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ON DISCRETE FOURIER ANALYSIS OF AMPLITUDE AND PHASE MODULATED SIGNALS

Abstract. In this work the problem of characterization of the Discrete Fourier Transform (DFT) spectrum of an original complex-valued signal o_t, $t = 0, 1, \ldots, n-1$, modulated by random fluctuations of its amplitude and/or phase is investigated. It is assumed that the amplitude and/or phase of the signal at discrete times of observation are distorted by realizations of uncorrelated random variables or randomly permuted sequences of complex numbers. We derive the expected values and bounds on the variances of such distorted signal DFT spectra. It is shown that the modulation considered in general entails changes in the amplitude and/or phase of the DFT spectra expected values, which together with imposed random deviations with finite variances can vary the amplitudes of peaks existing in the original signal spectrum, and consequently similarity to the original signal spectrum can be significantly blurred.

1. Introduction. The Discrete Fourier Transform (DFT) based periodogram is a widely used tool for analyzing time series that can be decomposed as a sum of monochromatic oscillations plus noise. Important applications of the periodogram include detection of hidden periodicities and estimation of unknown oscillation parameters (amplitude and frequency). Periodogram analysis often yields satisfactory results [3]. For example, it is well known that very accurate frequency estimates of the sinusoidal components can be obtained from the local maxima of a periodogram [20].

If the time series of complex-valued signal observations at discrete equidistant times x_t, $t = 0, 1, \ldots, n-1$, is available, then its Discrete Fourier

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Transform is computed as follows [9]:

$$\tilde{x}_\nu = \frac{1}{n}\sum_{t=0}^{n-1} x_t \exp(-i2\pi\nu t/n) \tag{1}$$

for $\nu = 0, 1, \ldots, n-1$ and integer $n > 0$. As mentioned earlier it can be used to calculate the values of the periodogram

$$I_n(\lambda) = \left|\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} x_t \exp(-i\lambda t)\right|^2, \qquad \lambda \in [-\pi, \pi],$$

at the discrete frequencies $\lambda_\nu = 2\pi\nu/n$, $\nu = 0, 1, \ldots, n-1$.

Frequently, the well-known Fast Fourier Transform procedures are used to perform the relevant calculations [6], [15], [16]. Theoretical as well as numerical properties of the DFT are described in time series analysis textbooks [2], [4], [5], [11]. Certain statistical properties of spectrum estimation using the DFT are investigated in the works of Foster [7], [8], and some other aspects like periodogram smoothing are considered in [17].

The present work deals with the problem of applicability of this technique to spectrum estimation of signals which are subject to random or pseudo-random amplitude and/or phase modulation. The investigation of this problem is justified by the fact that all signals that are normally called "periodic" have some amplitude and phase variation from period to period. For example an active sonar system transmits a periodic pulse train to detect targets. The received pulses are not perfectly periodic due to random modulation of the pulses from scattering and attenuation [10]. Also geophysical signals related to El Niño phenomena are recognized as amplitude and phase modulated [1].

The concept of random and pseudo-random modulation modeling is described in Section 2. Theoretical results relating to the modulated signal DFT spectrum are presented both in the case of a noiseless signal (Section 3) and in the case of signal observations corrupted by uncorrelated random errors (Section 4).

2. Modulation modeling. Let us consider a finite duration time series of complex-valued signal measurements o_t at discrete equidistant times $t = 0, 1, \ldots, n-1$. We assume that the analyzed signal is of deterministic character and involves some regular oscillations. Such a signal can be represented for example by a sum of monochromatic oscillations $\sum_{k=1}^{K} A_k \exp(i\omega_k t + i\Phi_k)$, $t = 0, 1, \ldots, n-1$, with constant frequencies ω_k, amplitudes A_k, and phases Φ_k, $k = 1, \ldots, K$.

Now, let us assume that the amplitude and phase of the signal values at the observation times are distorted by fluctuations a_t and ϕ_t, respectively, according to the model

$$v_t = a_t \exp(i\phi_t) o_t = u_t o_t, \tag{2}$$

where $u_t = a_t \exp(i\phi_t)$, $t = 0, 1, \ldots, n-1$. Hence, we deal with an amplitude and phase modulated signal.

