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# EXISTENCE OF SOLUTIONS TO THE (rot, div)-SYSTEM IN $L_p$-WEIGHTED SPACES

*Abstract.* The existence of solutions to the elliptic problem $\operatorname{rot} v = w$, $\operatorname{div} v = 0$ in a bounded domain $\Omega \subset \mathbb{R}^3$, $v \cdot \bar{n}|_S = 0$, $S = \partial\Omega$ in weighted $L_p$-Sobolev spaces is proved. It is assumed that an axis $L$ crosses $\Omega$ and the weight is a negative power function of the distance to the axis. The main part of the proof is devoted to examining solutions of the problem in a neighbourhood of $L$. The existence in $\Omega$ follows from the technique of regularization.

**1. Introduction.** We consider the elliptic boundary value problem

$$\begin{aligned} &\operatorname{rot} v = w && \text{in } \Omega,\\ &\operatorname{div} v = 0 && \text{in } \Omega,\\ &v \cdot \bar{n} = b && \text{on } S, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $S = \partial\Omega$, $S \in C^2$, $\bar{n}$ is the unit outward vector normal to $S$ and the dot denotes the scalar product in $\mathbb{R}^3$.

For the solvability of problem (1.1), the following compatibility conditions have to be satisfied:

$$\int_S b(s)\, ds = 0, \tag{1.2}$$

$$\operatorname{div} w = 0. \tag{1.3}$$

Let $L$ be an axis passing through $\Omega$.

Our aim is to prove the existence of solutions to problem (1.1) in weighted Sobolev spaces with the weight equal to a power function of the distance to $L$. Therefore, we introduce the weighted Sobolev space $V^k_{p,-\mu}(\Omega)$ with the finite

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norm

$$\|u\|_{V^k_{p,-\mu}(\Omega)} = \Big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha_x u(x)|^p \varrho(x)^{p(-\mu-k+|\alpha|)}\,dx\Big)^{1/p} < \infty,$$

where $\varrho(x) = \operatorname{dist}\{x, L\}$, $p \in [1,\infty)$, $\mu \in \mathbb{R}_+$, $k \in \mathbb{N}\cup\{0\}$. Moreover, we introduce the notation

$$H^k_{-\mu}(\Omega) = V^k_{2,-\mu}(\Omega), \quad L_{p,-\mu}(\Omega) = V^0_{p,-\mu}(\Omega).$$

The main result of this paper is the following

Theorem 1.1. *Assume the compatibility conditions* (1.2), (1.3). *Assume that* $w \in V^k_{p,-\mu}(\Omega)$, $b \in V^{k+1-1/p}_{p,-\mu}(S)$, $\mu \in \mathbb{R}_+$, $\mu \notin \mathbb{Z}$, $k \in \mathbb{N}\cup\{0\}$, $p \in [2,\infty)$, $S \in C^2$. *Then there exists a solution to problem* (1.1) *such that* $v \in V^{k+1}_{p,-\mu}(\Omega)$ *and*

$$\|v\|_{V^{k+1}_{p,-\mu}(\Omega)} \le c(\|w\|_{V^k_{p,-\mu}(\Omega)} + \|b\|_{V^{k+1-1/p}_{p,-\mu}(S)}), \tag{1.4}$$

*where $c$ does not depend on $v, w, b$.*

To prove Theorem 1.1 we need [4] and the following result.

Theorem 1.2 (see [7]). *Assume the compatibility conditions* (1.2), (1.3). *Assume that* $w \in H^k_{-\mu}(\Omega)$, $b \in H^{k+1/2}_{-\mu}(S)$, $\mu \in \mathbb{R}_+$, $\mu \notin \mathbb{Z}$, $k \in \mathbb{N}\cup\{0\}$. *Then there exists a solution to problem* (1.1) *such that*

$$\|v\|_{H^{k+1}_{-\mu}(\Omega)} \le c(\|w\|_{H^k_{-\mu}(\Omega)} + \|b\|_{H^{k+1/2}_{-\mu}(S)}), \tag{1.5}$$

*where $c$ does not depend on $v, w, b$.*

The main step in the proofs of Theorems 1.1, 1.2 is to obtain an estimate in weighted spaces in neighbourhoods of points of the axis $L$ because estimates in neighbourhoods at a positive distance to the axis are well known (see [1, 6]). We concentrate on local estimates, and a global estimate follows by applying the idea of regularization (see [7, 3]). Restricting our considerations to neighbourhoods of points of $L$ we showed in [7] that it is sufficient to examine problem (1.1) in neighbourhoods of interior points of $\Omega \cap L$. In this case problem (1.1) can be replaced by the elliptic problem (see [7])

$$\begin{aligned} -\Delta u &= f && \text{in } C_{R,a}, \\ u &= 0 && \text{on } \partial C_{R,a}, \end{aligned} \tag{1.6}$$

where $C_{R,a}$ is an axially symmetric cylinder with axis of symmetry $L$ located in $\Omega$. Introducing the cylindrical coordinates $r, \varphi, z$ by relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$, where $x_1, x_2, x_3$ are the Cartesian coordinates such as $L$ is the $x_3$-axis we define

$$C_{R,a} = \{x \in \mathbb{R}^3 : r < R,\ -a < z < a,\ \varphi \in [0,2\pi]\}, \quad C_{R,a} \cap S = \emptyset,$$

and the origin of the Cartesian coordinates is at an interior point of $\Omega \cap L$.

Now we describe the proofs of Theorems 1.1 and 1.2 underlining their differences. In [7] we examine problem (1.6) by applying the Fourier transforms and using the norms of weighted spaces also in terms of the Fourier transforms. This is connected with the fact that the $L_2$-theory is developed in [7]. In this paper we improve the regularity of solutions from [7] by applying the local regularization method (see [5]) and by using the Paley–Littlewood partition of unity so the methods of this paper are totally different from those in [7].

Up to now Theorem 1.1 has not been applied in proofs of existence of regular solutions to the Navier–Stokes equations (see references in [7]). However, in more delicate proofs based on the $L_p$-approach Theorem 1.1 will be needed.

