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STUDY OF A CONTACT PROBLEM WITH NORMAL COMPLIANCE AND NONLOCAL FRICTION

Abstract. We consider a static frictional contact between a nonlinear elastic body and a foundation. The contact is modelled by a normal compliance condition such that the penetration is restricted with unilateral constraint and associated to the nonlocal friction law. We derive a variational formulation and prove its unique weak solvability if the friction coefficient is sufficiently small. Moreover, we prove the continuous dependence of the solution on the contact conditions. Also we study the finite element approximation of the problem and obtain an error estimate.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. A first attempt to study contact problems within the framework of variational inequalities was made in [8]. Unilateral contact models occupy an important place in the theory of variational inequalities and their approximation by finite element methods (see [9, 13]). Numerical studies of the Signorini contact problem were made in [1, 2, 3, 12]. In particular, in [12] we can find a detailed analysis of elastic contact problems together with a numerical approach.

In this work, our goal is the analysis and numerical approximation of a frictional and unilateral contact problem in nonlinear elasticity. We assume that the contact is modelled by a normal compliance condition similar to the one in [11], so that the penetration is restricted with unilateral constraint and associated to the nonlocal friction law. Now, we want to point out the physical interest of the model studied here. Indeed, before the reference [11] appeared, it was well known that no restriction with unilateral constraint

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was done on the penetration in compliance models. However, according again to this reference, the method presented here considers a compliance model in which the compliance term does not necessarily represent an important perturbation of the original problem without contact. This will help us to study the models where a strictly limited penetration occurs by means of a limit procedure to the Signorini contact problem. Recall that a numerical study of a static frictional contact problem with normal compliance for elastic materials was made in [10, 15]. The novelty in the present paper is that we extend the results in [5, 15] to the case when the elasticity operator is nonlinear, strongly monotone and Lipschitz continuous. We suppose that the displacement field is of class H^2 (the standard Sobolev space of degree 2) and we deduce an $O(h^{3/4})$ error estimate where $h > 0$ stands for the discretization parameter.

The paper is structured as follows. In Section 2 the mechanical problem (Problem P_1) is formulated, some notation is presented and the variational formulation is established. In Section 3 we give an existence and uniqueness result under a smallness hypothesis on the friction coefficient. In Section 4 we prove a continuous dependence result. Finally, in Section 5 we study the finite element approximation of the displacement variational formulation. We establish the convergence of the finite element method and derive an error estimate under a regularity assumption on the solution.

2. Variational formulation. Consider an elastic body occupying a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$). The boundary Γ of Ω is partitioned into three measurable parts, $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where Γ_i, $i = 1, 2, 3$, are disjoint open sets and $\operatorname{meas}(\Gamma_1) > 0$. The body is subjected to volume forces of density ϕ_1, prescribed zero displacements and tractions ϕ_2 on the parts Γ_1 and Γ_2, respectively. On Γ_3 the body is in unilateral contact with a foundation following a nonlocal friction law [7, 14].

Under these conditions, the classical formulation of the mechanical problem is the following.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}\sigma + \phi_1 = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = \phi_2 \quad \text{on } \Gamma_2, \tag{2.4}$$

$$u_\nu \le g, \quad \sigma_\nu + p(u_\nu) \le 0, \quad (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \quad \text{on } \Gamma_3, \tag{2.5}$$

$$(2.6)\qquad \left.\begin{aligned} &|\sigma_\tau| \le \mu|R\sigma_\nu| \\ &|\sigma_\tau| < \mu|R\sigma_\nu| \Rightarrow u_\tau = 0 \\ &|\sigma_\tau| = \mu|R\sigma_\nu| \Rightarrow \exists\lambda \ge 0, \sigma_\tau = -\lambda u_\tau \end{aligned}\right\} \quad \text{on } \Gamma_3.$$

Here (2.1) is the elastic constitutive law in which σ denotes the stress tensor, $\varepsilon(u)$ is the small strain and F is a given nonlinear function. Next, (2.2) represents the equilibrium equation while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (2.6) represent the nonlocal Coulomb friction law where μ denotes the friction coefficient.