For example, let us assume that ϕ_t and a_t, $t = 0, 1, \ldots, n-1$, are realizations of independent identically distributed random variables, and amplitude distortions are independent of phase distortions. Let the distribution of ϕ_t be uniform on the interval $(-\Phi, \Phi)$, i.e. $\phi_t \sim U(-\Phi, \Phi)$, where $0 < \Phi \le \pi$, which gives immediately

$$m_\phi = E_\phi \exp(i\phi_t) = \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} \exp(ix)\, dx = \frac{\sin(\Phi)}{\Phi},$$

$$\sigma_\phi^2 = E_\phi |\exp(i\phi_t) - m_\phi|^2 = E_\phi |\exp(i\phi_t)|^2 - |m_\phi|^2 = 1 - m_\phi^2.$$

Clearly, $0 \le m_\phi < 1$, and $m_\phi = 0$ only for $\Phi = \pi$, so we also have $0 < \sigma_\phi^2 \le 1$. About the distribution of the real-valued random variables a_t we assume that $E_a a_t = 1$ and $E_a |a_t - E_a a_t|^2 = \sigma_a^2 \ge 0$ (if $\sigma_a = 0$ there is only phase modulation of the signal), which further implies, for $u_t = a_t \exp(i\phi_t)$,

$$\begin{aligned} m_u &= E_a E_\phi a_t \exp(i\phi_t) = E_a a_t E_\phi \exp(i\phi) = m_\phi, \\ \sigma_u^2 &= E_a E_\phi |a_t \exp(i\phi_t) - m_u|^2 \\ &= E_a |a_t|^2 E_\phi |\exp(i\phi_t)|^2 - |m_u|^2 = 1 + \sigma_a^2 - m_\phi^2, \end{aligned}$$

and obviously $\sigma_a^2 < \sigma_u^2 \le 1 + \sigma_a^2$. For instance, a_t can be uniformly distributed on the interval $(1 - A, 1 + A)$, where $A > 0$, and then $\sigma_a^2 = A^2/3$.

Next, we will consider the case of pseudo-random modulation, where $u_t = z_{\sigma(t)}$, $t = 0, 1, \ldots, n-1$, represent some permutation σ of the finite sequence of complex numbers $z_j = \varrho_j \exp(i\varphi_j)$, where $\varrho_j \ge 0$, $\varphi_j \in [0, 2\pi)$, $j = 0, 1, \ldots, n-1$. This case corresponds to the situation when the possible modulation series values are known but we do not know the order of their occurrence in time. It is assumed here that the permutation σ is drawn by simple random sampling with equal selection probabilities $1/n!$ from the set of all permutations of $\{0, 1, \ldots, n-1\}$.

According to our modulation model (2) we have to analyze the DFT spectrum of the signals of the form $v_t = u_t o_t$, $t = 0, 1, \ldots, n-1$, modulated by a time series of complex numbers u_t which represent realizations of complex-valued random variables. Indeed, the case of $u_t = \exp(i\phi_t)$ corresponds to phase modulation, the case of $u_t = a_t \exp(i\phi_t)$ to phase and amplitude modulation, and the case of real-valued modulation series $u_t = a_t$ to amplitude modulation.

In order to compute the DFT of the modulated signals of the form $v_t = u_t o_t$, $t = 0, 1, \ldots, n-1$, we apply the well-known circular convolution formula [9]:

$$\tilde{v}_\nu = \sum_{j+k=\nu \bmod n} \tilde{u}_j \tilde{o}_k = \sum_{\substack{j+k=\nu \text{ or} \\ j+k=n+\nu}} \tilde{u}_j \tilde{o}_k = \sum_{j=0}^{\nu} \tilde{u}_j \tilde{o}_{\nu-j} + \sum_{j=\nu+1}^{n-1} \tilde{u}_j \tilde{o}_{n+\nu-j} \tag{3}$$

for $\nu = 0, 1, \ldots, n-1$. Hence, if we want to analyze the DFT of the modulated signal $\tilde{v}_\nu$, it is necessary to characterize the statistical properties of the modulating series DFT $\tilde{u}_\nu$, $\nu = 0, 1, \ldots, n-1$. Such a characterization is given in the following two lemmas.