**2. Notation.** Using the cylindrical coordinates we introduce the weighted spaces

$$V^k_{p,-\mu}(\mathbb{R}^3) = \Big\{u : \Big(\sum_{|\alpha|\le k} \int_{\mathbb{R}^3} |D^\alpha_x u(x)|^p r^{p(-\mu-k+|\alpha|)}\,dx\Big)^{1/p} < \infty\Big\}.$$

**3. Localization of problem** (1.1)**.** We are looking for solutions to problem (1.1) in the form (see [7])

$$v = \nabla\varphi + u, \tag{3.1}$$

where $\varphi$ is a solution to the problem

$$\Delta\varphi = 0, \quad \bar{n}\cdot\nabla\varphi|_S = b \tag{3.2}$$

and $u$ satisfies

$$\begin{aligned} &\operatorname{rot} u = w \quad \text{in } \Omega,\\ &\operatorname{div} u = 0 \quad \text{in } \Omega,\\ &u\cdot\bar{n}|_S = 0. \end{aligned} \tag{3.3}$$

From [2] we know that $(3.3)_{2,3}$ imply the existence of a vector field $e$ such that

$$u = \operatorname{rot} e, \quad \operatorname{div} e = 0, \quad e\cdot\bar{\tau}|_S = 0, \tag{3.4}$$

where $\bar{\tau}$ is a tangent vector to $S$.

In view of (3.4) problem (3.3) takes the form

$$-\Delta e = w, \quad e\cdot\bar{\tau}|_S = 0, \quad \operatorname{div} e|_S = 0. \tag{3.5}$$

Localizing problems (3.2) and (3.5) by a smooth function from a partition of unity we obtain problem (1.6). The lower order terms which appear in this procedure will be estimated by applying the regularization method.

**4. Regularity near the axis $L$ in the $L_p$-approach.** After localization of problems (3.2) and (3.5) to a neighbourhood of an internal point of $L$ or a point where $L$ meets $S$ we can replace them by the problem

$$-\Delta u = f \quad \text{in } \Omega,\qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $\Omega = C_{R,a}$.

Lemma 4.1 (local regularity; see [5]). *Let $B_r = \{x_3 \in \mathbb{R} : |x_3| < r\}$, $\xi_i = \xi_i(x')$, $x' = (x_1, x_2)$, $\xi_i \in C_0^\infty(\mathbb{R}^2)$, $i = 1,2$, $\xi_1\xi_2 = \xi_1$,* $\operatorname{supp}\xi_2 \subset \{x' : c_1 < |x'| < c_2\}$. *Then for every $u \in W_p^{l+2}(\mathbb{R}^2 \times B_2)$ the following inequality holds:*

$$\|\xi_1 u\|_{W_p^{l+2}(\mathbb{R}^2\times B_1)} \le c(\|\xi_2 \Delta u\|_{W_p^l(\mathbb{R}^2\times B_2)} + \|\xi_2 u\|_{L_{p_1}(\mathbb{R}^2\times B_2)}), \tag{4.2}$$

*for any $p_1 \in [1,\infty]$, where $c$ does not depend on $u$.*

Let us introduce partitions of unity $\{\zeta_j\}_{j=-\infty}^{\infty}$, $\{\sigma_j\}_{j=-\infty}^{\infty}$ such that

$$\operatorname{supp}\zeta_j \subset \{x' : 2^{j-1} < |x'| < 2^{j+1}\},$$
$$\operatorname{supp}\sigma_j \subset \{x' : 2^{j-2} < |x'| < 2^{j+2}\},$$

$\sigma_j, \zeta_j \in C^\infty(\mathbb{R}^2)$, $\zeta_j\sigma_j = \zeta_j$, and for all multiindices $\alpha$,

$$|D^\alpha\zeta_j| + |D^\alpha\sigma_j| \le c_\alpha 2^{-j|\alpha|}.$$

Lemma 4.2. *Let $\beta \in \mathbb{R}$. Then for any $u \in W_p^{l+2}(\{2^{j-2} < |x'| < 2^{j+2}\} \times \mathbb{R}^1)$ the following inequality holds,*

$$\|\zeta_j u\|_{V_{p,\beta}^{l+2}(\mathbb{R}^2\times\mathbb{R}^1)} \le c(\|\sigma_j \Delta u\|_{V_{p,\beta}^{l}(\mathbb{R}^2\times\mathbb{R}^1)} + \|\sigma_j u\|_{L_{p,\beta-l-2}(\mathbb{R}^2\times\mathbb{R}^1)}), \tag{4.3}$$

*where $c$ does not depend on $u$ and $j$.*

*Proof.* Let $K = \{x' : 1 < |x'| < 2\}$, $B = \{x_3 : |x_3| < 2\}$, $K_\mu = \{x' : 2^\mu < |x'| < 2^{\mu+1}\}$, $B_\mu = \{x_3 : |x_3| < 2^{\mu+1}\}$. From (4.2) we have

$$\sum_{|\alpha|=0}^{l+2} \|D^\alpha(\zeta_j u)\|_{L_p(K\times B)} \le c \sum_{|\alpha|=0}^{l} \|D^\alpha(\sigma_j \Delta u)\|_{L_p(2K\times 2B)} + c\|\sigma_j u\|_{L_p(2K\times 2B)}, \tag{4.4}$$

where $2K = \{x' : 1/2 < |x'| < 4\}$, $2B = \{x_3 : |x_3| < 4\}$. In view of scaling $x \mapsto 2^\mu x$ we obtain

$$\sum_{|\alpha|=0}^{l+2} 2^{\mu(|\alpha|-2)} \|D^\alpha(\zeta_j u)\|_{L_p(K_\mu\times B_\mu)} \le c \sum_{|\alpha|=0}^{l} 2^{\mu|\alpha|} \|D^\alpha\sigma_j \Delta u\|_{L_p(2K_\mu\times 2B_\mu)} + c2^{-2\mu}\|\sigma_j u\|_{L_p(2K_\mu\times 2B_\mu)}. \tag{4.5}$$

Multiplying (4.5) by $2^{(\beta-l)\mu}$, taking the $p$th power, summing over $\mu$ and using that $r\sim 2^\mu$ on the set $K_\mu$, we obtain

$$(4.6)\qquad \sum_{|\alpha|=0}^{l+2}\|r^{\beta-l-2+|\alpha|}D^\alpha(\zeta_j u)\|_{L_p(\mathbb{R}^3)} \le c\sum_{|\alpha|=0}^{l}\|r^{\beta-l+|\alpha|}D^\alpha(\sigma_j\varDelta u)\|_{L_p(\mathbb{R}^3)}+c\|r^{\beta-l-2}\sigma_j u\|_{L_p(\mathbb{R}^3)}.$$

This implies (4.3) and concludes the proof.

