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LONG TIME ESTIMATE OF SOLUTIONS TO 3D NAVIER–STOKES EQUATIONS COUPLED WITH HEAT CONVECTION

Abstract. We examine the Navier–Stokes equations with homogeneous slip boundary conditions coupled with the heat equation with homogeneous Neumann conditions in a bounded domain in $\mathbb{R}^3$. The domain is a cylinder along the x_3 axis. The aim of this paper is to show long time estimates without assuming smallness of the initial velocity, the initial temperature and the external force. To prove the estimate we need however smallness of the L_2 norms of the x_3-derivatives of these three quantities.

1. Introduction. The aim of this paper is to derive a long time a priori estimate for some initial-boundary value problem for a system of the Navier–Stokes equations coupled with the heat equation. We assume the slip boundary conditions for the Navier–Stokes equations and the Neumann condition for the heat equation. We examine the problem in a straight finite cylinder. To obtain the estimate we follow the ideas from [7, 8, 10] and the solution considered remains close to a two-dimensional solution. The estimate is the first and most important step in proving the existence of solutions to the problem (see (1.1)) by the Leray–Schauder fixed point theorem (see the next paper of the authors [9]).

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We consider the following problem:

$$
(1.1)\qquad
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = \alpha(\theta) f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&\theta_{,t} + v\cdot\nabla\theta - \varkappa\Delta\theta = 0 && \text{in } \Omega^T,\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S^T = S\times(0,T),\\
&\bar n\cdot\bar v = 0 && \text{on } S^T,\\
&\bar n\cdot\nabla\theta = 0 && \text{on } S^T,\\
&v|_{t=0} = v(0),\quad \theta|_{t=0} = \theta(0) && \text{in } \Omega,
\end{aligned}
$$

where $x=(x_1,x_2,x_3)$ denote the Cartesian coordinates, $\Omega\subset\mathbb{R}^3$ is a cylindrical type domain parallel to the x_3 axis with arbitrary cross section, $S=\partial\Omega$, $v=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$ is the velocity of the fluid motion, $p=p(x,t)\in\mathbb{R}^1$ the pressure, $\theta=\theta(x,t)\in\mathbb{R}_+$ the temperature, $f=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ the external force field, $\bar n$ is the unit outward normal vector to the boundary S, $\bar\tau_\alpha$, $\alpha=1,2$, are tangent vectors to S and the dot denotes the scalar product in $\mathbb{R}^3$. We define the stress tensor by

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-p\mathbb{I},$$

where ν is the constant viscosity coefficient, $\mathbb{I}$ is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally $\varkappa$ is a positive heat conductivity coefficient.

We assume that $S=S_1\cup S_2$, where S_1 is the part of the boundary which is parallel to the x_3 axis and S_2 is perpendicular to that axis. More precisely,

$$
\begin{aligned}
S_1 &= \{x\in\mathbb{R}^3 : \varphi_0(x_1,x_2)=c_*,\ -b<x_3<b\},\\
S_2 &= \{x\in\mathbb{R}^3 : \varphi_0(x_1,x_2)<c_*,\ x_3 \text{ is equal either to } -b \text{ or } b\},
\end{aligned}
$$

where b,c_* are given positive numbers and $\varphi_0(x_1,x_2)$ describes a sufficiently smooth closed curve in the plane $x_3=\text{const}$. We can assume $\bar\tau_1=(\tau_{11},\tau_{12},0)$, $\bar\tau_2=(0,0,1)$ and $\bar n=(\tau_{12},-\tau_{11},0)$ on S_1. Assume that $\alpha\in C^2(\mathbb{R})$ and Ω^T satisfies the weak l-horn condition, where $l=(2,2,2,1)$ (see [2, Ch. 2, Sect. 8]).

To apply the simpler version of the Korn inequality we assume that Ω is not axially symmetric (see Lemma 2.1).

Assume that $\|\theta(0)\|_{L_\infty(\Omega)}<\infty$. Define

$$a:[0,\infty)\to[0,\infty),\qquad a(x)=\sup\{|\alpha(y)|+|\alpha'(y)| : |y|\le x\}$$

and assume that

$$(1.2)\qquad a(\theta(x))\le c_1,$$

where $c_1 = a(\|\theta(0)\|_{L_\infty(\Omega)})$. The inequality (1.2) is justified in view of Lemma 2.3, Remark 2.4 and the properties of the function $a(x)$. Let σ, ϱ be such that $5/3 < \sigma < \infty$, $5/3 < \varrho < \infty$, $5/\varrho - 5/\sigma < 1$.

Now we formulate the main result of this paper. Let

(1.3)
$$g = f_{,x_3}, \quad h = v_{,x_3}, \quad q = p_{,x_3}, \quad \vartheta = \theta_{,x_3}, \quad \chi = (\operatorname{rot} v)_3, \quad F = (\operatorname{rot} f)_3.$$

Assume the following conditions hold for all $t \le T$:

1. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1c_0\|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1\|F\|_{L_2(0,t;L_{6/5}(\Omega))}$ $+c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)} + \psi(c_0)$ $+ c_0^2(c_1\|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \le k_1 < \infty$,
2. $\|f\|_{L_2(0,t;L_3(\Omega))} \le k_2 < \infty$,
3. $\|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)} \le k_3 < \infty$,
4. $c_1\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2k_2^2} k_1 + c_1\|g\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}$ $\le k_4 < \infty$,
5. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)}$ $\le d < \infty$,
6. $c_1 + \|f\|_{L_\varrho(\Omega^t)} + \|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} \le k_5 < \infty$,

where c_0 is the constant from Lemma 2.3, $\psi(c_0)$ is the increasing function from Lemma 3.3 and $k_1, \ldots, k_5$ are constants.

Main Theorem. *For every fixed T, and given positive constants k_1–k_5, c_0, c_1 under the above assumptions* 1–6, *if the constant d in condition* 5 *is small enough, then there exists $B = B(k_1, \ldots, k_5, c_0, c_1) < \infty$ such that for any strong solution (v, p, θ) to problem* (1.1) *we have*

$$\|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} + \|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \le B, \tag{1.4}$$

$$\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} + \|\vartheta\|_{W_\sigma^{2,1}(\Omega^t)} \le B, \tag{1.5}$$

for all $t \le T$.

In the next paper [6], we use this result to prove the long time existence of regular solutions to (1.1).

Finally, we underline that a global a priori estimate guaranteeing existence of global regular solutions to (1.1) (see [9]) is possible under the restriction that the quantity d from assumption 5 is sufficiently small. This kind of assumption in the case of the Navier–Stokes equations only appeared in [7, 10]. Problem (1.1) in the case of inflow-outflow was generalized by Kacprzyk in [3, 4]. Papers [3, 4] base on [13], where the inflow-outflow problem for the Navier–Stokes motions in a cylindrical pipe is considered.

2. Preliminaries. In this section we introduce notation and basic estimates for weak solutions to problem (1.1).

2.1. Notation. We use isotropic and anisotropic Lebesgue spaces: $L_p(Q)$, $Q \in \{\Omega^T, S^T, \Omega, S\}$, $p \in [1,\infty]$, and $L_q(0,T;L_p(Q))$, $Q \in \{\Omega, S\}$, $p,q \in [1,\infty]$; and Sobolev spaces

$$W_q^{s,s/2}(Q^T), \quad Q \in \{\Omega, S\},\ q \in [1,\infty],\ s \in \mathbb{N} \cup \{0\},\ s \text{ even},$$

with the norm

$$\|u\|_{W_q^{s,s/2}(Q^T)} = \Big(\sum_{|\alpha|+2a \le s} \int_{Q^T} |D_x^\alpha \partial_t^a u|^q \, dx\, dt \Big)^{1/q},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $a, \alpha_i \in \mathbb{N} \cup \{0\}$.