In Lemma 2.1 the DFT spectrum of a finite sample od uncorrelated random variables with identical first and second moments is characterized.

LEMMA 2.1. *If complex-valued random variables Z_t, $t = 0, 1, \ldots, n-1$, are uncorrelated, and their mean values and variances satisfy the conditions $E_z Z_t = m_z$ and $E_z|Z_t - m_z|^2 = \sigma_z^2 < \infty$, then for $\nu, \mu = 0, 1, \ldots, n-1$,*

$$E_z \tilde{Z}_\nu = m_z \delta_{0\nu} \quad \text{and} \quad E_z(\tilde{Z}_\nu - E_z\tilde{Z}_\nu)(\overline{\tilde{Z}}_\mu - E_z\overline{\tilde{Z}}_\mu) = \frac{1}{n}\sigma_z^2 \delta_{\nu\mu},$$

where $\delta_{\nu\mu}$ denotes the Kronecker delta.

Proof. Since for any integer $k \neq 0, \pm n, \pm 2n, \ldots$, we have

$$\sum_{t=0}^{n-1} \exp(i2\pi kt/n) = \frac{1 - \exp(i2\pi k)}{1 - \exp(i2\pi k/n)} = 0, \tag{4}$$

the assumptions of the lemma yield for $\nu = 0, 1, \ldots, n-1$,

$$E_z \tilde{Z}_\nu = \frac{m_z}{n} \sum_{t=0}^{n-1} \exp(-i2\pi\nu t/n) = m_z \delta_{0\nu}.$$

The assumed zero correlation of the random variables Z_t, $t = 0, 1, \ldots, n-1$, together with equality (4) imply, for $\nu, \mu = 0, 1, \ldots, n-1$,

$$E_z(\tilde{Z}_\nu - E_z\tilde{Z}_\nu)(\overline{\tilde{Z}}_\mu - E_z\overline{\tilde{Z}}_\mu)$$
$$= \frac{1}{n^2} E_z \sum_{t=0}^{n-1} (Z_t - E_z Z_t) \exp(-i2\pi\nu t/n) \sum_{s=0}^{n-1} (\overline{Z}_s - E_z \overline{Z}_s) \exp(i2\pi\mu s/n)$$

$$= \frac{1}{n^2}\sum_{t=0}^{n-1}\sum_{s=0}^{n-1} E_z(Z_t - E_z Z_t)(\overline{Z}_s - E_z\overline{Z}_s)\exp(-i2\pi(\nu t - \mu s)/n)$$

$$= \frac{1}{n^2}\sum_{t=0}^{n-1}\sigma_z^2\exp(-i2\pi(\nu-\mu)t/n) = \frac{1}{n}\sigma_z^2\delta_{\nu\mu}. \ \blacksquare$$

The assertion on the covariance structure of $\tilde{Z}_\nu$, $\nu = 0,1,\ldots,n-1$, becomes more understandable if we notice that for $\lambda_\nu = 2\pi\nu/n$, $\nu = 0,1,\ldots,n-1$, the vectors $e_\nu = n^{-1/2}(1,\exp(i\lambda_\nu),\ldots,\exp(i\lambda_\nu(n-1)))^T$ form an orthonormal basis in $\mathbb{C}^n$.

In order to analyze the case of modulation by a permuted finite sequence of complex numbers we need the following lemma.

LEMMA 2.2. *Let z_j, $j = 0,1,\ldots,n-1$, be complex numbers and let σ be a randomly selected permutation of $\{0,1,\ldots,n-1\}$, with selection probability $1/n!$. If $c_t = z_{\sigma(t)}$, $t = 0,1,\ldots,n-1$, then for $\nu = 0,1,\ldots,n-1$,*

$$E_\sigma\tilde{c}_\nu = m_n\delta_{\nu 0}, \quad \textit{where} \quad m_n = \frac{1}{n}\sum_{j=0}^{n-1} z_j,$$

$$\mathrm{Var}_\sigma(\tilde{c}_\nu) = \frac{V_n}{n-1}(1-\delta_{\nu 0}), \quad \textit{where} \quad V_n = \frac{1}{n}\sum_{j=0}^{n-1}|z_j - m_n|^2,$$

$$\mathrm{Cov}_\sigma(\tilde{c}_\nu,\tilde{c}_\mu) = 0 \quad \textit{for } \nu \neq \mu,\ \nu,\mu = 0,1,\ldots,n-1.$$