Summing up (4.3) with respect to $j$ yields

$$(4.7)\qquad \|u\|_{V^{l+2}_{p,\beta}(\mathbb{R}^3)}\le c\|\varDelta u\|_{V^l_{p,\beta}(\mathbb{R}^3)}+c\|u\|_{V^0_{p,\beta-l-2}(\mathbb{R}^3)}.$$

Let

$$P(\partial_{x'},\partial_{x_3})=-\varDelta,\qquad P(\partial_{x'},\eta)=-\varDelta'+\eta^2,$$

where $\varDelta'=\partial^2_{x_1}+\partial^2_{x_2}$. Let us denote by $A(\eta): V^{l+2}_{p,\beta}(\mathbb{R}^2)\to V^l_{p,\beta}(\mathbb{R}^2)$ the operator of the problem

$$(4.8)\qquad \begin{aligned} &P(\partial_{x'},\eta)u=f &&\text{in } \mathbb{R}^2,\\ &u=0 &&\text{on } S^1_R,\end{aligned}$$

where $S^1_R=\{x':|x'|=R\}$.

First, we consider the problem

$$(4.9)\qquad \begin{aligned} &P(\partial_{x'},0)u=f &&\text{in } \mathbb{R}^2,\\ &u=0 &&\text{on } S^1_R.\end{aligned}$$

Passing to the polar coordinates $(r,\varphi)$, next to the variable $\tau=-\ln r$ and applying the Fourier transform with respect to $\tau$ we have

$$(4.10)\qquad \begin{aligned} &\bar P(\partial_\varphi,\lambda)\tilde u=\widetilde{e^{2\tau}f},\\ &u|_{\varphi=0}=u|_{\varphi=2\pi},\\ &u_{,\varphi}|_{\varphi=0}=u_{,\varphi}|_{\varphi=2\pi},\end{aligned}$$

and

$$\tilde u(\lambda,\varphi)=\int_{\mathbb{R}^1}u(\tau,\varphi)e^{-i\lambda\tau}\,d\tau.$$

On the line $\operatorname{im}\lambda=\beta-1\notin\mathbb{Z}$ there is no eigenvalue of problem (4.10). Moreover, $\ker A(\eta)$ and $\operatorname{coker} A(\eta)$ are trivial in the spaces $V^0_{2,\beta}(\mathbb{R}^2)$, $\beta\notin\mathbb{Z}$. Thus the operator $A(\eta)^{-1}$ is defined in $V^0_{2,\beta}(\mathbb{R}^2)$, $\beta\notin\mathbb{Z}$.

Lemma 4.3. *For any* $\eta \in \mathbb{R}^1 \setminus \{0\}$ *and any* $f \in V^0_{2,\beta}(\mathbb{R}^2)$, $\beta \notin \mathbb{Z}$, *the following inequality holds:*

$$\sum_{\nu=0}^{2} |\eta|^\nu \|\partial_\eta^\gamma A(\eta)^{-1} f\|_{V^{2-\nu}_{2,\beta}(\mathbb{R}^2)} \le c|\eta|^{-\gamma} \|f\|_{V^0_{2,\beta}(\mathbb{R}^2)}. \tag{4.11}$$

*Proof.* We have

$$\partial_\eta^\gamma A(\eta)^{-1} = \sum_{1\le q\le\gamma} \sum_{\alpha_1+\dots+\alpha_q=\gamma} c_{\alpha_1\dots\alpha_q} A^{-1}\partial_\eta^{\alpha_1} P A^{-1} \partial_\eta^{\alpha_2} P \dots A^{-1}\partial_\eta^{\alpha_q} P A^{-1},$$

where $c_{\alpha_1\dots\alpha_q}$ are some constant coefficients. Since

$$\sum_{\nu=0}^{2} |\eta|^\nu \|u\|_{V^{2-\nu}_{2,\beta}(\mathbb{R}^2)} \le c\|P(\partial_{x'},\eta)u\|_{V^0_{2,\beta}(\mathbb{R}^2)},$$

and

$$\|\partial_\eta^\gamma P(\partial_{x'},\eta) A(\eta)^{-1} f\|_{V^0_{2,\beta}(\mathbb{R}^2)} \le c\sum_{\nu=0}^{2-\gamma} |\eta|^\nu \|A(\eta)^{-1} f\|_{V^{2-\gamma-\nu}_{2,\beta}(\mathbb{R}^2)},$$

we obtain

$$\|\partial_\eta^\gamma P(\partial_{x'},\eta) A(\eta)^{-1} f\|_{V^0_{2,\beta}(\mathbb{R}^2)} \le c|\eta|^{-\gamma}\|f\|_{V^0_{2,\beta}(\mathbb{R}^2)}.$$

Therefore (4.11) is proved. This concludes the proof.

Lemma 4.4. *Let the assumptions of Lemma* 4.3 *be satisfied. Let* $\sigma_\mu, \zeta_\nu$ *be the functions introduced after Lemma* 4.1. *Then for all* $\eta \ne 0$ *the estimate*

$$\|\sigma_\mu A(\eta)^{-1}\zeta_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)\to V^0_{2,\beta}(\mathbb{R}^2)} \le c2^{-\varepsilon|\mu-\nu|+2\mu} \tag{4.12}$$

*holds for* $\beta \notin \mathbb{Z}$ *and* $\varepsilon > 0$ *sufficiently small.*

*Proof.* Observe that the operator $\sigma_\mu A(\eta)^{-1}\zeta_\nu$ maps functions with support in $\operatorname{supp}\zeta_\nu$ to functions with support in $\operatorname{supp}\sigma_\mu$.

First we examine the operator $r^{\pm\varepsilon}A(\eta)r^{\mp\varepsilon}$. Since $A(\eta) = -\Delta' + \eta^2$ and $A(\eta)$ maps $V^2_{2,\beta}(\mathbb{R}^2)$ into $V^0_{2,\beta}(\mathbb{R}^2)$ we have for small $\varepsilon$,

$$\begin{aligned}
&\|r^{\pm\varepsilon}A(\eta)r^{\mp\varepsilon}u\|_{V^0_{2,\beta}(\mathbb{R}^2)} = \|r^{\pm\varepsilon}(-\Delta'+\eta^2)r^{\mp\varepsilon}u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\\
&\quad\le \|(-\Delta'+\eta^2)u\|_{V^0_{2,\beta}(\mathbb{R}^2)} + \varepsilon\|r^{\mp\varepsilon}r^{\pm\varepsilon-1}\nabla' r\nabla' u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\\
&\qquad+\varepsilon(1-\varepsilon)\|r^{\mp\varepsilon}r^{\pm\varepsilon-2}|\nabla' r|^2 u\|_{V^0_{2,\beta}(\mathbb{R}^2)} + \varepsilon\|r^{\mp\varepsilon}r^{\pm\varepsilon-1}\Delta' r u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\\
&\quad\le \|A(\eta)u\|_{V^0_{2,\beta}(\mathbb{R}^2)} + c\varepsilon\|u\|_{V^1_{2,\beta-1}(\mathbb{R}^2)}.
\end{aligned}$$