We now want to explain the physical meaning of the unilateral conditions (2.5), which are of our main interest. Here σ_ν denotes the normal stress, u_ν is the normal displacement, $g \ge 0$ is given and p is a Lipschitz continuous increasing function. Indeed, if $u_\nu < 0$, i.e. if there is separation between the body and the foundation, then (2.5) combined with hypotheses (2.10) below on the function p shows that the reaction of the foundation vanishes (since $\sigma_\nu = 0$). If $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$, which means that the reaction of the foundation is uniquely determined by the normal displacement. If $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. We note that when $g = 0$, condition (2.5) becomes the classical Signorini contact condition without a gap,

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p = 0$, (2.5) is the Signorini contact condition with a gap,

$$u_\nu \le g, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu(u_\nu - g) = 0.$$

The last two conditions are used to model the contact with a perfectly rigid foundation.

In the study of Problem P_1 we shall adopt the following notation and hypotheses:

We denote by S_d the space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while '.' and $|\cdot|$ will represent the inner product and Euclidean norm on S_d and $\mathbb{R}^d$, respectively, i.e.

$$u.v = u_i v_i, \qquad |v| = (v.v)^{1/2}, \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau.\tau)^{1/2}, \quad \forall \sigma, \tau \in S_d.$$

Here and below, the indices i and j run between 1 and d, and the summation convention over repeated indices is adopted.

To proceed with the variational formulation, we need some function spaces:

$$H = (L^2(\Omega))^d, \qquad Q = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = (H^1(\Omega))^d.$$

H, Q are Hilbert spaces equipped with the respective inner products

$$(u, v)_H = \int_\Omega u_i v_i\, dx, \qquad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

The linearized strain tensor is defined as

$$\varepsilon(v) = (\varepsilon_{ij}(v)) = \left(\frac{1}{2}(v_{i,j} + v_{j,i})\right) \quad \forall v \in H_1.$$

For every $v \in H_1$ we also write v for the trace of v on Γ, and we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by $v_\nu = v.\nu$, $v_\tau = v - v_\nu \nu$. Similarly, σ_ν and σ_τ denote the normal and the tangential traces of a function $\sigma \in Q_1 = \{\tau \in Q : \operatorname{div}\tau \in H\}$. If σ is a regular function, then $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, and the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu.v\, da \quad \forall v \in H_1.$$

Next let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\}$$

and define the set of admissible displacement fields by

$$K = \{v \in V;\ v_\nu \le g \text{ a.e. on } \Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, the following Korn inequality holds [8]:

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.7}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and let $\|\cdot\|_V$ be the associated norm. It follows from (2.7) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent and $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by Sobolev's trace theorem, there exists a constant $d_\Omega > 0$ depending only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.8}$$

In the study of the mechanical problem P_1, we assume that the operator of elasticity F satisfies

$$(2.9)\quad \begin{cases} \text{(a)} & F : \Omega \times S_d \to S_d. \\ \text{(b)} & \text{There exists } M > 0 \text{ such that} \\ & |F(x, \varepsilon_1) - F(x, \varepsilon_2)| \le M|\varepsilon_1 - \varepsilon_2| \\ & \text{for all } \varepsilon_1, \varepsilon_2 \in S_d \text{ and a.e. } x \in \Omega. \\ \text{(c)} & \text{There exists } m > 0 \text{ such that} \\ & (F(x, \varepsilon_1) - F(x, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge m|\varepsilon_1 - \varepsilon_2|^2 \\ & \text{for all } \varepsilon_1, \varepsilon_2 \in S_d \text{ and a.e. } x \in \Omega. \\ \text{(d)} & \text{The mapping } x \mapsto F(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for all } \varepsilon \in S_d. \\ \text{(e)} & \text{The mapping } x \mapsto F(x, 0) \text{ belongs to } Q. \end{cases}$$

An example of a nonlinear elasticity operator F is given by

$$F(\xi) = E\xi + \frac{1}{\lambda}(\xi - P_U\xi) \quad \forall \xi \in S_d,$$

where $\lambda > 0$; $E : S_d \to S_d$ is a fourth order symmetric and positive definite tensor; U denotes a nonempty closed convex set in S_d; and P_U represents the projection mapping.

