APPLICATIONES MATHEMATICAE
32,1 (2005), pp. 1–16

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ON THE PRINCIPAL EIGENCURVE OF THE p-LAPLACIAN RELATED TO THE SOBOLEV TRACE EMBEDDING

Abstract. We prove that for any $\lambda \in \mathbb{R}$, there is an increasing sequence of eigenvalues $\mu_n(\lambda)$ for the nonlinear boundary value problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial u/\partial\nu = \lambda\varrho(x)|u|^{p-2}u + \mu|u|^{p-2}u & \text{on } \partial\Omega, \end{cases}$$

and we show that the first one $\mu_1(\lambda)$ is simple and isolated; we also prove some results about variations of the density ϱ and the continuity with respect to the parameter λ.

1. Introduction and notations. Let Ω be a smooth bounded domain in $\mathbb{R}^N$; $N \geq 1$; $1 < p < \infty$ and $\varrho \in L^\infty(\partial\Omega)$ with $\varrho \not\equiv 0$ which can change the sign; $\lambda, \mu \in \mathbb{R}$. We consider the following nonlinear boundary value problem:

$$\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \tag{1.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\varrho(x)|u|^{p-2}u + \mu|u|^{p-2}u \quad \text{on } \partial\Omega. \tag{1.2}$$

The p-Laplacian $\Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ occurs in many mathematical models of physical topics including glaciology, nonlinear diffusion and filtration problem (see [4, 17]), power-low materials [14], non-Newtonian fluids [3]. For a discussion of some physical background, see [10]. The nonlinear boundary condition (1.2) describes a flux through the boundary $\partial\Omega$ which depends on the solution itself. For physical motivation of such conditions see for example [16].

Observe that in the particular case $\mu = 0$ and $p = 2$, (1.1)–(1.2) becomes linear and it is known as the Steklov problem [7].

2000 *Mathematics Subject Classification*: 35P30, 35J20, 35J60.
Key words and phrases: p-Laplacian operator, principal eigencurve, nonlinear boundary conditions, Sobolev trace embedding.

Classical Dirichlet problems involving the p-Laplacian have been extensively studied by various authors in the cases $\lambda = 0$ or $\mu = 0$ (cf. e.g. [1, 2, 5, 10, 13, 18, 19]). For nonlinear boundary conditions such as (1.2), recently the authors of [8] studied the case of $\mu = 0$ and ϱ belonging to some $L^s(\partial\Omega)$, not necessarily essentially bounded, with a restrictive condition on its sign.

We set

$$\mu_1(\lambda) = \inf\Big\{ \|v\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho(x)|v|^p \, d\sigma : \\ v \in W^{1,p}(\Omega),\ \int_{\partial\Omega} |u|^p \, d\sigma = 1 \Big\}, \tag{1.3}$$

where $\|\cdot\|_{1,p}$ denotes the $W^{1,p}(\Omega)$-norm, i.e.,

$$\|v\|_{1,p} = (\|\nabla v\|_p^p + \|v\|_p^p)^{1/p}$$

and $\|\cdot\|_p$ is the L^p-norm, with σ being the $(N-1)$-dimensional Lebesgue measure. By the *principal* (or *first*) *eigencurve* of the p-Laplacian related to the Sobolev trace embedding, we understand the graph of the map $\mu_1 : \lambda \mapsto \mu_1(\lambda)$ from $\mathbb{R}$ into $\mathbb{R}$. In [12] the simplicity and isolation of the first eigencurve of the Dirichlet p-Laplacian was proved by extending a similar result shown by Binding and Huang in [6].

Our purpose is to obtain some results (known for the ordinary Dirichlet p-Laplacian) for nonlinear eigenvalue problems where two-parameter eigenvalues appear in the nonlinear boundary condition. We show that $\mu_1(\lambda)$ is simple and isolated for any $\lambda \in \mathbb{R}$. Note that to show the simplicity (uniqueness) result, we use a simple convexity argument by remarking that the energy functional associated to problem (1.1)–(1.2) is convex in u^p for nonnegative u, without using in any way $C^1(\Omega)$ and $L^\infty(\Omega)$ regularity of the eigenfunctions associated to (1.1)–(1.2). In this respect our procedure is new.

Observe that $\mu_1(0) = \lambda_1$ is the optimal reciprocal constant of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. For the particular case $\mu = 0$ and $\varrho \in L^s(\partial\Omega)$ (for a suitable s), the isolation and simplicity of the first eigenvalue of (1.1)–(1.2) were studied in [8]. The main objective of our work is to extend this result to any $\lambda \in \mathbb{R}$, by using new technical methods.

The rest of the paper is organized as follows. In Section 2, we establish some definitions and preliminaries. In Section 3, we use a variational method to prove the existence of a sequence of eigencurves of (1.1)–(1.2). In Section 4, we prove the simplicity and isolation results for each point of the first eigencurve. Finally, in Section 5, we show some results about variations of the weight as a direct application of the simplicity result.

2. Definitions. In this paper, all solutions are weak ones, i.e., $u \in W^{1,p}(\Omega)$ is a solution of (1.1)–(1.2) if for all $v \in W^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_{\Omega} |u|^{p-2}uv\, dx = \int_{\partial\Omega} (\lambda\varrho(x) + \mu)|u|^{p-2}uv\, d\sigma. \tag{2.1}$$

If $u \in W^{1,p}(\Omega) \setminus \{0\}$, then u is called an eigenfunction of (1.1)–(1.2) associated to the eigenpair (λ, μ).

Set

$$\mathcal{M} = \Big\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p\, d\sigma = 1\Big\}. \tag{2.2}$$

A *principal eigenfunction* of (1.1)–(1.2) is any eigenfunction $u \in \mathcal{M}, u \geq 0$ a.e. on $\overline{\Omega}$, associated to the pair $(\lambda, \mu_1(\lambda))$.