Conversely,

$$\begin{aligned}\|A(\eta)u\|_{V^0_{2,\beta}(\mathbb{R}^2)} &= \|r^{\mp\varepsilon}r^{\pm\varepsilon}A(\eta)u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\\ &= \|r^{\mp\varepsilon}r^{\pm\varepsilon}(-\Delta'+\eta^2)u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\\ &\le \|r^{\mp\varepsilon}(-\Delta'+\eta^2)r^{\pm\varepsilon}u\|_{V^0_{2,\beta}(\mathbb{R}^2)} + c\varepsilon\|u\|_{V^1_{2,\beta-1}(\mathbb{R}^2)}.\end{aligned}$$

Hence for small $\varepsilon$, the norms of the operators $r^{\pm\varepsilon}A(\eta)r^{\mp\varepsilon}$ and $A(\eta)$ are close to each other because

$$\Big|\sup_{\|u\|_{V^2_{2,\beta}(\mathbb{R}^2)}\le 1}\|r^{\pm\varepsilon}A(\eta)r^{\mp\varepsilon}u\|_{V^0_{2,\beta}(\mathbb{R}^2)} - \sup_{\|u\|_{V^2_{2,\beta}(\mathbb{R}^2)}\le 1}\|A(\eta)u\|_{V^0_{2,\beta}(\mathbb{R}^2)}\Big| \le c\varepsilon.$$

Therefore for $\beta\pm\varepsilon\notin\mathbb{Z}$ and $\varepsilon$ sufficiently small the operator $A(\eta)$ is an isomorphism from $V^2_{2,\beta\pm\varepsilon}(\mathbb{R}^2)$ into $V^0_{2,\beta\pm\varepsilon}(\mathbb{R}^2)$. Let $u_\mu = u\sigma_\mu$ and $f_\nu = f\zeta_\nu$. Then we examine the problem

$$P(\partial_{x'},\eta)u = f_\nu. \tag{4.13}$$

Since the operator $A(\eta)^{-1}$ is defined we have

$$u_\mu = \sigma_\mu A^{-1}(\eta)f_\nu. \tag{4.14}$$

In view of (4.11) we obtain

$$\begin{aligned}\|u_\mu\|_{V^2_{2,\beta}(\mathbb{R}^2)} &\le c2^{\varepsilon\mu}\|u_\mu\|_{V^2_{2,\beta-\varepsilon}(\mathbb{R}^2)}\\ &\le c2^{\varepsilon\mu}\|f_\nu\|_{V^0_{2,\beta-\varepsilon}(\mathbb{R}^2)} \le c2^{\varepsilon(\mu-\nu)}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)}.\end{aligned}$$

Since

$$\|u_\mu\|_{V^2_{2,\beta}(\mathbb{R}^2)} \ge c2^{-2\mu}\|u_\mu\|_{V^0_{2,\beta}(\mathbb{R}^2)}$$

we obtain

$$\|u_\mu\|_{V^0_{2,\mu}(\mathbb{R}^2)} \le c2^{\varepsilon(\mu-\nu)+2\mu}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)},$$

where for $\mu>\nu$ we replace $\varepsilon$ by $-\varepsilon$, $\varepsilon>0$ and for $\mu<\nu$ we take $\varepsilon$. Then we have

$$\|u_\mu\|_{V^0_{2,\beta}(\mathbb{R}^2)} \le c2^{-\varepsilon|\mu-\nu|+2\mu}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)}. \tag{4.15}$$

From (4.15) we derive (4.12). This concludes the proof.

LEMMA 4.5. *Let the assumptions of Lemma* 4.4 *be satisfied. Then the operator $P$ of the problem*

$$\begin{aligned}&P(\partial_{x'},\partial_{x_3})u = f,\\ &u|_{S_R} = 0\end{aligned} \tag{4.16}$$

*is an isomorphism*

$$P : V^2_{2,\beta}(\mathbb{R}^3) \to V^0_{2,\beta}(\mathbb{R}^3). \tag{4.17}$$

The lemma follows from the estimate

$$\|u\|_{V^2_{2,\beta}(\mathbb{R}^3)} \le c\|f\|_{V^0_{2,\beta}(\mathbb{R}^3)}, \quad \beta \notin \mathbb{Z},$$

and the fact that $\ker P$ in $V^2_{2,\beta}(\mathbb{R}^3)$ and $\operatorname{coker} P$ in $V^0_{2,\beta}(\mathbb{R}^3)$ are trivial.

To prove the next lemma we need the following Marcinkiewicz–Mikhlin type result (see [2, Ch. 11, §11]):

Lemma 4.6. *Let $L_p(\mathbb{R}^d;H)$ be the space of functions with the finite norm*

$$\|f\|_{L_p(\mathbb{R}^d;H)} = \Big(\int\limits_{\mathbb{R}^d} \|f(z)\|_H^p\,dz\Big)^{1/p} < \infty,$$

*where $H$ is a Hilbert space. Let $M(\xi)$, $\xi \in \mathbb{R}^d$, be a bounded function in $\mathbb{R}^d$ acting as a multiplier. Let for $s = 0,\dots,d$, $i_k \ne i_l$,*

$$|\xi|^s \bigg\|\frac{\partial^s M}{\partial\xi_{i_1}\dots\partial\xi_{i_s}}(\xi)\bigg\|_{H\to H} \le \text{const.}$$

*Then $F^{-1}_{\xi\to z}M(\xi)F_{z\to\xi}$, where $F$ is the Fourier transform in $\mathbb{R}^d$, is a continuous operator in $L_p(\mathbb{R}^d;H)$.*

Lemma 4.7. *Let the assumptions of Lemma 4.3 be satisfied. Let $u_\nu \in V^2_{2,\beta}(\mathbb{R}^3)$ be a solution to the problem*

$$P(\partial_{x'},\partial_{x_3})u_\nu = \zeta_\nu f.$$

*Then*

$$\int\limits_{\mathbb{R}^1}\Big(\int\limits_{\mathbb{R}^2} r^{2\beta}|\sigma_\mu(x')u_\nu(x',x_3)|^2\,dx'\Big)^{p/2}dx_3 \le c2^{-p\varepsilon|\mu-\nu|+2\mu p}\int\limits_{\mathbb{R}^1}\Big(\int\limits_{\mathbb{R}^2} r^{2\beta}|\zeta_\nu(x')f(x',x_3)|^2\,dx'\Big)^{p/2}dx_3, \tag{4.18}$$

*where $c$ does not depend on $u$ and $f$.*

*Proof.* We have

$$u_\nu = F^{-1}_{\eta\to x_3}A(\eta)^{-1}F_{x_3\to\eta}\zeta_\nu f,$$

where $F$ is the Fourier transform in $\mathbb{R}^1$.