We also assume as in [11] that the normal compliance function p satisfies

$$(2.10)\quad \begin{cases} \text{(a)} & p : \,]-\infty, g] \to \mathbb{R}; \\ \text{(b)} & \text{there exists } L_p > 0 \text{ such that} \\ & |p(r_1) - p(r_2)| \le L_p|r_1 - r_2| \text{ for all } r_1, r_2 \le g; \\ \text{(c)} & (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \text{ for all } r_1, r_2 \le g; \\ \text{(d)} & p(r) = 0 \text{ for all } r < 0. \end{cases}$$

The forces and the tractions are assumed to satisfy

$$(2.11)\qquad \phi_1 \in H, \quad \phi_2 \in (L^2(\Gamma_2))^d,$$

and we denote by f the element of V given by

$$(f, v)_V = (\phi_1, v)_H + (\phi_2, v)_{(L^2(\Gamma_2))^d} \quad \forall v \in V.$$

We suppose that the friction coefficient μ satisfies

$$(2.12)\qquad \mu \in L^\infty(\Gamma_3), \quad \mu \ge 0 \quad \text{a.e. on } \Gamma_3$$

and

(2.13) $R : H^{-1/2}(\Gamma) \to L^2(\Gamma_3)$ is a linear continuous mapping (see [5]),

where $H^{-1/2}(\Gamma)$ is the dual of $H^{1/2}(\Gamma)$. Next, we define

$$W = \{v \in H_1;\ \operatorname{div} \sigma(v) \in H\}.$$

Let $j_c : V \times V \to \mathbb{R}$, $j_\mu : W \times V \to \mathbb{R}$ and $j : (V \cap W) \times V \to \mathbb{R}$ be the functionals

$$j_c(v,w) = \int_{\Gamma_3} p(v_\nu) w_\nu \, da, \qquad j_\mu(v,w) = \int_{\Gamma_3} \mu |R\sigma_\nu(v)| \, |w_\tau| \, da,$$

$$j(v,w) = j_c(v,w) + j_\mu(v,w).$$

We note that if $v \in W$ then $\sigma(v) \in Q_1$ and $\sigma_\nu(v) \in H^{-1/2}(\Gamma)$. Thus, $j_\mu(v,\cdot)$ makes sense.

Now, in order to establish the weak formulation of Problem P_1, we assume u is a smooth function satisfying (2.1)–(2.6). Let $v \in V$ and multiply the equilibrium of forces (2.2) by $v - u$, integrate the result over Ω and use Green's formula to obtain

$$\int_\Omega \sigma(\varepsilon(v) - \varepsilon(u)) \, dx = \int_\Omega \phi_1.(v-u) \, dx + \int_\Gamma \sigma\nu.(v-u) \, da.$$

Taking into account the boundary conditions (2.3) and $v = 0$ on Γ_1, we get

$$\int_\Gamma \sigma\nu.(v-u) \, da = \int_{\Gamma_2} \phi_2.(v-u) \, da + \int_{\Gamma_3} \sigma\nu.(v-u) \, da.$$

Moreover,

$$\int_{\Gamma_3} \sigma\nu.(v-u) \, da = \int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu) \, da + \int_{\Gamma_3} \sigma_\tau.(v_\tau - u_\tau) \, da.$$

But from the frictional contact conditions (2.6) we have

$$\sigma_\tau.(v_\tau - u_\tau) + \mu|R\sigma_\nu(u)|(|v_\tau| - |u_\tau|) \ge 0 \qquad \forall v_\tau$$

and we see that

$$\int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu) \, da = \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))(v_\nu - u_\nu) \, da - \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu) \, da.$$

