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NOTE ON THE VARIANCE OF THE SUM OF GAUSSIAN FUNCTIONALS

Abstract. Let $(X_i, i = 1, 2, \ldots)$ be a Gaussian sequence with $X_i \in N(0, 1)$ for each i and suppose its correlation matrix $R = (\rho_{ij})_{i,j\geq 1}$ is the matrix of some linear operator $R : l_2 \to l_2$. Then for $f_i \in L^2(\mu)$, $i = 1, 2, \ldots,$ where μ is the standard normal distribution, we estimate the variation of the sum of the Gaussian functionals $f_i(X_i)$, $i = 1, 2, \ldots$.

1. Introduction. Let (X, Y) be a Gaussian random vector such that $X, Y \in N(0, 1)$ and $E(XY) = \rho$, $(|\rho| < 1)$. We denote by μ the normalized one-dimensional Gaussian measure, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\, x^2\right) dx.$$

In $L^2(\mu)$ we have the scalar product

$$(f, g)_\mu = \int_{\mathbb{R}} f(x)g(x)\, \mu(dx).$$

Introducing a random variable $Z \in N(0, 1)$ such that Z, Y are independent, we find that the Gaussian vectors (X, Y) and (U, Y) with $U = \rho Y + \sqrt{1-\rho^2}\, Z$ have the same joint distribution. Thus, for $f, g \in L^2(\mu)$ we have

$$E(f(X)g(Y)) = E(f(U)g(Y)) = E(P_\rho(Y)g(Y)), \tag{1.1}$$

where

$$P_\rho f(y) = E(f(U)\,|\,Y = y) = \int_{\mathbb{R}} f(\rho y + \sqrt{1-\rho^2}\, z)\, d\mu(z), \qquad y \in \mathbb{R},$$

is called the *Ornstein–Uhlenbeck operator*. The Ornstein–Uhlenbeck operator has a representation in terms of orthonormal Hermite polynomials

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$\{h_n\}_{n\geq 0} \subset L^2(\mu)$, namely

$$P_\rho f = \sum_{n=0}^{\infty} \rho^n (f, h_n)_\mu \, h_n, \qquad f \in L^2(\mu). \tag{1.2}$$

In particular,

$$P_\rho h_n = \rho^n \, h_n, \qquad n \geq 0.$$

From (1.2) we obtain Gebelein's inequality (see [G] and [DK]):

Proposition 1.1. *If $f \in L^2$ and $(f,1)_\mu = 0$, then*

$$\|P_\rho f\|_2 \leq |\rho| \cdot \|f\|_2, \tag{1.3}$$

or equivalently for any $g \in L^2$ and f as above,

$$|(P_\rho f, g)_\mu| \leq |\rho| \cdot \|f\|_2 \cdot \|g\|_2.$$

In both inequalities we have equality if and only if $f(x) = cx$.

Consider a Gaussian sequence $(X_i, i = 1, 2, \ldots)$ of random variables with $X_i \in N(0,1)$ for each i. It is assumed that the correlation matrix $R = (\rho_{ij})_{i,j\geq 1}$, where $\rho_{ij} = E(X_i X_j)$, $i, j = 1, 2, \ldots$, satisfies

$$C = \sup_{i\geq 1} \sum_{j\geq 1} |\rho_{ij}| < \infty. \tag{1.4}$$

It is evident that $C \geq 1$. The Frobenius Theorem (see [HLP]) implies that R is the matrix (in the standard basis) of a continuous linear operator (which we denote by the same letter) $R : l_p \to l_p$ for $1 \leq p \leq \infty$ with $\|R\| \leq C$. Hence, it is easily seen that for $C < 2$ the linear operator R is invertible. Using Gebelein's inequality (1.3), one can prove (see [BC1], [BC2], [V])

Lemma 1.1. *Let the Gaussian sequence $(X_i, i = 1, 2, \ldots)$ with $X_i \in N(0,1)$ for each i satisfy the hypothesis* (1.4) *and let $(f_i, i = 1, 2, \ldots) \subset L^2(\mu)$. Then for each $n \geq 1$ we have*

$$(2 - C) \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)) \leq \mathrm{Var}\Big(\sum_{i=1}^{n} f_i(X_i)\Big) \leq C \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)). \tag{1.5}$$

For $C \geq 2$ the left inequality in (1.5) holds trivially. In fact, we can say more: an inequality of the form

$$M \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)) \leq \mathrm{Var}\Big(\sum_{i=1}^{n} f_i(X_i)\Big), \tag{1.6}$$

where M is a positive constant, is not satisfied in general when $C \geq 2$.

