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BOUNDS FOR THE RANGE OF AMERICAN CONTINGENT CLAIM PRICES IN THE JUMP-DIFFUSION MODEL

Abstract. The problem of valuation of American contingent claims for a jump-diffusion market model is considered. Financial assets are described by stochastic differential equations driven by Gaussian and Poisson random measures. In such setting the money market is incomplete, thus contingent claim prices are not uniquely defined. For different equivalent martingale measures different arbitrage free prices can be derived. The problem is to find exact bounds for the set of all possible prices obtained in this way. The paper extends and improves some results of [1].

1. Introduction. The jump-diffusion market model described by a stochastic differential equation driven by Gaussian and Poisson random measures serves as an important example of the so-called *incomplete financial market model* in the sense of Harrison and Pliska's definition (see [4], [8], [12]). This means that for any such model there exists an infinite set of equivalent martingale measures for which one can construct American (or some other) contingent claim prices without inducing arbitrage opportunities.

For the last three decades investigations of this problem have been focused on some particular cases of such measures and corresponding prices of given contingent claims. The most important equivalent martingale measures to be mentioned are: the measure proposed by Merton (see [9]), easy to construct and appropriate for applications, and the so-called *optimal martingale measure* defined and constructed by Schweizer (see [13], and also [3], where optimal exercise times for market models with Schweizer measures have been constructed).

However, assuming given a full family of equivalent martingale measures, one can ask for a maximal range of corresponding prices, and to localize some

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of them, e.g. those mentioned above. The first attempt in this direction was made by Eberlein and Jacod in [2], who proved that for the underlying asset price process, which is defined by a purely discontinuous Levy process with unbounded jumps, the trivial bounds for European contingent claims do not exist. Later on, Bellamy and Jeanblanc in [1] obtained the exact bounds for the range (in the form of a bounded interval) of prices of European and American vanilla options for a market model governed by a Wiener process and a Poisson process with jumps of constant amplitude.

Our goal in this paper is to derive and prove in detail an explicit formula for the range of prices of American contingent claims for a market model governed by a Wiener process and a finite number of Poisson processes with jumps of random magnitude. The non-trivial lower bound expressed in terms of values of the Black–Scholes function is exact, and it can be attained by a subfamily of equivalent martingale measures indexed by a single numerical parameter.

2. The jump-diffusion model. Throughout the paper, we consider a financial market on which there are two underlying assets, $\boldsymbol{B}$ and $\boldsymbol{S}$, traded up to a fixed time $T > 0$. Introducing a stochastic model of such market we agree that the non-risky asset $\boldsymbol{B}$ (*bond*) is defined by the function $B_t = e^{rt}$ for $t \in [0, T]$, and the risky asset $\boldsymbol{S}$ (*stock*) is given as a stochastic process, which takes the initial value $S_0 > 0$ (*endowment*), and solves the following stochastic differential equation:

$$
(1) \qquad dS_t = S_{t-}\Big(\mu dt + \sigma dW_t + \sum_{i=1}^{k} U^i_{N_t^{\lambda_i}} dN_t^{\lambda_i}\Big), \qquad t \in (0, T],
$$

where

(i) $r > 0$ denotes a risk-free (i.e. deterministic) constant interest rate,
(ii) μ, σ, and $\{\lambda_i\}_{i=1}^k$, with fixed $k \geq 1$, are given positive constants;
(iii) for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a right-continuous filtration $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t\}_{t\in[0,T]}$, which contains all $\mathbb{P}$-negligible sets in $\mathcal{F}$, the process $\boldsymbol{W} = \{W_t : t \in [0, T]\}$ is an $\{\mathcal{F}_t\}$-Brownian motion, and $\boldsymbol{N}^{\lambda_i} = \{N_t^{\lambda_i} : t \in [0, T]\}$ are $\{\mathcal{F}_t\}$-Poisson processes with intensities λ_i for $i = 1, \ldots, k$;
(iv) for $i = 1, \ldots, k$, $\{U^i_j : j = 1, 2, \ldots\}$ is a sequence of i.i.d. square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with densities $f^i = f^i(x)$, whose supports are contained in $(a_i, b_i) \subset (-1, \infty)$;
(v) W_t, $N_t^{\lambda_i}$, and U^i_j are all independent on $(\Omega, \mathcal{F}, \mathbb{P})$, and the filtration $\{\mathcal{F}_t\}$ can be assumed to be an appropriately constructed augmentation of $\sigma(W_s, \sum_{j=1}^{N_s^{\lambda_1}} U^1_j, \ldots, \sum_{j=1}^{N_s^{\lambda_k}} U^k_j : s \leq t)$.

Further on we assume that the stock $\boldsymbol{S}$ provides dividends for its holder with constant rate $\delta \in [0, r)$.

REMARK 1. Recall that in the classical Black–Scholes stochastic model of a financial market a risky asset is described by the SDE

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad t \in (0, T]. \tag{2}$$

Quantitative and qualitative comparison of stochastic models of the financial market given as solutions to SDEs (1) and (2) and obtained through computer simulations can be found in [7].

LEMMA 1. *Define* $\lambda = \lambda_1 + \cdots + \lambda_k$, *and a density function*

$$f(x) = \frac{\lambda_1}{\lambda} f^1(x) + \cdots + \frac{\lambda_k}{\lambda} f^k(x), \tag{3}$$

for parameters $\{\lambda_i\}_{i=1}^k$ *and densities* $\{f_i\}_{i=1}^k$ *introduced above. Let* $U_1, U_2, \ldots$ *be a sequence of i.i.d. square integrable random variables which follow the law assigned by this density. Then the SDE* (1) *is equivalent to the stochastic differential equation*

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + U_{N_t^\lambda} dN_t^\lambda), \qquad t \in (0, T],$$

or

$$dS_t = S_{t-}(\mu^* dt + \sigma dW_t + U_{N_t^\lambda} dN_t^\lambda - \lambda\, \mathrm{E}^{\mathbb{P}}(U_1) dt), \qquad t \in (0, T], \tag{4}$$

where $\mu^* = \mu + \lambda\, \mathrm{E}^{\mathbb{P}}(U_1)$, *or is equivalent to the stochastic differential equation in the following integral form*:

$$S_t = S_0 + \int_0^t \mu^* S_{s-}\, ds + \int_0^t \sigma S_{s-}\, dW_s + \int_0^t \int_{\mathbb{R}} S_{s-} y\, \widetilde{v}(ds, dy), \qquad t \in (0, T], \tag{5}$$

where $\widetilde{v}(dt, dx) = v(dt, dx) - \lambda m(dx) dt$, $v(dt, dx) = \sum_{s \in (t, t+dt)} I_{(x, x+dx)}(U_{N_s^\lambda})$, *and* $m(dx) = f(x) dx$ *for* $f = f(x)$ *given by* (3).