Then we deduce that the function u satisfies the inequality

$$\langle F\varepsilon(u), \varepsilon(v-u)\rangle_Q + j(u,v) - j(u,u) \ge (f, v-u)_V + \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))(v_\nu - u_\nu) \, da \qquad \forall v \in V. \tag{2.14}$$

On the other hand,

$$\int_{\Gamma_3} (\sigma_\nu + p(u_\nu)(v_\nu - u_\nu) \, da = \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))((v_\nu - g) - (u_\nu - g)) \, da$$
$$= \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))(v_\nu - g) \, da - \int_{\Gamma_3} (\sigma_\nu + p(u_\nu)(u_\nu - g) \, da.$$

Then using the contact conditions (2.5) yields

$$\int_{\Gamma_3} (\sigma_\nu + p(u_\nu)(v_\nu - g) \, da \ge 0 \qquad \forall v \in K$$

and

$$\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))(u_\nu - g)\, da = 0.$$

Hence we deduce that

$$\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))(v_\nu - u_\nu)\, da \geq 0 \quad \forall v \in K. \tag{2.15}$$

Combining now (2.14) and (2.15) we obtain the following variational formulation of the mechanical problem P_1.

PROBLEM P_2. Find a displacement field $u \in K \cap W$ such that

$$\langle F\varepsilon(u), \varepsilon(v-u)\rangle_Q + j(u,v) - j(u,u) \geq (f, v-u)_V \quad \forall v \in K. \tag{2.16}$$

3. Existence and uniqueness of solution. The main result of this section is the existence and uniqueness of solution for the weak formulation P_2.

THEOREM 3.1. *Let* (2.9)–(2.13) *hold. Then there exists a constant* $\mu_0 > 0$ *such that Problem* P_2 *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

The proof will be carried out in several steps. It is based on fixed point arguments. Let $g \in C_+$ where C_+ is a nonempty closed subset of $L^2(\Gamma_3)$ defined as

$$C_+ = \{s \in L^2(\Gamma_3);\ s \geq 0 \text{ a.e. on } \Gamma_3\}$$

and let $j_g : V \to \mathbb{R}$ be the functional given by

$$j_g(v) = \int_{\Gamma_3} \mu g |v_\tau|\, da \quad \forall v \in V.$$

We now consider the following contact problem with given friction.

PROBLEM P_g. Find $u_g \in K$ such that

$$\langle F\varepsilon(u_g), \varepsilon(v-u_g)\rangle_Q + j_g(v) - j_g(u_g) \geq (f, v-u_g)_V \quad \forall v \in K. \tag{3.1}$$

We prove the following lemma.

LEMMA 3.2. *For any* $g \in C_+$, *Problem* P_g *has a unique solution.*

Proof. Let $A : V \to V$ be the operator given by $(Au, v)_V = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + j_c(u,v)$ for all $u, v \in V$. We use (2.8), (2.9)(b),(c), (2.10)(b)(c) to show that A is strongly monotone and Lipschitz continuous. The functional $j_g : V \to \mathbb{R}$ is a continuous seminorm; since K is a nonempty closed convex subset of V, it follows from the theory of elliptic variational inequalities (see [4]) that the inequality (3.1) has a unique solution. ■

We now consider the mapping $T : C_+ \to C_+$ defined as

$$T(g) = |R\sigma_\nu(u_g)|. \tag{3.2}$$

LEMMA 3.3. *There exists a constant $\mu_0 > 0$ such that T admits a unique fixed point g^* and u_{g^*} is a unique solution of Problem P_2.*

Proof. Let $g_1, g_2 \in C_+$. Using (3.2) we have

$$\|T(g_1) - T(g_2)\|_{L^2(\Gamma_3)} \le \|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\|_{L^2(\Gamma_3)}. \tag{3.3}$$

Thus, by (2.13) and Green's formula, there exists a constant $c_0 > 0$ such that

$$\|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\|_{L^2(\Gamma_3)} \le c_0 M \|u_{g_1} - u_{g_2}\|_V. \tag{3.4}$$

