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REGULARIZATION OF NONCOERCIVE CONSTRAINTS IN HENCKY PLASTICITY

Abstract. The aim of this paper is to find the largest lower semicontinuous minorant of the elastic-plastic energy of a body with fissures. The functional of energy considered is not coercive.

1. Introduction. The largest lower semicontinuous (l.s.c.) minorant (called the *l.s.c. regularization*) of a functional with linear growth at infinity is found in the papers [3], [6] and [9], under the assumption of local coercivity of the functional. The largest l.s.c. minorant of a non-coercive functional is found in [5], but the set $\mathcal{K}(x)$ of admissible stresses (defined in that paper) does not take into account the influence of fissures. However, the original elastic-plastic energy which describes a body with fissures is not locally coercive, because the energy on a fissure is not coercive.

The propagation of a crack is considered in [1]. The domain of the functional is the space of functions of *bounded variation* BV. Namely, in [1] the potential is elastic in all the body. Here we present the static problem for a fixed fissure, taking into account the plastic zone at the tip of the fissure.

The physical problem of energy concentrated on a fixed smooth surface Σ (contained in a domain Ω) is considered in [7]. The problem of a body with fissure is not studied in [7], because the authors assume "local coercivity" of the original functional (cf. [7, hypothesis (H4)]).

Here we are concerned with the situation where a Hencky elastic-perfectly plastic material has fissures. We apply the method from [5] to find the largest l.s.c. minorant of the original elastic-plastic energy.

We give only those fragments of the proofs which are different from those in [5]. The multifunction $\mathcal{K}$ (which describes the set of admissible stresses) and the elastic-plastic potential j take into account the influence of the

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fissure. In the functional $\mathbb{P}_\lambda^j$ of total elastic-plastic energy the work of volume forces is omitted. The bipolar functional $(\mathbb{P}_\lambda^j)^{**}$ is the l.s.c. regularization of $\mathbb{P}_\lambda^j$ in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ (see (3.18), (3.19) and [11]).

As in [5], if we assume that $\mathbb{P}_\lambda^j$ is globally coercive (cf. [5, (5.5)]), then we can prove that $(\mathbb{P}_\lambda^j)^{**}$ is the l.s.c. regularization of $\mathbb{P}_\lambda^j$ in the weak* BD topology, since the weak* BD topology and $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ are equal on bounded sets in $[BD(\Omega), \|\cdot\|_{BD}]$ (or on bounded sets in the space $[\mathbf{Y}^1(\overline{\Omega}), \|\cdot\|_{\mathbb{M}_b}]$, which is isomorphic to the former).

Note that $\widetilde{\mathbb{P}}_\lambda^j \neq \mathbb{P}_\lambda^j$, so *a priori* $(\widetilde{\mathbb{P}}_\lambda^j)^{**}$ is not the l.s.c. regularization of $\mathbb{P}_\lambda^j$.

2. Some basic definitions and theorems. Throughout this work Ω denotes a nonempty, bounded, open, connected set of class C^1 in $\mathbb{R}^n$. $C(\overline{\Omega}, \mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued continuous functions on $\overline{\Omega}$, while $C_0(\Omega, \mathbb{R}^m)$, or simply C_0, stands for the space of continuous functions which take the value 0 on the boundary $\mathrm{Fr}\,\Omega$ of Ω. Moreover C_c is the space of continuous functions with compact supports, and $\mathbb{M}_b(\Omega, \mathbb{R}^m)$, or $\mathbb{M}_b$, stands for the space of $\mathbb{R}^m$-valued, Radon bounded regular measures on Ω, equipped with the norm $\|\cdot\|_{\mathbb{M}_b(\Omega, \mathbb{R}^m)}$.

We will use the duality pairs $(\mathbb{M}_r, C_c)$ and $(\mathbb{M}_b, C_0)$, where $\mathbb{M}_r$ is the space of regular measures. For $\mathbf{g} = (g_1, \ldots, g_m) \in C(\overline{\Omega}, \mathbb{R}^m)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m) \in \mathbb{M}_b(\Omega, \mathbb{R}^m)$, we write $\int_\Omega \mathbf{g} \cdot \boldsymbol{\mu} \equiv \sum_{i=1}^m \int_\Omega g_i \mu_i$. Finally, $\mathcal{L}^0(\Omega, \mathbb{R}^m)_\mu$ stands for the set of μ-measurable functions from Ω into $\mathbb{R}^m$.

The scalar product of $\mathbf{z}$ and $\mathbf{z}^* \in \mathbb{R}^m$ is denoted by $\mathbf{z} \cdot \mathbf{z}^* = \sum_{i=1}^m z^i z_i^*$ and the scalar product of $\mathbf{w}$ and $\mathbf{w}^* \in \mathbb{R}^{m \times m} \equiv \mathbf{E}^m$ by $\mathbf{w} : \mathbf{w}^* = \sum_{ij=1}^m w^{ij} w_{ij}^*$, where $\mathbf{E}^m$ is the space of real $m \times m$ matrices. Moreover, $\mathbf{E}_s^m$ is the space of symmetric real $m \times m$ matrices.

If $F : X \to \mathbb{R} \cup \{\infty\}$ is a function defined on a vector space X, then F^* denotes its polar function (cf. [11]). For an arbitrary set C in X, $I_C(\cdot)$ stands for its indicator function ($I_C(x) = 0$ if $x \in C$ and $I_C(x) = \infty$ if $x \notin C$).

The notation $\mathrm{cl}_V(Z)$ stands for the closure of the set $Z \subset V$ in the topology of the space V. We set $\|[e_i]\|_{\mathbb{R}^m} \equiv \sum_{i=1}^m |e_i|$, where $[e_i] \in \mathbb{R}^m$. The tensor product (resp. symmetric tensor product) is denoted by $\otimes$ (resp. $\otimes_s$). The symmetric tensor product is given by the expression $(\mathbf{p} \otimes_s \boldsymbol{\nu})_{ij} \equiv (p_i \nu_j + p_j \nu_i)/2$.

