varrho} r^{3-n}|h|^2\,ds$$

$$\le \frac{c_0(G)}{2s}\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot\varrho^{2s}.$$

Thus from (6.10) we have the differential inequality (CP) of §2.4 with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho}\cdot\{1 - c_1(n,\lambda,a_*,p^*)(\delta+\varrho^2)\}, \ \forall\delta>0; \qquad \mathcal{N}(\varrho) \equiv 0;$$

$$(6.11) \qquad \mathcal{Q}(\varrho) = \frac{c_0\lambda}{2s}\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot\delta^{-1}\varrho^{2s-1}, \qquad \forall\delta>0$$

$$U_0 = C\left\{\int\limits_G (u^2 + (1+r^{4-n})f^2(x))\,dx + \int\limits_\Sigma r^{3-n}h^2(x)\,ds + \int\limits_{\partial G} r^{3-n}g^2(x)\,ds\right\},$$

by (5.4) with $\alpha = 4-n$.

1) $s > \lambda$. Choosing $\delta = \varrho^\varepsilon$, $\forall \varepsilon > 0$, we get

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - 2\lambda c_1(n, \lambda, a_*, p^*)(\varrho^{\varepsilon-1} + \varrho);$$

$$\mathcal{Q}(\varrho) = \frac{c_0 \lambda}{2s} \left(f_1^2 + \frac{1}{\gamma_0} g_1^2 + \frac{1}{\beta_0} h_1^2 \right) \cdot \varrho^{2s-1-\varepsilon}.$$

Now for $0 < \varrho < \tau < d$ we have

$$-\int_\varrho^\tau \mathcal{P}(s)\, ds = -2\lambda \ln\left(\frac{\tau}{\varrho}\right) + 2\lambda c_1 \left(\frac{s^\varepsilon}{\varepsilon} + \frac{s^2}{2}\right)\Big|_\varrho^\tau$$

$$\leq \ln\left(\frac{\varrho}{\tau}\right)^{2\lambda} + 2\lambda c_1 \left(\frac{d^\varepsilon}{\varepsilon} + \frac{d^2}{2}\right),$$

so

$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau)\, d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda} \exp\left\{2\lambda c_1 \left(\frac{d^\varepsilon}{\varepsilon} + \frac{d^2}{2}\right)\right\} = K_0 \left(\frac{\varrho}{d}\right)^{2\lambda};$$

$$\exp\left(-\int_\varrho^\tau \mathcal{P}(\tau)\, d\tau\right) \leq K_0 \left(\frac{\varrho}{\tau}\right)^{2\lambda},$$

where $K_0 = \exp\{2\lambda c_1(d^\varepsilon/\varepsilon + d^2/2)\}$. We have as well

$$\int_\varrho^d \mathcal{Q}(\tau) \exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\, d\sigma\right) d\tau$$

$$\leq \frac{\lambda c_0 K_0}{2s} \left(f_0^2 + \frac{1}{\gamma_0} g_0^2 + \frac{1}{\beta_0} h_0^2 \right) \varrho^{2\lambda} \int_\varrho^d \tau^{2s-2\lambda-\varepsilon-1}\, d\tau$$

$$\leq \frac{\lambda c_0 K_0}{2s} \left(f_0^2 + \frac{1}{\gamma_0} g_0^2 + \frac{1}{\beta_0} h_0^2 \right) \cdot \frac{d^{s-\lambda}}{s-\lambda} \varrho^{2\lambda},$$

since $s > \lambda$ and we can choose $\varepsilon = s - \lambda$. Therefore in our case $K_0 = \exp\left\{2\lambda c_1\left(\frac{d^{s-\lambda}}{s-\lambda} + \frac{d^2}{2}\right)\right\}$.

Now we apply Theorem 2.8. Then from (2.13), by the above inequalities and (2.5) for $\alpha = 4 - n$, we obtain the statement of (6.1) for $s > \lambda$.

2) $s = \lambda$. Taking in (6.11) any function $\delta(\varrho) > 0$ instead of $\delta > 0$, we obtain the problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda(1 - c_1\delta(\varrho))}{\varrho} - 2\lambda c_1 \varrho; \quad \mathcal{N}(\varrho) = 0;$$

$$\mathcal{Q}(\varrho) = \frac{c_0}{2} \left(f_0^2 + \frac{1}{\gamma_0} g_0^2 + \frac{1}{\beta_0} h_0^2 \right) \cdot \delta^{-1}(\varrho) \varrho^{2\lambda-1}.$$

We choose $\delta(\varrho) = \frac{1}{2\lambda \ln(ed/\varrho)}$, $0 < \varrho < d$, where e is the Euler number. Then

we obtain

$$-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)d\tau \leq \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + c_1\int\limits_{\varrho}^{d} \frac{d\tau}{\tau\ln(ed/\tau)} + \lambda c_1 d^2$$

$$= \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + c_1 \ln\ln\left(\frac{ed}{\varrho}\right) + \lambda c_1 d^2,$$

so

$$\exp\left(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)\,d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda} \ln^{c_1}\left(\frac{ed}{\varrho}\right) \exp(\lambda c_1 d^2).$$

In this case we also have

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma\right) d\tau$$

$$\leq \lambda c_0\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)e^{\lambda c_1 d^2}\cdot\varrho^{2\lambda}\ln^{1+c_1}\left(\frac{ed}{\varrho}\right)\int\limits_{\varrho}^{d} \frac{d\tau}{\tau}$$

$$\leq \lambda c_0\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)e^{\lambda c_1 d^2}\cdot\varrho^{2\lambda}\ln^{2+c_1}\left(\frac{ed}{\varrho}\right).$$

Now we apply Theorem 2.8, and from (2.13), by the inequalities obtained, we get

$$U(\varrho) \leq c_2\left(U_0 + f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\varrho^{2\lambda}\ln^c\frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e},$$

where $c = \text{const}(n, \lambda, a_*, p^*)$. Thus we proved the statement of (6.1) for $s = \lambda$.