Proof. The lemma follows from well known properties of Poisson processes and Poisson random measures (see [5], [11]). The main thing is to notice that the Poissonian terms in SDEs (4) and (5) are related by

$$U_{N_t^\lambda} dN_t^\lambda - \lambda\, \mathrm{E}^{\mathbb{P}}(U_1) dt = \int_{\mathbb{R}} y\, \widetilde{v}(dt, dy). \ \blacksquare$$

The financial market model defined by equivalent SDEs (1), (4) or (5) is called a *diffusion model with jumps of random magnitude.*

3. Equivalent martingale measures. In our setting, a probability measure $\mathbb{Q}$ on the space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})$ is called an *equivalent martingale measure* if it is equivalent to the probability $\mathbb{P}$ and if the process $\{e^{-rt+\delta t} S_t : t \in [0, T]\}$ is a martingale under $\mathbb{Q}$.

Our aim is to find a subfamily of measures $\mathbb{Q}$ for which we get the range of the underlying American options prices. First we restrict ourselves to those equivalent martingale measures $\mathbb{Q}$ whose Radon–Nikodym densities $\{L_t = \mathrm{E}^{\mathbb{P}}\big(\frac{d\mathbb{Q}}{d\mathbb{P}} \,\big|\, \mathcal{F}_t\big) : t \in [0,T]\}$ satisfy the regularity condition

$$\sup_{t\in[0,T]} \mathrm{E}^{\mathbb{P}}(L_t^2) < \infty. \tag{6}$$

Such equivalent martingale measures are described by the following lemma, proven e.g. in [1] and [10].

Lemma 2. *The set $\mathcal{Q}$ of martingale measures equivalent to $\mathbb{P}$, with Radon–Nikodym densities satisfying* (6) *consists of all probability measures $\mathbb{P}^\gamma$ for which the processes $\{L_t^\gamma = \mathrm{E}^{\mathbb{P}}\big(\frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \,\big|\, \mathcal{F}_t\big) : t \in [0,T]\}$ take the form of a product $L_t^\gamma = L_t^{\gamma,W} L_t^{\gamma,\widetilde{v}}$ of two Doléans–Dade stochastic exponentials*, *i.e. solutions of two linear SDEs*

$$L_t^{\gamma,W} = 1 + \int_0^t L_{s-}^{\gamma,W}\psi_s\, dW_s,$$

$$L_t^{\gamma,\widetilde{v}} = 1 + \int_0^t L_{s-}^{\gamma,\widetilde{v}} \int_{\mathbb{R}} \gamma(s,y)\,\widetilde{v}(ds,dy).$$

The processes $\boldsymbol{\psi} = \{\psi_t : t\in[0,T]\}$ and $\boldsymbol{\gamma} = \{\gamma(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ are predictable and linked by the equality

$$\mu^* - r + \delta + \psi_t\sigma + \lambda \int_{\mathbb{R}} \gamma(t,y)y\, m(dy) = 0, \tag{7}$$

and such that

$$1 + \gamma(t,x) > 0, \qquad \mathrm{E}^{\mathbb{P}}\bigg(\frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \,\bigg|\, \mathcal{F}_T\bigg) = 1.$$

The process $\boldsymbol{\psi}$ is called the *market price of diffusion risk*, while $\boldsymbol{\gamma}$ is the *market price of jump risk*.

Remark 2. Predictable processes $\boldsymbol{\gamma} = \{\gamma(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ from (7) such that the processes $\{e^{-rt+\delta t}S_t : t\in[0,T]\}$ are martingales under the corresponding measures $\mathbb{P}^\gamma$, can be considered as a set of indices parametrizing the full family of relevant martingale measures. However, for our purposes (see Section 4) it is enough to consider only the smaller set $\varGamma$ of processes $\boldsymbol{\gamma} = \{\gamma(x) : x\in\mathbb{R}\}$ that do not depend on time t, i.e. we assume from now on that $\mathbb{P}^\gamma \in \mathcal{Q}$ if and only if $\boldsymbol{\gamma} \in \varGamma$. To attain bounds for the range of all prices for a given contingent claim and fixed model of a financial market it is enough (and convenient, see relations (12)) to consider only constant $\gamma \in (-1,\infty)$ (see Theorem 3). However, e.g. the Schweizer optimal price was explicitly described by some nonconstant function $\gamma = \gamma(x)$ (see [13] and [3]) though there exists a constant γ corresponding to it.

For any $\boldsymbol{\gamma} \in \Gamma$ the process

$$W_t^\gamma = W_t - \int_0^t \psi_s \, ds \tag{8}$$

is a standard Brownian motion under the probability measure $\mathbb{P}^\gamma$ from $\mathcal{Q}$.