Proof. First, let us note that according to the well-known formula for the expectation of a random variable,

$$E_\sigma c_t = \sum_{j=0}^{n-1} P(\sigma(t)=j)E(z_{\sigma(t)} \mid \sigma(t)=j) = \frac{1}{n}\sum_{j=0}^{n-1} z_j = m_n$$

for $t = 0,1,\ldots,n-1$, and consequently by definition (1) and (4) we have

$$E_\sigma\tilde{c}_\nu = \frac{m_n}{n}\sum_{t=0}^{n-1}\exp(-i2\pi\nu t/n) = m_n\delta_{\nu 0}$$

for $\nu = 0,1,\ldots,n-1$. Further, since

$$E_\sigma|c_t|^2 = \sum_{j=0}^{n-1} P(\sigma(t)=j)E(|z_{\sigma(t)}|^2 \mid \sigma(t)=j) = \frac{1}{n}\sum_{j=0}^{n-1}|z_j|^2$$

we have

$$\mathrm{Var}_\sigma(c_t) = E_\sigma|c_t|^2 - |E_\sigma c_t|^2 = \frac{1}{n}\sum_{j=0}^{n-1}|z_j|^2 - |m_n|^2 = V_n \tag{5}$$

for $t = 0,1,\ldots,n-1$. Furthermore,

$$E_\sigma c_t\overline{c}_s = \sum_{\substack{j,k=0\\ j\neq k}}^{n-1} P(\sigma(t)=j,\, \sigma(s)=k)E(z_{\sigma(t)}\overline{z}_{\sigma(s)} \,|\, \sigma(t)=j,\, \sigma(s)=k)$$
$$= \frac{1}{n(n-1)}\sum_{\substack{j,k=0\\ j\neq k}}^{n-1} z_j\overline{z}_k = \frac{1}{n(n-1)}\sum_{j=0}^{n-1} z_j(n\overline{m}_n - \overline{z}_j)$$
$$= \frac{1}{n(n-1)}\Big[n^2|m_n|^2 - \sum_{j=0}^{n-1}|z_j|^2\Big],$$

which gives

$$\mathrm{Cov}_\sigma(c_t, c_s) = E_\sigma(c_t - E_\sigma c_t)(\overline{c}_s - E_\sigma\overline{c}_s) = E_\sigma c_t\overline{c}_s - E_\sigma c_t E_\sigma \overline{c}_s$$
$$= \frac{1}{n(n-1)}\Big[n^2|m_n|^2 - \sum_{j=0}^{n-1}|z_j|^2\Big] - |m_n|^2 = \frac{1}{(n-1)}\Big[|m_n|^2 - \frac{1}{n}\sum_{j=0}^{n-1}|z_j|^2\Big],$$

and finally

$$\mathrm{Cov}_\sigma(c_t, c_s) = -V_n/(n-1) \quad \text{for } s\neq t,\ s,t = 0,1,\ldots,n-1. \tag{6}$$

Moreover, for any permutation σ,

$$\tilde{c}_0 = \frac{1}{n}\sum_{t=0}^{n-1} c_t = \frac{1}{n}\sum_{t=0}^{n-1} z_{\sigma(t)} = \frac{1}{n}\sum_{j=0}^{n-1} z_j = m_n,$$

which immediately yields $\tilde{c}_0 = E_\sigma\tilde{c}_0$, $\mathrm{Var}_\sigma(\tilde{c}_0) = 0$, and $\mathrm{Cov}_\sigma(\tilde{c}_\nu, \tilde{c}_0) = 0$ for $\nu = 1,\ldots,n-1$. In view of the equalities (5), (6) and (4) we easily obtain, for $\nu,\mu = 1,\ldots,n-1$,