We define the following Banach spaces (see [13], [15]):

$$(2.1) \qquad LD(\Omega) \equiv \left\{ \mathbf{u} \in L^1(\Omega, \mathbb{R}^n) \;\middle|\; \varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \in L^1(\Omega),\ i, j = 1, \ldots, n \right\},$$

$$(2.2) \qquad BD(\Omega) \equiv \{\mathbf{u} \in L^1(\Omega, \mathbb{R}^n) \mid \varepsilon_{ij}(\mathbf{u}) \in \mathbb{M}_b(\Omega),\ i, j = 1, \ldots, n\},$$

with the natural norms

$$\|\mathbf{u}\|_{LD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{L^1},$$

(2.3)

$$\|\mathbf{u}\|_{BD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{\mathbb{M}_{\mathrm{b}}}.$$

There exists a continuous linear trace $\boldsymbol{\gamma}_B : BD(\Omega) \to L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ such that $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{u}_{|\mathrm{Fr}\,\Omega}$ for all $\mathbf{u} \in BD(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^n)$ (see [15]). We define

(2.4) $$W^n(\Omega, \mathrm{div}) \equiv \{\boldsymbol{\sigma} \in L^\infty(\Omega, \mathbf{E}^n_s) \mid \mathrm{div}\,\boldsymbol{\sigma} \in L^n(\Omega, \mathbb{R}^n)\}$$

endowed with the norm $\|\boldsymbol{\sigma}\|_{W^n(\Omega,\mathrm{div})} = \|\boldsymbol{\sigma}\|_{L^\infty(\Omega,\mathbf{E}^n_s)} + \|\mathrm{div}\,\boldsymbol{\sigma}\|_{L^n(\Omega,\mathbb{R}^n)}$ (cf. [15, Chap. 2, Sec. 7] and [4]). The distribution $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})$, where $\boldsymbol{\sigma} \in W^n(\Omega, \mathrm{div})$, $\mathbf{u} \in BD(\Omega)$, defined (for $\varphi_1 \in C^\infty_c(\Omega)$) by

(2.5) $$\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}), \varphi_1 \rangle_{D' \times D} = -\int_\Omega (\mathrm{div}\,\boldsymbol{\sigma}) \cdot \mathbf{u}\varphi_1\, dx - \int_\Omega \boldsymbol{\sigma} : (\mathbf{u} \otimes \nabla\varphi_1)\, dx,$$

is a bounded measure on Ω (see [15]).

ASSUMPTION 1. Ω and Ω_1 are bounded open connected sets of class C^1 in $\mathbb{R}^n$. Furthermore, $\Omega \subset\subset \Omega_1$.

There is a continuous linear map β_B from $W^n(\Omega, \mathrm{div})$ onto $L^\infty(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ such that for every $\boldsymbol{\sigma} \in C(\overline{\Omega}, \mathbf{E}^n_s)$, $\boldsymbol{\beta}_B(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_{|\mathrm{Fr}\,\Omega} \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ denotes the exterior unit vector normal to $\mathrm{Fr}\,\Omega$ (see [15]). Moreover, for all $\mathbf{u} \in BD(\Omega)$ and all $\boldsymbol{\sigma} \in W^n(\Omega, \mathrm{div})$, the following formula holds:

(2.6) $$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) + \int_\Omega (\mathrm{div}\,\boldsymbol{\sigma}) \cdot \mathbf{u}\, dx = \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\beta}_B(\boldsymbol{\sigma}) \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds.$$

3. Mathematical description of an elastic-plastic body with fissure. Let $\{T_i\}_{i\in I}$ be a finite family of bounded connected sets of class C^1 in $\mathbb{R}^n$. In this paper, the Lebesgue and Hausdorff measures on $\mathbb{R}^n$ and $\mathrm{Fr}\,\Omega$, $\mathrm{Fr}\,\Omega_1$, $\mathrm{Fr}\,T_i$ (for $i \in I$) are denoted by dx and ds, respectively. We assume that $ds(\mathrm{Fr}\,T_i \cap \mathrm{Fr}\,T_k) = 0$, $ds(\mathrm{Fr}\,\Omega \cap \mathrm{Fr}\,T_i) = 0$, $ds(\mathrm{Fr}\,\Omega_1 \cap \mathrm{Fr}\,T_i) = 0$ for every $i, k \in I$, $i \neq k$. The boundary of Ω is composed of Γ_0 and Γ_1 ($= \overline{\Gamma}_1$) such that Γ_0 and Γ_1 are Borel subsets of $\mathrm{Fr}\,\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $ds(\mathrm{Fr}\,\Omega - (\Gamma_0 \cup \Gamma_1)) = 0$.

Let S be a closed subset of $\overline{\Omega} \cap \bigcup_{i\in I} \mathrm{Fr}\,T_i$ such that $S = \mathrm{cl}\,\mathrm{int}\,S$ (where the interior is taken relative to $\bigcup_{i\in I} \mathrm{Fr}\,T_i$).

We consider an elastic-perfectly plastic body occupying Ω. In this body we have a fissure (or fissures) S.

Below we define the set $\mathcal{K}(x)$ of admissible stresses, for every $x \in \Omega - S$. Moreover, we define the elastic-plastic potential $(x, \boldsymbol{\varepsilon}(\mathbf{u})) \mapsto j(x, \boldsymbol{\varepsilon}(\mathbf{u}))$ for dx-almost every (dx-a.e.) $x \in \Omega - S$ (where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain tensor).

The set $\mathcal{K}(x)$ and potential $(x,\boldsymbol{\varepsilon}(\mathbf{u}))\mapsto j(x,\boldsymbol{\varepsilon}(\mathbf{u}))$ are defined in a special way on S. Namely, to prevent the overlapping of opposite edges of the fissure, we introduce the so-called non-penetration condition. We assume that penetration of the material is restricted (see (3.4)). The non-penetration condition on S is described by the potential j_1. The dual functional to $\mathbb{R}^n \ni \mathbf{z} \mapsto j_1(x,(\mathbf{z}\otimes_s \boldsymbol{\nu}))$ (cf. (3.4)) is the indicator function of the set A^s (cf. (3.5)) on the fissure. Finally, for $x\in S$, we define the potential $\mathbf{E}^n_s \ni \mathbf{w}\mapsto j(x,\mathbf{w})$ on the larger space $\mathbf{E}^n_s$ (since $\mathbb{R}^n\otimes_s\{\boldsymbol{\nu}\}\subset \mathbf{E}^n_s$ and $\mathbb{R}^n\otimes_s\{\boldsymbol{\nu}\}\neq \mathbf{E}^n_s$). Therefore the dual functional to j (on S) is the indicator function of the set $\{\boldsymbol{\sigma}\in\mathbf{E}^n_s \mid \operatorname{tr}\boldsymbol{\sigma}\le 0,\ \boldsymbol{\sigma}^D=\mathbf{0}\}$ of admissible stresses on S, where $\boldsymbol{\sigma}^D=\boldsymbol{\sigma}-\frac{1}{n}\boldsymbol{\delta}\operatorname{tr}\boldsymbol{\sigma}$ and $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$. The final definition of j on S describes the original non-penetration condition on S. Here the unilateral contact condition on S without friction is studied.