From Girsanov's theorem (see [10]) it follows that also any homogeneous compensated Poisson martingale measure

$$\widetilde{\upsilon}^\gamma(dt, dx) = \widetilde{\upsilon}(dt, dx) - \lambda\gamma(t, x)m(dx)dt \tag{9}$$

on $(\Omega, \mathcal{F}, \mathbb{P})$ can be viewed on $(\Omega, \mathcal{F}, \mathbb{P}^\gamma)$ as a compensated random measure

$$\widetilde{\upsilon}^\gamma(dt, dx) = \upsilon^\gamma(dt, dx) - \lambda^\gamma m^\gamma(dx)dt, \tag{10}$$

with $m^\gamma(dx) = f^\gamma(x)dx$, or can be identified with a stochastic differential represented by a homogeneous compensated Poisson process as follows:

$$\int_{\mathbb{R}} y \, \widetilde{\upsilon}^\gamma(dt, dy) = U^\gamma_{N_t^{\lambda^\gamma}} dN_t^{\lambda^\gamma} - \lambda^\gamma \, \mathrm{E}^{\mathbb{P}^\gamma}(U_1^\gamma) \, dt, \tag{11}$$

where the Poisson process $\boldsymbol{N}^{\lambda^\gamma} = \{N_t^{\lambda^\gamma} : t \in [0, T]\}$ and the i.i.d. sequence $\{U_j^\gamma : j = 1, 2, \dots\}$ of square integrable random variables U_j^γ are defined—in terms of data from Lemma 1—by the following relations:

$$\lambda^\gamma = \lambda \int_{\mathbb{R}} (1 + \gamma(y)) f(y) \, dy, \qquad f^\gamma(x) = \frac{(1 + \gamma(x)) f(x)}{\int_{\mathbb{R}} (1 + \gamma(y)) f(y) \, dy}. \tag{12}$$

After rewriting SDE (4) as

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \mu^* dt + \sigma\psi_t dt + \lambda \int_{\mathbb{R}} y\gamma(y) \, m(dy) \, dt \\ &\quad + \sigma dW_t - \sigma\psi_t dt + \int_{\mathbb{R}} y \, \widetilde{\upsilon}(dt, dy) - \lambda \int_{\mathbb{R}} y\gamma(y) \, m(dy) \, dt, \end{aligned}$$

and making use of relations (7)–(11), we see that the stochastic differential equation for the stock price $\boldsymbol{S} = \{S_t : t \in [0, T]\}$ with respect to $\mathbb{P}^\gamma \in \mathcal{Q}$ takes the following equivalent integral form:

$$\begin{aligned} S_t &= S_0 + (r - \delta) \int_{0+}^t S_{s-} \, ds + \sigma \int_{0+}^t S_{s-} \, dW_s^\gamma \\ &\quad + \int_{0+}^t S_{s-} \int_{\mathbb{R}} y \, \upsilon^\gamma(ds, dy) - \int_{0+}^t S_{s-} \int_{\mathbb{R}} y \lambda^\gamma \, m^\gamma(dy) \, du, \end{aligned} \tag{13}$$

where $\{\int_{0+}^t S_{s-} \int_{\mathbb{R}} y \, \lambda^\gamma \, m^\gamma(dy) \, ds : t \in [0, T]\}$ is a semimartingale with continuous paths and $\{\int_{0+}^t S_{s-} \int_{\mathbb{R}} y \, \upsilon^\gamma(ds, dy) : t \in [0, T]\}$ is a pure jump semimartingale.

Rewriting equation (13) in detail, we see now that the stock price process $\boldsymbol{S}$ solves the SDE reduced to the following differential form:

$$dS_t = S_{t-}(\mu^\gamma dt + \sigma dW_t^\gamma + U^\gamma_{N_t^{\lambda^\gamma}} dN_t^{\lambda^\gamma}), \tag{14}$$

where $\mu^\gamma = r - \delta - \lambda^\gamma \, \mathrm{E}^{\mathbb{P}^\gamma}(U_1^\gamma)$.

So, we conclude this section with the observation that the family $\{\mathbb{P}^\gamma\}_{\gamma\in\Gamma}$ of equivalent martingale measures is correctly constructed. We can derive different prices of American contingent claims for different measures $\mathbb{P}^\gamma$, working with the stochastic model of time evolution of the underlying risky asset price, which provides the same (independent of $\gamma \in \Gamma$) stochastic process $\boldsymbol{S} = \{S_t : t \in [0,T]\}$.

4. The range of contingent claim prices. In our framework the problem of American contingent claim pricing can be formulated as a stochastic nonstationary optimal stopping time problem for a linear SDE (14), describing time evolution of a given risky asset $\boldsymbol{S}$ with given probability space $(\Omega, \mathcal{F}, \mathbb{P}^\gamma, \{\mathcal{F}_t\}_{t\in[0,T]})$ and fixed probability $\mathbb{P}^\gamma \in \mathcal{Q}$.

This means that for any fixed $(t,x) \in [0,T] \times \mathbb{R}_+$ we have at our disposal the solution $\big\{S_s \equiv S_s^{t,x}(\omega) : s \in [t,T]\big\}$ of the problem

$$S_t = x, \quad dS_s = S_{s-}(\mu^\gamma ds + \sigma dW_s^\gamma + U^\gamma_{N_s^{\lambda^\gamma}} dN_s^{\lambda^\gamma}), \quad s \in (t,T], \tag{15}$$

and we can define the viable (i.e. arbitrage free) *American contingent claim price* $V^\gamma(t,x) \equiv V_\zeta^\gamma(t,x)$ in the form of a *value function*

$$V_\zeta^\gamma(t,x) := \sup_{\tau\in\mathcal{T}(t,T)} e^{rt} \, \mathrm{E}^{\mathbb{P}^\gamma}(e^{-r\tau}\zeta(S_\tau^{t,x})), \tag{16}$$

where $\mathcal{T}(t,T)$ denotes the set of all stopping times $\tau = \tau(\omega)$ for the space $(\Omega, \mathcal{F}, \mathbb{P}^\gamma, \{\mathcal{F}_t\}_{t\in[0,T]})$, with values in $[t,T]$. Here $\zeta = \zeta(x)$ defined for $x \in \mathbb{R}_+$ is a *pay-off function*. We assume throughout the paper that ζ multiplied by $\exp(-\mu|\ln(x)|)$ is a bounded function with bounded first derivative on $\mathbb{R}_+$.

It is known that always $V^\gamma(t,x) \geq \zeta(x)$, and that the *optimal stopping time* (i.e. *optimal exercise time*) for the problem (16) is the random variable

$$\tau_*^{V^\gamma}(t,x;\omega) := \inf\{s \in [t,T] : V^\gamma(s, S_s^{t,x}(\omega)) = \zeta(S_s^{t,x}(\omega))\}. \tag{17}$$

REMARK 3. To obtain, for example, an American put option price it is enough to choose $\zeta(x) = (K-x)^+$ for a fixed positive striking price K. In [3], [14] one can find price surfaces $\{V^\gamma(t,x) : (t,x) \in [0,T] \times \mathbb{R}_+\}$ and statistical estimates of the densities of the corresponding optimal exercise times $\tau_*^{V^\gamma}(t,x) = \tau_*^{V^\gamma}(t,x;\omega)$ for the American put option corresponding to the optimal Schweizer equivalent martingale measure $\mathbb{P}^\gamma \in \mathcal{Q}$, obtained from computer solution of the underlying free boundary problem (i.e. backward parabolic variational inequality with integro-differential operator) by linear programming methods, and from numerical approximation and Monte Carlo simulations of solutions to SDE (15).

