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LONG TIME EXISTENCE OF SOLUTIONS
TO 2D NAVIER–STOKES EQUATIONS
WITH HEAT CONVECTION

Abstract. Global existence of regular solutions to the Navier–Stokes equations for (v, p) coupled with the heat convection equation for θ is proved in the two-dimensional case in a bounded domain. We assume the slip boundary conditions for velocity and the Neumann condition for temperature. First an appropriate estimate is shown and next the existence is proved by the Leray–Schauder fixed point theorem. We prove the existence of solutions such that $v, \theta \in W_s^{2,1}(\Omega^T)$, $\nabla p \in L_s(\Omega^T)$, $s > 2$.

1. Introduction. We consider the problem

$$
\begin{aligned}
&v_t + v \cdot \nabla v - \operatorname{div} \boldsymbol{T}(v, p) = \alpha(\theta)g && \text{in } \Omega^T = \Omega \times (0, T), \\
&\operatorname{div} v = 0 && \text{in } \Omega^T, \\
&\theta_t + v \cdot \nabla \theta - \varkappa \Delta \theta = 0 && \text{in } \Omega^T, \\
&\bar{n} \cdot \boldsymbol{D}(v) \cdot \bar{\tau} = 0 && \text{on } S^T = S \times (0, T), \\
&v \cdot \bar{n} = 0 && \text{on } S^T, \\
&\bar{n} \cdot \nabla \theta = 0 && \text{on } S^T, \\
&v|_{t=0} = v_0 && \text{in } \Omega, \\
&\theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
$$

(1.1)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, Ω^T satisfies the weak horn condition (see [2, Sect. 8]) and is not axially symmetric, $S = \partial\Omega \in C^2$, $x = (x_1, x_2) \in \mathbb{R}^2$ are the Cartesian coordinates, $v = v(x, t) = (v_1(x, t), v_2(x, t)) \in \mathbb{R}^2$ the

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velocity of the fluid, $\theta = \theta(x,t) \in \mathbb{R}$ the temperature, $\nu > 0$ the constant viscosity coefficient, $\varkappa > 0$ the constant heat coefficient, $\bar{n}$ the unit outward vector normal to S, and $\bar{\tau}$ the unit tangent vector to S. By a dot we denote the scalar product in $\mathbb{R}^2$.

By $\boldsymbol{D}(v)$ we denote the dilatation tensor of the form

$$(1.2) \qquad \boldsymbol{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2},$$

and by $g = g(x,t)$ the external force. Finally, $\boldsymbol{T}(v,p)$ denotes the stress tensor of the form

$$(1.3) \qquad \boldsymbol{T}(v,p) = \nu\boldsymbol{D}(v) - p\boldsymbol{I},$$

where $\boldsymbol{I}$ is the unit matrix.

THEOREM 1.1. *Assume that $\alpha \in C^1(\mathbb{R})$, there exist constants $c_2 < c_3 < \infty$ such that $c_2 \leq \theta_0 \leq c_3$, $v_t(0) \in L_2(\Omega)$, $\theta_t(0) \in L_2(\Omega)$, $\operatorname{div} v_0 = 0$, $\theta_0 \in L_2(\Omega)$, $2 < s < \eta < \infty$, $2 < \eta$, $4/s - 2/\eta < 1$, $v_0 \in W_s^{2-2/s}(\Omega)$, $g \in L_s(\Omega^T)$. Moreover assume that $g_t \in L_2(0,T;L_l(\Omega))$, where $l > 1$ is arbitrarily close to 1, and $g \in L_2(0,T;L_r(\Omega))$, where $r > 2$ is arbitrarily close to 2. Then there exists a solution (v,p,θ) of problem (1.1) such $v, \theta \in W_s^{2,1}(\Omega^T)$, $\nabla p \in L_s(\Omega^T)$, and a constant C_* such that*

$$(1.4) \qquad \|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} + \|\theta\|_{W_s^{2,1}(\Omega^T)} \leq C_*,$$

and

$$c_2 \leq \theta \leq c_3.$$

2. Notation. Let us consider the Stokes problem

$$(2.1) \qquad \begin{aligned} v_t - \operatorname{div}\boldsymbol{T}(v,p) &= f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ \bar{n} \cdot \boldsymbol{D}(v) \cdot \bar{\tau} &= 0 && \text{on } S^T, \\ v \cdot \bar{n} &= 0 && \text{on } S^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned}$$