3) $0 < s < \lambda$. Now similar to case 1), by (6.11) we have

$$\exp\left(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)\,d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda(1-c_1\delta)} \cdot e^{\lambda d^2}.$$

In this case we also have

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma\right) d\tau$$

$$\leq \frac{\lambda c_0 e^{\lambda d^2}}{2s\delta}\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot\varrho^{2\lambda(1-c_1\delta)}\int\limits_{\varrho}^{d} \tau^{2s-2\lambda(1-c_1\delta)-1}\,d\tau$$

$$\leq c_0 c_3(n, \lambda, a_*, p^*, d, s)\left(f_0^2 + \frac{1}{\gamma_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot\varrho^{2s},$$

if we choose $\delta \in \left(0, \frac{\lambda-s}{\lambda c_1}\right)$.

Now we apply Theorem 2.8, and then from (2.13), by the above inequalities, we obtain

$$U(\varrho) \le c_4 \left(U_0 \varrho^{2\lambda(1-c_1\delta)} + \left(f_0^2 + \frac{1}{\gamma_0}\, g_0^2 + \frac{1}{\beta_0}\, h_0^2 \right) \cdot \varrho^{2s} \right)$$

$$\le c_5 \left(U_0 + f_0^2 + \frac{1}{\gamma_0}\, g_0^2 + \frac{1}{\beta_0}\, h_0^2 \right) \varrho^{2s}.$$

Thus we have proved the statement of (6.1) for $s < \lambda$. ∎

7. The power modulus of continuity at the conical point for weak solutions

Proof of Theorem 1.2. Define

(7.1)
$$\psi(\varrho) = \begin{cases} \varrho^\lambda & \text{if } s > \lambda, \\ \varrho^\lambda \ln^c(1/\varrho) & \text{if } s = \lambda, \\ \varrho^s & \text{if } s < \lambda, \end{cases}$$

for $0 < \varrho < d$, where $c > 0$ is determined by Theorem 6.1..

By Theorem 4.1 we have

(7.2) $\sup_{G_0^{\varrho/2}} |u(x)| \le C\{\varrho^{-n/2}\|u\|_{2,G_0^\varrho} + \varrho^{2(1-n/q)}\|f\|_{q/2,G_0^\varrho}$
$$+ \varrho(\|g\|_{\infty,\Gamma_0^\varrho} + \|h\|_{\infty,\Sigma_0^\varrho})\}$$

where $C = C(n, a_*, a^*, q, G)$ and $q > n$. Now by Theorem 6.1,

(7.3) $\quad \varrho^{-n/2}\|u\|_{2,G_0^\varrho} \le 2^{n/2} \Big(\int\limits_{G_0^\varrho} r^{-n} u^2(x)\, dx \Big)^{1/2}$

$$\le C \left(\|u\|_{2,G} + \|f\|_{2,G} + \|g\|_{2,\partial G} + \|h\|_{2,\Sigma_0} + f_0 + \frac{1}{\sqrt{\gamma_0}}\, g_0 + \frac{1}{\sqrt{\beta_0}}\, h_0 \right) \psi(\varrho).$$

Further, by assumption (c),

(7.4) $\quad \varrho^{2(1-n/q)}\|f\|_{q/2,G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_0^\varrho} + \|h\|_{\infty,\Sigma_0^\varrho})$
$$\le c \left(f_0 + \frac{1}{\sqrt{\gamma_0}}\, g_0 + \frac{1}{\sqrt{\beta_0}}\, h_0 \right) \psi(\varrho).$$

From (7.2)–(7.4) it follows that

$$\sup_{G_{\varrho/4}^{\varrho/2}} |u(x)| \le C \left(\|u\|_{2,G} + \|f\|_{2,G} + \|g\|_{2,\partial G} + \|h\|_{2,\Sigma_0} \right.$$
$$\left. + f_0 + \frac{1}{\sqrt{\gamma_0}}\, g_0 + \frac{1}{\sqrt{\beta_0}}\, h_0 \right) \psi(\varrho).$$

Putting now $|x| = \frac{1}{3}\varrho$ we finally obtain the desired estimate (1.4).