We say that a subset ω of $\overline{\Omega}$ has *μ-measure zero* if $dx(\omega)=0$ and $ds(\omega\cap S)=0$. Let $\mathcal{K}:\overline{\Omega}\to 2^{\mathbf{E}^n_s}$ be a multifunction.

ASSUMPTION 2 (cf. [4]). $\mathcal{K}(x)$ is a convex and closed subset of $\mathbf{E}^n_s$ for all $x\in\overline{\Omega}$. There exists $\mathbf{z}_0\in C^1(\overline{\Omega},\mathbf{E}^n_s)$ such that $\mathbf{z}_0(x)\in\mathcal{K}(x)$ for every $x\in\overline{\Omega}$. Moreover:

(i) if $\mathbf{z}(x)\in\mathcal{K}(x)$ for μ-almost every (μ-a.e.) $x\in\Omega$, $\mathbf{z}\in C(\overline{\Omega},\mathbf{E}^n_s)$ and $\mathbf{z}_{|\operatorname{int}\Omega}\in W^n(\Omega,\operatorname{div})$, then $\mathbf{z}(y)\in\mathcal{K}(y)$ for every $y\in\overline{\Omega}$;

(ii) for every $y\in\overline{\Omega}$ and every $\mathbf{w}\in\mathcal{K}(y)$ there exists $\mathbf{z}\in C(\overline{\Omega},\mathbf{E}^n_s)$ such that $\mathbf{z}_{|\operatorname{int}\Omega}\in W^n(\Omega,\operatorname{div})$, $\mathbf{z}(y)=\mathbf{w}$ and $\mathbf{z}(x)\in\mathcal{K}(x)$ for every $x\in\overline{\Omega}$.

DEFINITION 1 (cf. [11, Chap. 8, p. 232]). A mapping $j^*:\Omega\times\mathbf{E}^n_s\to\mathbb{R}\cup\{\infty\}$ is a *convex*, *normal integrand* if:

(a) the function $\mathbf{E}^n_s\ni\mathbf{w}^*\mapsto j^*(x,\mathbf{w}^*)$ is convex and l.s.c. for μ-a.e. $x\in\Omega$;

(b) there exists a Borel function $\widetilde{j^*}:\Omega\times\mathbf{E}^n_s\to\mathbb{R}\cup\{\infty\}$ such that $\widetilde{j^*}(x,\cdot)=j^*(x,\cdot)$ for μ-a.e. $x\in\Omega$.

Moreover, assume

$$\{\mathbf{w}^*\in\mathbf{E}^n_s \mid j^*(x,\mathbf{w}^*)<\infty\}=\mathcal{K}(x)\quad \text{for } \mu\text{-a.e. } x\in\Omega. \tag{3.1}$$

ASSUMPTION 3 (see [5]). For every $\widehat{r}>0$ there exists $c_{\widehat{r}}$ such that

$$\sup\Big\{\int_\Omega j^*(x,\mathbf{z}^*)\,dx \,\Big|\, \mathbf{z}^*\in L^\infty(\Omega,\mathbf{E}^n_s),\ \|\mathbf{z}^*\|_{L^\infty}<\widehat{r} \text{ and } \mathbf{z}^*(x)\in\mathcal{K}(x) \text{ for } dx\text{-a.e. } x\in\Omega\Big\}<c_{\widehat{r}}<\infty. \tag{3.2}$$

ASSUMPTION 4. There exist $\mathbf{u}^e \in LD(\Omega)$ and $q \in L^1(\Omega, \mathbb{R})$ such that $j^*(x, \mathbf{w}^*) \geq \boldsymbol{\varepsilon}(\mathbf{u}^e)(x) : \mathbf{w}^* + q(x)$ for μ-a.e. $x \in \Omega$ and every $\mathbf{w}^* \in \mathbf{E}_s^n$, and $\gamma_B(\mathbf{u}^e) = \mathbf{0}$ on $\mathrm{Fr}\,\Omega$.

The set $\mathcal{K}(x)$ denotes the elasticity convex domain at any point $x \in \Omega - S$.

A Borel set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a *Caccioppoli set* if $\sup\{\int_{\mathcal{C}} \operatorname{div} \widetilde{f}\, dx \mid \widetilde{f} \in C_0^1(\Omega_2, \mathbb{R}^n),\ \|\widetilde{f}(x)\|_{\mathbb{R}^n} \leq 1\ \forall x \in \Omega_2\} < \infty$ for all bounded open subsets Ω_2 of $\mathbb{R}^n$ (see [12]).

ASSUMPTION 5. $\Gamma_1 = \mathrm{Fr}\,\Omega \cap \mathcal{C}$ and $S = \bigcup_{i\in I}(\mathrm{Fr}\,T_i \cap \mathcal{C}_i)$, where $\mathcal{C} = \mathrm{cl}\,\mathrm{int}\,\mathcal{C} \subset \Omega_1$ and $\mathcal{C}_i = \mathrm{cl}\,\mathrm{int}\,\mathcal{C}_i \subset \Omega_1$ (for $i \in I$) are closed Caccioppoli sets, $ds(\mathrm{Fr}\,\Omega \cap \mathrm{Fr}\,\mathcal{C}) = 0$ and $ds(\mathrm{Fr}\,T_i \cap \mathrm{Fr}\,\mathcal{C}_i) = 0$ (for $i \in I$).

By a finite induction, from [15, Chap. 2, Lemma 2.2], we obtain the following decomposition of the measure $\boldsymbol{\varepsilon}(\mathbf{u})$, for every $\mathbf{u} \in BD(\Omega_1)$:

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathbf{u}) &= \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega-S} + (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u}))_{|S} \otimes_s \boldsymbol{\nu}\, ds \\ &\quad + (\boldsymbol{\gamma}_B^O(\mathbf{u}) - \boldsymbol{\gamma}_B^I(\mathbf{u}))_{|\mathrm{Fr}\,\Omega} \otimes_s \boldsymbol{\nu}\, ds + \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega_1 - \overline{\Omega}},\end{aligned} \tag{3.3}$$

where $\boldsymbol{\nu}$ denotes the exterior unit vector normal to $\mathrm{Fr}\,\Omega$ or the normal vector to $\mathrm{Fr}\,T_i$ for some $i \in I$ (cf. [2]). The inside trace $\boldsymbol{\gamma}_B^I : BD(\Omega) \to L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ and outside trace $\boldsymbol{\gamma}_B^O : BD(\Omega_1 - \overline{\Omega}) \to L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ are given by $\boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{u}_{|\,\mathrm{Fr}\,\Omega}$ for $\mathbf{u} \in BD(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^n)$, and $\boldsymbol{\gamma}_B^O(\mathbf{u}) = \mathbf{u}_{|\mathrm{Fr}\,\Omega}$ for $\mathbf{u} \in BD(\Omega_1 - \overline{\Omega}) \cap C(\Omega_1 - \Omega, \mathbb{R}^n)$ respectively. The traces $\boldsymbol{\gamma}_B^- : BD(T_i) \to L^1(\mathrm{Fr}\,T_i, \mathbb{R}^n)$ and $\boldsymbol{\gamma}_B^+ : BD(\mathbb{R}^n - T_i) \to L^1(\mathrm{Fr}\,T_i, \mathbb{R}^n)$ are defined for every $i \in I$, where $\boldsymbol{\nu}$ denotes the exterior unit vector normal to the boundary of T_i.