THEOREM 2.1 (proof similar to one in [1]). *Let $f \in L_q(\Omega^T)$, $v_0 \in W_q^{2-2/q}(\Omega)$, $\operatorname{div} v_0 = 0$, $S \in C^2$, $q \in (1,\infty)$. Then there exists a unique solution to problem (2.1) such that $v \in W_q^{2,1}(\Omega^T)$, $\nabla p \in L_q(\Omega^T)$ and*

$$(2.2) \qquad \|v\|_{W_q^{2,1}(\Omega^T)} + \|\nabla p\|_{L_q(\Omega^T)} \leq c(\|f\|_{L_q(\Omega^T)} + \|v_0\|_{W_q^{2-2/q}(\Omega)}).$$

THEOREM 2.2 (see [2, Sect. 10]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that Ω^T satisfies the weak horn condition, and let $u \in W_s^{2,1}(\Omega^T) \cap L_2(\Omega^T)$. Then the following interpolation inequality holds:*

$$(2.3) \qquad \|\nabla u\|_{L_q(\Omega^T)} \leq \varepsilon\|u\|_{W_s^{2,1}(\Omega^T)} + c(1/\varepsilon)\|\nabla u\|_{L_2(\Omega^T)},$$

for $s, q \in (1, \infty)$ satisfying $(n+2)/s - (n+2)/q < 1$, $s \le q$, where $c(a)$ denotes an increasing positive function of a. Moreover, ε does not depend on s, n, q but c does.

LEMMA 2.3 (Korn inequality, see [6]). *Assume that Ω is not invariant with respect to any rotation. Assume that*

$$(2.4) \qquad \|\boldsymbol{D}(v)\|_{L_2(\Omega)} < \infty, \qquad v \cdot \bar{n}|_S = 0, \qquad \operatorname{div} v = 0.$$

Then there exists a constant c_0 such that

$$(2.5) \qquad \|v\|_{H^1(\Omega)} \le c_0 \|\boldsymbol{D}(v)\|_{L_2(\Omega)},$$

where c_0 is independent of v.

3. Estimates. We show estimates for the temperature.

LEMMA 3.1. *Assume $\theta(0) \ge c_2$. Then for θ sufficiently regular we have*

$$(3.1) \qquad \theta(t) \ge c_2 \quad \text{for } t \ge 0.$$

Proof. Let $(\theta - c_2)_- = \min\{0, \theta - c_2\}$. Multiplying $(1.3)_3$ by $(\theta - c_2)_-$ integrating over Ω and using the boundary conditions we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (\theta - c_2)_-^2 \, dx + \varkappa \int_\Omega |\nabla(\theta - c_2)_-|^2 \, dx = 0.$$

Integrating with respect to time we have

$$\frac{1}{2} \|(\theta - c_2)_-(t)\|_{L_2(\Omega)}^2 + \varkappa \|\nabla(\theta - c_2)_-\|_{L_2(\Omega^t)}^2 = \frac{1}{2} \|(\theta - c_2)_-(0)\|_{L_2(\Omega)}^2.$$

Since $(\theta - c_2)_-(0) = 0$ we conclude the proof. ∎

REMARK 3.1. If $\theta(0) \ge 0$, then $\theta(t) \ge 0$ for $t \ge 0$.