$$\mathrm{Cov}_\sigma(\tilde{c}_\nu, \tilde{c}_\mu)$$
$$= \frac{1}{n^2}E_\sigma\sum_{t=0}^{n-1}(c_t - E_\sigma c_t)\exp(-i2\pi\nu t/n)\sum_{s=0}^{n-1}(\overline{c}_s - E_\sigma\overline{c}_s)\exp(i2\pi\mu s/n)$$
$$= \frac{1}{n^2}\sum_{t=0}^{n-1}\sum_{s=0}^{n-1}E_\sigma(c_t - E_\sigma c_t)(\overline{c}_s - E_\sigma\overline{c}_s)\exp(-i2\pi\nu t/n)\exp(i2\pi\mu s/n)$$
$$= \frac{V_n}{n^2}\sum_{t=0}^{n-1}\exp(-i2\pi(\nu-\mu)t/n) + \frac{V_n}{n^2(n-1)}\sum_{t=0}^{n-1}\exp(-i2\pi(\nu-\mu)t/n)$$
$$- \frac{V_n}{n^2(n-1)}\sum_{t=0}^{n-1}\sum_{s=0}^{n-1}\exp(-i2\pi\nu t/n)\exp(i2\pi\mu s/n)$$
$$= \frac{V_n}{n}\Big(1 + \frac{1}{n-1}\Big)\delta_{\nu\mu} - \frac{V_n}{n^2(n-1)}\sum_{t=0}^{n-1}\exp(-i2\pi\nu t/n)\sum_{s=0}^{n-1}\exp(i2\pi\mu s/n)$$
$$= \frac{V_n}{n-1}\delta_{\nu\mu}. \ \blacksquare$$

It follows from the proof of the above lemma that this time the modulating random variables $c_t = z_{\sigma(t)}$, $t = 0, 1, \ldots, n-1$, have identical mean values and variances but are correlated.

3. Modulated signal spectra. Formula (3) together with Lemmas 2.1 or 2.2 allow us to characterize the DFT spectra corresponding to the modulation models considered. Namely, in the lemmas and corollaries below we derive formulae for the mean values and variances of the random variables $\tilde{v}_\nu$, $\nu = 0, 1, \ldots, n-1$. In the proofs we use the equality (see [9])

$$\sum_{\nu=0}^{n-1} |\tilde{o}_\nu|^2 = \frac{1}{n} \sum_{t=0}^{n-1} |o_t|^2. \tag{7}$$

The corollaries concern the case of bounded signals o_t, $t = 0, 1, \ldots, n-1$, and bounded modulation sequences z_j, $j = 0, 1, \ldots, n-1$.

LEMMA 3.1. *Under the assumptions of Lemma* 2.1 *the DFT of the finite time series $r_t = Z_t o_t$, $t = 0, 1, \ldots, n-1$, satisfies, for $\nu = 0, 1, \ldots, n-1$,*

$$E_z \tilde{r}_\nu = m_z \tilde{o}_\nu \quad \textit{and} \quad E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 = \frac{\sigma_z^2}{n^2} \sum_{t=0}^{n-1} |o_t|^2.$$

Proof. By (3) the first assertion of Lemma 2.1 yields $E_z \tilde{r}_\nu = m_z \tilde{o}_\nu$ for $\nu = 0, 1, \ldots, n-1$. By the same formula

$$\tilde{r}_\nu - E_z \tilde{r}_\nu = \sum_{j=0}^{\nu} (\tilde{Z}_j - E_z \tilde{Z}_j) \tilde{o}_{\nu-j} + \sum_{j=\nu+1}^{n-1} (\tilde{Z}_j - E_z \tilde{Z}_j) \tilde{o}_{n+\nu-j}$$

and the second assertion of Lemma 2.1 implies

$$E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 = \frac{\sigma_z^2}{n} \Big[\sum_{j=0}^{\nu} |\tilde{o}_{\nu-j}|^2 + \sum_{j=\nu+1}^{n-1} |\tilde{o}_{n+\nu-j}|^2 \Big] = \frac{\sigma_z^2}{n} \sum_{\mu=0}^{n-1} |\tilde{o}_\mu|^2$$

for $\nu = 0, 1, \ldots, n-1$, which together with (7) completes the proof. ∎

COROLLARY 3.1. *If $|o_t| \le B < \infty$, $t = 0, 1, \ldots, n-1$, and the assumptions of Lemma* 2.1 *hold, then the DFT of the finite time series $r_t = Z_t o_t$, $t = 0, 1, \ldots, n-1$, satisfies, for $\nu = 0, 1, \ldots, n-1$,*

$$E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 \le \frac{\sigma_z^2 B^2}{n}.$$