In the case $q = 2$,

$$H^s(Q) = W_2^s(Q), \qquad H^{s,s/2}(Q^T) = W_2^{s,s/2}(Q^T), \qquad Q \in \{\Omega, S\}.$$

Moreover, $L_2(Q) = H^0(Q)$, $L_p(Q) = W_p^0(Q)$, $L_p(Q^T) = W_p^{0,0}(Q^T)$.

We define a space natural for the study of weak solutions to the Navier–Stokes and parabolic equations:

$$V_2^k(\Omega^T) = \Big\{ u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|u\|_{H^k(\Omega)} + \Big(\int_0^T \|\nabla u\|_{H^k(\Omega)}^2 \, dt \Big)^{1/2} < \infty \Big\}.$$

2.2. Weak solutions. By a *weak solution* to problem (1.1) we mean a pair $v \in V_2^0(\Omega^T)$, $\theta \in V_2^0(\Omega^T) \cap L_\infty(\Omega^T)$ satisfying the integral identities

$$(2.1)\qquad -\int_{\Omega^T} v \cdot \varphi_{,t} \, dx\, dt + \int_{\Omega^T} v \cdot \nabla v \cdot \varphi \, dx\, dt + \frac{\nu}{2} \int_{\Omega^T} \mathbb{D}(v) \cdot \mathbb{D}(\varphi) \, dx\, dt = \int_{\Omega^T} \alpha(\theta) f \cdot \varphi \, dx\, dt + \int_\Omega v(0)\varphi(0)\, dx,$$

$$(2.2)\qquad -\int_{\Omega^T} \theta \psi_{,t} \, dx\, dt + \int_{\Omega^T} v \cdot \nabla \theta \psi \, dx\, dt + \varkappa \int_{\Omega^T} \nabla\theta \cdot \nabla\psi \, dx\, dt = \int_\Omega \theta(0)\psi(0)\, dx,$$

for all $\varphi, \psi \in W_2^{1,1}(\Omega^T) \cap L_5(\Omega^T)$ such that $\varphi(T) = 0$, $\psi(T) = 0$, $\operatorname{div} \varphi = 0$, $\varphi \cdot \bar{n}|_S = 0$.

Lemma 2.1 (Korn inequality, see [12]). *Assume that*

$$E_\Omega(v) = \|\mathbb{D}(v)\|_{L_2(\Omega)}^2 < \infty, \qquad v \cdot \bar{n}|_S = 0, \qquad \operatorname{div} v = 0.$$

If Ω is not axially symmetric there exists a constant c_1 independent of v such that

$$(2.3)\qquad \|v\|_{H^1(\Omega)}^2 \le c_1 E_\Omega(v).$$

If Ω is axially symmetric, and $\eta = (-x_2, x_1, 0)$, $\alpha = \int_\Omega v \cdot \eta\, dx$, then there exists a constant c_2 independent of v such that

$$\|v\|^2_{H^1(\Omega)} \le c_2(E_\Omega(v) + |\alpha|^2). \tag{2.4}$$

Let us consider the problem

$$\begin{aligned} & h_{,t} - \operatorname{div} \mathbb{T}(h, q) = f && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & \bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ & h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega. \end{aligned} \tag{2.5}$$

LEMMA 2.2. *Let $f \in L_p(\Omega^T)$, $h(0) \in W_p^{2-2/p}(\Omega)$, $S_1 \in C^2$, $1 < p < \infty$. Then there exists a solution to problem* (2.5) *such that $h \in W_p^{2,1}(\Omega^T)$, $\nabla q \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that*

$$\|h\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)} \le c(\|f\|_{L_p(\Omega^T)} + \|h(0)\|_{W_p^{2-2/p}(\Omega)}). \tag{2.6}$$

The proof is similar to the proof in [1].

LEMMA 2.3. *Assume $v(0) \in L_2(\Omega)$, $\theta(0) \in L_\infty(\Omega)$, $f \in L_2(0, T; L_{6/5}(\Omega))$, $T < \infty$. Assume that Ω is not axially symmetric. Assume that there exist constants θ_*, θ^* such that $\theta_* < \theta^*$ and $\theta_* \le \theta_0(x) \le \theta^*$, $x \in \Omega$. Then there exists a weak solution to problem* (1.1) *such that $(v, \theta) \in V_2^0(\Omega^T) \times V_2^0(\Omega^T)$, $\theta \in L_\infty(\Omega^T)$ and*

$$\theta_* \le \theta(x, t) \le \theta^*, \qquad (x, t) \in \Omega^T, \tag{2.7}$$

and there exist positive constants c, c_0 independent of v and θ such that

$$\|v\|_{V_2^0(\Omega^T)} \le c(a(\|\theta_0\|_{L_\infty(\Omega)})\|f\|_{L_2(0,T;L_{6/5}(\Omega))} + \|v_0\|_{L_2(\Omega)}) \le c_0, \tag{2.8}$$

$$\|\theta\|_{V_2^0(\Omega^T)} \le c\|\theta_0\|_{L_2(\Omega)} \le c_0. \tag{2.9}$$

Proof. Estimate (2.7) follows from standard considerations (see [8, Lemmas 3.1, 3.2]). Estimates (2.8), (2.9) follow formally from $(1.1)_{1,3}$ by multiplying them by v and θ, respectively, integrating over Ω and $(0, t)$, $t \in (0, T)$, employing (2.7), $(1.1)_2$ and using the boundary and initial conditions $(1.1)_{4-7}$. Existence can be shown in the same way as in [5, Ch. 3, Sect. 1–5]. This concludes the proof. ∎

REMARK 2.4. If $\theta(0) \ge 0$, then $\theta(t) \ge 0$ for $t \ge 0$.

2.3. Auxiliary problems. To prove the existence of global regular solutions we recall the quantities introduced in (1.3),

$$h = v_{,x_3}, \qquad q = p_{,x_3}, \qquad g = f_{,x_3}, \qquad \vartheta = \theta_{,x_3}.$$

Differentiating $(1.1)_{1,2,4,5}$ with respect to x_3 and using [10, 13] yields

$$
(2.10)\qquad
\begin{aligned}
&h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + \alpha_\theta \vartheta f + \alpha g && \text{in } \Omega^T,\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&\bar n\cdot h = 0,\quad \bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\beta = 0,\quad \beta = 1,2, && \text{on } S_1^T,\\
&h_i = 0,\quad i=1,2,\quad h_{3,x_3} = 0 && \text{on } S_2^T,\\
&h|_{t=0} = h(0) && \text{in } \Omega.
\end{aligned}
$$

Let q and f_3 be given, Then $w = v_3$ is a solution to the problem

$$
(2.11)\qquad
\begin{aligned}
&w_{,t} + v\cdot\nabla w - \nu\Delta w = -q + \alpha(\theta) f_3 && \text{in } \Omega^T,\\
&\bar n\cdot\nabla w = 0 && \text{on } S_1^T,\\
&w = 0 && \text{on } S_2^T,\\
&w|_{t=0} = w(0) && \text{in } \Omega.
\end{aligned}
$$

Let $F = (\operatorname{rot} f)_3$, h, v, w be given. Then $\chi = (\operatorname{rot} v)_3$ is a solution to the problem (see [8, 10])

$$
(2.12)\qquad
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi\\
&\quad = \alpha_\theta(\theta_{,x_1} f_2 - \theta_{,x_2} f_1) + \alpha F && \text{in } \Omega^T,\\
&\chi = v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j} n_j) + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\
&\chi_{,x_3} = 0 && \text{on } S_2^T,\\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
$$

where the summation convention over repeated indices is assumed.

