APPLICATIONES MATHEMATICAE
32,3 (2005), pp. 243–261

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ON RANDOM SPLIT OF THE SEGMENT

Abstract. We consider a partition of the interval $[0,1]$ by two partition procedures. In the first a chosen piece of $[0,1]$ is split into halves, in the second it is split by uniformly distributed points. Initially, the interval $[0,1]$ is divided either into halves or by a uniformly distributed random variable. Next a piece to be split is chosen either with probability equal to its length or each piece is chosen with equal probability, and then the chosen piece is split by one of the above procedures. These actions are repeated indefinitely. We investigate the probability distribution of the lengths of the consecutive pieces after n splits.

1. Introduction. Kopociński [6] investigated a problem of Hugo Steinhaus (cf. [7]) regarding the population growth of a rod-shaped bacterium. The problem is as follows: initially, one part breaks off from the original bacillus becoming an independent bacillus. At each step one part breaks off from the longest bacillus. The length of the descendant is the shortest at the moment. Steinhaus stated that if the first partition is into mutually non-rational pieces then in any given generation at most three different lengths of bacillus exist. Moreover fractions of small, medium and large ones oscillate over time. Kopociński added a random component to Steinhaus' problem so that the number of different lengths is countable and the probability distribution of the lengths oscillates over time. Kopociński considered the following partition of the interval $[0,1]$:

- Step 1: $[0,1]$ is divided either into halves (*splitting in half*), or into two pieces by using a uniformly distributed random variable (*uniform split*).

2000 *Mathematics Subject Classification*: 60F05, 60F15.
Key words and phrases: random partition, uniform split, splitting in half, random choice, proportional choice, convergence in probability and almost sure, moments, covariance.

- Step 2: First, one of the two pieces is chosen, either each with probability 1/2 (*random choice*), or each with probability equal to its length (*proportional choice*). Then the chosen piece is divided either into halves, or into two pieces by a uniformly distributed random variable. And now there are three pieces.
- Step 3: Like step 2, first one of the three pieces is chosen, then the chosen piece is divided, giving four pieces.

This action of partitioning is continued indefinitely. Denote by n the current number of split points and by

$$\mathbb{D}_n = (D_{1,n}, D_{2,n}, \ldots, D_{n+1,n}) \tag{1}$$

the vector of lengths of the consecutive pieces after n splits. Under the above assumptions four ways of choosing and partitioning can be obtained. In each of them Kopociński investigated the probability distribution of $D_{1,n}$. But the consecutive pieces are not identically distributed except for one case (proportional choice, uniform split). Hence in the other three cases of nonidentically distributed pieces we are interested in the probability distribution of $D_{2,n}$. Moreover, in the case of proportional choice and splitting in half, which corresponds to Steinhaus' problem, we give recurrence relations for the probability distribution and moments of $D_{i,n}$ for any i, $1 \le i \le n+1$.

In Section 2 we give some notions which will be needed in the next sections. The types of partitioning considered in Sections 3, 4, 5, respectively are described in the titles of these sections. We investigate the random variables

$$Z_{i,n} = -\log_2 D_{i,n}, \qquad T_{i,n} = D_{i,n}^{-1}, \qquad 1 \le i \le n+1,\ n \ge 1. \tag{2}$$

For $R_m \sim U(0,1)$, $1 \le m \le n$, i.e. for a uniformly distributed random variable on the interval $[0,1]$, set

$$E_m = -\log_2 R_m, \qquad F_m = -\log_2(1-R_m). \tag{3}$$

2. Preliminaries. Denote by $H_k^{(r)}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, the harmonic number of order r, i.e.

$$H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}, \qquad r \ge 1 \tag{4}$$

(cf. [4, Chap. 6.3]). For $r > 1$ we use the Riemann Zeta function $\zeta(r)$ and the Hurwitz (generalized) Zeta function $\zeta(r,q)$ defined by

$$\zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r}, \qquad \zeta(r,q) = \sum_{i=0}^{\infty} \frac{1}{(q+i)^r}, \qquad q \ne 0, -1, \ldots$$

(cf. [3, 7.422.1, 7.421.1]). There are relations between the harmonic numbers and the Psi (or Digamma) function $\psi(x) := \psi^{(0)}(x)$ defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x),$$

and the derivatives of the Psi function (or the Polygamma functions)

$$\psi^{(r)}(x) = \frac{d^{r+1}}{dx^{r+1}} \log \Gamma(x) = \frac{d^r}{dx^r} \psi(x), \qquad r = 0, 1, 2 \ldots,$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$, $x > 0$, is the Gamma function. Since

$$H_k^{(1)} = \gamma + \psi(k+1)$$

(cf. [5, (5.13.5)]), where $\gamma := -\psi(1) = 0,57721566\ldots$ is the Euler–Mascheroni constant and

$$\psi^{(r-1)}(k+1) = (-1)^r (r-1)! \sum_{i=0}^{\infty} \frac{1}{(k+1+i)^r}$$

(cf. [3, 6.356]), we have

$$\zeta(r, k+1) = \zeta(r) - H_k^{(r)} = \frac{(-1)^r}{(r-1)!} \, \psi^{(r-1)}(k+1),$$

which implies

$$H_k^{(r)} = \frac{(-1)^r}{(r-1)!} (\psi^{(r-1)}(1) - \psi^{(r-1)}(k+1)) \tag{5}$$

(cf. [5, (6.2.4)]). It is known (cf. [3]) that

$$\psi(k+z) \sim \log k, \quad \text{or equivalently} \quad H_{k+z}^{(1)} \sim \log k, \qquad z \geq 1, \ k \to \infty, \tag{6}$$

and

$$\lim_{k \to \infty} \psi'(k+z) = 0 \quad \text{or} \quad \lim_{k \to \infty} H_k^{(r)} = \zeta(r), \qquad z \geq 1, \ r > 1. \tag{7}$$

We denote by $(x)_n$ the *Pochhammer symbol* (or the *shifted factorial*), i.e.

$$(x)_n = x(x+1) \cdots (x+n-1), \qquad n \geq 1,$$

and $(x)_n = 1$ for $n = 0$.