Now we consider the sets $G^{2\varrho}_{\varrho/4}$ and $G^{\varrho}_{\varrho/2} \subset G^{2\varrho}_{\varrho/4}$, $\varrho > 0$. We perform the change of variables $x = \varrho x'$ and $u(\varrho x') = \psi(\varrho)v(x')$. Then the function $v(x')$ satisfies

$$(L'') \begin{cases} a_i\Delta'v_i + p_i\varrho^2 v_i(x') = \dfrac{\varrho^2}{\psi(\varrho)}\,f_i(\varrho x'), & x' \in (G_i)^1_{1/2},\ i = 1,\dots,N; \\[2mm] [v(x')]_{(\Sigma_k)^1_{1/2}} = 0, & k = 1,\dots,N-1; \\[2mm] \left[a\dfrac{\partial v}{\partial n'_k}\right]_{(\Sigma_k)^1_{1/2}} + \dfrac{1}{|x'|}\,\beta_k(\phi)v(x') = \dfrac{\varrho}{\psi(\varrho)}\,h_k(\varrho x'), \\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad x' \in (\Sigma_k)^1_{1/2},\ k = 1,\dots,N-1; \\[2mm] \alpha(x')\cdot a\dfrac{\partial v}{\partial n'} + \dfrac{1}{|x'|}\,\gamma(\phi)v(x') = \dfrac{\varrho}{\psi(\varrho)}\,g(\varrho x'), & x' \in \Gamma^1_{1/2}. \end{cases}$$

By the Sobolev imbedding theorems,

$$(7.5) \qquad \sup_{x'\in G^1_{1/2}} |\nabla'v(x')| \le c\|v\|_{\mathbf{W}^{2,q}(G^1_{1/2})}, \qquad q > n.$$

By the local L^p a priori estimate [13], for the solution of the equation of (L'') in the domains $(G_i)^2_{1/4}$, $i = 1,\dots,N$, and near smooth portions of the boundaries $(\Sigma_k)^2_{1/4}$, $k = 1,\dots,N-1$, and $\Gamma^2_{1/4}$ we have

$$(7.6) \qquad \|v\|_{\mathbf{W}^{2,q}(G^1_{1/2})} \le c\Bigg\{\frac{\varrho^2}{\psi(\varrho)}\,\|f\|_{\mathbf{L}^q(G^2_{1/4})} + \frac{\varrho}{\psi(\varrho)}\,\|h\|_{\mathbf{W}^{1-1/q,q}(\Sigma^2_{1/4})}$$
$$+ \frac{\varrho}{\psi(\varrho)}\,\|g\|_{\mathbf{W}^{1-1/q,q}(\Gamma^2_{1/4})} + \|v\|_{\mathbf{L}^q(G^2_{1/4})}\Bigg\}.$$

Returning back to the variables x, from (7.5) and (7.6), it follows that

$$\sup_{G^{\varrho}_{\varrho/2}} |\nabla u| \le c\varrho^{-1}\{\varrho^{-n/q}\|u\|_{\mathbf{L}^q(G^{2\varrho}_{\varrho/4})} + \varrho^{2-n/q}\|f\|_{q,G^{2\varrho}_{\varrho/4}}$$
$$+ \varrho^{2-n/q}\|g\|_{\mathbf{V}^{1-1/q}_{q,0}(\Gamma^{2\varrho}_{\varrho/4})} + \varrho^{2-n/p}\|h\|_{\mathbf{V}^{1-1/q}_{q,0}(\Sigma^{2\varrho}_{\varrho/4})}\}$$

and

$$\varrho^{2-n/q}\|u\|_{\mathbf{V}^2_{q,0}(G^{\varrho}_{\varrho/2})} \le c\{\varrho^{-n/q}\|u\|_{\mathbf{L}^q(G^{2\varrho}_{\varrho/4})} + \varrho^{2-n/q}\|f\|_{q,G^{2\varrho}_{\varrho/4}}$$
$$+ \varrho^{2-n/q}\|g\|_{\mathbf{V}^{1-1/q}_{q,0}(\Gamma^{2\varrho}_{\varrho/4})} + \varrho^{2-n/q}\|h\|_{\mathbf{V}^{1-1/q}_{q,0}(\Sigma^{2\varrho}_{\varrho/4})}\}$$

or

$$\sup_{G^{\varrho}_{\varrho/2}} |\nabla u| \le c\varrho^{-1}\{|u|_{0,G^{2\varrho}_{\varrho/4}} + \|f\|_{\mathbf{V}^0_{q,2q-N}(G^{2\varrho}_{\varrho/4})}$$
$$+ \|g\|_{\mathbf{V}^{1-1/q}_{q,2q-n}(\Gamma^{2\varrho}_{\varrho/4})} + \|h\|_{\mathbf{V}^{1-1/q}_{q,2q-n}(\Sigma^{2\varrho}_{\varrho/4})}\}$$

and
$$\|u\|_{\mathbf{V}^2_{q,2q-n}(G^\varrho_{\varrho/2})} \le c\{|u|_{0,G^{2\varrho}_{\varrho/4}} + \|f\|_{\mathbf{V}^0_{q,2q-N}(G^{2\varrho}_{\varrho/4})}$$
$$+ \|g\|_{\mathbf{V}^{1-1/q}_{q,2q-n}(\Gamma^{2\varrho}_{\varrho/4})} + \|h\|_{\mathbf{V}^{1-1/q}_{q,2q-n}(\Sigma^{2\varrho}_{\varrho/4})}\}.$$

Hence, because of (1.4), (1.5) and assumption (c), the required results (1.6) and (1.7) follow.

Now we can make the statement of Theorem 5.2 more precise. In fact, the estimate (1.4) proved above allows us to consider in Theorem 5.2 the value $\varkappa = \lambda - 1$. As a result we obtain the last statement of our theorem and the estimate (1.9) with the best possible exponent that satisfies the inequality (1.8).