LEMMA 3.2. *Assume $\theta(0) \le c_3$. Then for θ sufficiently regular we have*

$$(3.2) \qquad \theta(t) \le c_3 \quad \text{for } t \ge 0.$$

Proof. Let $(\theta - c_3)_+ = \max\{0, \theta - c_3\}$. Multiplying $(1.1)_3$ by $(\theta - c_3)_+$, integrating over Ω, integrating by parts and using the boundary conditions yields

$$(3.3) \qquad \frac{1}{2} \frac{d}{dt} \int_\Omega (\theta - c_3)_+^2 \, dx + \varkappa \int_\Omega |\nabla(\theta - c_3)_+|^2 \, dx = 0.$$

Integrating with respect to time we obtain

$$(3.4) \qquad \frac{1}{2} \|(\theta - c_3)_+(t)\|_{L_2(\Omega)}^2 + \varkappa \int_{\Omega^t} |\nabla(\theta - c_3)_+|^2 \, dx \, dt'$$

$$= \frac{1}{2} \|(\theta_0 - c_3)_+(0)\|_{L_2(\Omega)}^2.$$

Since $(\theta_0 - c_3)_+(0) = 0$ we conclude the proof. ∎

LEMMA 3.3. *Assume that $\alpha \in C^1(\mathbb{R})$ and there exist constants $c_2 <$ $c_3 < \infty$ such that $c_2 \leq \theta_0 \leq c_3$. Moreover assume that $g_t \in L_2(0,T;L_l(\Omega))$, $l > 1$, $g \in L_2(0,T;L_r(\Omega))$, $r > 2$, $v_t(0) \in L_2(\Omega)$, $\theta_t(0) \in L_2(\Omega)$, $v_0 \in L_2(\Omega)$, $\theta_0 \in L_2(\Omega)$. Then*

$$(3.5) \quad \|v(t)\|^2_{L_2(\Omega)} + \nu \int_0^t \|v(t')\|^2_{H^1(\Omega)}\, dt'$$
$$\leq c_1 \|g\|^2_{L_2(0,t;L_r(\Omega))} + \|v(0)\|^2_{L_2(\Omega)} \equiv c_4,$$

and

$$(3.6) \quad \|v_t(t)\|^2_{L_2(\Omega)} + \|\theta_t(t)\|^2_{L_2(\Omega)} + \int_0^t (\|v_{t'}\|^2_{H^1(\Omega)} + \|\theta_{t'}\|^2_{H^1(\Omega)})\, dt'$$
$$\leq c e^{ct} \exp[c(\|v_0\|^2_{L_2(\Omega)} + \|\theta_0\|^2_{L_2(\Omega)} + c_1\|g\|^2_{L_2(0,t;L_r(\Omega))})]$$
$$\cdot (\|v_t(0)\|^2_{L_2(\Omega)} + \|\theta_t(0)\|^2_{L_2(\Omega)} + \|g_t\|^2_{L_2(0,t;L_l(\Omega))}) \equiv c_5.$$

Proof. Multiply $(1.1)_1$ by v and integrate over Ω. Using the boundary conditions and the Korn inequality we obtain

$$(3.7) \quad \frac{1}{2}\frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|v\|^2_{H^1(\Omega)} \leq c\|\alpha(\theta)\|_{L_\infty(\Omega)} \int_\Omega |gv|\, dx.$$

Since $\alpha \in C^1(\mathbb{R})$ Lemmas 3.1 and 3.2 imply that there exists a positive function $\varphi : \{(x,y) \in \mathbb{R}^2 : x \leq y\} \to \mathbb{R}_+$ such that

$$(3.8) \quad \sup_t \|\alpha(\theta)\|_{L_\infty(\Omega)} \leq \varphi(c_2,c_3).$$

By the Hölder and Young inequalities, (3.7) and (3.8) imply

$$(3.9) \quad \frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \nu\|v\|^2_{H^1(\Omega)} \leq c_1\|g\|^2_{L_r(\Omega)},$$

where

$$c_1 = c\varphi^2(c_2,c_3)$$

and $r > 1$. Integrating (3.9) with respect to time we have (3.5).