Define the following energy functionals on $W^{1,p}(\Omega)$:

$$\Phi_\lambda(u) = \frac{1}{p}\,\|u\|_{1,p}^p - \frac{\lambda}{p}\int_{\partial\Omega} \varrho(x)|u|^p\, d\sigma = \frac{1}{p}\,\|u\|_{1,p}^p + \Phi(u), \qquad \lambda \in \mathbb{R},$$

$$\Psi(u) = \frac{1}{p}\int_{\partial\Omega} |u|^p\, d\sigma.$$

It is clear that for any $\lambda \in \mathbb{R}$, the solutions of (1.1)–(1.2) are the critical points of Φ_λ restricted to $\mathcal{M}$. We shall deal with operators T acting from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))'$. T is said to belong to the class (S_+) if for any sequence v_n weakly convergent to v in $W^{1,p}(\Omega)$ with $\limsup_{n\to\infty}\langle Tv_n, v_n - v\rangle \leq 0$, it follows that $v_n \to v$ strongly in $W^{1,p}(\Omega)$, where $(W^{1,p}(\Omega))'$ is the dual of $W^{1,p}(\Omega)$ with respect to the pairing $\langle\cdot,\cdot\rangle$.

3. Existence results. We will use Lyusternik–Schnirelmann theory on C^1-manifolds (see [19]). It is clear that for any $\lambda \in \mathbb{R}$, the functional Φ_λ is even and bounded from below on $\mathcal{M}$. Indeed, if $u \in \mathcal{M}$, then

$$\Phi_\lambda(u) \geq \frac{1}{p}\,(\|u\|_{1,p}^p - |\lambda|\,\|\varrho\|_{\infty,\partial\Omega}).$$

So

$$\Phi_\lambda(u) \geq \frac{1}{p}\,(\lambda_1 - |\lambda|\,\|\varrho\|_{\infty,\partial\Omega}) > -\infty, \tag{3.1}$$

where $\lambda_1 = \mu_1(0)$ is the reciprocal of the optimal constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

By employing the Sobolev trace embedding, we deduce that:

- Ψ and Φ are weakly continuous,
- Ψ' and Φ' are compact.

The following lemma is the key to showing the existence.

LEMMA 3.1. *For any $\lambda \in \mathbb{R}$, we have*:

(i) $(\Phi_\lambda)'$ *maps bounded sets to bounded sets*;

(ii) *if $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $(\Phi_\lambda)'(u_n)$ converges strongly in $(W^{1,p}(\Omega))'$, then $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$*;

(iii) *the functional Φ_λ satisfies the Palais–Smale condition on $\mathcal{M}$, i.e., for $(u_n)_n \subset \mathcal{M}$, if $\Phi_\lambda(u_n)$ is bounded and*

$$(\Phi_\lambda)'(u_n) - c_n\Psi'(u_n) \to 0 \tag{3.2}$$

with $c_n = \langle (\Phi_\lambda)'(u_n), u_n\rangle / \langle \Psi'(u_n), u_n\rangle$, then $(u_n)_n$ has a subsequence convergent in $W^{1,p}(\Omega)$.

Proof. (i) Let $u, v \in W^{1,p}(\Omega)$. Then

$$\langle (\Phi_\lambda)'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega |u|^{p-2}uv\,dx + \int_{\partial\Omega} \varrho(x)|u|^{p-2}uv\,d\sigma.$$

By Hölder's inequality, we obtain

$$\begin{aligned}|\langle (\Phi_\lambda)'(u), v\rangle| &\le \Big(\int_\Omega |\nabla u|^{(p-1)p'}\,dx\Big)^{1/p'}\|\nabla v\|_p + \Big(\int_\Omega |u|^{(p-1)p'}\,dx\Big)^{1/p'}\|v\|_p \\ &\quad + |\lambda|\,\|\varrho\|_{\infty,\partial\Omega}\Big(\int_{\partial\Omega} |u|^{(p-1)p'}\,d\sigma\Big)^{1/p'}\|v\|_{p,\partial\Omega} \\ &= \|\nabla u\|_p^{p-1}\|\nabla v\|_p + \|u\|_p^{p-1}\|v\|_p + |\lambda|\,\|\varrho\|_{\infty,\partial\Omega}\|u\|_{p,\partial\Omega}^{p-1}\|v\|_{p,\partial\Omega}.\end{aligned}$$

Now, the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ ensures the existence of a constant $c > 0$ such that

$$\|w\|_{p,\partial\Omega} \le c\|w\|_{1,p} \quad \text{for any } w \in W^{1,p}(\Omega).$$

Hence we deduce that

$$|\langle (\Phi_\lambda)'(u), v\rangle| \le \|\nabla u\|_p^{p-1}\|\nabla v\|_p + \|u\|_p^{p-1}\|v\|_p + c^p|\lambda|\,\|\varrho\|_{\infty,\partial\Omega}\|u\|_{1,p}^{p-1}\|v\|_{1,p}.$$

It is clear that

$$\|\nabla u\|_p^{p-1}\|\nabla v\|_p + \|u\|_p^{p-1}\|v\|_p \le \|u\|_{1,p}^{p-1}\|v\|_{1,p}.$$

Combining the above inequalities, we conclude that

$$|\langle (\Phi_\lambda)'(u), v| \le (1 + c^p|\lambda|\,\|\varrho\|_{\infty,\partial\Omega})\|u\|_{1,p}^{p-1}\|v\|_{1,p}$$

for any $u, v \in W^{1,p}(\Omega)$. It follows that

$$\|(\Phi_\lambda)'(u)\| \le (1 + c^p|\lambda|\,\|\varrho\|_{\infty,\partial\Omega})\|u\|_{1,p}^{p-1},$$

where $\|\cdot\|$ denotes the norm of $(W^{1,p}(\Omega))'$. This implies (i).

(ii) We use condition (S_+) as follows. $(\Phi_\lambda)'(u_n)$ being strongly convergent to some $f \in (W^{1,p}(\Omega))'$, by a calculation we have

$$\langle Au_n, v\rangle = \langle -\Delta_p u_n, v\rangle + \int_\Omega |u_n|^{p-2}u_n v\,dx + \int_{\partial\Omega} |\nabla u_n|^{p-2}\nabla u_n \nu v\,d\sigma \tag{3.3}$$

for any $v \in W^{1,p}(\Omega)$, where A is the operator from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))'$ defined by

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_\Omega |u|^{p-2}uv\, dx.$$

This operator satisfies condition (S_+) because $-\Delta_p$ does (cf. [12]).