3. Random choice, uniform split. For a given i, $1 \leq i \leq n+2$, we consider $D_{i,n+1}$, the length of the ith piece after $n+1$ splits. After n splits there are $n+1$ pieces $D_{1,n}, \ldots, D_{n+1,n}$ and $D_{i,n+1}$ depends on which of these is chosen for the $(n+1)$th split. When $i = 1$, $D_{1,n+1}$ depends on whether $D_{1,n}$ or some other is chosen, and similarly when $i = n+2$, $D_{n+2,n+1}$ depends on whether $D_{n+1,n}$ or some other is chosen. Next, when $i = 2$ or $n+1$ there are three possibilities: for example, $D_{2,n+1}$ depends on whether $D_{1,n}$, $D_{2,n}$ or some other is chosen. And generally, for $2 < i < n+1$ there are four possibilities, according as the piece chosen is $D_{j,n}$ for $j < i-1$, $D_{i-1,n}$, $D_{i,n}$, or $D_{j,n}$ for $j > i$. We now define the random variable

$$J_n^{(i)} = \begin{cases} 3 & \text{if } D_{j,n} \text{ is split, } 1 \le j < i-1, \\ 2 & \text{if } D_{i-1,n} \text{ is split,} \\ 1 & \text{if } D_{i,n} \text{ is split,} \\ 0 & \text{if } D_{j,n} \text{ is split, } i < j \le n+1, \end{cases} \qquad n \ge i-1.$$

Then the probability distribution of $J_n^{(i)}$ is given by

$$\begin{gathered} P(J_n^{(i)} = 3) = \frac{i-2}{n+1}, \quad P(J_n^{(i)} = 2) = P(J_n^{(i)} = 1) = \frac{1}{n+1}, \\ P(J_n^{(i)} = 0) = 1 - \frac{i}{n+1}, \quad n \ge i-1,\ i \ge 2. \end{gathered} \tag{8}$$

Let $A_n^{(i)}$, $B_n^{(i)}$ and $C_n^{(i)}$ be the random variables defined as follows:

$$A_n^{(i)} = I[J_n^{(i)} = 2], \quad B_n^{(i)} = I[J_n^{(i)} = 1], \quad C_n^{(i)} = I[J_n^{(i)} = 0]$$

for $n \ge i-1$, where $I[A]$ denotes the indicator of the event A. Note that $A_1^{(i)}, A_2^{(i)}, \ldots, A_n^{(i)}$ are mutually independent. Similar statements are true for $B_j^{(i)}$, $C_j^{(i)}$, R_j, E_j and F_j, $1 \le j \le n$.

Referring to (2) and (3), Kopociński [6] showed that $D_{1,n+1}$ satisfies the following recursive formulae:

$$D_{1,1} = R_0, \quad D_{1,n+1} = \begin{cases} D_{1,n} & \text{if } J_n^{(1)}{=}0, \\ D_{1,n}R_n & \text{if } J_n^{(1)} = 1, \end{cases}$$

and

$$D_{1,n+1} = \prod_{j=0}^{n} R_j^{B_j^{(1)}}, \quad Z_{1,n+1} = \sum_{j=0}^{n} B_j^{(1)} E_j. \tag{9}$$

We are interested in $D_{2,n+1}$ in (1). For $n \ge 1$ the following recursive formulae are satisfied:

$$D_{1,1} = R_0, \quad D_{2,1} = 1 - R_0, \quad D_{2,n+1} = \begin{cases} (1-R_n)D_{1,n} & \text{if } J_n^{(2)} = 2, \\ R_n D_{2,n} & \text{if } J_n^{(2)} = 1, \\ D_{2,n} & \text{if } J_n^{(2)} = 0. \end{cases}$$

Hence by (9) for $n \ge 1$ we have

$$\begin{gathered} D_{2,1} = 1 - R_0, \quad D_{2,n+1} = (1-R_n)^{A_n^{(2)}} D_{1,n}^{A_n^{(2)}} R_n^{B_n^{(2)}} D_{2,n}^{1-A_n^{(2)}}, \\ Z_{2,1} = F_0, \quad Z_{2,n+1} = A_n^{(2)} F_n + A_n^{(2)} Z_{1,n} + B_n^{(2)} E_n + (1 - A_n^{(2)}) Z_{2,n}. \end{gathered} \tag{10}$$

Since $B_i^{(1)} = A_i^{(2)}$, $0 \le i \le n$, for $n \ge 1$ we get

$$
\begin{aligned}
D_{2,n+1} = & \prod_{j=0}^{n} (1-R_j)^{A_j^{(2)} \prod_{i=j+1}^{n}(1-A_i^{(2)})} \\
& \cdot R_j^{B_j^{(2)} \prod_{i=j+1}^{n}(1-A_i^{(2)}) + B_j^{(1)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n}(1-A_i^{(2)})}, \\
Z_{2,n+1} = & \sum_{j=0}^{n} \Big(A_j^{(2)} \prod_{i=j+1}^{n} (1-A_i^{(2)}) F_j + B_j^{(2)} \prod_{i=j+1}^{n} (1-A_i^{(2)}) E_j \\
& + A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1-A_i^{(2)}) E_j \Big),
\end{aligned} \tag{11}
$$

where $A_0^{(2)} = 1$ and $B_0^{(2)} = 0$, which can be proved by induction with respect to n using (10).

For $D_{i,n}$, $1 \le i \le n+1$, we get

Proposition 1. *Under the assumptions of random choice and uniform split, the following recursive formulae hold*:

$$D_{1,1} = R_0, \qquad D_{2,1} = 1 - R_0,$$

and

$$
D_{i,n+1} = \begin{cases} D_{i-1,n} & \text{if } J_n^{(i)} = 3, \\ (1-R_n) D_{i-1,n} & \text{if } J_n^{(i)} = 2, \\ R_n D_{i,n} & \text{if } J_n^{(i)} = 1, \\ D_{i,n} & \text{if } J_n^{(i)} = 0, \end{cases} \qquad n \ge i-1.
$$

Write $c = (\log 2)^{-1}$. From the above relations we get formulae for the expectation, variance and covariance of $Z_{1,n}$ and $Z_{2,n}$.

Proposition 2. *Under the assumptions of random choice and uniform split the expectation and variance of* $Z_{1,n+1}$ *are given by*

$$
\begin{aligned}
\mathrm{E} Z_{1,n+1} &= c(\psi(n+2) + \gamma), \\
\sigma^2 Z_{1,n+1} &= c^2 \left(2\psi(n+2) + \psi'(n+2) + 2\gamma - \frac{\pi^2}{6} \right) \quad \text{(cf. [6])}.
\end{aligned}
$$

The expectation and variance of $Z_{2,n+1}$ *are given by*

$$\mathrm{E} Z_{2,n+1} = c\left(\psi(n+2) + \frac{n}{n+1} + \gamma \right), \tag{12}$$

$$\sigma^2 Z_{2,n+1} = c^2 \left(2\psi(n+2) + \psi'(n+2) + \frac{n^2}{(n+1)^2} + 2\gamma - \frac{\pi^2}{6} \right). \tag{13}$$

The covariance of the random variables $Z_{1,n+1}$ *and* $Z_{2,n+1}$ *has the form*

$$\text{(14)}\qquad \operatorname{Cov}(Z_{1,n+1}, Z_{2,n+1}) = c^2\left(\frac{2n+3}{n+1}\,\psi(n+2) + \frac{2n+3}{n+1}\,\gamma + \psi'(n+2) - 1 - \frac{\pi^2}{3}\right).$$