Multiplying $(1.1)_3$ by θ, integrating over Ω^T and using the boundary conditions we obtain

$$(3.10) \quad \|\theta\|^2_{L_2(\Omega)} + 2\varkappa\|\nabla\theta\|^2_{L_2(\Omega^t)} = \|\theta_0\|^2_{L^2(\Omega)}.$$

Differentiating $(1.1)_1$ with respect to t, multiplying by v_t, integrating over Ω and then applying the Hölder inequality, an interpolation inequality and the fact that

$$|\alpha_\theta(\theta)| \leq \varphi_1(c_2,c_3) \equiv c_4$$

we obtain

$$(3.11) \qquad \frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|v_t\|^2_{H^1(\Omega)}$$

$$\leq c[\|\nabla v_t\|_{L_2(\Omega)} \cdot \|v_t\|_{L_2(\Omega)} + \|v_t\|^2_{L_2(\Omega)}]\|\nabla v\|_{L_2(\Omega)}$$

$$+ c\|g_t\|^2_{L_l(\Omega)} + c\|g\|^2_{L_r(\Omega)}\|\theta_t\|^2_{L_s(\Omega)},$$

where $1/r + 1/s = 1/\bar{l}$ and $\bar{l} > 1$.

Differentiating $(1.1)_3$ with respect to t, multiplying by θ_t, integrating over Ω and then applying the Hölder inequality and interpolation inequality we have

$$(3.12) \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega \theta_t^2\,dx + \varkappa\|\theta_t\|^2_{H^1(\Omega)} \leq \int_\Omega |v_t|\,|\theta_t|\,|\nabla\theta|dx + c\|\theta_t\|^2_{L_2(\Omega)}$$

$$\leq \|v_t\|_{L_4(\Omega)}\|\theta_t\|_{L_4(\Omega)}\|\nabla\theta\|_{L_2(\Omega)} + c\|\theta_t\|^2_{L_2(\Omega)}$$

$$\leq c(\|\nabla v_t\|^{1/2}_{L_2(\Omega)}\|v_t\|^{1/2}_{L_2(\Omega)}\|\nabla\theta_t\|^{1/2}_{L_2(\Omega)}\|\theta_t\|^{1/2}_{L_2(\Omega)}$$

$$+ \|\nabla v_t\|^{1/2}_{L_2(\Omega)}\|v_t\|^{1/2}_{L_2(\Omega)}\|\theta_t\|_{L_2(\Omega)}$$

$$+ \|v_t\|_{L_2(\Omega)}\|\nabla\theta_t\|^{1/2}_{L_2(\Omega)}\|\theta_t\|^{1/2}_{L_2(\Omega)}$$

$$+ \|v_t\|_{L_2(\Omega)}\|\theta_t\|_{L_2(\Omega)})\|\nabla\theta\|_{L_2(\Omega)} + c\|\theta_t\|^2_{L_2(\Omega)}.$$

Let

$$y(t) = \|v_t\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)},$$
$$Y(t) = \|v_t\|^2_{H^1(\Omega)} + \|\theta_t\|^2_{H^1(\Omega)},$$
$$Z(t) = \|\nabla v\|^2_{L_2(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)}.$$

Then (3.11) and (3.12) imply

$$(3.13) \qquad \frac{d}{dt}y + Y \leq cY^{1/2}y^{1/2}Z^{1/2}$$

$$+ c(\|\theta_t\|^2_{L_2(\Omega)} + \|g_t\|^2_{L_l(\Omega)} + \|g\|^2_{L_r(\Omega)}\|\theta_t\|^2_{L_s(\Omega)}).$$

By the Young inequality we get

$$(3.14) \qquad \frac{d}{dt}y + Y \leq cyZ + c(\|\theta_t\|^2_{L_2(\Omega)} + \|g_t\|^2_{L_l(\Omega)} + \|g\|^2_{L_r(\Omega)}\|\theta_t\|^2_{L_s(\Omega)}).$$

Setting $s = 2$ in (3.14) yields

$$(3.15) \qquad \frac{d}{dt}y + Y \leq cy(Z + 1 + \|g\|^2_{L_r(\Omega)}) + c\|g_t\|^2_{L_l(\Omega)}$$

Integrating (3.15) with respect to time and employing (3.5) and (3.10) we obtain (3.6). ∎

LEMMA 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Moreover,
assume $v_0 \in W_s^{2-2/s}(\Omega)$ and $g \in L_s(\Omega^T)$ for some s, $1 < s < \infty$. Then*

$$(3.16) \quad \|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} + \|\theta\|_{W_s^{2,1}(\Omega^T)}$$
$$\leq c(\|g\|_{L_s(\Omega^T)} + \|v_0\|_{W_s^{2-2/s}(\Omega)} + \|\theta_0\|_{W_s^{2-2/s}(\Omega)}) + c(c_4, c_5).$$