If we take $v = u_n - u$ in (3.3) we obtain

$$\langle Au_n, u_n - v\rangle = \langle -\Delta_p u_n, u_n - v\rangle + \int_\Omega |u_n|^{p-2}u_n(u_n - u)\, dx$$
$$+ \int_{\partial\Omega} |\nabla u_n|^{p-2}\nabla u_n \nu(u_n - u)\, d\sigma.$$

Introducing $(\Phi_\lambda)'(u_n)$, we deduce that

$$\langle Au_n, u_n - u\rangle = \langle (\Phi_\lambda)'(u_n) - f, u_n - u\rangle + \langle f, u_n - u\rangle - \langle (\Phi_\lambda)'(u_n), u_n - u\rangle.$$

Using the compactness of Φ', we find that as $n \to \infty$,

$$\limsup_{n\to\infty}\, \langle Au_n, u_n - u\rangle \geq 0.$$

Hence $u_n \to u$ strongly in $W^{1,p}(\Omega)$, by condition (S_+).

(iii) From (3.1) we deduce that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Thus, without loss of generality, we can assume that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ for some $u \in W^{1,p}(\Omega)$. It follows that $\Psi'(u_n) \to \Psi'(u)$ in $(W^{1,p}(\Omega))'$ and $p\Psi(u) = 1$, because $p\Psi(u_n) = 1$ for all $n \in \mathbb{N}^*$. Hence $u \in \mathcal{M}$. Since $(u_n)_n$ is bounded, (i) ensures that $\{(\Phi_\lambda)'(u_n)\}$ is bounded. By a calculation we deduce via (3.2) that $\{(\Phi_\lambda)'(u_n)\}$ converges strongly in $(W^{1,p}(\Omega))'$. Consequently, from (ii) we conclude that $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$. ∎

Set $\Gamma_k = \{K \subset \mathcal{M} : K$ symmetric, compact and $\gamma(K) = k\}$, where $\gamma(K) = k$ is the genus of K, i.e., the smallest integer k such that there is an odd continuous map from K to $\mathbb{R}^k \setminus \{0\}$.

Next, we establish our existence result.

THEOREM 3.1. *For any $\lambda \in \mathbb{R}$ and any integer $k \in \mathbb{N}^*$,*

$$\mu_k(\lambda) := \inf_{K\in\Gamma_k} \max_{u\in K} \Phi_\lambda(u)$$

is a critical value of Φ_λ restricted to $\mathcal{M}$. More precisely, there exists $u_k(\lambda) \in \mathcal{M}$ such that

$$\mu_k(\lambda) = p\Phi_\lambda(u_k(\lambda)) = \max_{u\in K} p\Phi_\lambda(u)$$

and $(u_k(\lambda), \mu_k(\lambda))$ is a solution of (1.1)–(1.2). Moreover,

$$\mu_k(\lambda) \to \infty \qquad \textit{as } k \to \infty.$$

Proof. In view of [19], we need only prove that $\Gamma_k \neq \emptyset$ for any $k \in \mathbb{N}^*$, and the last assertion.

Indeed, since $W^{1,p}(\Omega)$ is separable, there exist $(e_i)_{i\geq 1}$ linearly dense in $W^{1,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_j = \emptyset$ if $i \neq j$, where $\operatorname{supp} e_i$ denotes the support of e_i. We can suppose that $e_i \in \mathcal{M}$ (if not we take $e'_i = e_i/p\Psi(e_i)$). For $k \in \mathbb{N}^*$, define $\mathcal{F}_k = \operatorname{span}\{e_1, \ldots, e_k\}$. Then $\mathcal{F}_k$ is a vector subspace and $\dim \mathcal{F}_k = k$. If $v \in \mathcal{F}_k$, then there exist $\alpha_1, \ldots, \alpha_k$ in $\mathbb{R}$ such that $v = \sum_{i=1}^k \alpha_i e_i$. Thus $\Psi(v) = \sum_{i=1}^k |\alpha_i|^p \Psi(e_i) = p^{-1} \sum_{i=1}^k |\alpha_i|^p$, because $\Psi(e_i) = 1$ for $i = 1, \ldots, k$. It follows that the map $v \mapsto (p\Psi(v))^{1/p}$ is a norm on $\mathcal{F}_k$. Hence, there is a constant $c > 0$ so that

$$c\|v\|_{1,p} \leq (p\Psi(v))^{1/p} \leq \frac{1}{c}\,\|v\|_{1,p}, \qquad \forall v \in \mathcal{F}_k.$$

That is,

$$c\|v\|_{1,p} \leq \Big(\int_{\partial\Omega} |v|^p \, d\sigma\Big)^{1/p} \leq \frac{1}{c}\,\|v\|_{1,p}, \qquad \forall v \in \mathcal{F}_k.$$

This implies that the set

$$\mathcal{V} = \mathcal{F}_k \cap \{v \in W^{1,p}(\Omega) : \|v\|_{p,\partial\Omega} \leq 1\}$$

is bounded, because $\mathcal{V} \subset B(0, 1/c) = \{v \in W^{1,p} : \|v\|_{1,p} \leq 1/c\}$. Moreover $\mathcal{V}$ is a symmetric bounded neighborhood of the origin 0. Consequently, from Proposition 2.3 of [19], we deduce that $\gamma(\mathcal{F}_k \cap \mathcal{M}) = k$. Then $\mathcal{F}_k \cap \mathcal{M} \in \Gamma_k$ (because $\mathcal{F}_k \cap \mathcal{M}$ is compact, since it equals the boundary of $\mathcal{V}$).