By Lemma 4.6 $F^{-1}_{\eta\to x_3}M(\eta)F_{x_3\to\eta}$, where $M(\eta) = \sigma_\mu A(\eta)^{-1}\zeta_\nu$, is a continuous operator in $L_p(\mathbb{R}^1;V^0_{2,\beta}(\mathbb{R}^2))$. Then by (4.12) we obtain (4.18). This concludes the proof.

Lemma 4.8. *Let the assumptions of Lemma 4.3 be satisfied. Let $u_\nu$ be defined in the assumptions of Lemma 4.7. Then for $p \ge 2$ we have*

$$\int_{\mathbb{R}^3} r^{p(\beta-1)-2}|\zeta_\mu(x')u_\nu(x',x_3)|^p\,dx'\,dx_3 \le c2^{-\varepsilon|\mu-\nu|p}\int_{\mathbb{R}^3} r^{p(\beta+1)-2}|\zeta_\nu(x')f(x',x_3)|^p\,dx'\,dx_3, \tag{4.19}$$

*where c does not depend on u and f.*

*Proof.* By the Hölder inequality the r.h.s. of (4.18) is estimated by

$$\begin{aligned} &c2^{-p\varepsilon|\mu-\nu|+2\mu p}\int_{\mathbb{R}^1} dx_3\Big[|\operatorname{supp}\zeta_\nu|^{(p-2)/p}\Big(\int_{\mathbb{R}^2} r^{p\beta}|\zeta_\nu f|^p\,dx'\Big)^{2/p}\Big]^{p/2} \\ &\quad\le c2^{-p\varepsilon|\mu-\nu|+2\mu p}\int_{\mathbb{R}^1} dx_3\,2^{\nu(p-2)}\int_{\mathbb{R}^2} r^{p\beta}|\zeta_\nu f|^p\,dx' \\ &\quad\le c2^{-p\varepsilon|\mu-\nu|+2\mu p}\int_{\mathbb{R}^1} dx_3\int_{\mathbb{R}^2} r^{p(\beta+1)-2}|\zeta_\nu f|^p\,dx', \end{aligned} \tag{4.20}$$

where we used that on $\operatorname{supp}\zeta_\nu$ we have $r\in(2^{\nu-1},2^{\nu+1})$.

Divide $\mathbb{R}^1$ into segments $Q_j$ of length $2^\mu$ each. By $2Q_j$ we denote a segment with length $2^{\mu+1}$ which contains $Q_j$.

By the Hölder inequality we have

$$\begin{aligned} \int_{2Q_j}\int_{\mathbb{R}^2}|\sigma_\mu u_\nu|\,dx'\,dx_3 &\le \int_{2Q_j}|\operatorname{supp}\sigma_\mu|^{1/2}\Big(\int_{\mathbb{R}^2}|\sigma_\mu u_\nu|^2\,dx'\Big)^{1/2}dx_3 \\ &\le c\int_{2Q_j}2^\mu\,dx_3\Big(\int_{\mathbb{R}^2}|\sigma_\mu u_\nu|^2\,dx'\Big)^{1/2}\equiv I_1, \end{aligned}$$

where we used that

$$\operatorname{supp}\sigma_\mu\subset\{x'\in\mathbb{R}^2: 2^{\mu-2}<|x'|<2^{\mu+2}\}. \tag{4.21}$$

Continuing,

$$I_1\le c\int_{2Q_j}2^{\mu-\beta\mu}\Big(\int_{\mathbb{R}^2}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{1/2}dx_3\equiv I_2,$$

where we used (4.21) again. Continuing,

$$\begin{aligned} \Big(\int_{2Q_j}\int_{\mathbb{R}^2}|\sigma_\mu u_\nu|\,dx'\,dx_3\Big)^p &\le 2^{(1-\beta)\mu p}\Big[\int_{2Q_j}dx_3\Big(\int_{\mathbb{R}^2}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{1/2}\Big]^p \\ &\le c2^{(1-\beta)\mu p}\Big(\int_{2Q_j}1^{p'}\,dx_3\Big)^{p/p'}\Big(\int_{2Q_j}dx_3\Big(\int_{\mathbb{R}^2}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{p/2}\Big)\equiv I_3, \end{aligned}$$

where $1/p+1/p'=1$, $p/p'=p-1$. Hence, we have

$$I_3=c2^{(1-\beta)\mu p+\mu(p-1)}\int_{2Q_j}dx_3\Big(\int_{\mathbb{R}^3}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{p/2}.$$

Therefore, we get

(4.22)
$$\Big(\int\limits_{2Q_j}\int\limits_{\mathbb{R}^2}|\sigma_\mu u_\nu|\,dx'\,dx_3\Big)^p\le c2^{-\mu p(\beta-2)-\mu}\int\limits_{2Q_j}dx_3\Big(\int\limits_{\mathbb{R}^2}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{p/2}.$$

From Lemma 4.1 for $|\mu-\nu|>3$ we obtain the inequality

$$\int\limits_{Q_j}\int\limits_{\mathbb{R}^2}|\zeta_\mu u_\nu|^p\,dx'\,dx_3\le c2^{3\mu(1-p)}\Big(\int\limits_{2Q_j}\int\limits_{\mathbb{R}^2}|\sigma_\mu u_\nu|\,dx'\,dx_3\Big)^p \tag{4.23}$$

by a scaling argument.

From (4.22) and (4.23) and for $|\mu-\nu|>3$ we obtain

$$\int\limits_{\mathbb{R}^3}|\zeta_\mu u_\nu|^p\,dx'\,dx_3\le c2^{2\mu-\mu p(\beta-1)}\int\limits_{\mathbb{R}}\Big(\int\limits_{\mathbb{R}^2}r^{2\beta}|\sigma_\mu u_\nu|^2\,dx'\Big)^{p/2}dx_3. \tag{4.24}$$

In the case $|\mu-\nu|\le 3$ we have to add on the r.h.s. of (4.22) the expression

$$c2^{2\mu p}\int\limits_{2Q_j}\int\limits_{\mathbb{R}^2}|\zeta_\nu f|^p\,dx'\,dx_3.$$

Hence, in this case, we have to add the same term on the r.h.s. of (4.24). Using (4.18) we obtain (4.19). This concludes the proof.