Proof. From (3.5) and (3.6) we have

$$(3.17) \quad \|v_t\|_{L_\infty(0,t;L_2(\Omega))} + \|\nabla v\|_{L_2(\Omega^t)} + \|v\|_{L_\infty(0,t;L_2(\Omega))} \leq c(c_4, c_5), \quad t \leq T.$$

Hence $v \in H^1(\Omega^T)$. Since $\Omega \subset \mathbb{R}^2$ we also have

$$(3.18) \quad \|v\|_{L_6(\Omega^T)} \leq c(c_4, c_5).$$

Now we want to increase regularity described by (3.17). For this purpose we
consider the problem

$$(3.19) \quad \begin{aligned}
& v_t - \nu \Delta v + \nabla p = -v \cdot \nabla v + \alpha g, \\
& \operatorname{div} v = 0, \\
& \bar{n} \cdot \boldsymbol{D}(v) \cdot \bar{\tau}|_{S^T} = 0, \\
& v \cdot \bar{n}|_{S^T} = 0, \\
& v|_{t=0} = v_0.
\end{aligned}$$

To apply Theorem 2.1 we examine

$$\|v \cdot \nabla v\|_{L_s(\Omega^T)} \leq \|v\|_{L_{s\lambda_1}(\Omega^T)} \|\nabla v\|_{L_{s\lambda_2}(\Omega^T)} \equiv I_1,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Assuming $s\lambda_1 = 6$ we apply (2.3) with $\frac{4}{s} - \frac{4}{s\lambda_2} < 1$,
i.e. $\frac{4}{s\lambda_1} < 1$, which is satisfied. Hence in view of (3.18) and (2.3) we have

$$I_1 \leq c(c_4, c_5) \|\nabla v\|_{L_{s\lambda_2}(\Omega^T)} \leq \varepsilon \|v\|_{W_s^{2,1}(\Omega^T)} + c(1/\varepsilon, c_4, c_5) \|\nabla v\|_{L_2(\Omega^T)}.$$

Assuming that $g \in L_s(\Omega^T)$ we apply Theorem 2.1. Then we have

$$\|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \leq \varepsilon \|v\|_{W_s^{2,1}(\Omega^T)} + c(1/\varepsilon, c_4, c_5) \|\nabla v\|_{L_2(\Omega^T)}$$
$$+ c(\|\theta\|_{L_\infty(\Omega^T)}) \|g\|_{L_s(\Omega^T)} + c\|v_0\|_{W_s^{2-2/s}(\Omega)}.$$

Assuming that ε is sufficiently small in view of (3.1) and (3.2) we obtain
$v \in W_s^{2,1}(\Omega^T)$ for $1 < s < 6$. Since

$$\|u\|_{L_\infty(\Omega^T)} \leq \|u\|_{W_s^{2,1}(\Omega^T)} \quad \text{for } 2 < s,$$

we have $\|v\|_{L_\infty(\Omega^T)} \leq c(c_4, c_5)$. Assume $1 < s < \infty$. To apply Theorem 2.1
we examine

$$\|v \cdot \nabla v\|_{L_s(\Omega^T)} \leq \|v\|_{L_\infty(\Omega^T)} \|\nabla v\|_{L_s(\Omega^T)} \equiv I_2.$$

Hence in view of (2.3) we have

$$I_2 \leq c(c_4, c_5) \|\nabla v\|_{L_s(\Omega^T)} \leq \varepsilon \|v\|_{W_s^{2,1}(\Omega^T)} + c(1/\varepsilon, c_4, c_5) \|\nabla v\|_{L_2(\Omega^T)}.$$

Applying Theorem 2.1, assuming that ε is sufficiently small and using the estimate for θ we get

$$\|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \le c(\|g\|_{L_s(\Omega^T)} + \|v_0\|_{W_s^{2-2/s}(\Omega)}) + c(c_4, c_5).$$

Similarly we obtain estimates for θ (see [3, Ch. 4, Sect. 9, Th. 9.1], [5, Theorem 17]).