LEMMA 3.2. *Under the assumptions of Lemma* 2.2 *the DFT of the finite time series $s_t = z_{\sigma(t)} o_t$, $t = 0, 1, \ldots, n-1$, satisfies, for $\nu = 0, 1, \ldots, n-1$,*

$$E_\sigma \tilde{s}_\nu = m_n \tilde{o}_\nu, \quad \textit{where} \quad m_n = \frac{1}{n}\sum_{j=0}^{n-1} z_j,$$

$$E_\sigma|\tilde{s}_\nu - E_\sigma \tilde{s}_\nu|^2 \le \frac{V_n}{n-1}\,\frac{1}{n}\sum_{t=0}^{n-1}|o_t|^2, \quad \textit{where} \quad V_n = \frac{1}{n}\sum_{j=0}^{n-1}|z_j - m_n|^2.$$

Proof. As in the proof of the previous lemma, equality (3) and the assertions of Lemma 2.2 ensure that $E_\sigma \tilde{s}_\nu = m_n \tilde{o}_\nu$ and

$$E_\sigma|\tilde{s}_\nu - E_\sigma \tilde{s}_\nu|^2 = \frac{V_n}{n-1}\sum_{\substack{\mu=0\\ \mu\ne\nu}}^{n-1}|\tilde{o}_\mu|^2$$

for $\nu = 0, 1, \ldots, n-1$, and the proof is complete in view of (7). ∎

Corollary 3.2. *If $|o_t| \le B < \infty$, $t = 0, 1, \ldots, n-1$, and the assumptions of Lemma 2.2 hold, then the DFT of the finite time series $s_t = z_{\sigma(t)} o_t$, $t = 0, 1, \ldots, n-1$, satisfies, for $\nu = 0, 1, \ldots, n-1$,*

$$E_\sigma|\tilde{s}_\nu - E_\sigma \tilde{s}_\nu|^2 \le \frac{V_n B^2}{n-1}, \quad \textit{where} \quad V_n = \frac{1}{n}\sum_{j=0}^{n-1}|z_j - m_n|^2.$$

Bounded signals $|o_t| \le B < \infty$, $t = 0, 1, \ldots, n-1$, are of course of primary interest in this work since we intend to investigate spectra of regular oscillations of stationary character, modulated by random amplitude and phase fluctuations. Moreover, boundedness of the modulating sequence z_j, $j = 0, 1, \ldots, n-1$, ensures that $V_n/(n-1) \to 0$ as $n \to \infty$. This will hold for the phase modulating sequences $z_j = \exp(i\varphi_j)$, $j = 0, 1, \ldots, n-1$. However, the simple example of the amplitude modulating sequence $z_j = 2^j$, $j = 0, 1, \ldots, n-1$, for which $V_n \sim 3^{-1}4^n/n$, shows that for unbounded modulating sequences we may have $V_n/(n-1) \to \infty$, and then our bound on spectrum variances in Corollary 3.2 is not useful. Hence, we formulate the relevant corollary.

Corollary 3.3. *If $|o_t| \le B < \infty$, $t = 0, 1, \ldots, n-1$, and $|z_j| \le C < \infty$, $j = 0, 1, \ldots, n-1$, and the assumptions of Lemma 2.2 hold, then the DFT of the finite time series $s_t = z_{\sigma(t)} o_t$, $t = 0, 1, \ldots, n-1$, satisfies, for $\nu = 0, 1, \ldots, n-1$,*

$$E_\sigma|\tilde{s}_\nu - E_\sigma \tilde{s}_\nu|^2 \le \frac{C^2 B^2}{n-1}.$$

Lemmas 3.1 and 3.2 show that modulation according to our models can make both the amplitudes and phases of the modulated signal spectrum mean values $E_z \tilde{r}_\nu$ or $E_\sigma \tilde{s}_\nu$, $\nu = 0, 1, \ldots, n-1$, differ from the amplitudes and phases of the original signal spectrum $\tilde{o}_\nu$, $\nu = 0, 1, \ldots, n-1$. The amplitudes of all spectrum mean values are multiplied by the constant factor $|m_z|$