To complete the proof, it suffices to show that for any $\lambda \in \mathbb{R}$, $\mu_k(\lambda) \to \infty$ as $k \to \infty$. Indeed, let $(e_n, e^*_j)_{n,j}$ be a biorthogonal system such that $e_n \in W^{1,p}(\Omega)$, $e^*_j \in (W^{1,p}(\Omega))'$, the $(e_n)_n$ are linearly dense in $W^{1,p}(\Omega)$, and the $(e^*_j)_j$ are total in $(W^{1,p}(\Omega))'$. For any $k \in \mathbb{N}^*$ set

$$\mathcal{F}^{\perp}_{k-1} = \overline{\operatorname{span}(e_{k+1}, e_{k+2}, \ldots)}.$$

Observe that $K \cap \mathcal{F}^{\perp}_{k-1} \neq \emptyset$ for any $K \in \Gamma_k$ (by Proposition 2.3(g) of [19]).

Now, we claim that

$$t_k := \inf_{K\in\Gamma_k} \sup_{K\cap\mathcal{F}^{\perp}_{k-1}} p\Phi_\lambda(u) \to \infty \qquad \text{as } k \to \infty.$$

Indeed, to obtain a contradiction, assume that for k large enough there is $u_k \in \mathcal{F}^{\perp}_{k-1}$ with $\int_{\partial\Omega} |u_k|^p \, d\sigma = 1$ such that

$$t_k \leq p\Phi_\lambda(u_k) \leq M$$

for some $M > 0$ independent of k. Then

$$\|u_k\|^p_{1,p} - \lambda \int_{\partial\Omega} \varrho(x)|u_k|^p \, d\sigma \leq M.$$

Hence

$$\|u_k\|^p_{1,p} \leq M + \lambda\|\varrho\|_{\infty,\partial\Omega} < \infty.$$

This implies that $(u_k)_k$ is bounded in $W^{1,p}(\Omega)$. Taking a subsequence if necessary, we can suppose that (u_k) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. By our choice of $\mathcal{F}_{k-1}^{\perp}$, we have $u_k \rightharpoonup 0$ in $W^{1,p}(\Omega)$ because $\langle e_n^*, e_k\rangle = 0$ for all $k \geq n$. This contradicts the fact that $\int_{\partial\Omega} |u_k|^p d\sigma = 1$ for all k, and the claim is proved.

Finally, since $\mu_k(\lambda) \geq t_k$ we conclude that $\mu_k(\lambda) \to \infty$ as $k \to \infty$, and the proof is complete. ∎

4. Simplicity and isolation of $\mu_1(\lambda)$

4.1. *Simplicity.* First, observe that solutions of (1.1)–(1.2), by the well-known advanced regularity, belong to $C^{1,\alpha}(\overline{\Omega})$ (see [20]).

LEMMA 4.1. *Eigenfunctions u associated to $\mu_1(\lambda)$ are either positive or negative in Ω. Moreover if $u \in C^{1,\alpha}(\Omega)$ then $u \neq 0$ in $\overline{\Omega}$.*

Proof. Let u be an eigenfunction associated to $\mu_1(\lambda)$. Since $\Phi_\lambda(|u|) \leq \Phi_\lambda(u)$ and $\Psi(|u|) = \Psi(u)$, it follows from (1.3) that $|u|$ is also an eigenfunction associated to $\mu_1(\lambda)$. Using Harnack's inequality (cf. [14]), we deduce that $|u| > 0$ in Ω. By regularity u is defined in the whole of $\overline{\Omega}$. In fact $|u| > 0$ in $\overline{\Omega}$ because $(\partial u/\partial\nu)(x_0) < 0$ for any $x_0 \in \partial\Omega$ with $u(x_0) = 0$, by Hopf's Lemma (see [21]). ∎

THEOREM 4.1 (Uniqueness). *For any $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ defined by* (1.3) *is a simple eigenvalue, i.e., the set of eigenfunctions associated to $(\lambda, \mu_1(\lambda))$ is $\{tu_1(\lambda) : t \in \mathbb{R}\}$, where $u_1(\lambda)$ denotes the principal eigenfunction associated to $(\lambda, \mu_1(\lambda))$.*

Proof. By Theorem 3.1 it is clear that $\mu_1(\lambda)$ is an eigenvalue of the problem (1.1)–(1.2) for any $\lambda \in \mathbb{R}$. Let u and v be two eigenfunctions associated to $(\lambda, \mu_1(\lambda))$ such that $u, v \in \mathcal{M}$. Thus in virtue of Lemma 4.1 we can assume that u and v are positive.

Note that the mappings $W^{1,p}(\Omega) \ni w \mapsto \|\nabla w\|_p^p$, $w \mapsto \int_{\partial\Omega} |w|^p\, d\sigma$ and $w \mapsto \int_{\partial\Omega} \varrho(x)|w|^p\, d\sigma$ are linear functionals in w^p, for $w^p \geq 0$. Hence if we consider

$$w = \left(\frac{u^p + v^p}{2}\right)^{1/p},$$

then it belongs to $W^{1,p}(\Omega)$ and $\int_{\partial\Omega} |w|^p\, d\sigma = 1$. Consequently, w is admissible in the definition of $\mu_1(\lambda)$. On the other hand, by the convexity of $\chi \mapsto |\chi|^p$ we have the inequalities

$$\begin{aligned}
\text{(4.1)} \quad \int_\Omega |\nabla w|^p\, dx &= \frac{1}{2}\int_\Omega (|u^{p-1}\nabla u + v^{p-1}\nabla v|^p (u^p + v^p)^{1-p})\, dx \\
&= \frac{1}{2}\int_\Omega \left|\frac{u^p}{u^p + v^p}\,\frac{\nabla u}{u} + \frac{v^p}{v^p + u^p}\,\frac{\nabla v}{v}\right|^p (u^p + v^p)^{1-p}\, dx
\end{aligned}$$

$$\leq \frac{1}{2}\int_{\Omega}\left(\frac{u^p}{u^p+v^p}\left|\frac{\nabla u}{u}\right|^p + \frac{v^p}{v^p+u^p}\left|\frac{\nabla v}{v}\right|^p\right)dx$$
$$\leq \frac{1}{2}\int_{\Omega}(|\nabla u|^p + |\nabla v|^p)\,dx.$$

By the choice of u and v, we deduce that

$$\left|t\,\frac{\nabla u}{u} + (1-t)\,\frac{\nabla v}{v}\right|^p = t\left|\frac{\nabla u}{u}\right|^p + (1-t)\left|\frac{\nabla v}{v}\right|^p \tag{4.2}$$

with $t = u^p/(u^p+v^p)$.