To prove the next lemma we need a result on operators in Banach spaces. Let $\mathcal{E}_0(\mathbb{R}^3),\mathcal{E}_1(\mathbb{R}^3)$ be Banach spaces of functions defined on $\mathbb{R}^3$, closed under pointwise multiplication with functions from $C_0^\infty(\mathbb{R}^2\setminus\{0\})$. Let $\{\zeta_j\}_{j=-\infty}^\infty$ be the partition of unity described above.

Assume that there exist $p$ and $q$, $1\le p\le q\le\infty$, such that for all $u\in\mathcal{E}_0$, $v\in\mathcal{E}_1$ the following inequalities hold:

$$\|u\|_{\mathcal{E}_0}\le c\Big(\sum_{j=-\infty}^\infty\|\zeta_j u\|_{\mathcal{E}_0}^q\Big)^{1/q}, \tag{4.25}$$

$$\|v\|_{\mathcal{E}_1}\ge c\Big(\sum_{j=-\infty}^\infty\|\zeta_j v\|_{\mathcal{E}_1}^p\Big)^{1/p}, \tag{4.26}$$

where $\|\cdot\|_{\mathcal{E}_i}$ is the norm of $\mathcal{E}_i$, $i=0,1$.

Lemma 4.9. *Let $\theta:\mathcal{E}_1\to\mathcal{E}_0$ be a linear operator defined on functions with compact supports such that for some $\varepsilon>0$ and all $\mu,\nu\in\mathbb{Z}$,*

$$\|\zeta_\mu\theta\zeta_\nu v\|_{\mathcal{E}_0}\le ce^{-\varepsilon|\mu-\nu|}\|\zeta_\nu v\|_{\mathcal{E}_1} \tag{4.27}$$

*for all $v\in\mathcal{E}_1(\mathbb{R}^3)$. Then for any $v\in\mathcal{E}_1(\mathbb{R}^3)$ with compact support,*

$$\|\theta v\|_{\mathcal{E}_0(\mathbb{R}^3)}\le c\|v\|_{\mathcal{E}_1(\mathbb{R}^3)}, \tag{4.28}$$

*where $c$ does not depend on $v$.*

*Proof.* From (4.25),

$$\|\theta v\|_{\mathcal{E}_0(\mathbb{R}^3)} = \Big\|\theta\Big(\sum_{\nu=-\infty}^{\infty}\zeta_\nu v\Big)\Big\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\Big(\sum_{\mu=-\infty}^{\infty}\Big\|\sum_{\nu=-\infty}^{\infty}\xi_\mu\theta\xi_\nu v\Big\|^q_{\mathcal{E}_0(\mathbb{R}^3)}\Big)^{1/q}$$
$$\le c\Big[\sum_{\mu=-\infty}^{\infty}\Big(\sum_{\nu=-\infty}^{\infty}\|\xi_\mu\theta\xi_\nu v\|_{\mathcal{E}_0(\mathbb{R}^3)}\Big)^q\Big]^{1/q}.$$

Applying (4.27) yields

$$\|\theta v\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\sum_{\mu=-\infty}^{\infty}\Big(\sum_{\nu=-\infty}^{\infty}e^{-\varepsilon|\mu-\nu|}\|\zeta_\nu v\|_{\mathcal{E}_1(\mathbb{R}^3)}\Big)^q\Big]^{1/q}.$$

Since the operator of discrete convolution with the kernel $\{e^{-|j|\varepsilon}\}_{j=-\infty}^{\infty}$ is continuous from $l_p$ to $l_q$ for $q \ge p$, we have

$$\|\theta v\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\Big(\sum_{\nu=-\infty}^{\infty}\|\zeta_\nu v\|^p_{\mathcal{E}_1(\mathbb{R}^3)}\Big)^{1/p}.$$

Using now (4.26) we obtain (4.28). This concludes the proof.

Lemma 4.10. *Let the assumptions of Lemma* 4.4 *be satisfied. Let* $u \in V^2_{2,\beta}(\mathbb{R}^3)$ *be a solution of problem* (4.16). *Then for* $p \ge 2$ *the inequality*

$$\int_{\mathbb{R}^3} r^{p(\beta-1)-2}|u(x',x_3)|^p\,dx'\,dx_3 \le c\int_{\mathbb{R}^3} r^{p(\beta+1)-2}|f(x',x_3)|^p\,dx'\,dx_3 \tag{4.29}$$

*is valid, where* $c$ *does not depend on* $u$ *and* $f$.

*Proof.* We apply Lemma 4.9 to the inequality of Lemma 4.8 with $q = p$, $\mathcal{E}_0 = V^0_{p,\beta-1-2/p}(\mathbb{R}^3)$, $\mathcal{E}_1 = V^0_{p,\beta+1-2/p}(\mathbb{R}^3)$ and $\theta : V^0_{2,\beta}(\mathbb{R}^3) \to V^2_{2,\beta}(\mathbb{R}^3)$ is the inverse to the operator of problem (4.16). Then (4.19) becomes the inequality (4.27). Then Lemma 4.9 implies (4.28), which is exactly (4.29). This concludes the proof.

Lemma 4.11. *Let the assumptions of Lemma* 4.4 *hold. Let* $f \in V^l_{2,\varkappa}(\mathbb{R}^3) \cap V^l_{p,\varkappa}(\mathbb{R}^3)$, $p \ge 2$. *Then for solutions* $u \in V^{l+2}_{2,\varkappa}(\mathbb{R}^3)$ *of problem* (4.16) *the following inequality is valid:*

$$\|u\|_{V^{l+2}_{p,\varkappa}(\mathbb{R}^3)} \le c\|f\|_{V^l_{p,\varkappa}(\mathbb{R}^3)}, \tag{4.30}$$

*where* $c$ *does not depend on* $u$ *and* $f$.

*Proof.* From (4.7) with $\beta = \varkappa$ we have

$$\|u\|_{V^{l+2}_{p,\varkappa}(\mathbb{R}^3)} \le c(\|f\|_{V^l_{p,\varkappa}(\mathbb{R}^3)} + \|u\|_{V^0_{p,\varkappa-l-2}(\mathbb{R}^3)}).$$

From Lemma 4.10 and from the assumption that $f \in V^0_{p,\varkappa-l}$ we have $p(\beta+1) - 2 = p(\varkappa-l)$, so $\beta = \varkappa - l - 1 + 2/p$, so $p(\beta-1)-2 = p(\varkappa-l-2)$. Hence,

$$\|u\|_{V^0_{p,\varkappa-l-2}(\mathbb{R}^3)} \le c\|f\|_{V^0_{p,\varkappa-l}(\mathbb{R}^3)} \le c\|f\|_{V^l_{p,\varkappa}(\mathbb{R}^3)}.$$

Thus (4.30) holds. This concludes the proof.