Here our main goal is to provide bounds for the function $V^\gamma = V^\gamma(t,x)$. In order to get them we need—by analogy with (15)–(17)—more information on the American claim prices in the classical Black–Scholes framework, given by the stochastic model

$$\widehat{S}_0 = S_0, \quad d\widehat{S}_t = \widehat{S}_t((r-\delta)dt + \sigma dW_t), \quad t \in (0,T].$$

In this setting there exist a unique equivalent martingale measure and a unique arbitrage free price, so—without notational changes—we can use

$$\widehat{S}_t = x, \quad d\widehat{S}_s = \widehat{S}_s((r-\delta)ds + \sigma dW_s), \quad s \in (t,T],$$

in order to define the American contingent claim price as a *Black–Scholes function* given by

$$\mathcal{G}(t,x) \equiv \mathcal{G}_\zeta(t,x) := \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\,\mathrm{E}(e^{-r\tau}\zeta(\widehat{S}^{t,x}_\tau)) \quad \text{for } t \in [0,T], \tag{18}$$

and the corresponding optimal stopping time $\tau^{\mathcal{G}}_*$ as

$$\tau^{\mathcal{G}}_* \equiv \tau^{\mathcal{G}}_*(t,x) := \inf\{s \in [t,T] : \mathcal{G}(s,\widehat{S}^{t,x}_s) = \zeta(\widehat{S}^{t,x}_s)\}.$$

Now, let us introduce on the space of $\mathcal{C}^{1,2}$-regular functions $u = u(t,x)$ the following *backward parabolic operator*

$$\mathcal{L}_{\mathcal{BS}}(u)(t,x) := \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2\,\frac{\partial^2 u}{\partial x^2}(t,x) + (r-\delta)x\,\frac{\partial u}{\partial x}(t,x), \tag{19}$$

with the degenerating elliptic part as $x \downarrow 0$.

It is well understood (for constructive details see [14], [15]) that the pay-off function $\mathcal{G} = \mathcal{G}(t,x)$ and the so-called *free boundary* $b^{\mathcal{G}} = b^{\mathcal{G}}(t)$ can be obtained as a solution of the backward parabolic variational inequality determined by the operator (19), and with the initial conditon (at time $t = T$)

$$\mathcal{G}(T,x) = \zeta(x).$$

The free boundary is a continuous function on $[0,T]$. Assuming that ζ is monotonic, we see that so is the free boundary. It separates the domain $[0,T]\times\mathbb{R}_+$ of $\mathcal{G}$ into the *continuation region* $\mathcal{C}^{\mathcal{G}}$ and the *stopping region* $\mathcal{S}^{\mathcal{G}}$:

$$\mathcal{C}^{\mathcal{G}} := \{(t,x) \in (0,T]\times\mathbb{R}_+ : \mathcal{G}(t,x) > \zeta(x)\},$$
$$\mathcal{S}^{\mathcal{G}} := \{(t,x) \in (0,T]\times\mathbb{R}_+ : \mathcal{G}(t,x) = \zeta(x)\}.$$

If ζ is convex, then the corresponding Black–Scholes function $\mathcal{G} = \mathcal{G}(t,x)$ has the following properties:

(i) $\mathcal{G}$ is convex with respect to x for all $t \in [0, t]$,
(ii) $\mathcal{G}$ has bounded partial derivative $\partial\mathcal{G}/\partial x$ for all $(t, x) \in (0, T] \times \mathbb{R}_+$ and (for some $C > 0$)

$$\left|\frac{\partial\mathcal{G}}{\partial x}(t, x)\right| \le C, \tag{20}$$

(iii) $\mathcal{G} \in \mathcal{C}^{1,2}$ and for all $(t, x) \in \mathcal{C}^{\mathcal{G}}$,

$$\mathcal{L}_{\mathcal{BS}}(e^{-rt}\mathcal{G}(t, x)) = 0, \tag{21}$$

(iv) for any $(t, x) \in [0, T] \times \mathbb{R}_+$ and any $y \in \mathbb{R}$,

$$\mathcal{A}(\mathcal{G}(t, x); y)) \ge 0. \tag{22}$$

The proofs of these properties are given in [6], [8] and [10].

For further use we need another operator, defined by

$$\mathcal{A}(u(t, x); y) := u(t, (1 + y)x) - u(t, x) - xy\,\frac{\partial u}{\partial x}(t, x), \tag{23}$$

from which we construct a few integral operators, e.g.

$$\mathcal{L}^{\gamma}_{\mathcal{A}}(u)(t, x) := \int_0^t\int_{\mathbb{R}} \mathcal{A}(u(s, x); y)\, \upsilon^{\gamma}(ds, dy).$$

We also need the random variable

$$\tau^{\mathcal{G}} \equiv \tau^{\mathcal{G}}(t, x) := \inf\{s \in [t, T] : \mathcal{G}(s, S_s^{t,x}) = \zeta(S_s^{t,x})\}, \tag{24}$$

which is a stopping time from $\mathcal{T}(t, T)$.

To end these preliminary remarks, notice that the solution to SDE (14) can be described explicitly as a stochastic exponential (see e.g. [11])

$$S_s = S_t \exp\Big(\mu^{\gamma}s + \sigma W_s^{\gamma} - \frac{1}{2}\sigma^2 s + \sum_{j=1}^{N_s^{\lambda^{\gamma}}} \ln(1 + U_j^{\gamma})\Big), \qquad s \in [t, T].$$

It is not difficult to see that for any $a \ge 1$,

$$\begin{aligned}\mathrm{E}^{\mathbb{P}^{\gamma}}(S_s)^a = (S_t)^a \exp\big(a\mu^{\gamma}s + \tfrac{1}{2}a^2\sigma^2 s^2 - \tfrac{1}{2}a\sigma^2 s\big)&\\ \times \exp(-\lambda^{\gamma}(1 - \mathrm{E}^{\mathbb{P}^{\gamma}}((1 + U_1^{\gamma})^a))s) < \infty,& \qquad s \in [t, T].\end{aligned} \tag{25}$$

Let us also recall the martingale properties of the most important stochastic integrals we need in the proofs.