Now, we claim that $u = v$ a.e. on $\overline{\Omega}$. Indeed, consider the auxiliary function

$$F(\chi_1,\chi_2) = |t\chi_1 + (1-t)\chi_2|^p - t|\chi_1|^p + (1-t)|\chi_2|^p.$$

Since $t \neq 0$, the critical points of F are the solutions of the system

$$\frac{\partial F(\chi_1,\chi_2)}{\partial \chi_1} = pt(|t\chi_1 + (1-t)\chi_2|^{p-2}(t\chi_1 - |\chi_1|^{p-2}\chi_1) = 0, \tag{4.3}$$

$$\frac{\partial F(\chi_1,\chi_2)}{\partial \chi_2} = p(t-1)(|t\chi_1 + (1-t)\chi_2|^{p-2}(t\chi_1 - |\chi_2|^{p-2}\chi_2) = 0. \tag{4.4}$$

Thus (4.2)–(4.4) imply that $(\chi_1 = \nabla u/u, \chi_2 = \nabla v/v)$ is a solution of the above system. Therefore

$$\left|\frac{\nabla u}{u}\right|^{p-2}\frac{\nabla u}{u} = \left|\frac{\nabla v}{v}\right|^{p-2}\frac{\nabla v}{v}.$$

Hence

$$\frac{\nabla u}{u} = \frac{\nabla v}{v} \quad \text{a.e. in } \overline{\Omega}.$$

This implies easily that $u = cv$ for some positive constant c. By normalization we conclude that $c = 1$. ■

REMARK 4.1. Various proofs of the uniqueness result were given in the Dirichlet p-Laplacian case by using $C^{1,\alpha}$-regularity and L^∞-estimation of the first eigenfunctions and by applying either Picone's identity (cf. [1]) or Díaz–Saá's inequality (cf. [2, 9, 11]) or an abstract inequality (cf. [15]).

4.2. *Isolation*

PROPOSITION 4.1. *For any $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is the unique eigenvalue associated to λ, having an eigenfunction not changing its sign on the boundary $\partial\Omega$.*

Proof. Fix $\lambda \in \mathbb{R}$ and let $u_1(\lambda)$ be the principal eigenfunction associated to $(\lambda, \mu_1(\lambda))$. Suppose that there exists an eigenfunction v corresponding to a pair (λ, μ) with $v \geq 0$ on $\partial\Omega$ and $v \in \mathcal{M}$. By the Maximum Principle,

$v > 0$ on $\overline{\Omega}$. To simplify the notation, set $u = u_1(\lambda)$. For $\varepsilon > 0$ small enough, write

$$u_\varepsilon = u + \varepsilon, \qquad v_\varepsilon = v + \varepsilon, \tag{4.5}$$

$$\phi(u_\varepsilon, v_\varepsilon) = \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}}. \tag{4.6}$$

It is clear that $\phi(u_\varepsilon, v_\varepsilon) \in W^{1,p}(\Omega)$ and it is an admissible test function in (1.1)–(1.2). Thus we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi(u_\varepsilon, v_\varepsilon)\,dx + \int_\Omega u^{p-1}\phi(u_\varepsilon, v_\varepsilon)\,dx = \int_{\partial\Omega} (\lambda\varrho(x) + \mu_1(\lambda))u^{p-1}\phi(u_\varepsilon, v_\varepsilon)\,d\sigma \tag{4.7}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi(u_\varepsilon, v_\varepsilon)\,dx + \int_\Omega v^{p-1}\phi(u_\varepsilon, v_\varepsilon)\,dx = \int_{\partial\Omega} (\lambda\varrho(x) + \mu))v^{p-1}\phi(u_\varepsilon, v_\varepsilon)\,d\sigma. \tag{4.8}$$

From (4.7) and (4.8), we deduce by calculation that

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi(u_\varepsilon, v_\varepsilon)\,dx + \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi(u_\varepsilon, v_\varepsilon)\,dx \\ &\qquad + \int_\Omega |v|^{p-2}v\phi(u_\varepsilon, v_\varepsilon)\,dx \\ &= \int_{\partial\Omega} \lambda\varrho(x)\biggl(\Bigl(\frac{u}{u_\varepsilon}\Bigr)^{p-1} - \Bigl(\frac{v}{v_\varepsilon}\Bigr)^{p-1}\biggr)(u_\varepsilon^p - v_\varepsilon^p)\,d\sigma \\ &\quad + \mu_1(\lambda)\int_{\partial\Omega} u^{p-1}\biggl[u_\varepsilon - \Bigl(\frac{v_\varepsilon}{u_\varepsilon}\Bigr)^{p-1} v_\varepsilon\biggr]\,d\sigma + \mu\int_{\partial\Omega} u^{p-1}\biggl[v_\varepsilon - \Bigl(\frac{u_\varepsilon}{v_\varepsilon}\Bigr)^{p-1} u_\varepsilon\biggr]\,d\sigma. \end{aligned} \tag{4.9}$$

On the other hand, by a long calculation again, we obtain

$$\nabla\phi(u_\varepsilon, v_\varepsilon) = \biggl\{1 + (p-1)\Bigl(\frac{v_\varepsilon}{u_\varepsilon}\Bigr)^p\biggr\}\nabla u_\varepsilon - p\Bigl(\frac{v_\varepsilon}{u_\varepsilon}\Bigr)^{p-1}\nabla v_\varepsilon \tag{4.10}$$

and

$$\int_\Omega [u^{p-1}\phi(u_\varepsilon, v_\varepsilon) + v^{p-1}\phi(u_\varepsilon, v_\varepsilon)]\,dx = \int_\Omega \biggl[\Bigl(\frac{u}{u_\varepsilon}\Bigr)^{p-1} - \Bigl(\frac{v}{v_\varepsilon}\Bigr)^{p-1}\biggr](u_\varepsilon^p - v_\varepsilon^p)\,dx. \tag{4.11}$$