Differentiating $(1.1)_{3,6,7}$ with respect to x_3 yields

$$
(2.13)\qquad
\begin{aligned}
&\vartheta_{,t} + v\cdot\nabla\vartheta + h\cdot\nabla\theta - \varkappa\Delta\vartheta = 0 && \text{in } \Omega^T,\\
&\bar n\cdot\nabla\vartheta = 0 && \text{on } S_1^T,\\
&\vartheta = 0 && \text{on } S_2^T,\\
&\vartheta|_{t=0} = \vartheta(0) && \text{in } \Omega.
\end{aligned}
$$

LEMMA 2.5. *Assume that* $\mathbb{D}(h) \in L_2(\Omega)$, $h\cdot\bar n|_S = 0$, $\operatorname{div} h = 0$ *and* $\Omega \subset \mathbb{R}^3$. *Then h satisfies the inequality*

$$
(2.14)\qquad \|h\|_{H^1(\Omega)} \le c\|\mathbb{D}(h)\|_{L_2(\Omega)},
$$

where c is a constant independent of h.

Proof. To show (2.14) we examine the expression

$$
\int_\Omega |\mathbb{D}(h)|^2\,dx = \int_\Omega (h_{i,x_j} + h_{j,x_i})^2\,dx = \int_\Omega (2h_{i,x_j}^2 + 2h_{i,x_j}h_{j,x_i})\,dx,
$$

where the second expression under the last integral is

$$\int_\Omega h_{i,x_j}h_{j,x_i}\,dx=\int_\Omega (h_{i,x_j}h_j)_{,x_i}\,dx-\int_\Omega h_{i,x_ix_j}h_j\,dx=\int_{S_1\cup S_2} n_ih_{i,x_j}h_j\,dS$$
$$=-\int_{S_1} n_{i,x_j}h_ih_j\,dS_1+\int_{S_2} n_ih_{i,x_j}h_j\,dS_2=-\int_{S_1} n_{i,x_j}h_ih_j\,dS_1.$$

From the above considerations we have

$$\|\nabla h\|^2_{L_2(\Omega)}\le c\int_\Omega |\mathbb{D}(h)|^2\,dx+c\|h\|^2_{L_2(S_1)}. \tag{2.15}$$

By the trace theorem

$$\|\nabla h\|^2_{L_2(\Omega)}\le c(\|\mathbb{D}(h)\|^2_{L_2(\Omega)}+\|h\|^2_{L_2(\Omega)}). \tag{2.16}$$

From [11] we have

$$\|h\|_{L_2(\Omega)}\le \delta\|\nabla h\|_{L_2(\Omega)}+M\|\mathbb{D}(h)\|_{L_2(\Omega)}, \tag{2.17}$$

where δ can be chosen sufficiently small and $M=M(\delta)$ is some constant. From (2.15)–(2.17) we have

$$\|\nabla h\|^2_{L_2(\Omega)}\le c\|\mathbb{D}(h)\|^2_{L_2(\Omega)}. \tag{2.18}$$

From (2.18) and (2.17) we obtain (2.14). This concludes the proof. ∎

Let us consider the elliptic problem

$$\begin{aligned} &v_{2,x_1}-v_{1,x_2}=\chi &&\text{in } \Omega\subset\mathbb{R}^2,\\ &v_{1,x_1}+v_{2,x_2}=-h_3 &&\text{in } \Omega\subset\mathbb{R}^2,\\ &v\cdot\bar n=0 &&\text{on } S=\partial\Omega, \end{aligned} \tag{2.19}$$

where x_3 is treated as a parameter.

Lemma 2.6. *Let $\Omega\subset\mathbb{R}^2$. Assume that $\chi,h_3\in L_2(\Omega)$. Then there exists a solution to problem* (2.19) *such that $v\in H^1(\Omega)$ and*

$$\|v\|_{H^1(\Omega)}\le c(\|\chi\|_{L_2(\Omega)}+\|h_3\|_{L_2(\Omega)}). \tag{2.20}$$

Assume that $\chi,h_3\in H^1(\Omega)$. Then the solution is such that $v\in H^2(\Omega)$ and

$$\|v\|_{H^2(\Omega)}\le c(\|\chi\|_{H^1(\Omega)}+\|h_3\|_{H^1(\Omega)}). \tag{2.21}$$

Proof. To solve problem (2.19) we introduce potentials φ,ψ such that

$$v_1=\varphi_{,x_1}+\psi_{,x_2},\qquad v_2=\varphi_{,x_2}-\psi_{,x_1}. \tag{2.22}$$

Using representation (2.22) we see that $(2.19)_3$ takes the form

$$\bar n\cdot\nabla\varphi+\bar\tau\cdot\nabla\psi=0\quad\text{on } S, \tag{2.23}$$

where $\bar n\perp TS$, $\bar\tau\in TS$. The potentials φ and ψ are determined up to an arbitrary constant. Moreover, to determine the potential we split the boundary condition (2.23) into two boundary conditions

$$\begin{aligned} &\bar n\cdot\nabla\varphi|_S=0,\\ &\bar\tau\cdot\nabla\psi|_S=0\ \Rightarrow\ \psi|_S=0. \end{aligned}\tag{2.24}$$

Given $v=(v_1,v_2)$ we calculate φ and ψ from the problems

$$\begin{aligned} &\Delta\varphi=v_{1,x_1}+v_{2,x_2}\quad \text{in } \Omega,\\ &\bar n\cdot\nabla\varphi|_S=0,\\ &\int_\Omega\varphi\,dx=0 \end{aligned}\tag{2.25}$$

and

$$\begin{aligned} &\Delta\psi=v_{1,x_2}-v_{2,x_1}\\ &\psi|_S=0. \end{aligned}\tag{2.26}$$

In view of (2.25), (2.26) problem (2.19) takes the form

$$\begin{aligned} &\Delta\psi=\chi, &&\psi|_S=0,\\ &\Delta\varphi=-h_3, &&\bar n\cdot\nabla\varphi|_S=0, \quad \int_\Omega\varphi\,dx=0. \end{aligned}\tag{2.27}$$

Solving problem (2.27) we have the estimates

$$\|\psi\|_{H^2(\Omega)}\le c\|\chi\|_{L_2(\Omega)},\quad \|\varphi\|_{H^2(\Omega)}\le c\|h_3\|_{L_2(\Omega)}.\tag{2.28}$$

Hence in view of (2.22) we get (2.20).

For more regular χ and h_3 we also have the estimates

$$\|\psi\|_{H^3(\Omega)}\le c\|\chi\|_{H^1(\Omega)},\quad \|\varphi\|_{H^3(\Omega)}\le c\|h_3\|_{H^1(\Omega)}.\tag{2.29}$$

Then (2.29) implies (2.21). This concludes the proof. ∎

Now we formulate the result on local existence of solutions to problem (1.1) with regularity allowed by the regularity of data formulated in the Main Theorem.