8. Examples: Eigenvalue transmission problem in a composite plane domain with an angular point. We study the transmission boundary value problem (L). Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G that is a smooth curve everywhere except at the origin $\mathcal{O} \in \partial G$. Near $\mathcal{O}$ it is a fan that consists of N corners with vertices at $\mathcal{O}$. Thus

$$G = \bigcup_{i=1}^{N} G_i; \quad \partial G = \bigcup_{j=0}^{N+1} \Gamma_j; \quad \Sigma = \bigcup_{k=1}^{N-1} \Sigma_k.$$

Here Σ_k, $k = 1, \dots, N-1$, are rays that divide G into angular domains G_i, $i = 1, \dots, N$. Let ω_i be the aperture at the vertex $\mathcal{O}$ in G_i, $i = 1, \dots, N$. Define $\theta_k = \omega_1 + \cdots + \omega_k$. If (r, ω) are the polar coordinates of (x, y), then $\omega = 0$ is the equation of the ray Γ_0 and $\omega = \theta_N$ is the equation of the ray Γ_{N+1}. Let $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ be the curvilinear portion of ∂G.
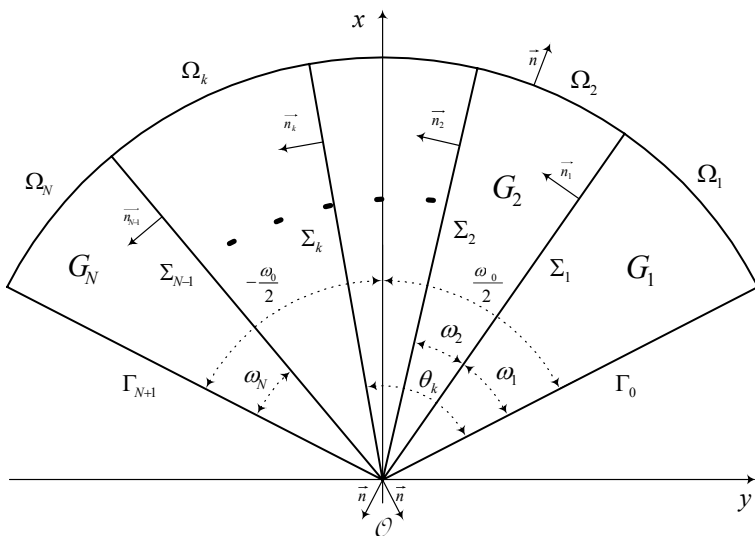

Fig. 2

8.1. *Three-media transmission problem.* Our goal is the derivation of the eigenvalue equation that corresponds to our transmission problem for $N = 3$. Let S^1 be the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$. We define $\Omega_i = G_i \cap S^1$, $i = 1, 2, 3$. The eigenvalue problem is the following (see also [11, 12]):

$$(8.1) \quad \begin{cases} \psi_i'' + \lambda^2 \psi_i(\omega) = 0, \quad \omega \in \Omega_i \ (i = 1, 2, 3); \\ \psi_1(\omega_1) = \psi_2(\omega_1); \quad \psi_3(\theta_2) = \psi_2(\theta_2); \\ a_2 \psi_2'(\omega_1) - a_1 \psi_1'(\omega_1) + \beta_1 \psi_1(\omega_1) = 0; \\ a_3 \psi_3'(\theta_2) - a_2 \psi_2'(\theta_2) + \beta_2 \psi_2(\theta_2) = 0; \\ \alpha_0 \psi_1'(0) - \gamma_0 \psi_1(0) = 0; \\ \alpha_3 \psi_3'(\theta_3) - \gamma_3 \psi_3(\theta_3) = 0, \end{cases}$$

where $\alpha_0 = \alpha|_{\Gamma_0} = \alpha|_{\omega=0}$, $\alpha_3 = \alpha|_{\Gamma_4} = \alpha|_{\omega=\theta_3}$, $\gamma_0 = \gamma(0)$, $\gamma_3 = \gamma(\theta_3)$.

We find the general solution of (8.1):

$$\psi_i(\omega) = A_i \cos(\lambda\omega) + B_i \sin(\lambda\omega),$$
$$\psi_i'(\omega) = -\lambda A_i \sin(\lambda\omega) + \lambda B_i \cos(\lambda\omega) \quad (i = 1, 2, 3),$$

where A_i, B_i $(i = 1, 2, 3)$ are any constants. From the boundary condition of (8.1) we obtain a homogeneous algebraic system of six linear equations for A_i, B_i $(i = 1, 2, 3)$. The determinant of the system must be zero for a nontrivial solution to exist. This gives the required eigenvalue equation:

$$(8.2) \quad [\lambda^2 \alpha_3 a_3 (\beta_1 \gamma_0 + \lambda^2 \alpha_0 a_1) - \gamma_3 (\beta_1 \beta_2 \gamma_0 + \lambda^2 \alpha_0 \beta_2 a_1 - \lambda^2 \gamma_0 a_2^2)]$$
$$\cdot \sin(\lambda\omega_1) \sin(\lambda\omega_2) \sin(\lambda\omega_3)$$
$$+ \lambda \cdot [\lambda^2 \alpha_3 a_3 (\beta_1 \alpha_0 - \gamma_0 a_1) - \gamma_3 (\beta_1 \beta_2 \alpha_0 - \gamma_0 \beta_2 a_1 - \lambda^2 \alpha_0 a_2^2)]$$
$$\cdot \cos(\lambda\omega_1) \sin(\lambda\omega_2) \sin(\lambda\omega_3)$$
$$+ \lambda \cdot [\gamma_3 a_3 (\beta_1 \gamma_0 + \lambda^2 \alpha_0 a_1) + \alpha_3 (\beta_1 \beta_2 \gamma_0 + \lambda^2 \alpha_0 \beta_2 a_1 - \lambda^2 \gamma_0 a_2^2)]$$
$$\cdot \sin(\lambda\omega_1) \sin(\lambda\omega_2) \cos(\lambda\omega_3)$$
$$+ \lambda^2 \cdot [\gamma_3 a_3 (\beta_1 \alpha_0 - \gamma_0 a_1) + \alpha_3 (\beta_1 \beta_2 \alpha_0 - \gamma_0 \beta_2 a_1 - \lambda^2 \alpha_0 a_2^2)]$$
$$\cdot \cos(\lambda\omega_1) \sin(\lambda\omega_2) \cos(\lambda\omega_3)$$
$$+ \lambda a_2 \cdot [\gamma_3 (\beta_2 \gamma_0 + \lambda^2 \alpha_0 a_1 + \beta_1 \gamma_0) - \lambda^2 \alpha_3 \gamma_0 a_3] \cdot \sin(\lambda\omega_1) \cos(\lambda\omega_2) \sin(\lambda\omega_3)$$
$$+ \lambda^2 a_2 \cdot [\gamma_3 (\beta_2 \alpha_0 + \alpha_0 \beta_1 - \gamma_0 a_1) - \lambda^2 \alpha_3 \alpha_0 a_3] \cdot \cos(\lambda\omega_1) \cos(\lambda\omega_2) \sin(\lambda\omega_3)$$
$$- \lambda^2 a_2 \cdot [\gamma_0 \gamma_3 a_3 + \alpha_3 (\beta_2 \gamma_0 + \lambda^2 \alpha_0 a_1 + \beta_1 \gamma_0)] \cdot \sin(\lambda\omega_1) \cos(\lambda\omega_2) \cos(\lambda\omega_3)$$
$$- \lambda^3 a_2 \cdot [\alpha_0 \gamma_3 a_3 + \alpha_3 (\beta_2 \alpha_0 + \alpha_0 \beta_1 - a_1 \gamma_0)] \cdot \cos(\lambda\omega_1) \cos(\lambda\omega_2) \cos(\lambda\omega_3) = 0.$$

We consider the special cases of boundary conditions.

- *Dirichlet problem*: $\alpha_0 = \alpha_3 = \beta_1 = \beta_2 = 0$; $\gamma_0 = \gamma_3 = 1$. Then

$$a_2^2 \cdot \sin(\lambda\omega_1) \sin(\lambda\omega_2) \sin(\lambda\omega_3) - a_1 a_3 \cdot \cos(\lambda\omega_1) \sin(\lambda\omega_2) \cos(\lambda\omega_3)$$
$$- a_1 a_2 \cdot \cos(\lambda\omega_1) \cos(\lambda\omega_2) \sin(\lambda\omega_3) - a_2 a_3 \cdot \sin(\lambda\omega_1) \cos(\lambda\omega_2) \cos(\lambda\omega_3) = 0.$$

In the isotropic case $(a_1 = a_2 = a_3)$ we hence obtain the well known result:

$$\sin(\lambda\theta_3) = 0, \quad \text{so} \quad \lambda_n = \frac{\pi n}{\theta_3}, \quad n = 1, 2, \ldots.$$

Corollary: $\lambda = \pi/\theta_3 > 1$ if $\omega_1 + \omega_2 + \omega_3 < \pi$.

• *Neumann problem*: $\alpha_0 = \alpha_3 = 1$; $\beta_1 = \beta_2 = \gamma_0 = \gamma_3 = 0$. Then

$$a_2^2 \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) + a_2 a_3 \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)$$
$$+ a_1 a_2 \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) - a_1 a_3 \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) = 0.$$

In the isotropic case $(a_1 = a_2 = a_3)$ we hence obtain the well known result:

$$\sin(\lambda\theta_3) = 0, \quad \text{so} \quad \lambda_n = \frac{\pi n}{\theta_3}, \quad n = 0, 1, 2, \ldots.$$

Corollary: $\lambda = \pi/\theta_3 > 1$ if $\omega_1 + \omega_2 + \omega_3 < \pi$.

• *Mixed problem*: $\alpha_0 = \gamma_3 = 1$, $\alpha_3 = \beta_1 = \beta_2 = \gamma_0 = 0$. Then

$$a_2^2 \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) + a_1 a_3 \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)$$
$$+ a_1 a_2 \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) - a_2 a_3 \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) = 0.$$

In the isotropic case $(a_1 = a_2 = a_3)$ we hence obtain the well known result:

$$\cos(\lambda\theta_3) = 0, \quad \text{so} \quad \lambda_n = \frac{\pi(2n - 1)}{2\theta_3}, \quad n = 1, 2, \ldots.$$

Corollary: $\lambda = \pi/2\theta_3 > 1$ if $\omega_1 + \omega_2 + \omega_3 < \pi/2$.