We take $v = u_{g_2}$ in the inequality equivalent to (3.1) with $g = g_1$, and $v = u_{g_1}$ in the inequality equivalent to (3.1) with $g = g_2$, and add the results to obtain

$$\begin{aligned} &\langle F\varepsilon(u_{g_1}) - F\varepsilon(u_{g_2}), \varepsilon(u_{g_1}) - \varepsilon(u_{g_2})\rangle_Q \\ &\qquad + \int_{\Gamma_3} (p(u_{g_{2\nu}}) - p(u_{g_{1\nu}}))(u_{g_{2\nu}} - u_{g_{1\nu}})\, da \\ &\le \int_{\Gamma_3} \mu(g_1 - g_2)(|u_{g_{2\tau}}| - |u_{g_{1\tau}}|)\, da. \end{aligned} \tag{3.5}$$

Keeping in mind (2.10)(c), we have

$$\int_{\Gamma_3} (p(u_{g_{2\nu}}) - p(u_{g_{1\nu}}))(u_{g_{2\nu}} - u_{g_{1\nu}})\, da \ge 0$$

and therefore from (3.5) we deduce that

$$\|u_{g_1} - u_{g_2}\|_V \le \|\mu\|_{L^\infty(\Gamma_3)} \frac{d_\Omega}{m} \|g_1 - g_2\|_{L^2(\Gamma_3)}.$$

Hence using (3.4), we get

$$\|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\|_{L^2(\Gamma_3)} \le c_0 M \|\mu\|_{L^\infty(\Gamma_3)} \frac{d_\Omega}{m} \|g_1 - g_2\|_{L^2(\Gamma_3)}$$

and so by (3.2),

$$\|T(g_1) - T(g_2)\|_{L^2(\Gamma_3)} \le c_0 M \|\mu\|_{L^\infty(\Gamma_3)} \frac{d_\Omega}{m} \|g_1 - g_2\|_{L^2(\Gamma_3)}.$$

Let now

$$\mu_0 = m/c_0 M d_\Omega.$$

Then for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, the mapping T is a contraction, so it admits a unique fixed point g_*, and u_{g^*} is a unique solution to (2.16). ∎

Next denote $u_{g^*} = u$.

REMARK 3.4. As $u_g \in W$ for all $g \in C_+$, we have $u \in W$.

4. Continuous dependence. Next, we investigate the behaviour of the weak solution to Problem P_1 with respect to perturbations of the normal compliance function p. For every $\alpha \geq 0$, let p^α be a perturbation of p which satisfies (2.10) with Lipschitz constant L_p^α. Let us also introduce the functionals j^α, which are obtained by replacing p by p^α in j. We now consider the following problem.

PROBLEM P_α. For every $\alpha \geq 0$, find a displacement field $u^\alpha \in K \cap W$ such that

$$\langle F\varepsilon(u^\alpha), \varepsilon(v-u^\alpha)\rangle_Q + j(u^\alpha, v) - j(u^\alpha, u^\alpha) \geq (f, v-u^\alpha)_V \quad \forall v \in K. \tag{4.1}$$

Using Theorem 3.1 we deduce that for each $\alpha \geq 0$ Problem P_α has a unique solution u^α for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. Suppose now that the contact function p^α satisfies the following assumption:

There exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{cases} \text{(a)} \ |p^\alpha(r) - p(r)| \leq \varphi(\alpha)|r| \quad \forall r \leq g, \\ \text{(b)} \ \lim\limits_{\alpha \to 0} \varphi(\alpha) = 0. \end{cases} \tag{4.2}$$

THEOREM 4.1. *Under the assumption* (4.2) *we have*

$$u^\alpha \to u \quad \textit{strongly in } V \textit{ as } \alpha \to 0. \tag{4.3}$$

Proof. Let $\alpha \geq 0$. From (2.16) and (4.1) we get

$$\langle F\varepsilon(u^\alpha) - F\varepsilon(u), \varepsilon(u^\alpha - u)\rangle_Q \leq j(u, u^\alpha) - j(u, u) + j^\alpha(u^\alpha, u) - j^\alpha(u^\alpha, u^\alpha). \tag{4.4}$$