PROPOSITION 1. *Let a stochastic point process identified with a Poisson random measure υ with compensator q belong to the class QL, i.e. be quasi left-continuous. Suppose a process $\{g(t, x) : (t, x) \in [0, \infty) \times \mathbb{R}\}$ is $\{\mathcal{F}_t\}$-predictable and $\mathrm{E}(\int_0^t \int_{\mathbb{R}} |g(s, y)|\, q(ds, dy)) < \infty$ for all $t > 0$. Then the process $\{\int_0^t \int_{\mathbb{R}} g(s, y)\, \widetilde{\upsilon}(ds, dy) : t \in [0, T]\}$ is an $\{\mathcal{F}_t\}$-martingale.*

PROPOSITION 2. *Let* $\mathbf{\Phi} = \{\Phi_t : t \in [0,T]\}$ *be an* $\{\mathcal{F}_t\}$*-adapted and measurable stochastic process such that* $\mathrm{E}(\int_0^t \Phi_s^2\, ds) < \infty$ *for all* $t > 0$*. Then the stochastic integral* $\{\int_0^t \Phi_s\, dW_s : t \in [0,T]\}$ *is an* $\{\mathcal{F}_t\}$*-martingale.*

Now we are in a position to prove the main theorems providing bounds and the range of viable prices for American type financial derivatives.

THEOREM 1. *Let* $\mathbb{P}^\gamma \in \mathcal{Q}$ *be a martingale measure equivalent to the probability* $\mathbb{P}$, *and let* $V^\gamma(t, S_t)$ *be the arbitrage free price of an American type derivative asset defined by* (16). *Suppose the pay-off function* ζ *is convex and such that* $\zeta(0) = 0$, $0 \le \zeta(x) \le x$, *for* $x \in \mathbb{R}_+$. *Then for all* $\boldsymbol{\gamma} \in \Gamma$,

$$\mathcal{G}(t, S_t) \le V^\gamma(t, S_t) \le S_t \quad \textit{for any } t \in [0,T]. \tag{26}$$

Proof. The main task is to prove the inequality $\mathcal{G}(t, S_t) \le V^\gamma(t, S_t)$. The argument is of rather technical nature, but it is based on some, not obviously known facts from the theory of point processes, semimartingales and semimartingale stochastic integrals.

The main idea is to make use of the stopping time (24), i.e. to work with realisations (trajectories) of the stochastic process $\boldsymbol{S} = \{(s, S_s) : s \in [t,T]\}$, locating them in the sets $\mathcal{C}^{\mathcal{G}}$ and $\mathcal{S}^{\mathcal{G}}$.

If $(t, S_t) \in \mathcal{C}^{\mathcal{G}}$, i.e. $t < \tau^{\mathcal{G}}$, then we make use of the equality $\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) = \zeta(\tau^{\mathcal{G}})$. Let us begin with the obvious estimate

$$\begin{aligned} V^\gamma(t, S_t) &= \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^\gamma}(e^{-r\tau}\zeta(S_\tau) \mid S_t) \\ &\ge e^{rt}\, \mathrm{E}^{\mathbb{P}^\gamma}(e^{\tau^{\mathcal{G}}}\zeta(S_{\tau^{\mathcal{G}}}) \mid S_t) = e^{rt}\, \mathrm{E}^{\mathbb{P}^\gamma}(e^{\tau^{\mathcal{G}}}\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) \mid S_t). \end{aligned}$$

Then, applying the Itô formula (Theorem 33 in [11]) to the process $\boldsymbol{S}$ and the $\mathcal{C}^{1,2}$-regular function $f(t,x) := e^{-rt}\mathcal{G}(t,x)$, we arrive at

$$\begin{aligned} &e^{-r\tau^{\mathcal{G}}}\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) - e^{-rt}\mathcal{G}(t, S_t) \\ &= \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial t}(e^{-rs}\mathcal{G}(s, S_{s-}))\, ds + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s, S_{s-}))\, dS_s \\ &\quad + \frac{1}{2}\int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial^2}{\partial t^2}(e^{-rs}\mathcal{G}(s, S_{s-}))\, d[I,I]_s^c + \frac{1}{2}\int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial^2}{\partial x^2}(e^{-rs}\mathcal{G}(s, S_{s-}))\, d[S,S]_s^c \\ &\quad + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial^2}{\partial t \partial x}(e^{-rs}\mathcal{G}(s, S_{s-}))\, d[I,S]_s^c \\ &\quad + \sum_{t < s \le \tau^{\mathcal{G}}} \Big\{ \Delta(e^{-rs}\mathcal{G}(s, S_s)) - \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s, S_{s-}))\Delta S_s \Big\}, \end{aligned} \tag{27}$$

where $\boldsymbol{I} = \{I_s\}_{s\in[0,T]}$ stands for the function (deterministic process) of the form $I_s := s$.