The correlation coefficient of $Z_{1,n+1}$ *and* $Z_{2,n+1}$ *is given by*

$$\varrho_n = \frac{H^{(1)}_{n+1} + \frac{1}{n+1}H^{(1)}_{n+1} - H^{(2)}_{n+1} - 1 - \frac{\pi^2}{6}}{\sqrt{(2H^{(1)}_{n+1} - H^{(2)}_{n+1})\big(2H^{(1)}_{n+1} - H^{(2)}_{n+1} + \frac{n^2}{(n+1)^2}\big)}}.$$

Proof. First note that

$$\text{(15)}\qquad \mathrm{E}E^r_j = \mathrm{E}F^r_j = r!c^r,$$

$$\text{(16)}\qquad \mathrm{E}E_jF_j = c^2\left(2 - \frac{\pi^2}{6}\right).$$

The formula (12) can be proved by induction with respect to n or directly from (11). The inductive proof will be presented here. For $n = 1$ by (10), (15) and (8) we have $\mathrm{E}Z_{2,2} = 2c = c(H^{(1)}_2 + 1/2)$, as required. Now assume that (12) is true for n; we show that it is true for $n+1$. By (10), (15) and (8) we get

$$\begin{aligned}\mathrm{E}Z_{2,n+1} &= \frac{c}{n+1} + \frac{1}{n+1}\,\mathrm{E}Z_{1,n} + \frac{c}{n+1} + \left(1 - \frac{1}{n+1}\right)\mathrm{E}Z_{2,n}\\ &= c\left(H^{(1)}_n + \frac{1}{n+1} + \frac{n}{n+1}\right),\end{aligned}$$

upon using the formula for the expectation of $Z_{1,n}$. Since $H^{(1)}_{n+1} = H^{(1)}_n + \frac{1}{n+1}$, we find that (12) holds.

The formulae for the variance and covariance follow directly from (11). Using the identity $(\sum_{j=0}^n a_j)^2 = \sum_{j=0}^n a_j^2 + 2\sum_{m=1}^n\sum_{j=0}^{m-1} a_ja_m$, we get $\mathrm{E}Z^2_{2,n+1} := S_1 + S_2$, where

$$\begin{aligned}S_1 := \sum_{j=0}^n \Big\{ &\mathrm{E}(A^{(2)}_j)^2\mathrm{E}F^2_j \prod_{i=j+1}^n \mathrm{E}(1 - A^{(2)}_i)^2 + \mathrm{E}(B^{(2)}_j)^2\mathrm{E}E^2_j \prod_{i=j+1}^n \mathrm{E}(1 - A^{(2)}_i)^2\\ &+ \mathrm{E}(A^{(2)}_j)^2 \sum_{k=j+1}^n \mathrm{E}(A^{(2)}_k)^2 \prod_{i=k+1}^n \mathrm{E}(1 - A^{(2)}_i)^2\mathrm{E}E^2_j\Big\}\end{aligned}$$

and

$$S_2 := 2\sum_{m=1}^{n}\sum_{j=0}^{m-1} \mathrm{E}\Big\{A_j^{(2)} \prod_{i=j+1}^{n}(1-A_i^{(2)})F_j + B_j^{(2)} \prod_{i=j+1}^{n}(1-A_i^{(2)})E_j + A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n}(1-A_i^{(2)})E_j\Big\}\Big\{A_m^{(2)} \prod_{i=m+1}^{n}(1-A_i^{(2)})F_m + B_m^{(2)} \prod_{i=m+1}^{n}(1-A_i^{(2)})E_m + A_m^{(2)} \sum_{k=m+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n}(1-A_i^{(2)})E_m\Big\}.$$

For S_1, we see that by (8) and (15),

$$S_1 = 2c^2\sum_{j=0}^{n}\frac{1}{n+1} + 2c^2\sum_{j=1}^{n}\frac{1}{n+1} + 2c^2\sum_{j=0}^{n}\frac{1}{j+1}\sum_{k=j+1}^{n}\frac{1}{n+1},$$

as $\mathrm{E}\prod_{i=j+1}^{n}(1-A_i^{(2)}) = \prod_{i=j+1}^{n}\frac{i}{i+1} = \frac{j+1}{n+1}$. Hence by (4),

$$S_1 = 2c^2\left(H_{n+1}^{(1)} + \frac{n}{n+1}\right).$$

Now for S_2, multiplying the expressions in brackets and using the property

$$\mathrm{E}A_i^{(2)}B_i^{(2)} = 0 \quad\text{and}\quad \mathrm{E}A_i^{(2)}(1-A_i^{(2)}) = 0, \tag{17}$$

we see that

$$S_2 = 2\sum_{m=1}^{n}\sum_{j=0}^{m-1}\Big\{\mathrm{E}A_j^{(2)}B_m^{(2)}\prod_{i=j+1}^{n}(1-A_i^{(2)})F_jE_m + \mathrm{E}B_j^{(2)}B_m^{(2)}\prod_{i=j+1}^{n}(1-A_i^{(2)})E_jE_m\Big\}$$
$$+ 2\sum_{m=1}^{n}\sum_{j=0}^{m-1}\mathrm{E}A_j^{(2)}\sum_{k=j+1}^{n}A_k^{(2)}\prod_{i=k+1}^{n}(1-A_i^{(2)})E_jA_m^{(2)}\prod_{i=m+1}^{n}(1-A_i^{(2)})F_m$$
$$+ 2\sum_{m=1}^{n}\sum_{j=0}^{m-1}\mathrm{E}A_j^{(2)}\sum_{k=j+1}^{n}A_k^{(2)}\prod_{i=k+1}^{n}(1-A_i^{(2)})E_jB_m^{(2)}\prod_{i=m+1}^{n}(1-A_i^{(2)})E_m$$
$$+ 2\sum_{m=1}^{n}\sum_{j=0}^{m-1}\mathrm{E}A_j^{(2)}\sum_{k=j+1}^{n}A_k^{(2)}\prod_{i=k+1}^{n}(1-A_i^{(2)})E_jA_m^{(2)}\sum_{k=m+1}^{n}A_k^{(2)} \cdot \prod_{i=k+1}^{n}(1-A_i^{(2)})E_m =: 2s_{21}+2s_{22}+2s_{23}+2s_{24},$$

say. For s_{21}, by (8) and (15) we see that

$$s_{21} = \sum_{m=1}^{n} \sum_{j=0}^{m-1} \Big\{ \mathrm{E} A_j^{(2)} B_m^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) F_j E_m + \mathrm{E} B_j^{(2)} B_m^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) E_j E_m \Big\}$$

$$= c^2 \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{1}{(n+1)m} + c^2 \sum_{m=2}^{n} \sum_{j=1}^{m-1} \frac{1}{(n+1)m}$$

and then using (4) we get

$$s_{21} = c^2 \left(\frac{2n}{n+1} - \frac{H_{n+1}^{(1)}}{n+1} + \frac{1}{(n+1)^2} \right).$$