We have

$$\begin{aligned} &j(u, u^\alpha) - j(u, u) + j^\alpha(u^\alpha, u) - j^\alpha(u^\alpha, u^\alpha) \\ &\quad = (j_c(u, u^\alpha) - j_c(u, u)) + (j_c^\alpha(u^\alpha, u) - j_c^\alpha(u^\alpha, u^\alpha)) \\ &\qquad + (j_\mu(u, u^\alpha) - j_\mu(u, u)) + (j_\mu^\alpha(u^\alpha, u) - j_\mu^\alpha(u^\alpha, u^\alpha)) \\ &\quad = \int_{\Gamma_3} (p(u_\nu) - p(u_\nu^\alpha))(u_\nu^\alpha - u_\nu)\, da + \int_{\Gamma_3} (p(u_\nu^\alpha) - p^\alpha(u_\nu^\alpha))(u_\nu^\alpha - u_\nu)\, da \\ &\qquad + \int_{\Gamma_3} \mu(|R\sigma_\nu(u)| - |R\sigma_\nu(u^\alpha)|)(|u_\tau^\alpha| - |u_\tau|)\, da. \end{aligned} \tag{4.5}$$

Then by (2.10)(c) and (4.5), it follows from (4.4) that

$$\begin{aligned} \langle F\varepsilon(u^\alpha) - F\varepsilon(u), \varepsilon(u^\alpha - u)\rangle_Q \leq &\int_{\Gamma_3} (p(u_\nu^\alpha) - p^\alpha(u_\nu^\alpha))(u_\nu^\alpha - u_\nu)\, da \\ &+ \int_{\Gamma_3} \mu(|R\sigma_\nu(u)| - |R\sigma_\nu(u^\alpha)|)(|u_\tau^\alpha| - |u_\tau|)\, da. \end{aligned}$$

Using now (4.2)(a), we estimate the first term of the right hand side as

$$\int_{\Gamma_3} (p(u_\nu^\alpha) - p^\alpha(u_\nu^\alpha))(u_\nu^\alpha - u_\nu)\, da \le d_\Omega^2 \varphi(\alpha)\|u^\alpha\|_V \|u^\alpha - u\|_V, \tag{4.6}$$

while by (2.8) and (2.13) the other term is estimated as

$$\int_{\Gamma_3} \mu(|R\sigma_\nu(u)| - |R\sigma_\nu(u^\alpha)|)(|u_\tau^\alpha| - |u_\tau|)\, da \le \|\mu\|_{L^\infty(\Gamma_3)} M c_0 d_\Omega \|u - u^\alpha\|_V^2. \tag{4.7}$$

Hence using (2.9)(c), it follows from (4.6) and (4.7) that

$$m\|u^\alpha - u\|_V^2 \le d_\Omega^2 \varphi(\alpha)\|u^\alpha\|_V \|u^\alpha - u\|_V + \|\mu\|_{L^\infty(\Gamma_3)} M c_0 d_\Omega \|u - u^\alpha\|_V^2.$$

This implies

$$(m - \|\mu\|_{L^\infty(\Gamma_3)} M c_0 d_\Omega)\|u^\alpha - u\|_V \le d_\Omega^2 \frac{\varphi(\alpha)\|f\|_V}{m}. \tag{4.8}$$

As

$$m - \|\mu\|_{L^\infty(\Gamma_3)} M c_0 d_\Omega > 0,$$

by going to the limit in (4.8) using (4.2)(b), one obtains (4.3).