LEMMA 2.7. *Let the assumptions of the Main Theorem hold. Then for any $A>0$ there exists $t_*>0$ and a solution (v,θ,p) to problem* (1.1) *such that $v\in W_\varrho^{2,1}(\Omega^{t_*})$, $\theta\in W_\varrho^{2,1}(\Omega^{t_*})$, $\nabla p\in L_\varrho(\Omega^{t_*})$, $h\in W_\sigma^{2,1}(\Omega^{t_*})$, $\nabla q\in L_\sigma(\Omega^{t_*})$ and*

$$\begin{aligned} &\|h\|_{W_\sigma^{2,1}(\Omega^{t_*})}+\|\nabla q\|_{L_\sigma(\Omega^{t_*})}+\|\vartheta\|_{W_\sigma^{2,1}(\Omega^{t_*})}\le A,\\ &\|v\|_{W_\varrho^{2,1}(\Omega^{t_*})}+\|\theta\|_{W_\varrho^{2,1}(\Omega^{t_*})}+\|\nabla p\|_{L_\varrho(\Omega^{t_*})}\le A, \end{aligned}$$

where $\varrho,\sigma\in(5/3,\infty)$ satisfy $5/\varrho-5/\sigma<1$.

Consider the problem

$$u_{,t} - \nu\Delta u = 0,$$
$$u|_S = \varphi,$$
$$u|_{t=0} = 0.$$

LEMMA 2.8. *Assume that* $\varphi \in L_q(0,T;L_p(S))$, $p,q \in [1,\infty]$. *Then* $u \in L_q(0,T;L_p(\Omega))$ *and*

$$\|u\|_{L_q(0,T;L_p(\Omega))} \le c\|\varphi\|_{L_q(0,T;L_p(S))}.$$

Assume that $\varphi \in W_2^{1/2,1/4}(S^T)$. *Then* $u \in W_2^{1,1/2}(\Omega^T)$ *and*

$$\|u\|_{W_2^{1,1/2}(\Omega^T)} \le c\|\varphi\|_{W_2^{1/2,1/4}(S^T)}.$$

3. Estimates

LEMMA 3.1. *Let the assumptions of Lemma* 2.3 *be satisfied. Moreover assume that* $f\in L_2(0,T;L_3(\Omega))$, $f_3\in L_3(0,T;L_{4/3}(S_2))$, $g\in L_2(0,T;L_{6/5}(\Omega))$, $h(0)\in L_2(\Omega)$, $\vartheta(0)\in L_2(\Omega)$, $\nabla v\in L_2(0,T;L_3(\Omega))$, $\nabla\theta\in L_2(0,T;L_3(\Omega))$. *Assume that* h *and* ϑ *are sufficiently regular solutions to* (2.10), (2.13). *Let* $c_1 = a(\|\theta_0\|_{L_\infty})$ *and moreover* $h\in L_\infty(0,T;L_3(\Omega))$. *Then*

$$\begin{aligned}(3.1)\qquad &\|h\|_{V_2^0(\Omega^T)} + \|\vartheta\|^2_{V_2^0(\Omega^t)} \le c\exp(cc_1^2\|f\|^2_{L_2(0,t;L_3(\Omega))})\\ &\cdot[c_0^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + c_1^2\|g\|^2_{L_2(0,t;L_{6/5}(\Omega))} + c_1^2\|f_3\|^2_{L_2(0,t;L_{4/3}(S_2))}\\ &+\|h(0)\|^2_{L_2(\Omega)} + \|\vartheta(0)^2_{L_2(\Omega)}],\qquad t\le T.\end{aligned}$$

Let, additionally, $v,\theta\in L_2(0,T;W_3^1(\Omega))$. *Then*

$$\begin{aligned}(3.2)\qquad &\|h\|^2_{V_2^0(\Omega^t)} + \|\vartheta\|^2_{V_2^0(\Omega^t)} \le c\exp[c(\|\nabla v\|^2_{L_2(0,t;L_3(\Omega))}\\ &+\|\nabla\theta\|^2_{L_2(0,t;L_3(\Omega))} + c_1^2\|f\|^2_{L_2(0,t;L_3(\Omega))})]\cdot[c_1^2\|g\|^2_{L_2(0,t;L_{6/5}(\Omega))}\\ &+c_1^2\|f_3\|^2_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|^2_{L_2(\Omega)} + \|\vartheta(0)\|^2_{L_2(\Omega)}],\quad t\le T.\end{aligned}$$

Proof. Multiplying (2.10) by h, integrating over Ω and using Lemma 2.5 yields

$$\begin{aligned}(3.3)\qquad \frac{1}{2}\frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \nu\|h\|^2_{H^1(\Omega)} &\le c\int_\Omega |h\cdot\nabla v\cdot h|\,dx + c\int_\Omega |\alpha_\theta\vartheta f h|\,dx\\ &+ c\int_\Omega |\alpha g h|\,dx + c\int_{S_2}|\alpha f_3 h_3|\,dx_1\,dx_2\end{aligned}$$

where the first term on the r.h.s. is estimated by

$$\varepsilon_1\|h\|^2_{L_6(\Omega)} + c(1/\varepsilon_1)\|\nabla v\|^2_{L_2(\Omega)}\|h\|^2_{L_3(\Omega)},$$

the second by

$$\varepsilon_2\|h\|^2_{L_6(\Omega)} + c(1/\varepsilon_2)a^2(\|\theta_0\|_{L_\infty(\Omega)})\|\vartheta f\|^2_{L_{6/5}(\Omega)},$$

the third by

$$\varepsilon_3\|h\|^2_{L_6(\Omega)} + c(1/\varepsilon_3)a^2(\|\theta_0\|_{L_\infty(\Omega)})\|g\|^2_{L_{6/5}(\Omega)}$$

and the fourth by

$$\varepsilon_4\|h\|^2_{H^1(\Omega)} + c(1/\varepsilon_4)a^2(\|\theta_0\|_{L_\infty(\Omega)})\|f_3\|_{L_{4/3}(S_2)}.$$

Assuming that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are sufficiently small we obtain

$$\begin{aligned}(3.4)\qquad \frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \nu\|h\|^2_{H^1(\Omega)} &\le c(\|\nabla v\|^2_{L_2(\Omega)}\|h\|^2_{L_3(\Omega)}\\ &\quad + c_1^2(\|\vartheta\|^2_{L_2(\Omega)}\|f\|^2_{L_3(\Omega)} + \|g\|^2_{L_{6/5}(\Omega)} + \|f_3\|^2_{L_{4/3}(S_2)})).\end{aligned}$$

Multiplying (2.13) by ϑ and integrating over Ω yields

$$\begin{aligned}(3.5)\qquad \frac{1}{2}\frac{d}{dt}\|\vartheta\|^2_{L_2(\Omega)} + \varkappa\|\vartheta\|^2_{H^1(\Omega)} &\le c\int_\Omega |h\cdot\nabla\theta\vartheta|\,dx\\ &\le \varepsilon\|\vartheta\|^2_{L_6(\Omega)} + c(1/\varepsilon)\|h\|^2_{L_3(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}.\end{aligned}$$

For sufficiently small ε we have

$$(3.6)\qquad \frac{d}{dt}\|\vartheta\|^2_{L_2(\Omega)} + \varkappa\|\vartheta\|^2_{H^1(\Omega)} \le c\|h\|^2_{L_3(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}.$$

Adding (3.4) and (3.6), integrating with respect to time and using (2.8) and (2.9) we obtain (3.1).