4. Existence. For $\eta > 2$ define

$$\mathcal{M}(\Omega^T) = \{(v, \theta) \in [L_\infty(0, T; W_\eta^1(\Omega))]^2\}.$$

Let us consider the problems

(4.1)
$$\begin{aligned}
&v_t - \operatorname{div} \boldsymbol{T}(v, p) = -\lambda[\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})g], \\
&\operatorname{div} v = 0, \\
&v \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \boldsymbol{D}(v) \cdot \bar{\tau}|_S = 0, \\
&v|_{t=0} = v_0
\end{aligned}$$

and

(4.2)
$$\begin{aligned}
&\theta_t - \varkappa \Delta \theta = -\lambda \tilde{v} \cdot \nabla \tilde{\theta}, \\
&\bar{n} \cdot \nabla \theta|_S = 0, \\
&\theta|_{t=0} = \theta_0,
\end{aligned}$$

where $\lambda \in [0, 1]$ is a parameter and $\tilde{v}, \tilde{\theta}$ are treated as given functions.

LEMMA 4.1. *Let $\alpha \in C^1(\mathbb{R})$. Let*

$$\begin{aligned}
&(\tilde{v}, \tilde{\theta}) \in \mathcal{M}(\Omega^T), \quad \eta > 2, \\
&g \in L_s(\Omega^T), \\
&v_0 \in W_s^{2-2/s}(\Omega), \quad 2 < s < \eta, \\
&S \in C^2, \quad \frac{4}{s} - \frac{2}{\eta} < 1.
\end{aligned}$$

Then there exists a unique solution to problem (4.1) such that

$$v \in W_s^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\eta^1(\Omega))$$

and

$$\begin{aligned}
\|v\|_{L_\infty(0,T;W_\eta^1(\Omega))} &\le c\|v\|_{W_s^{2,1}(\Omega^T)} \le c(\lambda\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2 \\
&\quad + \lambda\psi(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_s(\Omega^T)} + \|v_0\|_{W_s^{2-2/s}(\Omega)}).
\end{aligned}$$

Moreover the imbedding $W_s^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\eta^1(\Omega))$ is compact for $4/s - 2/\eta < 1$, so for $\eta > 2$ we have $s > 2$.

Proof. We have

$$\|\tilde{v}\cdot\nabla\tilde{v}\|_{L_s(\Omega^T)} \le c\|\tilde{v}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{v}\|_{L_\eta(\Omega^T)} \le c\|\tilde{v}\|^2_{L_\infty(0,T;W^1_\eta(\Omega))}$$
$$\le c\|(\tilde{v},\tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)}.$$

Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(a) = \sup\{|\alpha(x)| : -a \le x \le a\} \quad \text{for } a \ge 0.$$

Then

$$\|\alpha(\tilde{\theta})g\|_{L_s(\Omega^T)} \le \psi(c\|\tilde{\theta}\|_{L_\infty(0,T;W^1_\eta(\Omega))})\|g\|_{L_s(\Omega^T)}$$
$$\le \psi(c\|(\tilde{v},\tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_s(\Omega^T)}.$$

By Theorem 2.1 the proof is complete. ∎

LEMMA 4.2. *Assume that*

$$(\tilde{v},\tilde{\theta}) \in \mathcal{M}(\Omega^T), \quad \eta > 2, \quad \eta > s > 1, \quad \theta_0 \in W_s^{2-2/s}(\Omega).$$

Then there exists a unique solution to problem (4.2) *such that*

$$\theta \in W_s^{2,1}(\Omega^T) \subset L_\infty(0,T;W^1_\eta(\Omega))$$

and

$$\|\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))} \le c\|\theta\|_{W_s^{2,1}(\Omega^T)} \le c(\lambda\|(\tilde{v},\tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)} + \|\theta_0\|_{W_s^{2-2/s}(\Omega)}).$$

Proof. We have

$$\|\tilde{v}\cdot\nabla\tilde{\theta}\|_{L_s(\Omega^T)} \le \|\tilde{v}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\theta}\|_{L_\eta(\Omega^T)}$$
$$\le c\|\tilde{v}\|_{L_\infty(0,T;W^1_\eta(\Omega))}\|\tilde{\theta}\|_{L_\infty(0,T;W^1_\eta(\Omega))} \le c\|(\tilde{v},\tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)}.$$