• *Robin problem*: $\alpha_0 = 1$, $\alpha_3 = -1$; $\beta_1 = \beta_2 = 0$.

$$(\lambda^2 a_1 a_3 - \gamma_0\gamma_3 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)$$
$$- \lambda \cdot (\gamma_3 a_2^2 + \gamma_0 a_1 a_3) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)$$
$$- \lambda \cdot (\gamma_3 a_1 a_3 + \gamma_0 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)$$
$$+ (\gamma_0\gamma_3 a_1 a_3 - \lambda^2 a_2^2) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)$$
$$- \lambda a_2 \cdot (\gamma_3 a_1 + \gamma_0 a_3) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)$$
$$+ a_2 \cdot (\gamma_3\gamma_0 a_1 - \lambda^2 a_3) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)$$
$$+ a_2 \cdot (\gamma_0\gamma_3 a_3 - \lambda^2 a_1) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)$$
$$+ \lambda a_2 \cdot (\gamma_3 a_3 + a_1\gamma_0) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) = 0.$$

In the isotropic case $(a_1 = a_2 = a_3)$ we hence obtain the well known result (see [3, Example 1, §10.1.7])

$$\tan(\lambda\theta_3) = \frac{\lambda(\gamma_0 + \gamma_3)}{\lambda^2 - \gamma_0\gamma_3}.$$

8.2. *Two-media transmission problem.* Setting in the above derived equations $\omega_2 = 0$, $\beta_2 = 0$ we get the corresponding results for domains with two different media $(N = 2)$. The eigenvalue problem in this case has the form

$$(8.3) \quad \begin{cases} \psi_i'' + \lambda^2 \psi_i(\omega) = 0, \quad \omega \in \Omega_i \ (i = 1, 2); \\ \psi_1(\omega_1) = \psi_2(\omega_1); \\ a_2 \psi_2'(\omega_1) - a_1 \psi_1'(\omega_1) + \beta_1 \psi_1(\omega_1) = 0; \\ \alpha_0 \psi_1'(0) - \gamma_0 \psi_1(0) = 0; \\ \alpha_2 \psi_2'(\theta_2) - \gamma_2 \psi_2(\theta_2) = 0. \end{cases}$$

The eigenvalue equation is

$$(8.4) \qquad [\lambda^2(\alpha_2\gamma_0 a_2 - \alpha_0\gamma_2 a_1) - \beta_1\gamma_0\gamma_2] \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)$$
$$+ \lambda(\lambda^2\alpha_0\alpha_2 a_2 - \alpha_0\beta_1\gamma_2 + \gamma_0\gamma_2 a_1) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)$$
$$+ \lambda(\lambda^2\alpha_0\alpha_2 a_1 + \lambda\beta_1\alpha_2\gamma_0 + \gamma_0\gamma_2 a_2) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)$$
$$+ \lambda^2(\alpha_0\gamma_2 a_2 + \beta_1\alpha_0\alpha_2 - \alpha_2\gamma_0 a_1) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2) = 0.$$

Here we consider in detail the 2-dimensional transmission problem with two different media $(\omega_1 = \omega_2 = \omega_0/2)$ for the Laplace operator in an angular symmetric domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, the domain G lies inside the corner

$$G_0 = \{(r, \omega) \mid r > 0; \ -\omega_0/2 < \omega < \omega_0/2\}, \quad \omega_0 \in (0, 2\pi);$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\omega_0/2$ and $\omega = \omega_0/2$. We define $\Gamma_{\pm} = \{(r, \omega) \mid r > 0; \ \omega = \pm\omega_0/2\}$, $\Sigma_0 = \{(r, \omega) \mid r > 0; \ \omega = 0\}$ and put $\sigma(\omega)|_{\Sigma_0} = \sigma(0) = \sigma = $ const ≥ 0, $\gamma(\omega)|_{\omega = \pm\omega_0/2} = \gamma_{\pm} = $ const > 0. We consider the following problem:

$$(8.5) \quad \begin{cases} \Delta u_{\pm} = f_{\pm}(x), & x \in G_{\pm}; \\ [u]_{\Sigma_0} = 0, \quad \left[a\dfrac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \dfrac{1}{|x|}\,\sigma u(x) = h(x), & x \in \Sigma_0, \\ \alpha_{\pm}\dfrac{\partial u_{\pm}}{\partial \vec{n}} + \dfrac{1}{|x|}\,\gamma_{\pm} u_{\pm}(x) = g_{\pm}(x), & x \in \Gamma_{\pm} \setminus \mathcal{O}, \end{cases}$$

where

$$\alpha_{\pm} = \begin{cases} 0 & \text{if } \Gamma_{\pm} \in \mathcal{D}, \\ \pm 1 & \text{if } \Gamma_{\pm} \notin \mathcal{D}. \end{cases}$$

It is well known that the homogeneous problem $(f(x)=0, \ h(x)=0, \ g(x)=0)$ has a solution of the form $u(r, \omega) = r^{\lambda}\psi(\omega)$, where λ is an eigenvalue and $\psi(\omega)$ is a regular eigenfunction associated to this λ of the problem

$$(8.6) \quad \begin{cases} \psi_+'' + \lambda^2 \psi_+(\omega) = 0, \quad \omega \in (0, \omega_0/2); \\ \psi_-'' + \lambda^2 \psi_-(\omega) = 0, \quad \omega \in (-\omega_0/2, 0); \\ \psi_+(0) = \psi_-(0); \quad a_+\psi_+'(0) - a_-\psi_-'(0) = \sigma\psi(0); \\ \pm\alpha_{\pm}\psi'(\pm\omega_0/2) + \gamma_{\pm}\psi(\pm\omega_0/2) = 0. \end{cases}$$

1) *Case $\lambda = 0$, $\sigma > 0$.* In this case the solution of our equations has the form $\psi_\pm(\omega) = A_\pm \cdot \omega + B_\pm$. From the boundary conditions we obtain $B_+ = B_- = B$ and for A_+, A_-, B we have the system

$$\begin{cases} a_+ A_+ - a_- A_- - \sigma B = 0, \\ \left(\alpha_+ + \dfrac{\omega_0}{2} \gamma_+ \right) A_+ + \gamma_+ B = 0, \\ -\left(\alpha_- + \dfrac{\omega_0}{2} \gamma_- \right) A_- + \gamma_- B = 0. \end{cases}$$