We can replace inequalities (3.4) and (3.6) by

$$\begin{aligned}(3.7)\qquad \frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \nu\|h\|^2_{H^1(\Omega)} &\le c(\|\nabla v\|^2_{L_3(\Omega)}\|h\|^2_{L_2(\Omega)}\\ &\quad + c_1^2(\|\vartheta\|^2_{L_2(\Omega)}\|f\|^2_{L_3(\Omega)} + \|g\|^2_{L_{6/5}(\Omega)} + \|f_3\|^2_{L_{4/3}(S_2)}))\end{aligned}$$

and

$$(3.8)\qquad \frac{d}{dt}\|\vartheta\|^2_{L_2(\Omega)} + \varkappa\|\vartheta\|^2_{H^1(\Omega)} \le c\|\nabla\theta\|^2_{L_3(\Omega)}\|h\|^2_{L_2(\Omega)}.$$

Adding (3.7) and (3.8), and integrating the sum with respect to time, yields (3.2). This ends the proof. ■

To obtain an estimate for solutions to problem (2.12) we introduce a function $\tilde\chi : \Omega\times[0,T]\to\mathbb{R}$ as a solution to the problem

$$(3.9)\qquad \begin{aligned} &\tilde\chi_{,t} - \nu\Delta\tilde\chi = 0 && \text{in } \Omega^T,\\ &\tilde\chi = \chi_* && \text{on } S_1^T,\\ &\tilde\chi_{,x_3} = 0 && \text{on } S_2^T,\\ &\tilde\chi|_{t=0} = 0 && \text{in } \Omega.\end{aligned}$$

Then the function

$$(3.10)\qquad \chi' = \chi - \tilde\chi$$

satisfies

$$
\begin{aligned}
(3.11)\qquad & \chi'_{,t} + v\cdot\nabla\chi' - h_3\chi' + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi' \\
&\quad = \alpha_\theta(\theta_{,x_1} f_2 - \theta_{,x_2} f_1) + \alpha F - v\cdot\nabla\tilde\chi + h_3\tilde\chi && \text{in } \Omega^T, \\
& \chi' = 0 && \text{on } S_1^T, \\
& \chi'_{,x_3} = 0 && \text{on } S_2^T, \\
& \chi'|_{t=0} = \chi(0) && \text{in } \Omega.
\end{aligned}
$$

LEMMA 3.2. *Let the assumptions of Lemma* 2.3 *be satisfied. Moreover assume that* $h, f \in L_\infty(0,T;L_3(\Omega))$, $F \in L_2(0,T;L_{6/5}(\Omega))$, $v' = (v_1, v_2) \in L_\infty(0,T;H^{1/2+\varepsilon}(\Omega)) \cap W_2^{1,1/2}(\Omega^T)$, $\chi(0) \in L_2(\Omega)$, *and* $\varepsilon_7 > 0$ *is arbitrarily small. Assume that* (v,θ) *is a sufficiently regular solution to* (1.1). *Then for the solution* χ *to* (2.12) *we have*

$$
\begin{aligned}
(3.12)\quad \|\chi\|^2_{V_2^0(\Omega^t)} &\le c\big(c_0^2 \sup_t \|h\|^2_{L_3(\Omega)} + c_1^2 c_0^2 \sup_t \|f\|^2_{L_3(\Omega)} \\
&\quad + c_1^2 \|F\|^2_{L_2(0,t;L_{6/5}(\Omega))} + c_0^2\varepsilon_7^2 \|v'\|^2_{L_\infty(0,t;H^1(\Omega))} \\
&\quad + \|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} + \|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|\chi(0)\|^2_{L_2(\Omega)} \\
&\quad + (c_0^2 c^2(1/\varepsilon_7) + \sup_t \|h\|^2_{L_3(\Omega)}) \\
&\quad \times (a^2(\|\theta_0\|_{L_\infty(\Omega^t)})\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|v_0\|^2_{L_2(\Omega)})\big)
\end{aligned}
$$

for all $t \le T$.

Proof. Multiplying $(3.11)_1$ by χ', integrating over Ω, and using boundary conditions $(3.11)_{2,3}$, $(1.1)_5$ and $(1.1)_2$, we obtain

$$
\begin{aligned}
(3.13)\qquad \frac{1}{2}\frac{d}{dt}\|\chi'\|^2_{L_2(\Omega)} + \nu\|\nabla\chi'\|^2_{L_2(\Omega)} &= \int_\Omega h_3\chi'^2\,dx \\
&\quad - \int_\Omega (h_2 w_{,x_1} - h_1 w_{,x_2})\chi'\,dx + \int_\Omega \alpha_\theta(\theta_{,x_1} f_2 - \theta_{,x_2} f_1)\chi'\,dx \\
&\quad + \int_\Omega \alpha F\chi'\,dx - \int_\Omega v\cdot\nabla\tilde\chi\chi'\,dx + \int_\Omega h_3\tilde\chi\chi'\,dx.
\end{aligned}
$$

Now we estimate the terms on the r.h.s. of the above equality. Let $x' = (x_1, x_2)$. The first term is estimated by

$$
\varepsilon_1\|\chi'\|^2_{L_6(\Omega)} + \frac{c}{\varepsilon_1}\|\chi'\|^2_{L_2(\Omega)}\|h_3\|^2_{L_3(\Omega)},
$$

the second by

$$
\varepsilon_2\|\chi'\|^2_{L_6(\Omega)} + \frac{c}{\varepsilon_2}\|h\|^2_{L_3(\Omega)}\|w_{,x'}\|^2_{L_2(\Omega)},
$$

and the third by

$$\varepsilon_3\|\chi'\|^2_{L_6(\Omega)}+\frac{c}{\varepsilon_3}c_1^2\|\theta_{,x}\|^2_{L_2(\Omega)}\|f\|^2_{L_3(\Omega)},$$

where we have used (1.2), and the fourth by

$$\varepsilon_4\|\chi'\|^2_{L_6(\Omega)}+\frac{c}{\varepsilon_4}c_1^2\|F\|^2_{L_{6/5}(\Omega)},$$

where we have also used (1.2).

To estimate the fifth term on the r.h.s. of (3.13) we integrate it by parts and use $(1.1)_{2,5}$. Then it takes the form

$$I\equiv\int_\Omega v\cdot\nabla\chi'\tilde\chi\,dx.$$

Hence

$$|I|\le\varepsilon_5\|\nabla\chi'\|^2_{L_2(\Omega)}+\frac{c}{\varepsilon_5}\|v\|^2_{L_6(\Omega)}\|\tilde\chi\|^2_{L_3(\Omega)}.$$

Finally, the last term on the r.h.s. of (3.13) is bounded by

$$\varepsilon_6\|\chi'\|^2_{L_6(\Omega)}+\frac{c}{\varepsilon_6}\|h\|^2_{L_3(\Omega)}\|\tilde\chi\|^2_{L_2(\Omega)}.$$

Using the above estimates in (3.13), assuming that $\varepsilon_1,\dots,\varepsilon_6$ are sufficiently small, integrating the result with respect to time and using (2.8)–(2.9) we obtain

$$\begin{aligned}(3.14)\qquad \|\chi'\|^2_{V_2^0(\Omega^t)}&\le c(\sup_t\|h\|^2_{L_3(\Omega)}\|\chi'\|^2_{L_2(0,t;L_2(\Omega))}\\&+c_0^2\sup_t\|h\|^2_{L_3(\Omega)}+c_1^2c_0^2\sup_t\|f\|^2_{L_3(\Omega)}+c_1^2\|F\|^2_{L_2(0,t;L_{6/5}(\Omega))}\\&+c_0^2\|\tilde\chi\|^2_{L_\infty(0,t;L_3(\Omega))}+\sup_t\|h\|^2_{L_3(\Omega)}\|\tilde\chi\|^2_{L_2(0,t;L_2(\Omega))}+\|\chi(0)\|^2_{L_2(\Omega)}).\end{aligned}$$

In view of (2.8) we have $\|\chi\|_{L_2(\Omega^t)}\le cc_0$.