Below we take into account the influence of the fissure.

The original potential on the fissure (or fissures) is described by

$$j_1(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) = \begin{cases} \infty & \text{if } \mathrm{tr}(\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu} < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

for ds-a.e. $x \in S$. Define

$$\begin{aligned} A^s = \{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \sigma_{ij}((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu})_{ij} \leq 0\ \forall \mathbf{u} \text{ such that} \\ \mathrm{tr}((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) = (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \cdot \boldsymbol{\nu} \geq 0\}. \end{aligned} \tag{3.5}$$

We find that the potential on the fissure, dual to the original one, is given by

$$\begin{aligned} j_1^*(x, \boldsymbol{\sigma}) &= \sup\Big\{ \sum_{i,j=1}^{n} \sigma_{ij}((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu})_{ij} \\ &\quad - j_1(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \,\Big|\, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \in L^1(S, \mathbb{R}^n)\Big\} \\ &= \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in A^s, \\ \infty & \text{otherwise,} \end{cases} \end{aligned} \tag{3.6}$$

for ds-a.e. $x \in S$ and every $\boldsymbol{\sigma} \in \mathbf{E}_s^n$.

LEMMA 1. *Let* $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{n}\boldsymbol{\delta}\operatorname{tr}\boldsymbol{\sigma}$ *where* $\delta_{ij} = 1$ *for* $i = j$ *and* $\delta_{ij} = 0$ *for* $i \neq j$. *Then*

$$\{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\} \subset A^s. \tag{3.7}$$

Proof. Follows from the decomposition

$$\sum_{i,j=1}^{n} \sigma_{ij}((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu})_{ij} = \frac{1}{n}\operatorname{tr}\boldsymbol{\sigma}\operatorname{tr}((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) + \sum_{i,j=1}^{n} \sigma_{ij}^D((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu})_{ij}^D. \ \blacksquare \tag{3.8}$$

LEMMA 2. *If* $\{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\} \subset \mathcal{K}_1$, *then*

$$\begin{aligned} j_1(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) &= (j_1^* + I_{\mathcal{K}_1})^*(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \\ &= j_1^{**}(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \end{aligned} \tag{3.9}$$

for ds-*a.e.* $x \in S$ *and all* $\mathbf{u} \in BD(\Omega)$, *where* $\mathcal{K}_1$ *is any convex and closed set in* $\mathbf{E}_s^n$, *and* $I_{\mathcal{K}_1}(\cdot)$ *is the indicator function of* $\mathcal{K}_1$.

Proof. By (3.5) and (3.7) we obtain $\{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\} \subset A^s \cap \mathcal{K}_1 \subset A^s$. Then by (3.6) and (3.8) we have

$$\begin{aligned} &j_1(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \\ &\quad = \sup\{\boldsymbol{\sigma} : ((\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\} \\ &\quad \leq (j_1^* + I_{\mathcal{K}_1})^*(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \\ &\quad \leq j_1^{**}(x, (\boldsymbol{\gamma}_B^+(\mathbf{u}) - \boldsymbol{\gamma}_B^-(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \end{aligned} \tag{3.10}$$

for ds-a.e. $x \in S$ and all $\mathbf{u} \in BD(\Omega)$. Since $j_1 \geq j_1^{**}$ we obtain (3.9). ■

ASSUMPTION 6. The inclusion $\{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\} \subset \mathcal{K}(x)$ holds for every $x \in \overline{\Omega}$.

DEFINITION 2. Let

$$j(x, \mathbf{w}) = j^{**}(x, \mathbf{w}) = \sup\{\mathbf{w} : \mathbf{w}^* - j^*(x, \mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^n\} \tag{3.11}$$

for μ-a.e. $x \in \mathbf{E}_s^n$, where j^* is defined by Definition 1, Assumptions 3 and 4. Among the functions j and $\mathcal{K}$ defined by Definition 1, Assumptions 2, 3, 4 and 6 there are also functions which satisfy

$$j(x, \mathbf{w}) = \begin{cases} \infty & \text{if } \operatorname{tr}\mathbf{w} < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.12}$$

for ds-a.e. $x \in S$ and every $\mathbf{w} \in \mathbf{E}_s^n$, and $\mathcal{K}(x) = \{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr}\boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\}$ for every $x \in S$.

The above defined multifunction $\mathcal{K}$ and potential j describe the elastic-plastic body with fissure (or fissures) S. Moreover, $\mathcal{K}$ and j satisfy Definition 1, Assumption 2, 3, 4, 6 and expression (3.1), for μ-a.e. $x \in \Omega$. We have $\mathcal{K}(x) = \{\boldsymbol{\sigma} \in \mathbf{E}_s^n \mid \operatorname{tr} \boldsymbol{\sigma} \leq 0,\ \boldsymbol{\sigma}^D = \mathbf{0}\}$ for every $x \in S$.

Since j^* is a normal integrand, j is a convex normal integrand (cf. (3.11) and [11]). The dual potential j^* satisfies

$$j^*(x, \mathbf{w}^*) = \sup\{\mathbf{w} : \mathbf{w}^* - j(x, \mathbf{w}) \mid \mathbf{w} \in \mathbf{E}_s^n\} \tag{3.13}$$

for μ-a.e. $x \in \Omega$. Define the *recession function* $j_\infty : \overline{\Omega} \times \mathbf{E}_s^n \to \mathbb{R} \cup \{\infty\}$ by

$$j_\infty(x, \mathbf{w}) = \sup\{\mathbf{w} : \mathbf{w}^* - I_{\mathcal{K}}(x, \mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^n\} \tag{3.14}$$

for $x \in \overline{\Omega}$ and $\mathbf{w} \in \mathbf{E}_s^n$.