Since $A_+^2 + A_-^2 + B^2 \neq 0$, the determinant of the system must be zero, that is,

$$(8.7) \qquad \sigma \left(\alpha_+ + \frac{\omega_0}{2} \gamma_+ \right) \left(\alpha_- + \frac{\omega_0}{2} \gamma_- \right) + a_+ \gamma_+ \left(\alpha_- + \frac{\omega_0}{2} \gamma_- \right)$$
$$+ a_- \gamma_- \left(\alpha_+ + \frac{\omega_0}{2} \gamma_+ \right) = 0.$$

Thus if (8.7) holds, then $\lambda = 0$ and the corresponding eigenfunction is

$$\psi(\omega) =$$
$$\begin{cases} -\gamma_+ \left(\alpha_- + \dfrac{\omega_0}{2} \gamma_- \right) \left(\omega + \dfrac{a_+}{\sigma} \right) - \dfrac{a_- \gamma_-}{\sigma} \left(\alpha_+ + \dfrac{\omega_0}{2} \gamma_+ \right), & \omega \in (0, \omega_0/2), \\ \gamma_- \left(\alpha_+ + \dfrac{\omega_0}{2} \gamma_+ \right) \left(\omega - \dfrac{a_-}{\sigma} \right) - \dfrac{a_+ \gamma_+}{\sigma} \left(\alpha_- + \dfrac{\omega_0}{2} \gamma_- \right), & \omega \in (-\omega_0/2, 0). \end{cases}$$

2) *Case $\lambda \neq 0$.* In this case the solution of our equations has the form

$$\psi_\pm(\omega) = A_\pm \cos(\lambda \omega) + B_\pm \sin(\lambda \omega).$$

From the boundary conditions we obtain $A_+ = A_- = A$ and for A, B_+, B_- we have the system

$$\begin{cases} \sigma A - \lambda a_+ B_+ + \lambda a_- B_- = 0, \\ \left(\gamma_+ \cos \dfrac{\lambda \omega_0}{2} - \lambda \alpha_+ \sin \dfrac{\lambda \omega_0}{2} \right) A + \left(\gamma_+ \sin \dfrac{\lambda \omega_0}{2} + \lambda \alpha_+ \cos \dfrac{\lambda \omega_0}{2} \right) B_+ = 0, \\ \left(\gamma_- \cos \dfrac{\lambda \omega_0}{2} - \lambda \alpha_- \sin \dfrac{\lambda \omega_0}{2} \right) A - \left(\gamma_- \sin \dfrac{\lambda \omega_0}{2} + \lambda \alpha_- \cos \dfrac{\lambda \omega_0}{2} \right) B_- = 0. \end{cases}$$

Since $A^2 + B_+^2 + B_-^2 \neq 0$, the determinant must be zero; this means that λ is defined from the transcendental equation

$$(8.8) \quad \sigma(\lambda^2 \alpha_+ \alpha_- + \gamma_+ \gamma_-) + \lambda^2 (a_+ - a_-)(\alpha_- \gamma_+ - \alpha_+ \gamma_-)$$
$$+ \lambda[\sigma(\alpha_- \gamma_+ + \alpha_+ \gamma_-) + (a_+ + a_-)(\gamma_+ \gamma_- - \lambda^2 \alpha_+ \alpha_-)] \sin(\lambda \omega_0)$$
$$+ [\sigma(\lambda^2 \alpha_+ \alpha_- - \gamma_+ \gamma_-) + \lambda^2 (a_+ + a_-)(\alpha_- \gamma_+ + \alpha_+ \gamma_-)] \cos(\lambda \omega_0) = 0.$$

Now we investigate each type of boundary conditions separately.

The Dirichlet problem: $\alpha\pm = 0$. The equation (8.8) takes the form $\sigma(1 - \cos(\lambda\omega_0)) + \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0$. Hence

$$
\lambda = \begin{cases}
\dfrac{\pi}{\omega_0} & \text{if } \sigma = 0, \\
\text{the least positive root of } \tan\dfrac{\lambda\omega_0}{2} = -\dfrac{a_+ + a_-}{\sigma}\cdot\lambda & \text{if } \sigma \neq 0,
\end{cases}
$$

and the corresponding eigenfunction is

$$
\psi(\omega) = \begin{cases}
\sin\lambda(\omega_0/2 - \omega), & \omega \in (0, \omega_0/2), \\
\sin\lambda(\omega_0/2 + \omega), & \omega \in (-\omega_0/2, 0).
\end{cases}
$$

The Neumann problem: $\gamma\pm = 0$. The equation (8.8) takes the form $\sigma(1 + \cos(\lambda\omega_0)) - \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0$. Hence $\lambda = \min\{\lambda^*, \pi/\omega_0\}$, where λ^* is the least positive root of the transcendental equation

$$
\tan\frac{\lambda\omega_0}{2} = \frac{\sigma}{a_+ + a_-}\cdot\frac{1}{\lambda}.
$$

For $\lambda = \pi/\omega_0$ we find the corresponding eigenfunction

$$
\psi(\omega) = \begin{cases}
a_-\sin(\pi\omega/\omega_0), & \omega \in (0, \omega_0/2), \\
a_+\sin(\pi\omega/\omega_0), & \omega \in (-\omega_0/2, 0).
\end{cases}
$$