Consider the following simple example: Let $(Y_i, i = 1, 2, \ldots) \subset N(0,1)$ be a sequence of independent Gaussian random variables. Let $a, b \in \mathbb{R}$ be such that $a^2 + b^2 = 1$ and define

$$X_{3k-2} = -Y_{2k}, \quad X_{3k-1} = a\, Y_{2k-1} - b Y_{2k}, \quad X_{3k} = a Y_{2k-1} + b\, Y_{2k}, \qquad k \geq 1.$$

Moreover, we put

$$f_{3k-2}(t)=2bt,\quad f_{3k-1}(t)=-t,\quad f_{3k}(t)=t,\quad t\in\mathbb{R},\ k\geq 1.$$

It is easy to check that

$$C=\sup_{i\geq 1}\sum_{j\geq 1}|\rho_{ij}|=1+|b|+\max\{|b|,|1-2b^2|\}\geq 2$$

and

$$\mathrm{Var}\Big(\sum_{i=1}^{3n}f_i(X_i)\Big)=0\quad\text{and}\quad\sum_{i=1}^{3n}\mathrm{Var}(f_i(X_i)>0,\qquad n\geq 1.$$

2. Main result. In this section we are going to prove the inequality (1.5) under a slightly weaker condition than (1.4). First let us introduce some notations. For a given correlation matrix $R=(\rho_{ij})_{i,j\geq 1}$, we put

$$R_n^{(m)}=(\rho_{ij}^m)_{1\leq i,j\leq n},\qquad m,n\geq 1,$$

and let $\lambda_{n,1}^{(m)}$ and $\lambda_{n,n}^{(m)}$ denote the least and the greatest of the eigenvalues of the matrix $R_n^{(m)}$. By the Schur lemma (see [B]) the matrix $R_n^{(m)}$ is nonnegative definite. Hence, the eigenvalues $\lambda_{n,1}^{(m)}$ are nonnegative. For the matrix $R_n=R_n^{(1)}$ we use the well known decomposition

$$R_n=U_nD_nU_n^T,$$

where

$$D_n=\begin{pmatrix}\lambda_{n,1}^{(1)} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \lambda_{n,n}^{(1)}\end{pmatrix}$$

is a diagonal matrix with eigenvalues $\lambda_{n,i}^{(1)}$, $i=1,\ldots,n$, of R_n on the main diagonal. The matrix $U_n=(u_{n,ij})_{1\leq i,j\leq n}$ is an orthogonal matrix and T stands for transposition. It follows that

$$\rho_{ij}=\sum_{k=1}^{n}\lambda_{n,k}^{(1)}u_{n,ik}u_{n,jk},\qquad 1\leq i,j\leq n. \tag{2.1}$$

Now we can state the following simple result.

Lemma 2.1. *Fix $n\geq 1$. Then the least and the greatest eigenvalues of the matrix $R_n^{(m)}$ are monotonic with respect to m, i.e.*

$$\lambda_{n,1}^{(m+1)}\geq\lambda_{n,1}^{(m)}\quad\text{and}\quad\lambda_{n,n}^{(m+1)}\leq\lambda_{n,n}^{(m)},\qquad\text{for } m=1,2,\ldots. \tag{2.2}$$

Proof. Since the matrix $R_n^{(m+1)}$ is symmetric, we have

$$\lambda_{n,1}^{(m+1)} = \inf_{\|c\|=1} (R_n^{(m+1)} c, c) = \inf_{\|c\|=1} \sum_{i,j=1}^{n} \rho_{ij}^{m+1} c_i c_j, \tag{2.3}$$

where $c = (c_1, \ldots, c_n) \in l_2^n$ and l_2^n is the n-dimensional Euclidean space with the scalar product denoted here by $(\cdot, \cdot)$. From (2.3) and (2.1) we conclude that for every $c = (c_1, \ldots, c_n) \in l_2^n$ with $\|c\| = 1$ we have