Let $\boldsymbol{\zeta} \in \mathbb{M}_b(\Omega, \mathbf{E}_s^n)$. We recall that $|\boldsymbol{\zeta}|$ is the total variation measure associated with $\boldsymbol{\zeta}$, and the density of $\boldsymbol{\zeta}$ with respect to $|\boldsymbol{\zeta}|$ will be denoted by $d\boldsymbol{\zeta}/d|\boldsymbol{\zeta}|$. Let $\boldsymbol{\zeta} = \boldsymbol{\zeta}_a(x)\, dx + \boldsymbol{\zeta}_s$ be the Lebesgue decomposition of $\boldsymbol{\zeta}$ into the absolutely continuous and singular parts with respect to dx. If a bounded measure $\boldsymbol{\zeta} \in \mathbb{M}_b(\Omega, \mathbf{E}_s^n)$ is absolutely continuous with respect to dx, then we write $\boldsymbol{\zeta} \in L^1(\Omega, \mathbf{E}_s^n)$.

Let $\mathbf{f} \in L^n(\Omega, \mathbb{R}^n)$ and $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$. In this paper we consider the functional

$$BD(\Omega) \ni \mathbf{u} \mapsto P_{\lambda,j}(\mathbf{u}) = \lambda F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})), \tag{3.15}$$

where

$$\lambda F(\mathbf{u}) \equiv -\lambda L(\mathbf{u}) + I_{C_a}(\mathbf{u}), \qquad L(\mathbf{u}) \equiv \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx + \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds, \tag{3.16}$$

and $C_a \equiv \{\mathbf{u} \in BD(\Omega) \mid \boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}$. The functional $G_j : \mathbb{M}_b(\Omega, \mathbf{E}_s^n) \to \mathbb{R} \cup \{\infty\}$ is given by

$$G_j(\boldsymbol{\zeta}) \equiv \begin{cases} \displaystyle\int_{\Omega - S} j(x, \boldsymbol{\zeta})\, dx + \int_S j(x, \boldsymbol{\zeta})\, ds & \text{if } \boldsymbol{\zeta}_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}_s^n) \\ & \text{and } \boldsymbol{\zeta}_{|S} \in L^1(S, \mathbf{E}_s^n), \\ \infty & \text{if } \boldsymbol{\zeta} \text{ is not absolutely continuous with} \\ & \text{respect to } dx \text{ in } \Omega \text{ or to } ds \text{ in } S, \end{cases} \tag{3.17}$$

(cf. (3.3)).

The formula (3.15) describes the total elastic-perfectly plastic energy of a body occupying the given subset Ω of $\mathbb{R}^n$. This body is subjected to the volume forces $\mathbf{f} \in L^n(\Omega, \mathbb{R}^n)$ and boundary forces $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$. The constant $\lambda \geq 0$, $\lambda < \infty$ is the load multiplier (see [15]). The body is clamped on Γ_0. Moreover, in Ω there is a fissure (or fissures) S.

Assumption 7. There exists $\boldsymbol{\sigma}_0 \in C(\overline{\Omega}, \mathbf{E}_s^n)$ such that $\boldsymbol{\sigma}_{0|\operatorname{int}\Omega} \in W^n(\Omega, \operatorname{div})$, $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda \mathbf{g}$ on Γ_1 and $\boldsymbol{\sigma}_0(x) \in \mathcal{K}(x)$ for μ-a.e. $x \in \Omega$.

By Assumption 7, the boundary force $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$ is a regular function.

We consider the spaces $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ given by

$$\begin{aligned}(3.18)\qquad \mathbf{Y}^1(\overline{\Omega}) \equiv \{\mathbf{M} \in \mathbb{M}_b(\overline{\Omega}, \mathbf{E}^n_s) \mid \exists \mathbf{u}_1 \in BD(\Omega_1),\\ \boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}} = \mathbf{M},\ \mathbf{u}_{1|\Omega_1 - \overline{\Omega}} = \mathbf{0}\},\end{aligned}$$

$$(3.19)\qquad C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \equiv \{\sigma \in C(\overline{\Omega}, \mathbf{E}^n_s) \mid \boldsymbol{\sigma}_{|\Omega} \in W^n(\Omega, \mathrm{div})\}.$$

These are topological vector spaces in duality defined by the bilinear pairing

$$(3.20)\qquad \langle \mathbf{M}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1 \times C} = \int_{\overline{\Omega}} \boldsymbol{\sigma} : \mathbf{M} = \sum_{i,j=1}^{n} \int_{\overline{\Omega}} \sigma_{ij} M^{ij}$$

(cf. [5, Remark 2]). We say that a net $\{\mathbf{M}_k\}_{k\in K} \subset \mathbf{Y}^1(\overline{\Omega})$ converges to $\mathbf{M}_0$ in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s))$ if $\langle(\mathbf{M}_k - \mathbf{M}_0), \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} \to 0$ for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$.

The functional $\mathbb{P}^j_\lambda : \mathbf{Y}^1(\overline{\Omega}) \to \mathbb{R} \cup \{\infty\}$ of the original elastic-plastic energy is defined by

$$\begin{aligned}(3.21)\quad \mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \equiv -\int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds + \int_{\Omega - S} j(x, \boldsymbol{\varepsilon}(\mathbf{u}))\, dx\\ + \int_S j(x, \boldsymbol{\varepsilon}(\mathbf{u}))\, ds + \int_{\Gamma_0} I_{\{\boldsymbol{\gamma}^I_B(\mathbf{u})\otimes_s \boldsymbol{\nu} = 0\}}(\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds\end{aligned}$$

if $\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$ and $\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = 0$, where $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda \mathbf{g}$ on Γ_1, and $\mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \equiv \infty$ otherwise. In $\mathbb{P}^j_\lambda$ the work of the volume forces is omitted. The expression $\mathbb{P}^j_\lambda$ is obtained from $P_{\lambda,j}$ by means of the formula $\int_{\Gamma_1} \boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds = \int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$ (which holds for every $\mathbf{u} \in BD(\Omega)$, cf. (3.15) and [5, (3.16)]).

We assume that there exists $\widetilde{\mathbf{u}} \in BD(\Omega_1)$ with $\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$ and $\mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}) < \infty$.