It is not difficult to check that that $[I,I]^c \equiv 0$ and $[I,S]^c \equiv 0$, so from representation (13) of $\boldsymbol{S}$ we get $\Delta S_s = S_{s-}\int_{\mathbb{R}} y\,\upsilon^\gamma(ds,dy)$, and then

$$d[S,S]^c_s = d[S,S]_s - (\Delta S_s)^2 = d[S,S]_s - \Big(S_{s-}\int_{\mathbb{R}} y\,\upsilon^\gamma(ds,dy)\Big)^2,$$

$$d[S,S]_s = \sigma^2 S^2_{s-}\,ds + \Big(S_{s-}\int_{\mathbb{R}} y\,\upsilon^\gamma(ds,dy)\Big)^2,$$

$$d[S,S]^c_s = \sigma^2 S^2_{s-}\,ds.$$

Thus, a more explicit form of (27) is as follows:

$$\begin{aligned}(28)\qquad & e^{-r\tau^{\mathcal{G}}}\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) - e^{-rt}\mathcal{G}(t,S_t)\\
&= \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial t}(e^{-rs}\mathcal{G}(s,S_{s-}))\,ds + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s,S_{s-}))S_{s-}(r-\delta)\,ds\\
&\quad + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s,S_{s-}))S_{s-}\sigma\,dW^\gamma_s\\
&\quad + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s,S_{s-}))S_{s-}\int_{\mathbb{R}} y\,\widetilde{\upsilon}^\gamma(ds,dy)\\
&\quad + \frac{1}{2}\int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial^2}{\partial x^2}(e^{-rs}\mathcal{G}(s,S_{s-}))\sigma^2 S^2_{s-}\,ds\\
&\quad + \sum_{t<s\le\tau^{\mathcal{G}}}\Big\{\Delta(e^{-rs}\mathcal{G}(s,S_s)) - \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s,S_{s-}))\Delta S_s\Big\}.\end{aligned}$$

Noticing that $\mathcal{G}(s, S_{s-}+\Delta S_s) = \int_{\mathbb{R}} \mathcal{G}(s, S_{s-}+S_{s-}y)\,\upsilon^\gamma(ds,dy)$, we get

$$\begin{aligned}&\Delta(e^{-rs}\mathcal{G}(s,S_s)) - \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s,S_{s-}))\Delta S_s\\
&\qquad = e^{-rs}\mathcal{G}(s,S_{s-}+\Delta S_s) - e^{-rs}\mathcal{G}(s,S_{s-}) - e^{-rs}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})\Delta S_s\\
&\qquad = e^{-rs}\int_{\mathbb{R}} \mathcal{G}(s,S_{s-}+S_{s-}y)\,\upsilon^\gamma(ds,dy) - e^{-rs}\int_{\mathbb{R}}\mathcal{G}(s,S_{s-})\,\upsilon^\gamma(ds,dy)\\
&\qquad\quad - e^{-rs}\int_{\mathbb{R}}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})S_{s-}y\,\upsilon^\gamma(ds,dy)\\
&\qquad = e^{-rs}\int_{\mathbb{R}}\mathcal{A}(\mathcal{G}(s,S_{s-});y)\,\upsilon^\gamma(ds,dy).\end{aligned}$$

From (19) and (21) we derive

$$
\begin{aligned}
0 &= \int_{t+}^{\tau^{\mathcal{G}}} \mathcal{L}_{\mathcal{BS}}(e^{-rs}\mathcal{G}(s, S_{s-}))\,ds \\
&= \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial t}(e^{-rs}\mathcal{G}(s, S_{s-}))\,ds + \int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial}{\partial x}(e^{-rs}\mathcal{G}(s, S_{s-}))S_{s-}(r-\delta)\,ds \\
&\quad + \frac{1}{2}\int_{t+}^{\tau^{\mathcal{G}}} \frac{\partial^2}{\partial x^2}(e^{-rs}\mathcal{G}(s, S_{s-}))\sigma^2 S_{s-}^2\,ds.
\end{aligned}
$$

Thus (27) takes the form

$$
\begin{aligned}
(29)\qquad & e^{-r\tau^{\mathcal{G}}}\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) - e^{-rt}\mathcal{G}(t, S_t) \\
&= \int_{t+}^{\tau^{\mathcal{G}}} e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\Big(\sigma\,dW_s^{\gamma} + \int_{\mathbb{R}} y\,\widetilde{\upsilon}^{\gamma}(ds, dy)\Big) \\
&\quad + \int_{t+}^{\tau^{\mathcal{G}}} e^{-rs}\int_{\mathbb{R}} \mathcal{A}(\mathcal{G}(s, S_{s-}); y)\,\upsilon^{\gamma}(ds, dy) \\
&= \int_{t+}^{\tau^{\mathcal{G}}} e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma\,dW_s^{\gamma} \\
&\quad + \int_{t+}^{\tau^{\mathcal{G}}}\int_{\mathbb{R}} \Big(e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})y + e^{-rs}\mathcal{A}\big(\mathcal{G}(s, S_{s-}); y\big)\,\widetilde{\upsilon}^{\gamma}(ds, dy)\Big) \\
&\quad + \int_{t+}^{\tau^{\mathcal{G}}} e^{-rs}\int_{\mathbb{R}} \mathcal{A}(\mathcal{G}(s, S_{s-}); y)\lambda^{\gamma}\,m^{\gamma}(dy)\,ds.
\end{aligned}
$$

Now we show that the first two integrals on the right hand side above are $\mathbb{P}^{\gamma}$-martingales. Indeed, the process $\{e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma : s \in [0,T]\}$ is adapted and measurable. Therefore, to prove that $\big\{\int_{t+}^{z} e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma\,dW_s^{\gamma} : z \in [t,T]\big\}$ is a $\mathbb{P}^{\gamma}$-martingale it is enough to check that for any $z \in (t,T]$,

$$
\mathrm{E}^{\mathbb{P}^{\gamma}}\bigg(\int_{t+}^{z}\Big(e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma\Big)^2\,ds\bigg) < \infty
$$

(see Proposition 2). The estimate

$$
\Big(e^{-rs}S_{s-}\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma\Big)^2 \le C^2\sigma^2 S_{s-}^2
$$

follows directly from (20), so

$$\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}\Big(e^{-rs}S_{s-}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})\sigma\Big)^2 ds\Big) \le C^2\sigma^2\,\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}S_{s-}^2\,ds\Big) = C^2\sigma^2\int_{t+}^{z}\mathrm{E}^{\mathbb{P}^\gamma}(S_{s-}^2)\,ds.$$

From (25) it follows that the function $s\mapsto \mathrm{E}^{\mathbb{P}^\gamma}(S_{s-}^2)$ is integrable on $(t,z]$. The assumptions of Proposition 2 are satisfied for all $z\in(t,T]$. Therefore, the process $\big\{\int_{t+}^{z}e^{-rs}S_{s-}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})\sigma\,dW_s^\gamma : z\in[t,T]\big\}$ is a martingale with respect to $\mathbb{P}^\gamma$.