$$\begin{aligned} \sum_{i,j=1}^{n} \rho_{ij}^{m+1} c_i c_j &= \sum_{i,j=1}^{n} \rho_{ij}^{m} \rho_{ij} c_i c_j \\ &= \sum_{i,j=1}^{n} \rho_{ij}^{m} \sum_{k=1}^{n} \lambda_{n,k}^{(1)} u_{n,ik} u_{n,jk} c_i c_j = \sum_{k=1}^{n} \lambda_{n,k}^{(1)} \Big(\sum_{i,j=1}^{n} \rho_{ij}^{m} c_i u_{n,ik} c_j u_{n,jk} \Big) \\ &\ge \sum_{k=1}^{n} \lambda_{n,k}^{(1)} \sum_{i=1}^{n} c_i^2 u_{n,ik}^2 \inf_{\|b\|=1} (R_n^{(m)} b, b) = \lambda_{n,1}^{(m)}, \end{aligned} \tag{2.4}$$

since

$$\sum_{k=1}^{n} \lambda_{n,k}^{(1)} \sum_{i=1}^{n} c_i^2 u_{n,ik}^2 = \sum_{i=1}^{n} c_i^2 \sum_{k=1}^{n} \lambda_{n,k}^{(1)} u_{n,ik}^2 = 1$$

by (2.1). Taking the infimum in (2.4) over all $c = (c_1, \ldots, c_n) \in l_2^n$ with $\|c\| = 1$ we obtain the first inequality of (2.2). The proof of the second one runs similarly. ∎

We can now formulate our main result.

THEOREM 2.1. *Let $(X_i, i = 1, 2, \ldots)$ be a Gaussian sequence with $X_i \in N(0,1)$ for each i and suppose its correlation matrix $R = (\rho_{ij})_{i,j\ge 1}$ is the matrix of some operator $R : l_2 \to l_2$. Then for $f_i \in L^2(\mu)$, $i = 1, 2, \ldots,$ and for every $n \ge 1$ we have*

$$\lambda_{\min} \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)) \le \mathrm{Var}\Big(\sum_{i=1}^{n} f_i(X_i) \Big) \le \lambda_{\max} \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)), \tag{2.5}$$

where

$$\lambda_{\min} = \inf_{\|x\|=1} (Rx, x), \qquad \lambda_{\max} = \sup_{\|x\|=1} (Rx, x).$$

REMARK. Let us point out that the assumption concerning the correlation matrix $R = (\rho_{ij})_{i,j\ge 1}$ of the sequence $(X_i, i = 1, 2, \ldots)$ is slightly weaker than the hypothesis (1.4). To see this, consider the following example: Let $(Y_i, i = 1, 2, \ldots) \subset N(0,1)$ be a sequence of independent Gaussian random variables and define

$$X_1 = aY_1 + \sum_{j=2}^{\infty} Y_j/j, \quad \text{where} \quad a = \sqrt{2 - \pi^2/6},$$
$$X_i = Y_i \quad \text{for } i \geq 2.$$

It follows immediately that the correlation matrix $R = (\rho_{ij})_{i,j\geq 1}$ of the sequence $(X_i, i = 1, 2, \ldots)$ is the matrix of some linear operator $R : l_2 \to l_2$ and the hypothesis (1.4) is not satisfied.

Proof of Theorem 1.1. First we prove the left inequality of (2.5). Without loss of generality we assume that $E(f_i(X_i)) = 0$, $i = 1, 2, \ldots$. If $\lambda_{\min} = 0$ then the inequality holds trivially. Assume that $\lambda_{\min} \neq 0$. Expanding each f_i, $i \geq 1$, with respect to the Hermite basis in $L^2(\mu)$ we obtain

$$f_i = \sum_{k=1}^{\infty} c_{ik} h_k, \qquad \|f_i\|_{\mu}^2 = \sum_{k=1}^{\infty} c_{ik}^2, \qquad i = 1, 2, \ldots. \tag{2.6}$$