4. Lower semicontinuous regularization. In this section the l.s.c. regularization of the functional $\mathbb{P}^j_\lambda$ is found, where the space $\mathbf{Y}^1(\overline{\Omega})$ is endowed with the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s))$. Unfortunately, the explicit formula for the bipolar functional $(\mathbb{P}^j_\lambda)^{**}$ is not found directly. Therefore a modification $\widetilde{\mathbb{P}}^j_\lambda$ of $\mathbb{P}^j_\lambda$ is defined below, and only $(\widetilde{\mathbb{P}}^j_\lambda)^{**}$ is found explicitly. In Theorem 11 it is shown that $(\widetilde{\mathbb{P}}^j_\lambda)^{**} = (\mathbb{P}^j_\lambda)^{**}$. The reasoning given below is a modification of the method of [5]. Only those fragments of the proofs which are different from those in [5] are given. In the functionals $\mathbb{P}^j_\lambda$ and $\widetilde{\mathbb{P}}^j_\lambda$ the work of the volume forces is omitted (cf. [5, Section 5]).

Because of the duality between $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$, we define a functional $(\mathbb{P}^j_\lambda)^* : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R} \cup \{\infty\}$ by

$$(4.1)\quad (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}) = \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - \mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1), \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s) \text{ and } \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}\}.$$

We say that $(\mathbb{P}^j_\lambda)^*$ is the dual functional to $\mathbb{P}^j_\lambda$ with respect to the duality between $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ (see [11, pp. 16–18]). The bidual functional $(\mathbb{P}^j_\lambda)^{**} : \mathbf{Y}^1(\overline{\Omega}) \to \mathbb{R}\cup\{\infty\}$ is defined by

$$(4.2)\quad (\mathbb{P}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)\}.$$

The space $\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ is isomorphic to $\{-\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu} \in L^1(\mathrm{Fr}\,\Omega, \mathbf{E}^n_s) \mid \mathbf{u} \in BD(\Omega)\}$ (cf. (3.3)). It follows that the bilinear form between $\mathbb{M}_b(\Omega, \mathbf{E}^n_s) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ is given by

$$(4.3)\quad \langle(\widetilde{\mathbf{w}}, -\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}), \boldsymbol{\sigma}\rangle_1 \equiv \int_\Omega \boldsymbol{\sigma} : \widetilde{\mathbf{w}} + \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (-\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$$

for $\widetilde{\mathbf{w}} \in \mathbb{M}_b(\Omega, \mathbf{E}^n_s)$, $-\boldsymbol{\gamma}^I_B(\mathbf{u}) ds \otimes_s \boldsymbol{\nu} \in \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$. Therefore a net $\{\boldsymbol{\sigma}_\delta\}_{\delta\in D} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ converges to $\boldsymbol{\sigma}_0 \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ in the topology

$$(4.4)\quad \sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), L^1_\mu(\Omega, \mathbf{E}^n_s) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega})$$

if $\langle(\widetilde{\mathbf{w}}, -\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}), \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_\delta\rangle_1 \to 0$ for every $\widetilde{\mathbf{w}}_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$, every $\mathbf{h} \in L^1(S, \mathbb{R}^n)$ where $\widetilde{\mathbf{w}}_{|S} = \mathbf{h} \otimes_s \boldsymbol{\nu}$, and every $-\boldsymbol{\gamma}^I_B(\mathbf{u}) ds \otimes_s \boldsymbol{\nu} \in \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$. The extension $\widetilde{\mathbb{P}}^j_\lambda$ of $\mathbb{P}^j_\lambda$ onto the space $\mathbb{M}_b(\Omega, \mathbf{E}^n_s) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ is given by

$$(4.5)\quad \widetilde{\mathbb{P}}^j_\lambda(\mathbf{w}, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}, -\boldsymbol{\gamma}_B(\mathbf{u}) ds \otimes_s \boldsymbol{\nu}) \equiv -\int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds + \int_{\Omega - S} j(x, \mathbf{w})\, dx + \int_S j(x, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds + \int_{\Gamma_0} I_{\{\boldsymbol{\gamma}_B(\mathbf{u})\otimes_s\boldsymbol{\nu} = 0\}}(\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$$

if $\mathbf{w} \in L^1(\Omega - S, \mathbf{E}^n_s)$ and $\mathbf{u} \in BD(\Omega)$, where $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda\mathbf{g}$ on Γ_1, and $\widetilde{\mathbb{P}}^j_\lambda(\mathbf{w}, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}, -\boldsymbol{\gamma}_B(\mathbf{u}) ds \otimes_s \boldsymbol{\nu}) \equiv \infty$ otherwise.

By duality between $\mathbb{M}_b(\Omega, \mathbf{E}^n_s) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$, we define the dual functional $(\widetilde{\mathbb{P}}^j_\lambda)^* : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R}\cup\{\infty\}$ (cf. (4.3)). It is given by

$$(4.6)\quad (\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) = \sup\Big\{ \int_{\Omega - S} \boldsymbol{\sigma} : \mathbf{w}\, dx + \int_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}) ds - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\beta}_B(\boldsymbol{\sigma}) \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds - \widetilde{\mathbb{P}}^j_\lambda(\mathbf{w}, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}, -\boldsymbol{\gamma}_B(\mathbf{u}) ds \otimes_s \boldsymbol{\nu}) \,\Big|\, \mathbf{w} \in L^1(\Omega - S, \mathbf{E}^n_s),\ \mathbf{u} \in BD(\Omega)\Big\}.$$

The bidual functional $(\widetilde{\mathbb{P}}^j_\lambda)^{**} : \mathbf{Y}^1(\overline{\Omega}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(4.7)\quad (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega-S}, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}, -\boldsymbol{\gamma}^I_B(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu})$$
$$= (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = \sup\Big\{\int_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Omega-S} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega-S}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\beta}_B(\boldsymbol{\sigma}) \cdot \boldsymbol{\gamma}^I_B(\mathbf{u})\, ds - (\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) \;\Big|\; \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)\Big\}$$

for $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} = (\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega-S}, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}, -\boldsymbol{\gamma}^I_B(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu}) \in \mathbf{Y}^1(\overline{\Omega})$ (cf. (3.3) and [5, (3.16)]).

LEMMA 3 (see [5] and [4]). *For every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ we have $(\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) \geq (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma})$. Moreover, $(\widetilde{\mathbb{P}}^j_\lambda)^{**}(\mathbf{M}) \leq (\mathbb{P}^j_\lambda)^{**}(\mathbf{M})$ for every $\mathbf{M} \in \mathbf{Y}^1(\overline{\Omega})$.*

PROPOSITION 4. *The functional $(\widetilde{\mathbb{P}}^j_\lambda)^{**}$ is given by the expression*

$$(4.8)\quad (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = -\int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Gamma_0} j_\infty(x, -\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds + \int_S j(x, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Omega-S} j(x, \boldsymbol{\varepsilon}(\mathbf{u})_a)\, dx + \int_{\Omega-S} j_\infty(x, d\boldsymbol{\varepsilon}(\mathbf{u})_s / d|\boldsymbol{\varepsilon}(\mathbf{u})_s|)\, d|\boldsymbol{\varepsilon}(\mathbf{u})_s|$$

for $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$, where $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda\mathbf{g}$ on Γ_1 and $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u})_a dx + \boldsymbol{\varepsilon}(\mathbf{u})_s$ is the Lebesgue decomposition of $\boldsymbol{\varepsilon}(\mathbf{u})$ into absolutely continuous and singular parts with respect to dx.