For s_{22}, after changing the order of summation, by (8) and (15) we get

$$s_{22} = \sum_{j=0}^{n-1} \sum_{m=j+1}^{n} \mathrm{E} A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_j A_m^{(2)} \prod_{i=m+1}^{n} (1 - A_i^{(2)}) F_m$$

$$= \sum_{j=0}^{n-1} \mathrm{E} A_j^{(2)} \sum_{k=j+1}^{n} (A_k^{(2)})^2 \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_j F_m$$

$$= c^2 \sum_{j=0}^{n-1} \frac{1}{j+1} \sum_{k=j+1}^{n} \frac{1}{n+1} = c^2 (H_{n+1}^{(1)} - 1).$$

Since $\mathrm{E}(1 - A_i^{(2)})^2 = \mathrm{E}(1 - A_i^{(2)}) = \frac{i}{i+1}$ we have

$$s_{23} = \sum_{m=1}^{n} \sum_{j=0}^{m-1} \mathrm{E} A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_j B_m^{(2)} \prod_{i=m+1}^{n} (1 - A_i^{(2)}) E_m$$

$$= c^2 \sum_{m=1}^{n} \sum_{j=0}^{m-1} \mathrm{E} A_j^{(2)} \sum_{k=j+1}^{m-1} \mathrm{E} A_k^{(2)} \mathrm{E} B_m^{(2)} \prod_{i=k+1}^{n} \mathrm{E}(1 - A_i^{(2)})$$

$$= c^2 \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{m-(j+1)}{(n+1)(j+1)m} = c^2 \left(\frac{1}{n+1} \sum_{m=1}^{n} H_m^{(1)} - \frac{n}{n+1} \right),$$

and

$$s_{24} = \sum_{m=1}^{n} \sum_{j=0}^{m-1} \mathrm{E} A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_j A_m^{(2)} \cdot \sum_{k=m+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_m$$

$$= c^2 \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{1}{j+1} \mathrm{E}A_m^{(2)} \sum_{k=m+1}^{n} \mathrm{E}(A_k^{(2)})^2 \prod_{i=k+1}^{n} \mathrm{E}(1 - A_i^{(2)})^2$$

$$= c^2 \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{1}{(j+1)(m+1)} - \frac{c^2}{n+1} \sum_{m=0}^{n} H_m^{(1)},$$

which by the identity

$$\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{kj} = \frac{1}{2} (H_n^{(1)})^2 - \frac{1}{2} H_n^{(2)} \tag{18}$$

(cf. [4]) gives

$$s_{24} = c^2 \left(\frac{1}{2} (H_{n+1}^{(1)})^2 - \frac{1}{2} H_{n+1}^{(2)} - \frac{1}{n+1} \sum_{m=1}^{n} H_m^{(1)} \right).$$

Hence

$$S_2 = c^2 \left((H_{n+1}^{(1)})^2 + 2H_{n+1}^{(1)} - \frac{2}{n+1} - \frac{2}{n+1} H_{n+1}^{(1)} + \frac{2}{(n+1)^2} - H_{n+1}^{(2)} \right),$$

which gives

$$\mathrm{E}Z_{2,n+1}^2 = c^2 \left((H_{n+1}^{(1)})^2 + 4H_{n+1}^{(1)} + 2 \frac{n^2}{(n+1)^2} - \frac{2}{n+1} H_{n+1}^{(1)} - H_{n+1}^{(2)} \right)$$

and

$$\sigma^2 Z_{2,n+1} = c^2 \left(2H_{n+1}^{(1)} - H_{n+1}^{(2)} + \frac{n^2}{(n+1)^2} \right),$$

and by (2) and (5) proves (13).

Similarly for

$$\mathrm{E}Z_{1,n+1}Z_{2,n+1} = \mathrm{E} \sum_{j=0}^{n} \sum_{m=0}^{n} A_m^{(2)} E_m \Big\{ A_j^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) F_j$$
$$+ B_j^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) E_j + A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) E_j \Big\},$$

using the equality

$$\sum_{j=0}^{n} \sum_{m=0}^{n} a_j a_m = \sum_{j=1}^{n} \sum_{m=0}^{j-1} a_j a_m + \sum_{j=0}^{n} a_j^2 + \sum_{j=0}^{n-1} \sum_{m=j+1}^{n} a_j a_m,$$

and property (17) we get

$$
\begin{aligned}
\mathrm{E}Z_{1,n+1}Z_{2,n+1} &= \sum_{j=1}^{n}\sum_{m=0}^{j-1}\mathrm{E}A_m^{(2)}(A_j^{(2)}+B_j^{(2)})\prod_{i=j+1}^{n}(1-A_i^{(2)})\mathrm{E}E_m\mathrm{E}F_j \\
&+\sum_{j=0}^{n}\mathrm{E}(A_j^{(2)})^2\prod_{i=j+1}^{n}(1-A_i^{(2)})\mathrm{E}E_jF_j \\
&+\sum_{j=1}^{n}\sum_{m=0}^{j-1}\mathrm{E}A_m^{(2)}A_j^{(2)}\sum_{k=j+1}^{n}A_k^{(2)}\prod_{i=j+1}^{n}(1-A_i^{(2)})\mathrm{E}E_m\mathrm{E}E_j \\
&+\sum_{j=0}^{n}\mathrm{E}A_j^{(2)}\sum_{k=j+1}^{n}A_k^{(2)}\prod_{i=j+1}^{n}(1-A_i^{(2)})\mathrm{E}E_j^2 \\
&+\sum_{j=0}^{n}\sum_{m=j+1}^{n}\mathrm{E}A_j^{(2)}(A_m^{(2)})^2\prod_{i=m+1}^{n}(1-A_i^{(2)})\mathrm{E}E_j\mathrm{E}E_m \\
&+\sum_{j=0}^{n}\sum_{m=j+1}^{n}\mathrm{E}A_j^{(2)}A_m^{(2)}\sum_{k=m+1}^{n}A_k^{(2)}\prod_{i=j+1}^{n}(1-A_i^{(2)})\mathrm{E}E_j\mathrm{E}E_m.
\end{aligned}
$$