Therefore (4.9), (4.10) and (4.11) yield

$$\int_{\Omega}\left[\left\{1+(p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p}\right\}|\nabla u_\varepsilon|^p+\left\{1+(p-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p}\right\}|\nabla v_\varepsilon|^p\right]dx \tag{4.12}$$

$$+\int_{\Omega}\left[-p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}|\nabla v_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v_\varepsilon+p\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v_\varepsilon\right]dx$$

$$=J_\varepsilon+K_\varepsilon-I_\varepsilon$$

with

$$I_\varepsilon=\int_{\Omega}\left(\left(\frac{u}{u_\varepsilon}\right)^{p-1}-\left(\frac{v}{v_\varepsilon}\right)^{p-1}\right)(u_\varepsilon^p-v_\varepsilon^p)\,dx, \tag{4.13}$$

$$J_\varepsilon=\lambda\int_{\partial\Omega}\varrho(x)\left(\left(\frac{u}{u+\varepsilon}\right)^{p-1}-\left(\frac{v}{v+\varepsilon}\right)^{p-1}\right)(u_\varepsilon^p-v_\varepsilon^p)\,d\sigma, \tag{4.14}$$

$$K_\varepsilon=\mu_1(\lambda)\int_{\partial\Omega}u^{p-1}\left[u_\varepsilon-\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}v_\varepsilon\right]d\sigma \tag{4.15}$$

$$+\mu\int_{\partial\Omega}u^{p-1}\left[v_\varepsilon-\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}u_\varepsilon\right]d\sigma.$$

It is clear that $I_\varepsilon\ge 0$. Now, thanks to the inequalities of Lindqvist [15], we can distinguish two cases according to the value of p.

Case 1: $p\ge 2$. From (4.12) we have

$$J_\varepsilon+K_\varepsilon\ge\frac{1}{2^{p-2}-1}\int_{\Omega}\left(\frac{1}{(u+1)^p}+\frac{1}{(v+1)^p}\right)|u\nabla v-v\nabla u|^p\,dx\ge 0. \tag{4.16}$$

Case 2: $1<p<2$. Then

$$J_\varepsilon+K_\varepsilon\ge c(p)\int_{\Omega}\frac{uv(u^p+v^p)}{(v|\nabla u|+u|\nabla v|+1)^{2-p}}\,|u\nabla v-v\nabla u|^2\,dx\ge 0, \tag{4.17}$$

where the constant $c(p)>0$ is independent of u,v,λ and $\mu_1(\lambda)$.

The Dominated Convergence Theorem implies that

$$\lim_{\varepsilon\to0^+}J_\varepsilon=\lim_{\varepsilon\to0^+}K_\varepsilon=(\mu_1(\lambda)-\mu)\int_{\partial\Omega}(u^p-v^p)\,d\sigma=0,$$

because

$$\int_{\partial\Omega}u^p d\sigma=\int_{\partial\Omega}v^p d\sigma=1. \tag{4.18}$$

Now, letting $\varepsilon\to0^+$ in (4.16) and (4.17), we arrive at

$$u\nabla v=v\nabla u\quad\text{a.e. on }\Omega.$$

Thus

$$\nabla\left(\frac{u}{v}\right) = 0 \quad \text{a.e. on } \Omega.$$

Hence, there exists $t > 0$ such that $u = tv$ a.e. on Ω. By continuity $u = v$ a.e. in $\overline{\Omega}$; and by the normalization (4.18) we deduce that $t = 1$ and $u = v$ a.e. on $\partial\Omega$. This implies that $u = v$ a.e. on $\overline{\Omega}$. Finally, we conclude that $\mu = \mu_1(\lambda)$. ∎

REMARK 4.2. We can also show Proposition 4.1 by using Picone's identity. A similar result was given in [8] in the particular case $\lambda = 0$.

COROLLARY 4.1. *For any $\lambda \in \mathbb{R}$, if u is an eigenfunction associated to a pair (λ, μ) with $\mu \neq \mu_1(\lambda)$, then u changes its sign on the boundary $\partial\Omega$. Moreover*,

$$\min(|\partial\Omega^-|, |\partial\Omega^+|) \geq c_{p^*}^{-N}(|\lambda|\,\|\varrho\|_{\infty,\partial\Omega} + |\mu|)^{-\eta}, \tag{4.19}$$

where $\eta = N/p$ if $1 < p < N$ and $\eta = 2$ if $p > N$, c_{p^*} is the best constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, and $|\partial\Omega^\pm|$ denotes the $(N-1)$-dimensional measure of $\partial\Omega^\pm$. Here $p^* = p(N-1)/(N-p)$ is the critical Sobolev exponent and $\partial\Omega^\pm = \{x \in \overline{\Omega} : u(x) \gtrless 0\}$.

Proof. Set $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. It follows from (2.1), where we put $v = u^-$, that

$$\int_\Omega |\nabla u^-|^p\,dx + \int_\Omega |u^-|^p\,dx = \int_{\partial\Omega} (\lambda\varrho(x) + \mu)|u^-|^p\,d\sigma.$$

Thus

$$\begin{aligned}\|u^-\|_{1,p} &\leq (|\lambda|\,\|\varrho\|_{\infty,\partial\Omega} + |\mu|) \int_{\partial\Omega^-} |u^-|^p\,d\sigma \\ &\leq (|\lambda|\,\|\varrho\|_{\infty,\partial\Omega} + |\mu|)|\partial\Omega^-|^{p/N}\Big(\int_{\partial\Omega} |u^-|^{p^*}\Big)^{p/p^*}.\end{aligned}$$

By the Sobolev embedding $W^{1,p}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we deduce that

$$|\partial\Omega^-| \geq c_{p^*}^{-N}(|\lambda|\,\|\varrho\|_{\infty,\partial\Omega} + |\mu|)^{-\eta}.$$

The same holds for $\partial\Omega^+$ by taking $v = u^+$ in (2.1). Hence the estimate (4.19) follows. ∎

REMARKS 4.1. (i) The right-hand side of (4.19) is positive because $\varrho \not\equiv 0$ and if $\lambda = 0$ then μ is an eigenvalue of the p-Laplacian related to the trace embedding, so $\mu - \lambda_1 > 0$, where λ_1 is the first eigenvalue of (1.1)–(1.2) in the case $\lambda = 0$.