Proof. By [14, Theorem 3A and Proposition 2M] we obtain $(\widetilde{\mathbb{P}}^j_\lambda)^*$. From [8, Theorem 1] we get (4.8) (see also [5, Proposition 7]). We note that in the functional $(\widetilde{\mathbb{P}}^j_\lambda)^*$ the normal integrand over S is given by the expression

$$\sup\Big\{\int_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds - \int_S j(x, (\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \;\Big|$$
$$\mathbf{u} \in BD(\Omega)\Big\} = \int_S I_{A^s}(\boldsymbol{\sigma}(x))\, ds$$

(cf. (3.5), (3.6), (3.8) and (3.9)). ∎

LEMMA 5. *For every $\mathbf{u} \in BD(\Omega_1)$ such that $\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega-S} \in L^1(\Omega - S, \mathbf{E}^n_s)$, $\mathbf{u}_{|\Omega_1-\overline{\Omega}} = \mathbf{0}$ and $\boldsymbol{\gamma}^I_B(\mathbf{u})_{|\Gamma_0} = \mathbf{0}$, we have $(\mathbb{P}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = \mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}})$.*

Proof. By Lemma 3, we have $(\widetilde{\mathbb{P}}^j_\lambda)^{**}(\mathbf{M}) \leq (\mathbb{P}^j_\lambda)^{**}(\mathbf{M}) \leq \mathbb{P}^j_\lambda(\mathbf{M})$ for every $\mathbf{M} \in \mathbf{Y}^1(\overline{\Omega})$. Therefore, by (4.8), we get the assertion. ∎

LEMMA 6 (see [5, Lemma 9]). *For every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *and every* $\boldsymbol{\sigma}_s \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *such that* $\operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}$, *we have* $(\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}) = (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s)$.

We say that a net $\{\boldsymbol{\sigma}_k\}_{k\in K} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ converges to $\widehat{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ in the topology

$$\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), L^1(\Omega - S, \mathbf{E}^n_s) \times (L^1(S, \mathbb{R}^n) \otimes_s \boldsymbol{\nu}) \times \{\varphi \in \mathbf{Y}^1(\overline{\Omega})_{|\operatorname{Fr}\Omega} \mid \varphi_{|\Gamma_0} = \mathbf{0}\}) \tag{4.9}$$

if

$$\int_{\Omega - S} (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : \mathbf{w}\, dx + \int_S (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : (\mathbf{p}^1 \otimes_s \boldsymbol{\nu})\, ds + \int_{\Gamma_1} (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : (\mathbf{p} \otimes_s \boldsymbol{\nu})\, ds \to 0 \tag{4.10}$$

for every $\mathbf{w} \in L^1(\Omega, \mathbf{E}^n_s)$, $\mathbf{p}^1 \in L^1(S, \mathbb{R}^n)$ and $\mathbf{p} \in L^1(\Gamma_1, \mathbb{R}^n)$.

LEMMA 7. *Let* $\widehat{f} : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R}$ *be a linear functional, continuous in the topology* (4.9), *such that* $\widehat{f}(\boldsymbol{\sigma}_s) = 0$ *for every* $\boldsymbol{\sigma}_s \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *with* $\operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}$ *in* Ω. *Then there exists* $\widetilde{\mathbf{u}} \in BD(\Omega)$ *such that for every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$,

$$\widehat{f}(\boldsymbol{\sigma}) = \int_{\Omega - S} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})\, dx + \int_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\widetilde{\mathbf{u}}) - \boldsymbol{\gamma}^-_B(\widetilde{\mathbf{u}})) \otimes_s \boldsymbol{\nu})\, ds - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds, \tag{4.11}$$

$\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) = \mathbf{0}$ *on* Γ_0 *and* $\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$.

Proof. Since $\widehat{f}$ is continuous in the topology (4.9), by Theorem V.3.9 of [10] there exist $\mathbf{m} \in L^1(\Omega - S, \mathbf{E}^n_s)$, $\mathbf{m}_1 \in L^1(S, \mathbb{R}^n)$ and $\widehat{\mathbf{u}} \in BD(\Omega)$ such that $\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) = \mathbf{0}$ on Γ_0 and $\widehat{f}(\boldsymbol{\sigma}) = \int_{\Omega - S} \boldsymbol{\sigma} : \mathbf{m}\, dx + \int_S \boldsymbol{\sigma} : (\mathbf{m}_1 \otimes_s \boldsymbol{\nu})\, dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds$ for all $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$. Next, we proceed similarly to the proof of Lemma 10 in [5]. ∎

Let $Q : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$Q(\boldsymbol{\sigma}) = \inf\{(\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s) \mid \boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}^n_s) \text{ and } \operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}\}. \tag{4.12}$$

PROPOSITION 8. *For every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *we have*

$$(\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}) = \mathrm{cl}_{(4.9)}\, Q(\boldsymbol{\sigma}), \tag{4.13}$$

where $\mathrm{cl}_{(4.9)}\, Q$ *denotes the largest minorant of* Q *which is l.s.c. in the topology* (4.9) (*i.e.* $\mathrm{cl}_{(4.9)}\, Q$ *is the l.s.c. regularization of* Q *in* (4.9)).