5. Finite element approximation. In this section we study the finite element approximation of the variational problem P_1. Let $h \to 0_+$ and let Ω be a polygonal domain in $\mathbb{R}^2$. Then the boundary Γ consists of line segments. We also assume that the sets $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_3$ and $\overline{\Gamma}_2 \cap \overline{\Gamma}_3$ contain only a finite number of points. Write

$$\overline{\Gamma}_3 = \bigcup_{i=1}^{I} \overline{\Gamma}_{3,i}$$

with each $\overline{\Gamma}_{3,i}$ being a line segment. We define the finite-element space

$$V_h = \{v_h \in V \cap (C^0(\overline{\Omega}))^2;\ v_h|_T \in [\mathbf{P}_1(T)]^2,\ \forall T \in \mathcal{T}_h\}$$

where $\mathcal{T}_h$ is a regular triangulation on $\overline{\Omega}$ (see [5]) such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$ and $\mathbf{P}_1(T)$ denotes the set of all polynomials of global degree less than or equal to one with the definition domain T. We suppose that each triangulation is compatible with the boundary decomposition $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$; that is, each point where the boundary condition changes is a node of a set T. Now, before we establish the error estimate for the finite element approximation, we need the following standard interpolation error estimates (see [6]):

$$\begin{aligned} &\|\pi_h v - v\|_V \le c_1 h \|v\|_{(H^2(\Omega))^2}, \\ &\|\pi_h v - v\|_{(L^2(\Gamma_3))^2} \le c_1 h^{3/2} \|v\|_{(H^2(\Omega))^2}, \end{aligned} \tag{5.1}$$

for every $v \in V \cap (H^2(\Omega))^2$, where $\pi_h v$ denotes the V_h-interpolant of the function v. We define the nonempty closed convex set K_h by

$$K_h = \{v_h \in V_h;\ v_{h\nu} \le g \text{ a.e. on } \Gamma_3\}.$$

We remark that $K_h \subset K$ and formulate the discrete problem as

PROBLEM P_h. Find $u_h \in K_h$ such that

$$(5.2)\qquad \langle F\varepsilon(u_h), \varepsilon(v_h - u_h)\rangle_Q + j(u_h, v_h) - j(u_h, u_h) \ge (f, v_h - u_h)_V \quad \forall v_h \in K_h.$$

Under the assumptions of Theorem 3.1, the discrete inequality (5.2) has a unique solution $u_h \in K_h$ for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. Now, to obtain an error estimate, we need to make an additional hypothesis on the regularity of solution. Namely, assume

$$(5.3)\qquad u \in (H^2(\Omega))^2 \cap K.$$

Then we have the following proposition.

THEOREM 5.1. *Suppose that the conditions* (5.1) *and* (5.3) *hold and that* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. *Then*

$$(5.4)\qquad \|u - u_h\|_V \le ch^{3/4}\|u\|_{(H^2(\Omega))^2},$$

where c is a positive constant independent of h.

Proof. Taking $v = u_h$ in (2.16) and $v_h = \pi_h u$ in (5.2) we obtain

$$(5.5)\qquad m\|u - u_h\|_V^2 \le \langle F\varepsilon(u) - F\varepsilon(u_h), \varepsilon(u - u_h)\rangle_Q$$
$$\le \langle F\varepsilon(u_h) - F\varepsilon(u), \varepsilon(\pi_h u - u)\rangle_Q + \langle F\varepsilon(u), \varepsilon(\pi_h u - u)\rangle_Q - (f, \pi_h u - u)_V$$
$$+ j(u, u_h) + j(u_h, \pi_h u) - j(u_h, u_h) - j(u, \pi_h u) + j(u, \pi_h u) - j(u, u).$$

We have

$$(5.6)\qquad |\langle F\varepsilon(u_h) - F\varepsilon(u), \varepsilon(\pi_h u - u)\rangle_Q| \le M\|u - u_h\|_V \|u - \pi_h u\|_V.$$

Taking into account (5.3) and using Green's formula, we get

$$(5.7)\qquad \langle F\varepsilon(u), \varepsilon(\pi_h u - u)\rangle_Q - (f, \pi_h u - u)_V = \int_{\Gamma_3} \sigma(u)\nu.(\pi_h u - u)\, da$$
$$\le \|\sigma.\nu\|_{L^2(\Gamma)} \|\pi_h u - u\|_{(L^2(\Gamma_3))^2} \le c_2\|u\|_{(H^2(\Omega))^2} \|\pi_h u - u\|_{(L^2(\Gamma_3))^2}.$$