Using (3.10) and this fact we obtain from (3.14) the inequality

$$\begin{aligned}(3.15)\qquad \|\chi\|^2_{V_2^0(\Omega^t)}&\le c(c_0^2\sup_t\|h\|^2_{L_3(\Omega)}+c_1^2c_0^2\sup_t\|f\|^2_{L_3(\Omega)}\\&+c_1^2\|F\|^2_{L_2(0,t;L_{6/5}(\Omega))}+c_0^2\|\tilde\chi\|^2_{L_\infty(0,t;L_3(\Omega))}+\sup_t\|h\|^2_{L_3(\Omega)}\|\tilde\chi\|^2_{L_2(\Omega^t)}\\&+\|\tilde\chi\|_{V_2^0(\Omega^t)}+\|\chi(0)\|^2_{L_2(\Omega)}).\end{aligned}$$

Since $\tilde\chi$ is a solution of (3.9) and χ_* is described by $(2.12)_2$ we have the following estimates, by Lemma 2.8:

$$\int_0^t \|\tilde\chi(t')\|^2_{L_2(\Omega)}\,dt' \le c\int_0^t \|v'(t')\|^2_{L_2(S)}\,dt' \le c\int_0^t \|v'(t')\|^2_{H^1(\Omega)}\,dt' \le c(a^2(\|\theta_0\|_{L_\infty(\Omega)})\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|v_0\|^2_{L_2(\Omega)}),$$

(3.16)
$$\|\tilde\chi\|_{L_\infty(0,t;L_3(\Omega))} \le c\|v'\|_{L_\infty(0,t;L_3(S))} \le \varepsilon_7\|v'\|_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon_7)\|v'\|_{L_\infty(0,t;L_2(\Omega))},$$

$$\|\tilde\chi\|^2_{V_2^0(\Omega^t)} \le c\Big(\|\tilde\chi\|^2_{L_\infty(0,t;L_2(\Omega))} + \int_0^t \|\tilde\chi(t')\|^2_{H^1(\Omega)}\,dt'\Big) \le c\|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} + c\|v'\|^2_{W_2^{1,1/2}(\Omega^t)}, \quad \varepsilon > 0.$$

Employing (3.16) in (3.15) yields (3.12). This concludes the proof. ■

Let us consider the problem

$$
\begin{aligned}
&v_{2,x_1} - v_{1,x_2} = \chi && \text{in } \Omega',\\
&v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega',\\
&v'\cdot\bar n' = 0 && \text{on } S',
\end{aligned}
\tag{3.17}
$$

where $\Omega' = \Omega \cap \{x_3 = \text{const} \in (-a,a)\}$, $S' = S \cap \{x_3 = \text{const} \in (-a,a)\}$, x_3, t are treated as parameters, $\bar n' = (n_1, n_2)$.

LEMMA 3.3. *Let the assumptions of Lemmas* 2.3, 3.1, 3.2 *be satisfied. Assume that* (v,p,θ) *is a weak solution to problem* (1.1). *Assume that*

$$
\begin{aligned}
&c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1c_0\|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1\|F\|_{L_2(0,t;L_{6/5}(\Omega))}\\
&\quad + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)}\\
&\quad + c_0^2(c_1\|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) + \psi(c_0) \le k_1 < \infty,\\
&\|f\|_{L_2(0,t;L_3(\Omega))} \le k_2 < \infty
\end{aligned}
\tag{3.18}
$$

for all $t \le T$. *Then*

$$\|v'\|^2_{V_2^1(\Omega^t)} \le c[e^{cc_1^2k_2^2}(c_0^2\|h\|^2_{L_\infty(0,t,L_3(\Omega))} + \psi(c_0)k_1^2) + \|v'\|^2_{L_2(\Omega;H^{1/2}(0,t))}] \tag{3.19}$$

for all $t \le T$, *where* $v' = (v_1, v_2)$ *and* ψ *is an increasing positive function.*

Proof. Assuming that ε_7 is sufficiently small, in view of (3.1), (3.12) and Lemma 2.6 we obtain for solutions to problem (3.17) the inequality (see [11])

$$\|v'\|^2_{L_{10}(\Omega^T)} \le c\|v'\|^2_{V_2^1(\Omega^t)} \le c[e^{cc_1^2k_2^2}(c_0^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + \psi(c_0)k_1^2) + \|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} + \|v'\|^2_{W_2^{1,1/2}(\Omega^t)}], \tag{3.20}$$

where ε is an arbitrarily small number and (3.18) was used. By interpolation inequalities,

$$\|v'\|_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} \le \varepsilon_1 \|v'\|_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon_1)\|v'\|_{L_\infty(0,t;L_2(\Omega))},$$

and

$$\|v'\|_{W_2^{1,1/2}(\Omega^t)} = \|v'\|_{L_2(0,t;H^1(\Omega))} + \|v'\|_{L_2(\Omega;H^{1/2}(0,t))},$$

where

$$\|v'\|_{L_2(0,t;H^1(\Omega))} \le ck_1.$$

Then we obtain (3.19) from (3.20) for sufficiently small ε_1. This concludes the proof. ■

Let us consider problem $(1.1)_{1,2,4,5,7}$ in the form

$$\begin{aligned} &v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla' v - wh + \alpha(\theta)f && \text{in } \Omega^T,\\ &\operatorname{div} v = 0 && \text{in } \Omega^T,\\ &v\cdot\bar n = 0, \quad \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } S^T,\\ &v|_{t=0} = v_0 && \text{in } \Omega. \end{aligned} \tag{3.21}$$

where $v'\cdot\nabla' = v_1\partial_{x_1} + v_2\partial_{x_2}$.

LEMMA 3.4. *Assume that* (v,θ) *is a weak solution to problem* (1.1). *Let the assumptions of Lemma* 3.3 *be satisfied. Let*

$$\begin{gathered} \|f\|_{L_2(\Omega^t)} + \|v_0\|_{H^1(\Omega)} \le k_3 < \infty,\\ H(t) = \|h\|_{L_\infty(0,t;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^t)} < \infty, \end{gathered}$$

for all $t \le T$. *Then there exists a constant* $c_2 = c_2(c_0,c_1)$ *such that the solution* v *to problem* (3.21) *satisfies*

$$\|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \le c_2 e^{cc_1^2k_2^2}(H+1+k_1+k_3)^2 + ck_3, \quad t\le T. \tag{3.22}$$

The proof is the same as the proof of Lemma 3.3 in [7].

Finally, we obtain an estimate for h.