Proof. We proceed similarly to Steps 1–5 of the proof of Proposition 11 in [5]. We say that a net $\{\boldsymbol{\sigma}_k\}_{k\in K} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ is convergent to $\widehat{\boldsymbol{\sigma}} \in$

$C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ in the topology

$$\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \{\boldsymbol{\varphi} \in \mathbf{Y}^1(\overline{\Omega}) \mid \exists \mathbf{u} \in BD(\Omega_1), \boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varphi}, \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s), \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}, \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}) \tag{4.14}$$

if

$$\int\limits_{\Omega - S} (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : \boldsymbol{\varepsilon}(\mathbf{u})\, dx + \int\limits_S (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds - \int\limits_{\mathrm{Fr}\,\Omega} (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : (\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \to 0 \tag{4.15}$$

for every $\mathbf{u} \in BD(\Omega)$ such that $\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$ and $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0}$ on Γ_0. The l.s.c. regularization of $(\widetilde{\mathbb{P}}^j_\lambda)^*$ in the topology (4.14) (denoted by $\mathrm{cl}_c(\widetilde{\mathbb{P}}^j_\lambda)^*$) is given by

$$\begin{aligned}\mathrm{cl}_c(\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) &= \sup\Big\{ \int\limits_{\Omega - S} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})\, dx - \int\limits_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \\ &\quad + \int\limits_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds - (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \;\Big|\; \\ &\qquad \mathbf{u} \in BD(\Omega_1), \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s), \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}, \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0 \Big\} \\ &= \sup\Big\{ \int\limits_{\Omega - S} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})\, dx + \int\limits_S \boldsymbol{\sigma} : ((\boldsymbol{\gamma}^+_B(\mathbf{u}) - \boldsymbol{\gamma}^-_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \\ &\quad - \int\limits_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds - \mathbb{P}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \;\Big|\; \mathbf{u} \in BD(\Omega_1), \\ &\qquad \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s), \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}, \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0 \Big\} = (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma})\end{aligned} \tag{4.16}$$

for $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ (cf. Lemma 5). Similarly to the proof of Proposition 11 in [5], we obtain a contradiction. ■

LEMMA 9. *For every* $\widehat{r} > 0$, *the topology* (4.9) *is stronger than the topology* $\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \mathbf{Y}^1(\overline{\Omega}))$ *over the set* $\{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n} \le \widehat{r}\}$.

Proof. By [5, Lemma 12] the topology (4.12) defined in [5] is stronger than $\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \mathbf{Y}^1(\overline{\Omega}))$ over the set $\{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n} \le \widehat{r}\}$. Moreover, the topology (4.9) is stronger than the topology (4.12) from [5]. ■

PROPOSITION 10. *Let* $A_k \equiv \{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n} \le k\}$. *For every* $\widehat{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *and every* $k > \|\mathrm{div}\, \widehat{\boldsymbol{\sigma}}\|_{L^n}$,

$$(\mathbb{P}^j_\lambda)^*(\widehat{\boldsymbol{\sigma}}) = \mathrm{cl}_{A_k} Q(\widehat{\boldsymbol{\sigma}}), \tag{4.17}$$

where $\mathrm{cl}_{A_k} Q(\cdot)$ *is the l.s.c. regularization of the function* $\boldsymbol{\sigma} \mapsto Q(\boldsymbol{\sigma}) + I_{A_k}(\boldsymbol{\sigma})$ *in the topology* (4.9) *and* $I_{A_k}(\cdot)$ *is the indicator function of* A_k.

Proof. Step 1. Suppose there exist $\boldsymbol{\sigma}_1 \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ and constants $\delta_0 > 0$, $k > 0$ such that $k > \|\operatorname{div} \boldsymbol{\sigma}_1\|_{L^n}$ and $(\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0 < \mathrm{cl}_{A_k} Q(\boldsymbol{\sigma}_1)$. On account of Lemmas 3 and 6, it suffices to show that this assumption leads to a contradiction.

For every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ let

$$(4.18)\qquad (\widetilde{\mathbb{P}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \equiv \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in A_k\},$$

$$(4.19)\qquad (\widetilde{\mathbb{P}}^j_\lambda)^*_{\|A_k}(\boldsymbol{\sigma}) \equiv (\widetilde{\mathbb{P}}^j_\lambda)^*(\boldsymbol{\sigma}) + I_{A_k}(\boldsymbol{\sigma}) \quad \forall \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$$

(cf. [5, Proof of Proposition 13]). For every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ let

$$(4.20)\qquad \mathrm{cl}_c(\widetilde{\mathbb{P}}^j_\lambda)^*_{\|A_k}(\boldsymbol{\sigma}) = \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{P}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\ \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\},$$

where cl_c is the l.s.c. regularization (of the functional considered) in the topology (4.14). Then for every $\widehat{k} > 0$ such that $\|\operatorname{div} \boldsymbol{\sigma}_1\|_{L^n} < \widehat{k}$ we have

$$(4.21)\qquad \mathrm{cl}_c(\widetilde{\mathbb{P}}^j_\lambda)^*_{\|A_{\widehat{k}}}(\boldsymbol{\sigma}_1) = (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}_1)$$

(cf. (4.16)). Indeed,

$$(4.22)\qquad \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}_1\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{P}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\ \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}$$
$$= \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}_1\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\ \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}$$

if $k > \|\operatorname{div} \boldsymbol{\sigma}_1\|_{L^n}$, since $(\widetilde{\mathbb{P}}^j_\lambda)^{*k}$ is the supremum over all affine mappings $\mathbf{Y}^1(\overline{\Omega}) \ni \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \mapsto \langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} + \text{const}$ which are less than $(\widetilde{\mathbb{P}}^j_\lambda)$, and $\boldsymbol{\sigma} \in A_k$.

Step 2. Similarly to the proof of [5, Proposition 11], for every $k > 0$, there exists a linear functional $f_k : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R}$ given by

$$(4.23)\qquad f_k(\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}_k)\, dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\mathbf{u}_k) \otimes_s \boldsymbol{\nu})\, ds,$$

where $\boldsymbol{\gamma}_B(\mathbf{u}_k) = \mathbf{0}$ on Γ_0 and $\boldsymbol{\varepsilon}(\mathbf{u}_k)_{|\Omega - S} \in L^1(\Omega - S, \mathbf{E}^n_s)$ for every $k > 0$ (see Lemma 7). Moreover (by the proof of [5, Proposition 11]) for all $k > 0$ there exists $c_k \in \mathbb{R}$ such that

$$(4.24)\qquad (\mathbb{P}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0 < f_k(\boldsymbol{\sigma}_1) + c_k \quad \text{and} \quad f_k(\widetilde{\boldsymbol{\sigma}}) + c_k < \mathrm{cl}_{A_k} Q(\widetilde{\boldsymbol{\sigma}})$$

for every $\widetilde{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$. From (4.20), (4.21), (4.23) and (4.24) we obtain a contradiction. ∎

THEOREM 11. *For every $\boldsymbol{\varphi} \in \mathbf{Y}^1(\overline{\Omega})$ we have $(\widetilde{\mathbb{P}}^j_\lambda)^{**}(\boldsymbol{\varphi}) = (\mathbb{P}^j_\lambda)^{**}(\boldsymbol{\varphi})$.*

Proof. We prove this result similarly to that in [5, Theorem 14]. ∎

REMARK 1. If we assume that the functional considered is globally coercive then we easily obtain the existence theorem.

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