THEOREM 4.1. *Let* $\operatorname{Im}\lambda = \beta - 1 \notin \mathbb{Z}$. *Let* $f \in V^l_{p,\beta}(\mathbb{R}^3)$, $l \in \mathbb{N}$, $p \in (1,\infty)$, $\beta \in \mathbb{R}$. *Then there exists a solution to problem* (4.1) *such that* $u \in V^{l+2}_{p,\beta}(\mathbb{R}^3)$ *and*

$$\|u\|_{V^{l+2}_{p,\beta}(\mathbb{R}^3)} \le c\|f\|_{V^l_{p,\beta}(\mathbb{R}^3)}, \tag{4.31}$$

*where* $c$ *does not depend on* $u$ *and* $f$.

*Proof.* Let $\{f_\nu\}$ be a sequence of smooth functions with compact support in $\mathbb{R}^3 \setminus L$ which converges to $f$ in $V^l_{p,\beta}(\mathbb{R}^3)$. By Lemma 4.5 there exists a solution $u_\nu \in V^{l+2}_{2,\beta}(\mathbb{R}^3)$ to problem (4.1) with r.h.s. $f_\nu$. From (4.30) the sequence $\{u_\nu\}$ converges in $V^{l+2}_{p,\beta}(\mathbb{R}^3)$. This concludes the proof.

## 5. Existence in a bounded domain

*Proof of Theorem 1.1.* To prove Theorem 1.1 we introduce a partition of unity (see [7]). We distinguish four types of subdomains: $\Omega^{(1)}$—near an interior point of $\Omega \cap L$, $\Omega^{(2)}$—near the point where $L$ meets $S$, $\Omega^{(3)}$—near an interior point of $\Omega$ but at a positive distance from $L$, $\Omega^{(4)}$—near a point of $S$ at a positive distance from $L$. With each subdomain $\Omega^{(k)}$, $k = 1,2,3,4$, we connect a smooth function $\zeta^{(k)}$ which is equal to 1 in $\bar{w}^{(k)} \subset \Omega^{(k)}$ and vanishes outside of $\Omega^{(k)}$.

We shall restrict our considerations to problem (3.2) only, because problem (3.5) can be treated in the same way.

Let us extend the boundary condition $(3.2)_2$ by introducing a function $\tilde{b}$ such that

$$\bar{n} \cdot \nabla \tilde{b} = b. \tag{5.1}$$

Then the function

$$u = \varphi - \tilde{b} \tag{5.2}$$

is a solution to the problem

$$\begin{aligned} &\Delta u = f \quad \text{in } \Omega,\\ &\bar{n} \cdot \nabla u|_S = 0, \end{aligned} \tag{5.3}$$

where $f = -\Delta \tilde{b}$. Multiplying $(5.3)_1$ by $\zeta^{(1)}$ and defining $u^{(1)} = u\zeta^{(1)}$, $f^{(1)} = f \cdot \zeta^{(1)}$ we obtain, instead of (5.3), the problem

$$\begin{aligned} &\Delta u^{(1)} = f^{(1)} + 2\nabla\zeta^{(1)}\nabla u + \Delta\zeta^{(1)} u,\\ &u^{(1)}|_{\partial\Omega^{(1)}} = 0. \end{aligned} \tag{5.4}$$

Multiplying (5.3) by $\zeta^{(2)}$ we obtain

$$
\begin{aligned}
&\Delta u^{(2)} = f^{(2)} + 2\nabla\zeta^{(2)}\nabla u + \Delta\zeta^{(2)} u && \text{in } \Omega^{(2)} \times (0,T),\\
&\bar n \cdot \nabla u^{(2)} = u\bar n \cdot \nabla\zeta^{(2)} && \text{on } S \cap \bar\Omega^{(2)}.
\end{aligned}
\tag{5.5}
$$

Let us introduce a local coordinate system $y = \{y_1, y_2, y_3\}$ with the origin at the point where $L$ meets $S$ and such that $y_3 > 0$ describes points inside $\Omega$. Let $S^{(2)} = S \cap \bar\Omega^{(2)}$ be described by the relation

$$
y_3 = F(y_1, y_2). \tag{5.6}
$$

Introducing new coordinates

$$
\begin{aligned}
&z_i = y_i, \quad i = 1, 2,\\
&z_3 = y_3 - F(y_1, y_2),
\end{aligned}
\tag{5.7}
$$

we define a mapping $z = \Phi(y)$.

If problem (3.2) is formulated in coordinates $x = \{x_1, x_2, x_3\}$ we can pass to coordinates $y = \{y_1, y_2, y_3\}$ by a rotation and a translation. We denote this mapping by $y = Y(x)$. Hence

$$
\begin{aligned}
&z = (\Phi \circ Y)(x) \equiv \Psi(x),\\
&\hat\Omega^{(2)} = \Psi(\Omega^{(2)}), \quad \hat S^{(2)} = \Psi(S^{(2)}).
\end{aligned}
\tag{5.8}
$$

Introducing the notation

$$
\begin{aligned}
&\tilde u^{(2)}(z) = u^{(2)}(\Psi^{-1}(z)), \quad \tilde u(z) = u(\Psi^{-1}(z)),\\
&\nabla_\Psi = \frac{\partial z}{\partial x}\bigg|_{x=\Psi^{-1}(z)} \cdot \nabla_z = \Psi_x|_{x=\Psi^{-1}(z)} \cdot \nabla_z,\\
&\bar n_z = (0,0,1), \quad \bar n_\Psi = (F_{y_1}, F_{y_2}, -1)|_{y=\Phi^{-1}(z)},
\end{aligned}
$$

we can express problem (5.5) in the form

$$
\begin{aligned}
&\nabla_z^2 \tilde u^{(2)} = (\nabla_z^2 - \nabla_\Psi^2)\tilde u^{(2)} + 2\nabla_\Psi \tilde\zeta^{(2)} \nabla_\Psi \tilde u + \nabla_\Psi^2 \tilde\zeta^{(2)} \tilde u && \text{in } \hat\Omega^{(2)},\\
&\bar n_z \cdot \nabla_z \tilde u^{(2)} = (\bar n_z - \bar n_\Psi) \cdot \nabla_z \tilde u^{(2)} + \bar n_\Psi \cdot \nabla_z \tilde\zeta^{(2)} u && \text{on } \hat S^{(2)}.
\end{aligned}
\tag{5.9}
$$