The compensator of the martingale measure υ^γ is given by the formula $q^\gamma(ds,dy)=\lambda^\gamma m^\gamma(dy)ds$, where m^γ stands for a positive and finite measure on $\mathbb{R}$. So, the point process $(\boldsymbol{N}^{\lambda^\gamma},\{U_1^\gamma,U_2^\gamma,\dots\})$ belongs to the class of QL processes (see [5, Chapter II]). Let us define

$$M(s,y) := e^{-rs}S_{s-}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})y + e^{-rs}\mathcal{A}\big(\mathcal{G}(s,S_{s-});y\big).$$

The process $\boldsymbol{M}=\{M(s,y):(s,y)\in[0,T]\times\mathbb{R}\}$ is predictable. Therefore, in order to show that the stochastic process given by the stochastic integral $\{\int_{t+}^{z}\int_{\mathbb{R}}M(s,y)\,\widetilde{\upsilon}^\gamma(ds,dy): z\in(t,T]\}$ is a martingale for $\mathbb{P}^\gamma$, it is enough to prove that for any $z\in(t,T]$ we have

$$\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}\int_{\mathbb{R}}|M(s,y)|\,q^\gamma(ds,dy)\Big)<\infty$$

(see Proposition 1). Now, directly from the definition (23) of the operator $\mathcal{A}$ we get the estimate

$$\begin{aligned}|e^{rs}M(s,y)| &= \Big|S_{s-}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})y+\mathcal{A}(\mathcal{G}(s,S_{s-});y)\Big|\\ &= \Big|S_{s-}\frac{\partial\mathcal{G}}{\partial x}(s,S_{s-})y+\mathcal{G}(s,(1+y)S_{s-})-\mathcal{G}(s,S_{s-})-S_{s-}y\,\frac{\partial\mathcal{G}}{\partial x}(t,S_{s-})\Big|\\ &\le e^{rs}|\mathcal{G}(s,(1+y)S_{s-})-\mathcal{G}(s,S_{s-})|\le CS_{s-}|y|\end{aligned}$$

for some positive constant C. Thus we obtain

$$\begin{aligned}\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}\int_{\mathbb{R}}|M(s,y)|\,q^\gamma(ds,dy)\Big) &\le C\,\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}\int_{\mathbb{R}}S_{s-}|y|\lambda^\gamma\,m^\gamma(dy)\,ds\Big)\\ &= C\lambda^\gamma\,\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{z}S_{s-}\,ds\Big)\int_{\mathbb{R}}|y|f^\gamma(y)\,dy\\ &= C\lambda^\gamma\int_{t+}^{z}\mathrm{E}^{\mathbb{P}^\gamma}(S_{s-})\,ds\int_{\mathbb{R}}|y|f^\gamma(y)\,dy.\end{aligned}$$

Because the density $f^\gamma=f^\gamma(y)$ is positive only on a finite interval in $\mathbb{R}$, the integral $\int_{\mathbb{R}}|y|f^\gamma(y)\,dy$ is finite and the function $s\mapsto\mathrm{E}^{\mathbb{P}^\gamma}(S_{s-})$ is integrable on $(t, z]$. This means that the assumption of Proposition 1 is satisfied and the process $\{\int_{t+}^{z}\int_{\mathbb{R}} M(s,y)\,\widetilde{v}^{\gamma}(ds,dy) : z \in (t,T]\}$ is a martingale with respect to the probability $\mathbb{P}^{\gamma}$.

Concluding the investigation of both stochastic integrals above, which start from 0 at time $t \in [0,T]$, we see that

$$\mathrm{E}^{\mathbb{P}^{\gamma}}\Big(\int\limits_{t+}^{\tau^{\mathcal{G}}} e^{-rs} S_{s-}\,\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})\sigma\, dW_s^{\gamma}\,\Big|\, S_t\Big) = 0,$$

$$\mathrm{E}^{\mathbb{P}^{\gamma}}\bigg(\int\limits_{t+}^{\tau^{\mathcal{G}}}\int\limits_{\mathbb{R}}\bigg(e^{-rs} S_{s-}\,\frac{\partial \mathcal{G}}{\partial x}(s, S_{s-})y + e^{-rs}\mathcal{A}(\mathcal{G}(s,S_{s-});y)\bigg)\widetilde{v}^{\gamma}(ds,dy)\bigg|S_t\bigg) = 0.$$

Then, coming back to (29) we obtain

$$\begin{aligned}
&\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau^{\mathcal{G}}}\mathcal{G}(\tau^{\mathcal{G}}, S_{\tau^{\mathcal{G}}}) \,|\, S_t) - e^{-rt}\mathcal{G}(t,S_t) \\
&\qquad = \mathrm{E}^{\mathbb{P}^{\gamma}}\Big(\int\limits_{t+}^{\tau^{\mathcal{G}}} e^{-rs}\int\limits_{\mathbb{R}}\mathcal{A}(\mathcal{G}(s,S_{s-});y)\lambda^{\gamma}\, m^{\gamma}(dy)\, ds\,\Big|\, S_t\Big) \ge 0.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
V^{\gamma}(t,S_t) &\ge e^{rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau^{\mathcal{G}}}\zeta(S_{\tau^{\mathcal{G}}})\,\big|\, S_t) \\
&= \mathcal{G}(t,S_t) + e^{rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}\Big(\int\limits_{t+}^{\tau^{\mathcal{G}}} e^{-rs}\int\limits_{\mathbb{R}}\mathcal{A}\big(\mathcal{G}(s,S_{s-});y\big)\lambda^{\gamma}\, m^{\gamma}(dy)\, ds\,\Big|\, S_t\Big) \\
&\ge \mathcal{G}(t,S_t),
\end{aligned}$$

which ends the proof of the first inequality in (26) for the case of $(t,S_t) \in \mathcal{C}^{\mathcal{G}}$.