Hence by (8), (15) and (16) we obtain

$$
\begin{aligned}
\mathrm{E}Z_{1,n+1}Z_{2,n+1} &= 2c^2\sum_{j=1}^{n}\sum_{m=0}^{j-1}\frac{1}{(m+1)(n+1)}+\left(2-\frac{\pi^2}{6}\right)c^2 \\
&+c^2\sum_{j=1}^{n}\sum_{m=0}^{j-1}\frac{1}{(m+1)(j+1)}\sum_{k=j+1}^{n}\frac{1}{n+1}+c^2\sum_{j=0}^{n-1}\sum_{m=j+1}^{n}\frac{1}{(j+1)(n+1)} \\
&+2c^2\sum_{j=0}^{n}\frac{1}{j+1}\sum_{k=j+1}^{n}\frac{1}{n+1}+c^2\sum_{j=0}^{n-1}\sum_{m=j+1}^{n}\frac{1}{(j+1)(m+1)}\sum_{k=m+1}^{n}\frac{1}{n+1} \\
&= c^2(3H_{n+1}^{(1)}-1-\frac{\pi^2}{6}+(H_{n+1}^{(1)})^2-H_{n+1}^{(2)}),
\end{aligned}
$$

upon using (18) and the identity

$$
\sum_{j=1}^{n}H_j^{(1)}=(n+1)H_{n+1}^{(1)}-(n+1)
$$

(cf. [4]). Therefore,

$$
\mathrm{Cov}(Z_{1,n+1},Z_{2,n+1})=c^2\left(\frac{2n+3}{n+1}H_{n+1}^{(1)}-H_{n+1}^{(2)}-1-\frac{\pi^2}{6}\right),
$$

which by (2) and (5) proves (14). ∎

REMARK 1. By (6) and (7) we get

$$\mathrm{E}Z_{2,n} \sim \log_2 n, \qquad \sigma^2 Z_{2,n} \sim 2c\log_2 n, \qquad \varrho_n \to 1, \qquad n \to \infty.$$

Now we investigate the asymptotic behaviour of the distribution of $Z_{2,n}$.

LEMMA 1. *Under the assumptions of random choice and uniform split,*

$$\frac{Z_{2,n} - Z_{1,n}}{\sqrt{\log_2 n}} \xrightarrow{P} 0, \qquad n \to \infty,$$

where $\xrightarrow{P}$ *denotes convergence in probability.*

Proof. By Chebyshev's inequality and Proposition 2 we have

$$P\left(\left|\frac{Z_{2,n} - Z_{1,n}}{\sqrt{\log_2 n}}\right| \ge \varepsilon\right) \le \frac{\sigma^2 Z_{1,n} + \sigma^2 Z_{2,n} - 2\mathrm{Cov}(Z_{1,n}, Z_{2,n})}{\varepsilon^2 \log_2 n}$$
$$= \frac{c^2\varepsilon^{-2}\left(3 + \frac{\pi^2}{3} - \frac{2n+1}{(n+1)^2} - \frac{2}{n+1}H^{(1)}_{n+1}\right)}{\log_2 n} \to 0, \qquad n \to \infty. \ \blacksquare$$

THEOREM 1. *Under the assumptions of random choice and uniform split, the random variable* $Z_{2,n}$ *is asymptotically normal with expected value* $\log_2 n$ *and variance* $\frac{2}{\log 2}\log_2 n$*.*

Proof. Note that

$$\frac{Z_{2,n} - \log_2 n}{\sqrt{2c\log_2 n}} = \frac{Z_{2,n} - Z_{1,n}}{\sqrt{2c\log_2 n}} + \frac{Z_{1,n} - \log_2 n}{\sqrt{2c\log_2 n}}.$$

Since

$$\frac{Z_{1,n} - \log_2 n}{\sqrt{2c\log_2 n}} \to Z \sim N(0,1), \qquad n \to \infty,$$

the assertion follows from Lemma 1 and the Slucki theorem (cf. [1]). ■

To prove the next theorem we use Etemadi's method.

THEOREM 2. *Under the assumptions of random choice and uniform split,*

$$\frac{Z_{1,n}}{\log_2 n} \xrightarrow{a.s.} 1 \quad \textit{and} \quad \frac{Z_{2,n}}{\log n} \xrightarrow{a.s.} 1, \qquad n \to \infty.$$

Proof. Note that $\mathrm{E}\frac{Z_{1,n}}{\log_2 n} \to 1$, $n \to \infty$. Now let $\varepsilon > 0$. By Chebyshev's inequality we have

$$P\left(\left|\frac{Z_{1,n}}{\log_2 n} - \frac{\mathrm{E}Z_{1,n}}{\log_2 n}\right| \ge \varepsilon\right) \le \frac{\sigma^2 Z_{1,n}}{\varepsilon^2 \log_2^2 n} \le \frac{2c}{\varepsilon^2 \log_2 n},$$

which gives the rate of convergence in probability. We prove the almost sure convergence using Etemadi's method (cf. [2]). Let $\alpha > 1$ and $m_n = \lceil \alpha^{n^2} \rceil$ for $n \geq 1$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x (the notation from [4]), i.e. $\lceil x \rceil$ denotes the ceiling function of x. In what follows, C denotes a finite positive constant that can vary from step to step. Then

$$\sum_{n=1}^{\infty} P\left[\left|\frac{Z_{1,m_n}}{\log_2 m_n} - \frac{\mathrm{E} Z_{1,m_n}}{\log_2 m_n}\right| \geq \varepsilon\right] \leq C \sum_{n=1}^{\infty} \frac{1}{\log m_n} \leq C \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

The Borel–Cantelli lemma implies that

$$\frac{Z_{1,m_n}}{\log_2 m_n} \xrightarrow{\text{a.s.}} 1, \qquad n \to \infty.$$

Let $p(n)$ be such that $m_{p(n)} \leq n < m_{p(n)+1}$ for $n \geq 1$. Since the sequence $(Z_{1,n})$ is nondecreasing we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{Z_{1,n}}{\log_2 n} &\geq \liminf_{n\to\infty} \frac{Z_{1,m_{p_n}}}{\log_2 m_{p(n)}} \frac{\log_2 m_{p(n)}}{\log_2 m_{p(n)+1}} \\ &\geq \lim_{n\to\infty} \frac{Z_{1,m_{p_n}}}{\log_2 m_{p(n)}} \left(\frac{p(n)}{p(n)+1}\right)^2 = 1.\end{aligned}$$

Similarly, we get an analogous relation for the upper limit:

$$\begin{aligned}\limsup_{n\to\infty} \frac{Z_{1,n}}{\log_2 n} &\leq \limsup_{n\to\infty} \frac{Z_{1,m_{p_n+1}}}{\log_2 m_{p(n)+1}} \frac{\log_2 m_{p(n)+1}}{\log_2 m_{p(n)}} \\ &\leq \lim_{n\to\infty} \frac{Z_{1,m_{p(n)+1}}}{\log_2 m_{p(n)+1}} \left(\frac{p(n)+1}{p(n)}\right)^2 = 1.\end{aligned}$$

Thus $Z_{1,n}/\log_2 n \to 1$ almost surely. The proof of the statement for $Z_{2,n}$ is similar. ■

4. Random choice, splitting in half. Suppose that $A_n^{(2)}$, $B_n^{(1)}$ and $B_n^{(2)}$ satisfy the assumptions of the previous section. Referring to (2) and (3), the following recursive formulae hold:

$$D_{1,1} = \frac{1}{2}, \qquad D_{1,n+1} = \begin{cases} D_{1,n} & \text{if } J_n^{(1)} = 0, \\ D_{1,n} R_n & \text{if } J_n^{(1)} = 1, \end{cases}$$

and

$$D_{1,n+1} = \prod_{j=0}^{n} \left(\frac{1}{2}\right)^{B_j^{(1)}}, \qquad Z_{1,n+1} = \sum_{j=0}^{n} B_j^{(1)} \quad \text{(cf. [6])}.$$