(ii) An easy consequence of Corollary 4.1 is that the number of nodal components of each eigenfunction of (1.1)–(1.2) is finite.

Using Proposition 4.1 and Corollary 4.1, we can state the following important result.

THEOREM 4.2. *For any $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is isolated.*

5. Variations of the weight. Let $\mu_1(\lambda) = \mu_1(\varrho)$ and $u_1(\lambda) = u_1(\varrho)$ (to indicate the dependence on the weight ϱ).

THEOREM 5.1. *For any $\lambda \in \mathbb{R}$, if $(\varrho_k)_k$ is a sequence in $L^\infty(\partial\Omega)$ such that ϱ_k converges to ϱ in $L^\infty(\partial\Omega)$ with $\varrho \not\equiv 0$, then*

$$\lim_{k\to\infty} \mu_1(\varrho_k) = \mu_1(\varrho), \tag{5.1}$$

$$\lim_{k\to\infty} \|u_1(\varrho_k) - u_1(\varrho)\|_{1,p}^p = 0. \tag{5.2}$$

Proof. If $\lambda = 0$, the result is evident because $\mu_1(\varrho_k) = \mu_1(\varrho)$ for all $k \in \mathbb{N}^*$. If $\lambda \neq 0$, then for $v \in \mathcal{M}$,

$$\Big|\lambda \int_{\partial\Omega} (\varrho_k - \varrho)|v|^p \, d\sigma\Big| \le |\lambda| \, \|\varrho_k - \varrho\|_{\infty,\partial\Omega}.$$

By the convergence of ϱ_k to ϱ in $L^\infty(\partial\Omega)$, for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that for all $k \ge k_\varepsilon$,

$$\Big|\lambda \int_{\partial\Omega} (\varrho_k - \varrho)|v|^p \, d\sigma\Big| \le |\lambda| \, \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

This implies that

$$\lambda \int_{\partial\Omega} \varrho|v|^p \, d\sigma \le \varepsilon + \lambda \int_{\partial\Omega} \varrho_k|v|^p \, d\sigma, \tag{5.3}$$

$$\lambda \int_{\partial\Omega} \varrho_k|v|^p \, d\sigma \le \varepsilon + \lambda \int_{\partial\Omega} \varrho|v|^p \, d\sigma, \tag{5.4}$$

for any $v \in \mathcal{M}$, $\varepsilon > 0$ and $k \ge k_\varepsilon$.

On the other hand, we have $\varrho \not\equiv 0$. We take k_ε large enough so that $\varrho_k \not\equiv 0$. Thus

$$\mu_1(\varrho_k) \le \|v\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho_k|v|^p \, d\sigma.$$

Combining with (5.3) and (5.4), we obtain

$$\mu_1(\varrho_k) \le \|v\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho|v|^p \, d\sigma + \varepsilon.$$

Passing to the infimum over $v \in \mathcal{M}$, we find

$$\mu_1(\varrho_k) \le \mu_1(\varrho) + \varepsilon, \qquad \mu_1(\varrho) \le \mu_1(\varrho_k) + \varepsilon, \qquad \forall\varepsilon > 0 \ \forall k > k_\varepsilon.$$

Hence, we obtain the convergence (5.1).

For the strong convergence (5.2) we argue as follows. For k large enough, we have $\varrho_k \not\equiv 0$ and

$$\mu_1(\varrho_k) = \|u_1(\varrho_k)\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho_k (u_1(\varrho_k))^p \, d\sigma. \tag{5.5}$$

Thus

$$\|u_1(\varrho_k)\|_{1,p}^p \le |\mu_1(\varrho_k)| + |\lambda| \, \|\varrho_k\|_{\infty,\partial\Omega}.$$

From (5.1) and the convergence of ϱ_k to ϱ in $L^\infty(\partial\Omega)$, we deduce that $(u_1(\varrho_k))_k$ is a bounded sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is reflexive and compactly embedded in $L^p(\partial\Omega)$ we can extract a subsequence of $(u_1(\varrho_k))_k$, again labelled by k, such that $u_1(\varrho_k) \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $u_1(\varrho_k) \to u$ (strongly) in $L^p(\partial\Omega)$ as $k \to \infty$. We can also suppose that $u_1(\varrho_k) \to u$ in $L^p(\Omega)$. Passing to a subsequence if necessary, we can assume that $u_1(\varrho_k) \to u$ a.e. in $\overline{\Omega}$. Thus $u \ge 0$ a.e. in $\overline{\Omega}$. We will prove that $u \equiv u_1(\varrho)$. To do this, using the Dominated Convergence Theorem in $\partial\Omega$, we deduce that

$$\int_{\partial\Omega} \varrho_k (u_1(\varrho_k))^p d\sigma \to \int_{\partial\Omega} \varrho u^p \, d\sigma$$

as $k \to \infty$. By (5.5), (5.1) and the lower weak semicontinuity of the norm we obtain

$$\|u\|_{1,p}^p \le \mu_1(\varrho) + \lambda \int_{\partial\Omega} \varrho u^p \, d\sigma. \tag{5.6}$$

The normalization $\int_{\partial\Omega} u^p d\sigma = 1$ is proved. Moreover, $u \ge 0$ a.e. in $\overline{\Omega}$, because $u_1(\varrho_k) > 0$ in $\overline{\Omega}$. Thus u is an admissible function in the variational definition of $\mu_1(\lambda)$. So

$$\mu_1(\lambda) \le \|u\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho u^p d\sigma.$$

This and (5.6) yield

$$\mu_1(\varrho) = \|u\|_{1,p}^p - \lambda \int_{\partial\Omega} \varrho u^p d\sigma. \tag{5.7}$$

By the uniqueness of the principal eigenfunction associated to $\mu_1(\lambda)$, we must have $u \equiv u_1(\varrho)$. Consequently, the limit function $u_1(\varrho)$ is independent of the choice of the (sub)sequence. Hence, $u_1(\varrho_k)$ converges to $u_1(\varrho)$ at least in $L^p(\partial\Omega)$ and in $L^p(\Omega)$. To complete the proof of (5.2), it suffices to use Clarkson's inequalities related to uniform convexity of $W^{1,p}(\Omega)$. For this we distinguish two cases.