LEMMA 3.5. *Let the assumptions of Lemma* 3.4 *be satisfied. Let*

$$\begin{aligned} &c_1\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2k_2^2}k_1 + c_1\|g\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega^t)}\\ &\qquad\qquad + \|h(0)\|_{W_\sigma^{2-1/\sigma}(\Omega)} \le k_4 < \infty,\\ &c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)}\\ &\qquad\qquad + \|\vartheta(0)\|_{L_2(\Omega)} \le d < \infty,\\ &c_1\|f\|_{L_\varrho(\Omega^T)} + \|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} \le k_5 < \infty, \end{aligned} \tag{3.23}$$

for $t \le T$. Then for d sufficiently small there exists a constant A such that

$$\|h\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \le A, \quad 5/3 < \sigma \le 10/3, \quad t \le T, \tag{3.24}$$

$$\|\nabla p\|_{L_\varrho(\Omega^t)} + \|v\|_{W^{2,1}_\varrho(\Omega^t)} + \|\theta\|_{W^{2,1}_\varrho(\Omega^t)} \le \varphi(A) + ck_5, \quad 5/3 \le \varrho < 10, \quad t \le T, \tag{3.25}$$

where φ is some positive increasing function.

Proof. In view of Lemma 2.2 for solutions to problem (2.10) we have

$$\begin{aligned} \|h\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} &\le c(\|v \cdot \nabla h\|_{L_\sigma(\Omega^t)} + \|h \cdot \nabla v\|_{L_\sigma(\Omega^t)} \\ &\quad + \|\alpha_\theta \vartheta f\|_{L_\sigma(\Omega^t)} + \|\alpha g\|_{L_\sigma(\Omega^t)} + \|h(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}). \end{aligned} \tag{3.26}$$

In view of the imbedding

$$\|v\|_{L_{10}(\Omega^t)} + \|\nabla v\|_{L_{\frac{10}{3}}(\Omega^t)} \le c\|v\|_{W^{2,1}_2(\Omega^t)}. \tag{3.27}$$

and inequality (3.22) we estimate the first term on the r.h.s. of (3.26) by

$$\|v\|_{L_{10}(\Omega^t)}(\varepsilon_1\|h\|_{W^{2,1}_\sigma(\Omega^t)} + c(1/\varepsilon_1)\|h\|_{L_2(\Omega^t)})$$

and the second by

$$\|\nabla v\|_{L_{10/3}(\Omega^t)}(\varepsilon_2\|h\|_{W^{2,1}_\sigma(\Omega^t)} + c(1/\varepsilon_2)\|h\|_{L_2(\Omega^t)}).$$

In view of (2.6) and (1.2) the third and the fourth terms on the r.h.s. of (3.26) can be estimated by

$$cc_1(\|f\|_{L_\infty(\Omega^t)}\|\vartheta\|_{L_\sigma(\Omega^t)} + \|g\|_{L_\sigma(\Omega^t)}) \equiv I.$$

We use (3.1) with the notation of (3.18). Then we obtain

$$I \le cc_1(\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2k_2^2}(k_1 + c_0\|h\|_{L_\infty(0,t;L_3(\Omega))}) + \|g\|_{L_\sigma(\Omega^t)}),$$

where $\sigma \le 10/3$.

We will also use the interpolation

$$\|h\|_{L_\infty(0,t;L_3(\Omega))} \le \varepsilon_2\|h\|_{W^{2,1}_\sigma(\Omega^t)} + c(1/\varepsilon_3)\|h\|_{L_2(\Omega^t)}.$$

Employing the above estimates in (3.26), assuming that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are sufficiently small and using (3.22) we obtain

$$\begin{aligned} \|h\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} &\le \varphi(H)\|h\|_{L_2(\Omega^t)} \\ &\quad + cc_1(\|f\|_{L_\infty(\Omega^t)} e^{cc_1^2k_2^2}k_1 + \|g\|_{L_\sigma(\Omega^t)}) + c\|h(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}, \end{aligned} \tag{3.28}$$

where $5/3 < \sigma \le 10/3$, φ is an increasing positive function depending on H and on the constants $c_0, c_1, k_1, \dots, k_5$. Using the notation of $(3.23)_1$ we have

$$\|h\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \le \varphi(H)\|h\|_{L_2(\Omega^t)} + ck_4. \tag{3.29}$$

We want to estimate $\|h\|_{L_2(\Omega^t)}$ by applying (3.2). For this purpose we need to estimate $\|\nabla\theta\|_{L_2(0,t;L_3(\Omega))}$. Hence we consider problem $(1.1)_{3,6,7}$ and we are looking for solutions of this problem such that $\theta \in W_\varrho^{2,1}(\Omega^t)$ with ϱ so large that

$$\|\nabla\theta\|_{L_2(0,t;L_3(\Omega))} \le c\|\theta\|_{W_\varrho^{2,1}(\Omega^t)}. \tag{3.30}$$

We see that (3.30) holds for $\varrho \ge 5/3$. Considering problem $(1.1)_{3,6,7}$ we have

$$\|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \le c(\|v\cdot\nabla\theta\|_{L_\varrho(\Omega^t)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)}). \tag{3.31}$$

The first term on the r.h.s. is estimated by

$$\|v\|_{L_{\varrho\lambda_1}(\Omega^t)}\|\nabla\theta\|_{L_{\varrho\lambda_2}(\Omega^t)} \equiv I_1,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\varrho\lambda_1 = 10$.

We have the interpolation inequality

$$\|\nabla\theta\|_{L_{\varrho\lambda_2}(\Omega^t)} \le \varepsilon_4\|\theta\|_{W_\varrho^{2,1}(\Omega^t)} + c(1/\varepsilon_4)\|\theta\|_{L_2(\Omega^t)}$$

which holds for $\frac{5}{\varrho} - \frac{5}{\varrho\lambda_2} < 1$ so for $\frac{5}{\varrho\lambda_1} < 1$. Hence

$$I_1 \le \|v\|_{L_{10}(\Omega^t)}(\varepsilon_4\|\theta\|_{W_\varrho^{2,1}(\Omega^t)} + c(1/\varepsilon_4)\|\theta\|_{L_2(\Omega^t)}).$$

Using the estimate in (3.31), assuming that ε_4 is sufficiently small, and using (3.27) and (3.22), we obtain

$$\|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \le \varphi(H) + c\|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)}, \tag{3.32}$$

where $\varrho < 10$.

Similarly by Lemma 2.2 applied to (3.21) and (2.8) we obtain

$$\begin{aligned}\|v\|_{W_\varrho^{2,1}(\Omega^t)} &+ \|\nabla p\|_{L_\varrho(\Omega^t)} \\ &\le \varphi(H, c_0) + c_1\|f\|_{L_\varrho(\Omega^T)} + c\|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)}.\end{aligned} \tag{3.33}$$

Let us consider (3.29). In view of (3.2) we estimate the norm $\|h\|_{L_2(\Omega^t)}$, where

$$\|\nabla v\|_{L_2(0,t;L_3(\Omega))} + \|\nabla\theta\|_{L_2(0,t;L_3(\Omega))} \le \varphi(H) + ck_5.$$

Then (3.29) takes the form

$$\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \le \varphi(H)d + ck_4, \tag{3.34}$$

where φ is an increasing positive function.