We are interested in $D_{2,n+1}$, for which the following recursive formula holds for $n \geq 1$:

$$D_{1,1} = \frac{1}{2}, \qquad D_{2,1} = \frac{1}{2}, \qquad D_{2,n+1} = \begin{cases} \frac{1}{2} D_{1,n} & \text{if } J_n^{(2)} = 2, \\ \frac{1}{2} D_{2,n} & \text{if } J_n^{(2)} = 1, \\ D_{2,n} & \text{if } J_n^{(2)} = 0. \end{cases}$$

Hence by (9) we have

$$D_{2,1} = \frac{1}{2}, \quad D_{2,n+1} = \left(\frac{1}{2}\right)^{A_n^{(2)}} D_{1,n}^{A_n^{(2)}} \left(\frac{1}{2}\right)^{B_n^{(2)}} D_{2,n}^{1-A_n^{(2)}},$$

$$Z_{2,1} = 1, \qquad Z_{2,n+1} = A_n^{(2)} + A_n^{(2)} Z_{1,n} + B_n^{(2)} + (1 - A_n^{(2)}) Z_{2,n},$$

and

$$\begin{aligned} D_{2,n+1} = & \prod_{j=0}^{n} \left(\frac{1}{2}\right)^{A_j^{(2)} \prod_{i=j+1}^{n} (1-A_i^{(2)})} \\ & \cdot \left(\frac{1}{2}\right)^{B_j^{(2)} \prod_{i=j+1}^{n} (1-A_i^{(2)}) + B_j^{(1)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1-A_i^{(2)})}, \\ Z_{2,n+1} = & \sum_{j=0}^{n} \Big(A_j^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) + B_j^{(2)} \prod_{i=j+1}^{n} (1 - A_i^{(2)}) \\ & + A_j^{(2)} \sum_{k=j+1}^{n} A_k^{(2)} \prod_{i=k+1}^{n} (1 - A_i^{(2)}) \Big), \qquad n \geq 1, \end{aligned}$$

where $A_0^{(2)} = 1$ and $B_0^{(2)} = 0$.

In general, considering $D_{i,n}$, $1 \leq i \leq n+1$, we get

PROPOSITION 3. *Under the assumptions of random choice and splitting into halves, the following recursive formulae hold for* $n \geq i - 1$:

$$D_{1,1} = \frac{1}{2}, \qquad D_{2,1} = \frac{1}{2}, \qquad D_{i,n+1} = \begin{cases} D_{i-1,n} & \textit{if } J_n^{(i)} = 3, \\ \frac{1}{2} D_{i-1,n} & \textit{if } J_n^{(i)} = 2, \\ \frac{1}{2} D_{i,n} & \textit{if } J_n^{(i)} = 1, \\ D_{i,n} & \textit{if } J_n^{(i)} = 0. \end{cases}$$

The probability generating function of $Z_{1,n+1}$ is given by

$$g_{Z_{1,n+1}}(s) = \mathrm{E} s^{Z_{1,n+1}} = \prod_{j=0}^{n} \left(\frac{s}{j+1} + \frac{j}{j+1} \right) = \frac{(s)_{n+1}}{(n+1)!}.$$

Note that

$$\log g_{Z_{1,n+1}}(s) = \sum_{j=0}^{n} \log(s+j) - \log(n+1)!,$$

and differentiating both sides we have

$$g'_{Z_{1,n+1}}(s) = \frac{(s)_{n+1}}{(n+1)!} \sum_{j=0}^{n} \frac{1}{s+j},$$

$$g''_{Z_{1,n+1}}(s) = \frac{(n+1)!}{(s)_{n+1}}\, g'_{Z_{1,n+1}}(s) + \frac{(s)_{n+1}}{(n+1)!} \sum_{j=0}^{n} \frac{1}{(s+j)^2}.$$

Taking into account that $\mathrm{E}Z_{1,n+1} = g'_{Z_{1,n+1}}(1)$ and $\mathrm{E}Z_{1,n+1}(Z_{1,n+1}-1) = g''_{Z_{1,n+1}}(1)$ we can easily get the formulae for the expectation and variance of $Z_{1,n+1}$ given below in Proposition 6.

From the above probability generating function using the identity $(x)_n = \sum_{j=0}^{n} c(n,j)x^j$ (cf. [4]), where $c(n,j)$ are the unsigned Stirling numbers of the first kind, we get

PROPOSITION 4. *The rth moment of $Z_{1,n+1}$ is given by*

$$\mathrm{E}Z_{1,n+1}^r = \frac{1}{(n+1)!} \sum_{j=1}^{n+1} c(n+1,j) j^r.$$

Let $p_n(j) = P(Z_{1,n} = j)$. Kopociński [6] showed that

$$p_{n+1}(j) = p_n(j)\, \frac{n}{n+1} + p_n(j-1)\, \frac{1}{n+1}.$$

Hence one can prove

PROPOSITION 5. *Under the assumptions of random choice and splitting in half, the moments of $Z_{1,n}$ satisfy the following recurrence relation*:

$$\mathrm{E}Z_{1,n+1}^r = \mathrm{E}Z_{1,n}^r + \frac{1}{n+1} \sum_{i=0}^{r-1} \binom{r}{i} \mathrm{E}Z_{1,n}^i.$$

As in the proof of Proposition 2, one can show

PROPOSITION 6. *Under the assumptions of random choice and splitting in half, the expectation and variance of $Z_{1,n+1}$ are given by*

$$\mathrm{E}Z_{1,n+1} = \psi(n+2) + \gamma,$$

$$\sigma^2 Z_{1,n+1} = \psi(n+2) + \psi'(n+2) + \gamma - \frac{\pi^2}{6} \quad \text{(cf. [6])}.$$

The expectation and variance of $Z_{2,n+1}$ are given by

$$\mathrm{E}Z_{2,n+1} = \psi(n+2) + \gamma + \frac{n}{n+1},$$

$$\sigma^2 Z_{2,n+1} = \psi(n+2) + \psi'(n+2) - \frac{n}{(n+1)^2} + \gamma - \frac{\pi^2}{6}.$$

The covariance of the random variables $Z_{1,n+1}$ *and* $Z_{2,n+1}$ *has the form*

$$\operatorname{Cov}(Z_{1,n+1}, Z_{2,n+1}) = \frac{n+2}{n+1}\,\psi(n+2) + \psi'(n+2) + \frac{n+2}{n+1}\,\gamma - \frac{\pi^2}{6} - 1.$$