Then the proof is similar to that of Theorem 9.1 from [3, Ch. 4, Sect. 9] (see also [5, Th. 17]). ∎

To prove the existence of solutions to problem (1.1) we apply the Leray–Schauder fixed point theorem (see [4]). Therefore we introduce the mapping $\phi : [0,1] \times \mathcal{M}(\Omega^T) \to \mathcal{M}(\Omega^T)$, $(\lambda,\tilde{v},\tilde{\theta}) \mapsto \phi(\lambda,\tilde{v},\tilde{\theta}) = (v,\theta)$, where (v,θ) is a solution to problems (4.1)–(4.2). For $\lambda = 0$ we have the existence of a unique solution. For $\lambda = 1$ every fixed point is a solution to problem (1.1).

LEMMA 4.3. *Let the assumptions of Lemmas* 4.1 *and* 4.2 *be satisfied. Then the mappings* $\phi(\lambda,\cdot) : \mathcal{M}(\Omega^T) \to \mathcal{M}(\Omega^T)$, $\lambda \in [0,1]$, *are completely continuous.*

Proof. By Lemmas 4.1 and 4.2 the mappings $\phi(\lambda,\cdot)$, $\lambda \in [0,1]$, are compact. From this it follows that bounded sets in $\mathcal{M}(\Omega^T)$ are transformed into bounded sets in $\mathcal{M}(\Omega^T)$. Let $(\tilde{v}_i,\tilde{\theta}_i) \in \mathcal{M}(\Omega^T)$, $i=1,2$, be two given

elements. Then (v_i, θ_i), $i = 1, 2$, are solutions to the problems

$$
\begin{aligned}
&v_{it} - \operatorname{div} \boldsymbol{T}(v_i, p_i) = -\lambda(\tilde{v}_i \cdot \nabla \tilde{v}_i + \alpha(\tilde{\theta}_i)g), \\
&\operatorname{div} v_i = 0, \\
&\bar{n} \cdot \boldsymbol{D}(v_i) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot v_i|_S = 0, \\
&v_i|_{t=0} = v_0, \quad i = 1, 2,
\end{aligned}
$$

(4.3)

and

$$
\begin{aligned}
&\theta_{it} - \varkappa \Delta \theta_i = -\lambda \tilde{v}_i \cdot \nabla \tilde{\theta}_i, \\
&\bar{n} \cdot \nabla \theta_i|_S = 0, \\
&\theta_i|_{t=0} = \theta_0, \quad i = 1, 2.
\end{aligned}
$$

(4.4)

To show continuity we introduce the differences

(4.5) $$V = v_1 - v_2, \quad P = p_1 - p_2, \quad \mathcal{T} = \theta_1 - \theta_2$$

which are solutions to the problems

$$
\begin{aligned}
&V_t - \operatorname{div} \boldsymbol{T}(V, P) = -\lambda[\tilde{V} \cdot \nabla \tilde{v}_1 + \tilde{v}_2 \cdot \nabla \tilde{V} + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2))g] \\
&\operatorname{div} V = 0 \\
&V \cdot \bar{n}|_S = 0 \quad \bar{n} \cdot \boldsymbol{D}(V) \cdot \bar{\tau}|_S = 0, \\
&V|_{t=0} = 0
\end{aligned}
$$

(4.6)

and

$$
\begin{aligned}
&\mathcal{T}_t - \varkappa \Delta \mathcal{T} = -\lambda[\tilde{V} \cdot \nabla \tilde{\theta}_1 + \tilde{v}_2 \cdot \nabla \tilde{\mathcal{T}}] \\
&\bar{n} \cdot \nabla \mathcal{T}|_S = 0, \\
&\mathcal{T}|_{t=0} = 0,
\end{aligned}
$$