Case 1: $p \ge 2$. We have

$$\int_{\Omega} \left| \frac{\nabla u_1(\varrho_k) - \nabla u_1(\varrho)}{2} \right|^p dx + \int_{\Omega} \left| \frac{\nabla u_1(\varrho_k) + \nabla u_1(\varrho)}{2} \right|^p dx \le \frac{1}{2} \int_{\Omega} |\nabla u_1(\varrho_k)|^p \, dx + \frac{1}{2} \int_{\Omega} |\nabla u_1(\varrho)|^p \, dx$$

and

$$\mu_1(\varrho_k)\int\limits_{\partial\Omega}\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma \le \int\limits_{\Omega}\left|\frac{\nabla u_1(\varrho_k)+\nabla u_1(\varrho)}{2}\right|^p dx \\ -\lambda\int\limits_{\partial\Omega}\varrho_k\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma.$$

Moreover

$$\int\limits_{\Omega}\left|\frac{u_1(\varrho_k)-u_1(\varrho)}{2}\right|^p dx \le \int\limits_{\Omega}\left|\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right|^p dx+\frac{1}{2}\,\|u_1(\varrho_k)\|_p^p+\frac{1}{2}\,\|u_1(\varrho)\|_p^p.$$

Hence

$$\|u_1(\varrho_k)-u_1(\varrho)\|_{1,p}^p \\ \le -\mu_1(\varrho_k)\int\limits_{\partial\Omega}\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma-\lambda\int\limits_{\partial\Omega}\varrho_k\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma \\ +\frac{1}{2}\left(\mu_1(\varrho_k)-\lambda\int\limits_{\partial\Omega}\varrho_k(x)u_1(\varrho_k)\,d\sigma\right)+\frac{1}{2}\left(\mu_1(\varrho)-\lambda\int\limits_{\partial\Omega}\varrho u_1^p\,d\sigma\right).$$

Then, by using the Dominated Convergence Theorem we deduce that

$$\limsup_{k\to\infty}\|u_1(\varrho_k)-u_1(\varrho)\|_{1,p}^p=0.$$

Case 2: $1<p<2$. In this case, we have

$$\left\{\int\limits_{\Omega}\left|\frac{\nabla u_1(\varrho_k)-\nabla u_1(\varrho)}{2}\right|^p dx\right\}^{1/(p-1)}+\left\{\int\limits_{\Omega}\left|\frac{\nabla u_1(\varrho_k)+\nabla u_1(\varrho)}{2}\right|^p dx\right\}^{1/(p-1)} \\ \le\left\{\frac{1}{2}\int\limits_{\Omega}|\nabla u_1(\varrho_k)|^p\,dx+\frac{1}{2}\int\limits_{\Omega}|\nabla u_1(\varrho)|^p\,dx\right\}^{1/(p-1)}$$

and

$$\mu_1(\varrho_k)\int\limits_{\partial\Omega}\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma \le \int\limits_{\Omega}\left|\frac{\nabla u_1(\varrho_k)+\nabla u_1(\varrho)}{2}\right|^p \\ -\lambda\int\limits_{\partial\Omega}\varrho_k\left(\frac{u_1(\varrho_k)+u_1(\varrho)}{2}\right)^p d\sigma.$$

Hence, by definitions of $\mu_1(\varrho_k)$ and $\mu_1(\varrho)$, and the second Clarkson inequality we obtain the convergence (5.2). ∎

Corollary 5.1. *For any bounded domain Ω, the function $\lambda\mapsto\mu_1(\lambda)$ is differentiable on $\mathbb{R}$ and the function $\lambda\mapsto u(\lambda)$ is continuous from $\mathbb{R}$ into $W^{1,p}(\Omega)$. More precisely*

$$\mu_1'(\lambda_0)=-\int\limits_{\partial\Omega}\varrho(x)(u_1(\lambda_0))^p\,d\sigma,\qquad\forall\lambda_0\in\mathbb{R}.$$

Proof. Denote by $\mu_1(\lambda, \varrho)$ the principal eigenvalue associated with λ and the weight ϱ and by $u_1(\lambda, \varrho)$ the corresponding principal eigenfunction. Suppose that $\lambda_k \to \lambda_0$ in $\mathbb{R}$; then $h_k = \lambda_k \varrho \to \lambda_0 \varrho = h$ in $L^\infty(\partial\Omega)$. From Theorem 5.1 we deduce that

$$\mu_1(\lambda_k) = \mu_1(1, h_k) \to \mu_1(1, h) = \mu_1(\lambda_0)$$

and

$$u_1(\lambda_k) = u_1(1, h_k) \to u_1(1, h) = u_1(\lambda_0) \quad \text{in } W^{1,p}(\Omega).$$

For the differentiability, it suffices to use the variational characterization of $\mu_1(\lambda)$ and of $\mu_1(\lambda_0)$, so that we have

$$(\lambda - \lambda_0) \int_{\partial\Omega} \varrho(x)(u_1(\lambda))^p \, d\sigma \le \mu_1(\lambda) - \mu_1(\lambda_0) \le (\lambda_0 - \lambda) \int_{\partial\Omega} (u_1(\lambda_0))^p \, d\sigma$$

for any $\lambda, \lambda_0 \in \mathbb{R}$. ∎

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Received on 4.7.2003;
revised version on 12.7.2004 (1701)