The correlation coefficient of $Z_{1,n+1}$ *and* $Z_{2,n+1}$ *is given by*

$$\varrho_n = \frac{H^{(1)}_{n+1} + \frac{1}{n+1}H^{(1)}_{n+1} - H^{(2)}_{n+1} - 1}{\sqrt{\big(H^{(1)}_{n+1} - H^{(2)}_{n+1}\big)\big(H^{(1)}_{n+1} - H^{(2)}_{n+1} - \frac{n}{(n+1)^2}\big)}}.$$

Proof. The steps of the proof are the same as in the proof of Proposition 2. It is enough to replace E_j and F_j by 1, and then make similar calculations. We drop them here. Finally, we get

$$\mathrm{E}Z_{2,n+1} = H^{(1)}_{n+1} + \frac{n}{n+1}, \qquad \sigma^2 Z_{2,n+1} = H^{(1)}_{n+1} - H^{(2)}_{n+1} - \frac{n}{(n+1)^2},$$

and

$$\operatorname{Cov}(Z_{1,n+1}, Z_{2,n+1}) = \frac{n+2}{n+1}\,H^{(1)}_{n+1} - H^{(2)}_{n+1} - 1,$$

which by (2) and (5) gives the required assertions. ■

Remark 2.

$$\mathrm{E}Z_{2,n} \sim \log n, \qquad \sigma^2 Z_{2,n} \sim \log n, \qquad \varrho_n \to 1, \qquad n \to \infty.$$

Now we investigate the limit behaviour of $Z_{1,n}$ and $Z_{2,n}$.

Lemma 2. *Under the assumptions of random choice and splitting in half*,

$$\frac{Z_{2,n} - Z_{1,n}}{\sqrt{\log n}} \xrightarrow{P} 0, \qquad n \to \infty.$$

The next two theorems correspond to Theorems 1 and 2, and the proofs are similar.

Theorem 3. *Under the assumptions of random choice and splitting in half, the random variable* $Z_{2,n}$ *is asymptotically normal with expected value* $\log n$ *and variance* $\log n$.

Theorem 4. *Under the assumptions of random choice and splitting in half*,

$$\frac{Z_{1,n}}{\log n} \xrightarrow{a.s.} 1 \quad \text{and} \quad \frac{Z_{2,n}}{\log n} \xrightarrow{a.s.} 1, \qquad n \to \infty.$$

5. Proportional choice, splitting in half. We give recurrence relations for the probability distribution of $D_{i,n}$, $1 \le i \le n+1$, and also recursive formulae for the moments of $D_{i,n}$ and $T_{i,n}$. As a special case we can obtain the results for $D_{1,n}$ presented by Kopociński [6] and also for $D_{2,n}$.

Define $p^{(i)}_n(j) = P(D_{i,n} = 1/2^j)$ for fixed i, $1 \le i \le n+1$. Let

$$S_{0,n} = 0, \qquad S_{m,n} = S_{m-1,n} + D_{m,n}, \qquad 1 \le m \le n+1.$$

Proposition 7. *Under the assumptions of proportional choice and splitting in half, the random variable* $D_{i,n}$ *and its probability distribution* $p_n^{(i)}(j) = P(D_{i,n} = 1/2^j)$, $1 \le j \le n$, *satisfy the following recursive formulae for* $n \ge i-1$:

$$D_{1,1} = \frac{1}{2}, \qquad D_{2,1} = \frac{1}{2}, \qquad D_{i,n+1} = \begin{cases} D_{i-1,n} & \text{if } R_n < S_{i-2,m}, \\ \frac{1}{2} D_{i-1,n} & \text{if } S_{i-2,n} \le R_n < S_{i-1,n}, \\ \frac{1}{2} D_{i,n} & \text{if } S_{i-1,n} \le R_n < S_{i,n}, \\ D_{i,n} & \text{if } R_n \ge S_{i,n}, \end{cases}$$

and

$$p_{n+1}^{(i)}(j) = p_n^{(i-1)}(j) \mathrm{E} S_{i-2,n} + \frac{p_n^{(i-1)}(j-1)}{2^{j-1}} + \frac{p_n^{(i)}(j-1)}{2^{j-1}} + p_n^{(i)}(j)\left(1 - \frac{1}{2^j} - \mathrm{E} S_{i-1,n}\right).$$

Proof. Using the recursive formula for $D_{i,n+1}$ we have

$$p_{n+1}^{(i)}(j) = p_n^{(i-1)}(j) P(R_n < S_{i-2,m}) + p_n^{(i-1)}(j-1) P\left(S_{i-2,m} < R_n < S_{i-2,n} + \frac{1}{2^{j-1}}\right) + p_n^{(i)}(j-1) \cdot P\left(S_{i-1,m} < R_n < S_{i-1,n} + \frac{1}{2^{j-1}}\right) + p_n^{(i)}(j) P\left(R_n > S_{i-1,n} + \frac{1}{2^j}\right).$$

Since R_n is uniformly distributed on $(0,1)$ we see that

$$P\left(S_{i-1,m} < R_n < S_{i-1,n} + \frac{1}{2^{j-1}}\right) = \frac{1}{2^{j-1}}$$

and $P(R_n < S_{i-2,m}) = \mathrm{E} S_{i-2,m}$, which ends the proof. ∎

Corollary 1. *The random variable* $D_{1,n}$ *and its probability distribution* $p_n^{(1)}(j) = P(D_{1,n} = 1/2^j)$, $1 \le j \le n$, *satisfy the following recursive formulae for* $n \ge 1$:

$$D_{1,1} = \frac{1}{2}, \qquad D_{1,n+1} = \begin{cases} \frac{1}{2} D_{1,n} & \text{if } R_n < D_{1,n}, \\ D_{1,n} & \text{if } R_n \ge D_{1,n}, \end{cases}$$

and

$$p_1^{(1)}(1) = 1, \qquad p_{n+1}^{(1)}(j) = p_n^{(1)}(j-1)\, \frac{1}{2^{j-1}} + p_n^{(1)}(j)\left(1 - \frac{1}{2^j}\right).$$

The random variable $D_{2,n}$ *and its probability distribution*

$$p_n^{(2)}(j) = P(D_{2,n} = 1/2^j), \qquad 1 \le j \le n,$$

satisfy the following recursive formulae for $n \geq 1$:

$$D_{1,1} = \frac{1}{2}, \qquad D_{2,1} = \frac{1}{2}, \qquad D_{2,n+1} = \begin{cases} \frac{1}{2} D_{1,n} & \textit{if } R_n < D_{1,n}, \\ \frac{1}{2} D_{2,n} & \textit{if } D_{1,n} \leq R_n < D_{1,n} + D_{2,n}, \\ D_{2,n} & \textit{if } R_n \geq D_{1,n} + D_{2,n}, \end{cases}$$

and $p_1^{(1)}(1) = 1$,

$$p_{n+1}^{(2)}(j) = p_n^{(1)}(j-1)\,\frac{1}{2^{j-1}} + p_n^{(2)}(j-1)\,\frac{1}{2^{j-1}} + p_n^{(2)}(j)\left(1 - \frac{1}{2^j} - \mathrm{E}D_{1,n}\right).$$