(4.7)

where $\tilde{V} = \tilde{v}_1 - \tilde{v}_2$, $\tilde{\mathcal{T}} = \tilde{\theta}_1 - \tilde{\theta}_2$. Define $\psi_1 : [0, \infty) \to [0, \infty)$ by $\psi_1(a) = \sup\{|\alpha'(x)| : |x| \le a\}$ for $a \ge 0$. In view of [2] and [1, 3] we have

$$
\begin{aligned}
\text{(4.8)} \quad &\|V\|_{W_s^{2,1}(\Omega^T)} + \|\mathcal{T}\|_{W_s^{2,1}(\Omega^T)} \le c[\|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{v}_1\|_{L_s(\Omega^T)} \\
&+ \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{V}\|_{L_s(\Omega^T)} + c\psi_1(\max\{\|\tilde{\theta}_1\|, \|\tilde{\theta}_2\|\}) \|\tilde{\mathcal{T}}\|_{L_\infty(\Omega^T)} \|g\|_{L_s(\Omega^T)} \\
&+ \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\mathcal{T}}\|_{L_s(\Omega^T)} + \|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \theta_1\|_{L_s(\Omega^T)}] \\
&\qquad\qquad\qquad\qquad\qquad \le c(\|\tilde{V}\|_{\mathcal{M}(\Omega^T)} + \|\tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^T)}),
\end{aligned}
$$

so continuity of ϕ follows. ■

LEMMA 4.4. *Let the assumptions of Lemmas 4.1 and 4.2 be satisfied. Then for every bounded subset $\mathcal{M}_0$ of $\mathcal{M}(\Omega^T)$, the family of maps*

$$
\phi(\cdot, \tilde{v}, \tilde{\theta}) : [0, 1] \to \mathcal{M}(\Omega^T), \quad (\tilde{v}, \tilde{\theta}) \in \mathcal{M}_0,
$$

is uniformly equicontinuous.

Proof. Let $(\tilde{v}, \tilde{\theta}) \in \mathcal{M}_0$, $\lambda_i \in [0,1]$, $i = 1, 2$, $\lambda_1 \geq \lambda_2$, and v_i, θ_i be solutions to the problems

$$v_{it} - \operatorname{div} \boldsymbol{T}(v_i, p_i) = -\lambda_i(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})g),$$
$$\operatorname{div} v_i = 0,$$
$$\bar{n} \cdot \boldsymbol{D}(v_i) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot v_i|_S = 0,$$
$$v_i|_{t=0} = v_0, \quad i = 1, 2,$$

and

$$\theta_{it} - \varkappa \Delta \theta_i = -\lambda_i \tilde{v} \cdot \nabla \tilde{\theta},$$
$$\bar{n} \cdot \nabla \theta_i|_S = 0,$$
$$\theta_i|_{t=0} = \theta_0, \quad i = 1, 2.$$

To show uniform equicontinuity we introduce the differences

$$V = v_1 - v_2, \quad P = p_1 - p_2, \quad \mathcal{T} = \theta_1 - \theta_2$$

which are solutions to the problems

$$V_t - \operatorname{div} \boldsymbol{T}(V, P) = -(\lambda_1 - \lambda_2)(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})g),$$
$$\operatorname{div} V = 0,$$
$$\bar{n} \cdot \boldsymbol{D}(V) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot V|_S = 0,$$
$$V|_{t=0} = 0$$

and

$$\mathcal{T} - \varkappa \Delta \mathcal{T} = -(\lambda_1 - \lambda_2)\tilde{v} \cdot \nabla \tilde{\theta},$$
$$\bar{n} \cdot \nabla \mathcal{T}|_S = 0,$$
$$\mathcal{T}|_{t=0} = 0.$$

In view of Lemmas 4.1 and 4.2,

$$\|(V, \mathcal{T})\|_{\mathcal{M}(\Omega^T)} \leq c((\lambda_1 - \lambda_2)\|(\tilde{v}, \tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)}$$
$$+ (\lambda_1 - \lambda_2)\psi(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_s(\Omega^T)}),$$

so uniform equicontinuity of $\phi(\cdot, \tilde{v}, \tilde{\theta})$ follows. ∎

Proof of Theorem 1.1. In view of the above considerations the assumptions of the Leray–Schauder fixed point theorem are satisfied. Hence the main theorem is proved.

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