Let σ be such that

$$H = \|h\|_{L_\infty(0,t;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^t)} \le c\|h\|_{W_\sigma^{2,1}(\Omega^t)},$$

which holds for $\sigma > 5/3$. Then (3.34) takes the form

$$\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \le \varphi(\|h\|_{W_\sigma^{2,1}(\Omega^t)})d + ck_4. \tag{3.35}$$

Hence for d sufficiently small there exists a constant A such that

$$\|h\|_{W_\sigma^{2,1}(\Omega^t)} + \|\nabla q\|_{L_\sigma(\Omega^t)} \le A, \quad t \le T. \tag{3.36}$$

By (3.36), (3.32) and (3.33) the proof is complete. ■

Proof of the Main Theorem. Now we want to increase regularity described by (3.25). Assume $10 \le \varrho < \infty$. In view of [8, Theorem 2.1] for a solution v to problem (1.1) we have

$$\begin{aligned}\|v\|_{W_\varrho^{2,1}(\Omega^t)} &+ \|\nabla p\|_{L_\varrho(\Omega^t)} \\ &\le c(\|v\cdot\nabla v\|_{L_\varrho(\Omega^t)} + \|\alpha(\theta)f\|_{L_\varrho(\Omega^t)} + \|v_0\|_{W_\varrho^{2-2/\varrho}(\Omega)}).\end{aligned} \tag{3.37}$$

We estimate the first term on the r.h.s. of (3.37) by

$$\begin{aligned}\|v\|_{L_\infty(\Omega^t)}&\|\nabla v\|_{L_\varrho(\Omega^t)} \\ &\le c\|v\|_{W_5^{2,1}(\Omega^t)}(\varepsilon_1\|v\|_{W_\varrho^{2,1}(\Omega^t)} + c(1/\varepsilon_1)\|v\|_{L_2(\Omega^t)})\end{aligned} \tag{3.38}$$

and the second by

$$c_1\|f\|_{L_\infty(\Omega^t)}. \tag{3.39}$$

Assuming that ε_1 is sufficiently small and using (3.37)–(3.39) we obtain

$$\|v\|_{W_\varrho^{2,1}(\Omega^t)} + \|\nabla p\|_{L_\varrho(\Omega^t)} \le B_1, \tag{3.40}$$

where B_1 is a constant depending on the constants from the imbedding theorems and data.

Similarly by [5, Ch. 4, Sect. 9, Th. 9.1] we obtain

$$\|\theta\|_{W_\varrho^{2,1}(\Omega^t)} \le B_2. \tag{3.41}$$

Now we want to increase regularity described by (3.24). There exist $p' > \sigma$, $p'' > 5/2$ such that

$$\frac{5}{\varrho} - \frac{5}{p'} < 1, \quad \frac{5}{\varrho} - \frac{5}{p''} < 1.$$

Hence $p = \max\{p', p''\}$ satisfies

$$p > \sigma, \quad p > \frac{5}{2}, \quad \frac{5}{\varrho} - \frac{5}{p} < 1. \tag{3.42}$$

Similarly we can prove that there exists q such that

$$q > \sigma, \quad q > 5 \quad \text{and} \quad \frac{5}{\varrho} - \frac{5}{q} < 2. \tag{3.43}$$

Define $\bar p, \bar q$ by $1/p + 1/\bar p = 1/\sigma$, $1/q + 1/\bar q = 1/\sigma$. Assume $5/3 < \sigma < \infty$. In view of Theorem 2.1 for a solution to problem (2.10) we have

$$(3.44)\quad \begin{aligned}&\|h\|_{W^{2,1}_\sigma(\Omega^t)}+\|\nabla q\|_{L_\sigma(\Omega^t)}\\&\quad\le c(\|v\cdot\nabla h\|_{L_\sigma(\Omega^t)}+\|h\cdot\nabla v\|_{L_\sigma(\Omega^t)}+\|\alpha_\theta\vartheta f\|_{L_\sigma(\Omega^t)}\\&\quad\quad+\|\alpha g\|_{L_\sigma(\Omega^t)}+\|h(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}).\end{aligned}$$

By (3.42) and (3.43) we estimate the first term on the r.h.s. of (3.44) by

$$(3.45)\quad \begin{aligned}&\|v\|_{L_q(\Omega^t)}\|\nabla h\|_{L_{\bar q}(\Omega^t)}\\&\quad\le c\|v\|_{W^{2,1}_\varrho(\Omega^t)}(\varepsilon_2\|h\|_{W^{2,1}_\sigma(\Omega^t)}+c(\varepsilon_2)\|h\|_{L_2(\Omega^t)}),\end{aligned}$$

the second by

$$(3.46)\quad \begin{aligned}&\|\nabla v\|_{L_p(\Omega^t)}\|h\|_{L_{\bar p}(\Omega^t)}\\&\quad\le c\|v\|_{W^{2,1}_\varrho(\Omega^t)}(\varepsilon_3\|h\|_{W^{2,1}_\sigma(\Omega^t)}+c(\varepsilon_3)\|h\|_{L_2(\Omega^t)}),\end{aligned}$$

the third by

$$(3.47)\quad c_1\|f\|_{L_\infty(\Omega^t)}(\varepsilon_4\|\vartheta\|_{W^{2,1}_\sigma(\Omega^t)}+c(\varepsilon_4)\|\vartheta\|_{L_2(\Omega^t)}),$$

and the fourth by

$$(3.48)\quad c_1\|g\|_{L_\sigma(\Omega^t)}.$$

In view of [5, Ch. 4, Sect. 9, Th. 9.1] for any solution to problem (2.13) we have

$$(3.49)\quad \|\vartheta\|_{W^{2,1}_\sigma(\Omega^t)}\le c(\|v\cdot\nabla\vartheta\|_{L_\sigma(\Omega^t)}+\|h\cdot\nabla\theta\|_{L_\sigma(\Omega^t)}+\|\vartheta(0)\|_{W^{2-2/\sigma}_\sigma(\Omega^t)}).$$

By (3.42) and (3.43) we estimate the first term on the r.h.s. of (3.49) by

$$(3.50)\quad c\|v\|_{W^{2,1}_\varrho(\Omega^t)}(\varepsilon_5\|\vartheta\|_{W^{2,1}_\sigma(\Omega^t)}+c(\varepsilon_5)\|\vartheta\|_{L_2(\Omega^t)})$$

and the second by

$$(3.51)\quad c\|\theta\|_{W^{2,1}_\varrho(\Omega^t)}(\varepsilon_6\|h\|_{W^{2,1}_\sigma(\Omega^t)}+c(\varepsilon_6)\|h\|_{L_2(\Omega^t)}).$$

We choose r such that $5/3<r<10/3$ and $r\le\sigma$. By (3.1), the imbedding

$$\|h\|_{L_\infty(0,t;L_3(\Omega))}\le c\|h\|_{W^{2,1}_r(\Omega^t)}$$

and (3.24) there exists a constant B_3 depending on the constants in imbedding theorems and on the data such that

$$(3.52)\quad \|h\|_{L_2(\Omega^t)}+\|\vartheta\|_{L_2(\Omega^t)}\le B_3.$$

Assuming that $\varepsilon_2-\varepsilon_6$ are sufficiently small and using (3.44)–(3.52) we obtain

$$(3.53)\quad \|h\|_{W^{2,1}_\sigma(\Omega^t)}+\|\nabla q\|_{L_\sigma(\Omega^t)}+\|\vartheta\|_{W^{2,1}_\sigma(\Omega^t)}\le B_4,$$

where B_4 is some constant depending on the data. By (3.40), (3.41) and (3.53) the proof is finished. ∎

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