Proposition 8. *Under the assumptions of proportional choice and splitting in half, the moments of* $D_{i,n}$ *and* $T_{i,n}$, $1 \leq i \leq n+1$, *satisfy the following recurrence relations for* $r \geq 1$:

$$\mathrm{E}D_{i,n+1}^r = \mathrm{E}D_{i-1,n}^{r+1}\mathrm{E}S_{i-2,n} + \frac{1}{2^r}\mathrm{E}D_{i-1,n}^{r+1} + (1 - \mathrm{E}S_{i-1,n})\mathrm{E}D_{i,n}^r + \left(\frac{1}{2^r} - 1\right)\mathrm{E}D_{i,n}^{r+1},$$

and

$$\mathrm{E}T_{i,n+1}^r = \mathrm{E}T_{i-1,n}^r\mathrm{E}S_{i-2,n} + 2^r\mathrm{E}T_{i-1,n}^{r-1} + (1 - \mathrm{E}S_{i-1,n})\mathrm{E}T_{i,n}^r + (2^r - 1)\,\mathrm{E}T_{i,n}^{r-1}.$$

Corollary 2. *The moments of* $D_{1,n}$ *and* $T_{1,n}$ *satisfy the following recurrence relations*:

$$\mathrm{E}D_{1,n+1}^r = \mathrm{E}D_{1,n}^r + \left(\frac{1}{2^r} - 1\right)\mathrm{E}D_{1,n}^{r+1},$$

$$\mathrm{E}T_{1,n+1}^r = ET_{1,n}^r + (2^r - 1)\,ET_{1,n}^{r-1}, \qquad r \geq 1,$$

$$\mathrm{E}D_{1,1} = \frac{1}{2}, \qquad \mathrm{E}D_{1,n+1} = \mathrm{E}D_{1,n} - \frac{1}{2}\,\mathrm{E}D_{1,n}^2, \qquad \mathrm{E}T_{1,n} = n+1.$$

The moments of $D_{2,n}$ *and* $T_{2,n}$ *satisfy the following recurrence relations*:

$$\mathrm{E}D_{2,n+1}^r = \frac{1}{2^r}\mathrm{E}D_{1,n}^{r+1} + (1 - \mathrm{E}D_{1,n})\mathrm{E}D_{2,n}^r + \left(\frac{1}{2^r} - 1\right)\mathrm{E}D_{2,n}^{r+1},$$

$$\mathrm{E}T_{2,n+1}^r = 2^r\mathrm{E}D_{1,n}^{r-1} + (1 - \mathrm{E}D_{1,n})\mathrm{E}T_{2,n}^r + (2^r - 1)\mathrm{E}T_{2,n}^{r-1}, \qquad r \geq 1,$$

$$\mathrm{E}D_{2,1} = \frac{1}{2}, \qquad \mathrm{E}D_{2,n+1} = \frac{1}{2}\mathrm{E}D_{1,n}^2 + (1 - \mathrm{E}D_{1,n})\mathrm{E}D_{2,n} - \frac{1}{2}\mathrm{E}D_{2,n}^2,$$

$$\mathrm{E}D_{2,n+1} = \frac{1}{2}\Big(\prod_{j=1}^{n}(1 - \mathrm{E}D_{1,j}) + \sum_{j=1}^{n}\prod_{k=n+2-j}^{n}(1 - \mathrm{E}D_{1,k})(\mathrm{E}D_{1,n+1-j}^2 - \mathrm{E}D_{2,n+1-j}^2)\Big),$$

$$\begin{aligned}
\mathrm{E}T_{2,n+1} &= 3 + (1-\mathrm{E}D_{1,n})\mathrm{E}T_{2,n},\\
\mathrm{E}T_{2,n+1} &= 3\sum_{j=1}^{n}\prod_{k=n+2-j}^{n}(1-\mathrm{E}D_{1,k}) + 2\prod_{j=1}^{n}(1-\mathrm{E}D_{1,j}),\\
\mathrm{E}T_{2,n+1}^2 &= \frac{3}{2}n^2 + \frac{11}{2}n + 5 + (1-\mathrm{E}D_{1,n})\mathrm{E}T_{2,n}^2 + 3\mathrm{E}T_{2,n},\\
\sigma^2 T_{2,n+1} &= \frac{3}{2}n^2 + \frac{11}{2}n - 4 + (1-\mathrm{E}D_{1,n})\sigma^2 T_{2,n} + 3(2\mathrm{E}D_{1,n}-1)\mathrm{E}T_{2,n}\\
&\quad + \mathrm{E}D_{1,n}(1-\mathrm{E}D_{1,n})(\mathrm{E}T_{2,n})^2.
\end{aligned}$$

The next propositions can be proved using Jensen's inequality.

PROPOSITION 9. *Under the assumptions of proportional choice and splitting in half, the moments of* $D_{i,n}$ *and* $T_{i,n}$, $1 \le i \le n+1$, *satisfy*

$$ED_{i,n}^r \ge (ET_{i,n}^r)^{-1}, \qquad r \ge 1.$$

PROPOSITION 10. *Under the assumptions of proportional choice and splitting in half, the moments of* $Z_{i,n}$, $1 \le i \le n+1$, *satisfy*

$$\begin{aligned}
&\mathrm{E}Z_{i,n}^2 \le \log_2^2(\mathrm{E}T_{i,n}+1),\\
&\mathrm{E}(c_r + Z_{i,n})^r \le (c_r + \log_2(\mathrm{E}T_{i,n}))^r, \qquad r \ge 1,
\end{aligned}$$

where $c_r = (r-1)\log_2 e - 1$.

COROLLARY 3 ([6]). *The moments of* $Z_{1,n}$ *satisfy*

$$\begin{aligned}
&\mathrm{E}Z_{1,n}^2 \le \log_2^2(n+2),\\
&\mathrm{E}(c_r + Z_{1,n})^r \le (c_r + \log_2(n+1))^r, \qquad r \ge 1.
\end{aligned}$$

REMARK 3. Let $U_{i,n} = T_{i,n}/\mathrm{E}T_{i,n}$, $1 \le i \le n+1$. Then under the assumptions of proportional choice and splitting in half, the support of the random variable $Z_{i,n}$, $1 \le i \le n+1$, is $\{1,\ldots,n\}$ and $Z_{i,n}$ has the representation

$$Z_{i,n} - \log_2(\mathrm{E}T_{i,n}) = \log_2 U_{i,n}, \qquad \text{a.s.},$$

which implies that $U_{i,n}$ is not a degenerate random variable.

Acknowledgements. The authors are grateful to Professor B. Kopociński for many useful comments and suggestions which helped to improve the paper.

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Received on 21.3.2005;
revised version on 17.5.2005 (1777)