As

$$j(u, \pi_h u) - j(u, u) = (j_c(u, \pi_h u) - j_c(u, u)) + (j_\mu(u, \pi_h u) - j_\mu(u, u)),$$

by (2.8) and (2.10)(b), we deduce that

$$(5.8)\qquad j_c(u, \pi_h u) - j_c(u, u) \le \|p(u_\nu)\|_{L^2(\Gamma_3)} \|\pi_h u - u\|_{L^2(\Gamma_3)^2}$$
$$\le L_p\|u_\nu\|_{L^2(\Gamma_3)} \|\pi_h u - u\|_{(L^2(\Gamma_3))^2} \le c_3\|u\|_{(H^2(\Omega))^2} \|\pi_h u - u\|_{(L^2(\Gamma_3))^2}.$$

Following the same reasoning as previously we also have

$$(5.9)\qquad (j_\mu(u,\pi_h u) - j_\mu(u,u)) \le c_4\|u\|_{(H^2(\Omega))^2}\|\pi_h u - u\|_{(L^2(\Gamma_3))^2}.$$

Now since

$$\begin{aligned} j(u,u_h) + j(u_h,\pi_h u) - j(u_h,u_h) - j(u,\pi_h u) \\ = j_c(u,u_h) + j_c(u_h,\pi_h u) - j_c(u_h,u_h) - j_c(u,\pi_h u) \\ + j_\mu(u,u_h) + j_\mu(u_h,\pi_h u) - j_\mu(u_h,u_h) - j_\mu(u,\pi_h u), \end{aligned}$$

using (2.8), (2.10)(b) and (2.10)(c) one obtains

$$(5.10)\qquad \begin{aligned} j_c(u,u_h) + j_c(u_h,\pi_h u) - j_c(u_h,u_h) - j_c(u,\pi_h u) \\ \le L_p d_\Omega^2 \|u-u_h\|_V \|u - \pi_h u\|_V \end{aligned}$$

and

$$(5.11)\qquad \begin{aligned} j_\mu(u,u_h) + j_\mu(u_h,\pi_h u) - j_\mu(u_h,u_h) - j_\mu(u,\pi_h u) \\ \le \|\mu\|_{L^\infty(\Gamma_3)} c_0 M d_\Omega \|u-u_h\|_V^2. \end{aligned}$$

Substitution of (5.6)–(5.11) in (5.5) yields

$$(5.12)\qquad \begin{aligned} (m - \|\mu\|_{L^\infty(\Gamma_3)} c_0 M d_\Omega)\|u-u_h\|_V^2 \\ \le (M + L_p d_\Omega^2)\|u-\pi_h u\|_V \|u-u_h\|_V + c_5\|u\|_{(H^2(\Omega))^2}\|\pi_h u - u\|_{(L^2(\Gamma_3))^2}. \end{aligned}$$

Using Young's inequality

$$ab \le \frac{\delta a^2}{2} + \frac{b^2}{2\delta} \qquad \forall \delta > 0, \forall a,b \in \mathbb{R},$$

and having in mind that $\mu_0 = m/c_0 M d_\Omega$, for

$$\delta < \frac{(\mu_0 - \|\mu\|_{L^\infty(\Gamma_3)}) c_0 M d_\Omega}{M + L_p d_\Omega^2}$$

we deduce from (5.12) that

$$(5.13)\qquad \begin{aligned} \frac{\mu_0 - \|\mu\|_{L^\infty(\Gamma_3)}}{2}\|u-u_h\|_V^2 \\ \le c_6\|u-\pi_h u\|_V^2 + c_5\|u\|_{(H^2(\Omega))^2}\|\pi_h u - u\|_{(L^2(\Gamma_3))^2}. \end{aligned}$$

Therefore, from (5.13) and (5.1), the estimate (5.4) follows.

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