If $(t,S_t) \in \mathcal{S}^{\mathcal{G}}$, i.e. $t \ge \tau^{\mathcal{G}}$, then $\mathcal{G}(t,S_t) = \zeta(S_t)$, and we get

$$\begin{aligned}
V^{\gamma}(t,S_t) &= \sup_{\tau\in\mathcal{T}(t,T)} e^{rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\zeta(S_{\tau})\,|\, S_t) \\
&\ge e^{rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-rt}\zeta(S_t)\,|\, S_t) = \zeta(S_t) = \mathcal{G}(t,S_t).
\end{aligned}$$

It is easy to get the upper bound in (26). Thanks to the inequality $\zeta(x) \le x$, we obtain

$$\begin{aligned}
V^{\gamma}(t,S_t) &= \sup_{\tau\in\mathcal{T}(t,T)} e^{rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\zeta(S_{\tau})\,|\, S_t) \\
&\le \sup_{\tau\in\mathcal{T}(t,T)} e^{-\delta\tau+rt}\,\mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau+\delta\tau}S_{\tau}\,|\, S_t) = \sup_{\tau\in\mathcal{T}(t,T)} e^{-\delta\tau+\delta t}S_t = S_t.
\end{aligned}$$

This completes the proof. ■

THEOREM 2. *Let $\mathbb{P}^{\gamma} \in \mathcal{Q}$ be a martingale measure equivalent to the probability $\mathbb{P}$, and let $V^{\gamma}(t,S_t)$ be the arbitrage free price of an American type derivative asset defined by* (16). *Suppose the pay-off function ζ is convex and* *such that* $0 \le \zeta(x) \le \zeta(0)$ *for* $x \in \mathbb{R}_+$. *Then for all* $\boldsymbol{\gamma} \in \Gamma$,

$$\mathcal{G}(t, S_t) \le V^{\gamma}(t, S_t) \le \zeta(0) \quad \textit{for any } t \in [0, T]. \tag{30}$$

Proof. In comparison with Theorem 1 the only thing to do is to prove the upper bound in (30).

From the inequality $\zeta(x) \le \zeta(0)$ we get

$$\begin{aligned} V^{\gamma}(t, S_t) &= \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\zeta(S_\tau) \mid S_t) \\ &\le \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\zeta(0) \mid S_t) = \sup_{\tau \in \mathcal{T}(t,T)} e^{rt - r\tau}\zeta(0) = \zeta(0). \quad \blacksquare \end{aligned}$$

In [1] it is shown that the range of prices in question is a full interval (bounded, as we have seen above). Now, in view of Remark 2, we prove that the left end of this interval is identified exactly, and can be approached with constant indices $\gamma \in (-1, \infty)$.

THEOREM 3. *Let* $V^{\gamma}(t, S_t)$ *be the price of an American type derivative asset, defined by* (16) *for any constant* $\gamma \in (-1, \infty)$. *Suppose the pay-off function* $\zeta = \zeta(x)$ *is convex. Then*

$$\lim_{\gamma \to -1} V^{\gamma}(t, S_t) = \mathcal{G}(t, S_t).$$

Proof. From Theorem 1 we know that

$$\mathcal{G}(t, S_t) \le V^{\gamma}(t, S_t).$$

It is obvious that $\zeta(S_t) \le \mathcal{G}(t, S_t)$. Otherwise we would have an arbitrage opportunity on the market. So, we have

$$V^{\gamma}(t, S_t) = \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\zeta(S_\tau) \mid S_t) \le \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\mathcal{G}(\tau, S_\tau) \mid S_t).$$

Applying the Itô formula for any $\tau \in \mathcal{T}(t, T)$, we get

$$\begin{aligned} &e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}(e^{-r\tau}\mathcal{G}(\tau, S_\tau) | S_t) \\ &\quad = \mathcal{G}(t, S_t) + e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}\Big(\int_{t+}^{\tau} e^{-rs} \int_{\mathbb{R}} \mathcal{A}(\mathcal{G}(s, S_{s-}); y) \lambda^{\gamma}\, m^{\gamma}(dy)\, ds \,\Big|\, S_t \Big). \end{aligned}$$

It follows immediately from (12) that $\lambda^{\gamma} = \lambda(1 + \gamma)$ and $m^{\gamma}(dy) = f^{\gamma}(y)dy$ for γ constant, so

$$\begin{aligned} V^{\gamma}(t, S_t) &\le \mathcal{G}(t, S_t) \\ &+ \lambda(1+\gamma) \sup_{\tau \in \mathcal{T}(t,T)} e^{rt}\, \mathrm{E}^{\mathbb{P}^{\gamma}}\Big(\int_{t+}^{\tau} e^{-rs} \int_{\mathbb{R}} \mathcal{A}(\mathcal{G}(s, S_{s-}); y) f^{\gamma}(y)\, dy\, ds \,\Big|\, S_t \Big). \end{aligned} \tag{31}$$

It is easy to see (repeating an argument from the proof of Theorem 1) that $|\mathcal{A}(\mathcal{G}(s,S_{s-});y)| \le 2CS_{s-}|y|$, and then

$$\begin{aligned}\Big|e^{rt}\,\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{T} e^{-rs}\int_{\mathbb{R}} \mathcal{A}(\mathcal{G}(s,S_{s-});y) f^\gamma(y)\,dy\,ds\,\Big|\,S_t\Big)\Big| \\ &\le 2C\int_{\mathbb{R}} |y| f^\gamma(y)\,dy\,\mathrm{E}^{\mathbb{P}^\gamma}\Big(\int_{t+}^{T} S_{s-}ds\,\Big|\,S_t\Big) \\ &= 2C\int_{\mathbb{R}} |y| f^\gamma(y)\,dy \int_{t+}^{T} \mathrm{E}^{\mathbb{P}^\gamma}(S_{s-}\mid S_t)\,ds < \infty.\end{aligned}$$

Thus (31) yields $\lim_{\gamma\to-1} V^\gamma(t,S_t) = \mathcal{G}(t,S_t)$. ∎

5. Conclusions. It is quite obvious that the range of American contingent claim prices in the diffusion model with jumps of random magnitude presented and characterized in this paper is much too wide in comparison with real-life data from money markets. When performing appropriate computer experiments, it becomes evident that the "reasonable" range of prices would be much smaller, e.g. comparable with the smallest interval containing Merton and Schweizer prices (see [3], [14], or get some results with the help of the SDE-Solver software package from [7]). It is a challenging task to find and characterize subclasses of equivalent martingale measures $\mathbb{P}^\gamma$ in $\mathcal{Q}$ which would be appropriate for applications.

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