From (1.1) and (1.2) and the orthonormality of Hermite polynomials $\{h_n\}_{n\geq 1} \subset L^2(\mu)$ it follows that

$$E[h_n(X_i) h_m(X_j)] = \rho_{ij}^n \delta_m^n, \qquad n, m, i, j = 1, 2, \ldots, \tag{2.7}$$

where δ_m^n is the Kronecker delta. Combining (2.6) with (2.7) and using Lemma 2.1 we get

$$\begin{aligned}
\mathrm{Var}\Big(\sum_{i=1}^{n} f_i(X_i)\Big) &= E\Big(\sum_{i=1}^{n} f_i(X_i)\Big)^2 \\
&= \lim_{N\to\infty} E\Big(\sum_{i=1}^{n}\sum_{k=1}^{N} c_{ik} h_k(X_i)\Big)^2 = \lim_{N\to\infty} E\Big(\sum_{k=1}^{N}\sum_{i=1}^{n} c_{ik} h_k(X_i)\Big)^2 \\
&= \lim_{N\to\infty} \sum_{k,l=1}^{N} E\Big[\Big(\sum_{i=1}^{n} c_{ik} h_k(X_i)\Big)\Big(\sum_{j=1}^{n} c_{jl} h_l(X_j)\Big)\Big] \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} E\Big[\sum_{i=1}^{n} c_{ik} h_k(X_i)\Big]^2 = \lim_{N\to\infty} \sum_{k=1}^{N}\sum_{i,j=1}^{n} \rho_{ij}^k c_{ik} c_{jk} \\
&\geq \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_{n,1}^{(k)} \sum_{i=1}^{n} c_{ik}^2 \geq \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_{n,1}^{(1)} \sum_{i=1}^{n} c_{ik}^2 \\
&\geq \lambda_{\min} \sum_{i=1}^{n}\sum_{k=1}^{\infty} c_{ik}^2 = \lambda_{\min} \sum_{i=1}^{n} E\big[f_i(X_i)\big]^2 = \lambda_{\min} \sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)).
\end{aligned}$$

This proves the left inequality of (2.5). The proof of the right one is similar. ∎

Remark. Let us point out that under the assumptions of Theorem 2.1 the inequality (1.6) holds for all $f_i \in L^2(\mu)$, $i = 1, 2, \ldots$, with a positive constant M if and only if the operator $R : l_2 \to l_2$ is invertible. ∎

Adapting now the methods from [BC1] and [BC2] we can write the following two statements:

Lemma 2.2 (Borel–Cantelli Lemma). *Let $(X_i, i = 1, 2, \ldots)$ be a Gaussian sequence with $X_i \in N(0,1)$ for $i \geq 1$ and suppose its correlation matrix $R = (\rho_{ij})_{i,j\geq 1}$ is the matrix of some linear operator $R : l_2 \to l_2$. Then for every sequence of Borel sets $(A_i, i = 1, 2, \ldots)$ such that $\sum_{i=1}^{\infty} P\{X_i \in A_i\} = \infty$ we have $P\{X_i \in A_i \text{ i.o.}\} = 1$.* ∎

Theorem 2.2 (Strong Law of Large Numbers). *Let $(X_i, i = 1, 2, \ldots)$ be a Gaussian sequence with $X_i \in N(0,1)$ for $i \geq 1$ and suppose its correlation matrix $R = (\rho_{ij})_{i,j\geq 1}$ is the matrix of some linear operator $R : l_2 \to l_2$. Then for $f \in L^1(\mu)$ we have*

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \xrightarrow[n\to\infty]{} Ef(X_1) \qquad \text{a.s.} \ \blacksquare$$

References

[B] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.

[BC1] M. Beśka and Z. Ciesielski, *Ergodic type theorems for Gaussian systems*, Proc. Steklov Inst. Math. 248 (2005), 35–40.

[BC2] —, —, *Gebelein's inequality and its consequences*, in Banach Center Publ. 72, Inst. Math., Polish Acad. Sci., 2006, 11–23.

[DK] H. Dym and H. P. McKean, *Gaussian Processes, Function Theory and Inverse Spectral Problem*, Academic Press, 1976.

[G] H. Gebelein, *Das statistische Problem der Korrelation als Variations und Eigenwertproblem und sein Zusammenhang mit der Ausgleichsrechnung*, Z. Angew. Math. Mech. 21 (1941), 364–379.

[HLP] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1967.

[V] M. Veraar, *Correlation inequalities and applications to vector-valued Gaussian random variables and fractional Brownian motion*, Potential Anal. 30 (2009), 341–370.

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