For $\lambda = \lambda^*$ we find the corresponding eigenfunction

$$
\psi(\omega) = \begin{cases}
\cos\lambda^*(\omega - \omega_0/2), & \omega \in (0, \omega_0/2), \\
\cos\lambda^*(\omega + \omega_0/2), & \omega \in (-\omega_0/2, 0).
\end{cases}
$$

Mixed problem: $\alpha_+ = 1$, $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$. The equation (8.8) takes the form

$$
(8.9) \qquad \sigma\sin(\lambda\omega_0) + \lambda(a_+ + a_-)\cos(\lambda\omega_0) = \lambda(a_+ - a_-).
$$

In particular, if $\sigma = 0$, and $a_+ a_- > 0$ then

$$
\lambda = \frac{2}{\omega_0}\arctan\sqrt{a_-/a_+} > 1, \quad \text{if } \omega_0 < 2\arctan\sqrt{a_-/a_+},
$$

and the corresponding eigenfunction is

$$
\psi(\omega) = \begin{cases}
\cos(\lambda\omega) + \sqrt{a_-/a_+}\cdot\sin(\lambda\omega), & \omega \in (0, \omega_0/2), \\
\cos(\lambda\omega) + \sqrt{a_+/a_-}\cdot\sin(\lambda\omega), & \omega \in (-\omega_0/2, 0).
\end{cases}
$$

If λ is the least positive root of the transcendental equation (8.9), then the corresponding eigenfunction is

$$
\psi(\omega) = \begin{cases}
\sin(\lambda\omega_0/2)\cos\lambda(\omega - \omega_0/2), & \omega \in (0, \omega_0/2), \\
\cos(\lambda\omega_0/2)\sin\lambda(\omega + \omega_0/2), & \omega \in (-\omega_0/2, 0).
\end{cases}
$$

The Robin problem: $\alpha_\pm = a_\pm$; $\gamma_\pm \neq 0$. The equation (8.8) takes the form

$$\sigma(\lambda^2 a_+ a_- + \gamma_+ \gamma_-) + \lambda^2(a_+ - a_-)(a_- \gamma_+ - a_+ \gamma_-)$$
$$+ \lambda[\sigma(a_- \gamma_+ + a_+ \gamma_-) + (a_+ + a_-)(\gamma_+ \gamma_- - \lambda^2 a_+ a_-)]\sin(\lambda\omega_0)$$
$$+ [\sigma(\lambda^2 a_+ a_- - \gamma_+ \gamma_-) + \lambda^2(a_+ + a_-)(a_- \gamma_+ + a_+ \gamma_-)]\cos(\lambda\omega_0) = 0.$$

In particular, in the case of the problem without the interface ($a_+ = a_- = 1$, $\sigma = 0$) we obtain the least eigenvalue as the least positive root of the transcendental equation

$$(8.10) \qquad\qquad \tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-}$$

and the corresponding eigenfunction is

$$\psi(\omega) = \lambda\cos[\lambda(\omega - \omega_0/2)] - \gamma_+\sin[\lambda(\omega - \omega_0/2)]$$

(see [3, §10.1.7]).

In order to have $\lambda > 1$ we show that the condition $\gamma_\pm \geq \gamma_0 > \tan(\omega_0/2)$ from the assumption (b) of our theorems is justified. In fact, we rewrite the equation (8.10) in the equivalent form

$$(8.11) \qquad\qquad \lambda = \frac{1}{\omega_0}\left(\arctan\frac{\gamma_+}{\lambda} + \arctan\frac{\gamma_-}{\lambda}\right).$$

Hence

$$1 < \lambda < \frac{1}{\omega_0}(\arctan\gamma_+ + \arctan\gamma_-),$$

so

$$(8.12) \qquad \omega_0 < \arctan\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-} \quad \text{provided that } \gamma_+\gamma_- < 1.$$

But assumption (b) means that $\gamma_\pm \geq \gamma_0 > \tan(\omega_0/2)$. Hence

$$\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-} \geq \frac{2\gamma_0}{1 - \gamma_0^2} > \frac{2\tan\frac{\omega_0}{2}}{1 - \tan^2\frac{\omega_0}{2}} = \tan\omega_0, \quad \omega_0 < \frac{\pi}{2}.$$

Thus we get (8.12). In the case $\gamma_\pm \geq \gamma_0 > \tan(\omega_0/2) \geq 1$ for $\omega_0 \in [\pi/2, \pi)$ the inequality $\lambda > 1$ is fulfilled a fortiori, since the eigenvalues increase together with $\gamma(\phi)$ (see for example [6, Theorem 6, §2, Chapter VI]). In fact, $\lambda = 1$ is the solution of the equation (8.10) under $\gamma_\pm = \tan(\omega_0/2)$.

8.3. The function

$$u(r, \omega) = r^\lambda \ln\left(\frac{1}{r}\right)\psi(\omega)$$

with λ and $\psi(\omega)$ defined by (8.6) is a solution of the problem (8.5) with $f_\pm = -2a_\pm\lambda r^{\lambda-2}\psi(\omega)$, $h(x) = 0$, $g(x) = 0$ in the corner G_0. All assumptions of Theorem 1.1 are satisfied with $s = \lambda$. This example shows the precision of the assumptions on the right sides of (L) in Theorem 1.1.

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