Let $\tilde\eta^{(2)}$ be a function such that

$$
\frac{\partial}{\partial z_3}\tilde\eta^{(2)}\bigg|_{z_3=0} = (\bar n_z - \bar n_\Psi) \cdot \nabla_z \tilde u^{(2)} + \tilde u \bar n_\Psi \cdot \nabla_z \tilde\zeta^{(2)}. \tag{5.10}
$$

Then the function

$$
\tilde\psi^{(2)} = \tilde u^{(2)} - \tilde\eta^{(2)} \tag{5.11}
$$

is a solution to the problem

$$
(5.12)\qquad
\begin{aligned}
&\nabla_z^2\tilde\psi^{(2)} = \varDelta\tilde\eta^{(2)} + (\nabla_z^2 - \nabla_\Psi^2)\tilde u^{(2)} + 2\nabla_\Psi\tilde\zeta^{(2)}\nabla_\Psi\tilde u\\
&\qquad\qquad + \nabla_\Psi^2\tilde\zeta^{(2)}\tilde u \equiv \tilde F^{(2)} \quad \text{in } \hat\varOmega^{(2)},\\
&\frac{\partial}{\partial z_3}\tilde\psi^{(2)}\Big|_{z_3=0} = 0,\\
&\tilde\psi^{(2)}|_{\partial\hat\varOmega^{(2)}\setminus\{z:z_3=0\}} = 0,
\end{aligned}
$$

where

$$
\hat\varOmega^{(2)} = \{z\in\mathbb{R}^3 : (z_1^2+z_2^2)^{1/2} < R,\ 0<z_3<a,\ \varphi_z\in[0,2\pi]\},
$$

$\varphi_z = \operatorname{arctg}(\frac{z_2}{z_1})$. After reflection with respect to the plane $z_3=0$ problem (5.12) assumes the form

$$
(5.13)\qquad
\begin{aligned}
&-\nabla_z^2\tilde\psi'^{(2)} = \tilde F'^{(2)},\\
&\tilde\psi'^{(2)}|_{\partial\hat\varOmega'^{(2)}} = 0,
\end{aligned}
$$

where $\eta'$ means that $\eta'(z',z_3) = \eta(z',z_3)$ for $z_3>0$, $z'=(z_1,z_2)$ and $\eta'(z',z_3)=\eta(z',-z_3)$ for $z_3<0$, and

$$
\hat\varOmega'^{(2)} = \{z\in\mathbb{R}^3 : (z_1^2+z_2^2)^{1/2} < R,\ -a<z_3<a,\ \varphi_z\in[0,2\pi]\}.
$$

Applying Lemma 4.11 we obtain for solutions to problems (5.4) and (5.13) the estimate

$$
(5.14)\qquad
\begin{aligned}
\|u^{(k)}\|_{V^{l+2}_{p,-\mu}(\varOmega)} &\le c\|f^{(k)}\|_{V^l_{p,-\mu}(\varOmega)}\\
&\quad + c(\|\nabla u\|_{V^l_{p,-\mu}(\varOmega\cap\varOmega^{(k)})} + \|u\|_{V^l_{p,-\mu}(\varOmega\cap\varOmega^{(k)})}),
\end{aligned}
$$

where $k=1,2$ and in the case of problem (5.13) we used that $\operatorname{diam}\varOmega^{(2)}$ is sufficiently small.

In the case of the subdomains $\varOmega^{(3)}$ and $\varOmega^{(4)}$ we obtain problems similar to (5.4) and (5.9). Then instead of (5.14) we get

$$
(5.15)\qquad \|u^{(k)}\|_{W^{l+2}_p(\varOmega)} \le c\|f^{(k)}\|_{W^l_p(\varOmega)} + c(\|\nabla u\|_{W^l_p(\varOmega^{(k)})} + \|u\|_{W^l_p(\varOmega^{(k)})}).
$$

Moreover, for solutions to problem (5.3) we have

$$
(5.16)\qquad \|u\|_{W^{l+2}_p(\varOmega)} \le c\|f\|_{W^l_p(\varOmega)}.
$$

Let $\mu\in(0,1)$. Then summing up inequalities (5.14) and (5.15) and applying the Hardy inequality with estimate (5.16) to the last two terms on the r.h.s. of (5.15) we obtain

$$
(5.17)\qquad \|u\|_{V^{l+2}_{p,-\mu}(\varOmega)} \le c\|f\|_{V^l_{p,-\mu}(\varOmega)}.
$$

Now let $\mu\in(1,2)$. Then the last two terms on the r.h.s. of (5.14) can be estimated in view of the Hardy inequality and (5.17). Repeating the other arguments we obtain (5.17) for $\mu\in(1,2)$.

Continuing the above considerations and assuming that $u \in V^{l+2}_{p,-\mu}(\Omega)$, $\mu \in (k-1,k)$ we obtain (5.17) for $\mu \in (k,k+1)$, $k \in \mathbb{N}$.

Repeating the above considerations for problem (3.5) we obtain by (3.1) the following estimate for solutions to problem (1.1):

$$\|v\|_{V^{l+1}_{p,-\mu}(\Omega)} \le c(\|w\|_{V^{l}_{p,-\mu}(\Omega)} + \|b\|_{V^{l+1-1/p}_{p,-\mu}(S)}). \tag{5.18}$$

This holds for smooth functions vanishing sufficiently fast near $L$.

To prove Theorem 1.1 we use Theorem 1.2. Let $(w_\nu, b_\nu)$ be a sequence of smooth functions with compact support in $\bar{\Omega}\setminus L$ which converges to $(w,b) \in V^{k}_{p,-\mu}(\Omega) \times V^{k+1-1/p}_{p,-\mu}(S)$. In view of Theorem 1.2 we have the existence of an approximate solution of the problem

$$\begin{aligned} \operatorname{rot} v_\nu &= w_\nu, \\ \operatorname{div} v_\nu &= 0, \\ \bar{n}\cdot v_\nu &= b_\nu. \end{aligned} \tag{5.19}$$

By Theorem 1.2 there exists a solution to problem (5.19) such that $v_\nu \in H^{l+2}_{-\mu}(\Omega)$ with r.h.s. equal to $w_\nu$ and $b_\nu$. Using estimate (5.18) we find that the sequence converges in $V^{l+2}_{p,-\mu}(\Omega)$ and (1.4) holds. This concludes the proof.

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