or $|m_n|$, and their phases are changed by the constant additive distortion ϕ_z or Φ_n, where $m_z = |m_z|\exp(i\phi_z)$ or $m_n = |m_n|\exp(i\Phi_n)$, respectively. Phase distortion does not occur if m_z or m_n are real-valued as in the example described in the introduction or if $z_j = \varrho_j \geq 0$, $j = 0, 1, \ldots, n-1$. Such changes of the amplitudes and phases of the spectrum values are clearly of non-random character. However, there are also random effects of modulation which are characterized by variances of the random variables $\tilde{r}_\nu$, $\nu = 0, 1, \ldots, n-1$. If the modulating series Z_t, $t = 0, 1, \ldots, n-1$, involves realizations of uncorrelated random variables with identical mean values and variances, then according to Corollary 3.1 the variances $\mathrm{Var}_z(\tilde{r}_\nu)$, $\nu = 0, 1, \ldots, n-1$, decrease uniformly to zero as $n \to \infty$, whenever the original signal o_t, $t = 0, 1, \ldots, n-1$, is bounded. According to Corollary 3.3 the same property holds also in the case of a bounded modulating sequence z_j, $j = 0, 1, \ldots, n-1$, and bounded signals o_t, $t = 0, 1, \ldots, n-1$. Consequently in the two cases considered, the influence of the random distortions $\tilde{r}_\nu - E_z\tilde{r}_\nu$ or $\tilde{s}_\nu - E_\sigma\tilde{s}_\nu$, $\nu = 0, 1, \ldots, n-1$, on the modulated signal spectrum diminishes asymptotically as $n \to \infty$. This means that they will not blur completely the discrete spectrum mean values on which they are superimposed. Some small peaks present in the original signal spectrum can be smoothed due to amplitude and phase modulation but possibly larger ones will be still distinguishable.

Hence, any peaks present in the amplitude spectrum of the original bounded signal $|\tilde{o}_\nu|$, $\nu = 0, 1, \ldots, n-1$, may be less distinguishable in the modulated signal amplitude spectrum $|\tilde{r}_\nu|$ or $|\tilde{s}_\nu|$, $\nu = 0, 1, \ldots, n-1$, especially when $|m_z| < 1$ or $|m_n| < 1$, respectively. In the extreme case of $m_z = 0$ or $m_n = 0$ we have $E_z\tilde{r}_\nu = 0$ or $E_\sigma\tilde{s}_\nu = 0$, $\nu = 0, 1, \ldots, n-1$, and then the modulated signal spectrum will have purely stochastic character without any frequencies distinguished, so that any peaks present in the original signal spectrum will be completely blurred. For a bounded modulation sequence z_j, $j = 0, 1, \ldots, n-1$, and bounded signals o_t, $t = 0, 1, \ldots,$ $n-1$, this extreme effect can be asymptotically approached if $m_n \to 0$ as $n \to \infty$. This can occur if $z_j = \exp(i\varphi_j)$, $j = 0, 1, \ldots, n-1$, and the infinite sequence φ_j, $j = 0, 1, \ldots,$ is uniformly distributed in the interval $[0, 2\pi]$ [18].

4. Spectra of modulated noisy signal. Assume now that the time series of the original signal values o_t, $t = 0, 1, \ldots, n-1$, is corrupted by random observation errors, according to the model

$$y_t = o_t + \eta_t, \quad t = 0, 1, \ldots, n-1, \tag{8}$$

where η_t are realizations of uncorrelated complex-valued random variables having zero mean $E_\eta\eta_t = 0$ and finite second moment $\sigma_\eta^2 = E_\eta|\eta_t|^2 < \infty$.

Let us see what happens if the corrupted signal values are submitted to random or pseudo-random modulation of the same kind as above, i.e. $v_t = u_t y_t$, $t = 0, 1, \ldots, n-1$. In what follows we assume that the observation errors η_t, $t = 0, 1, \ldots, n-1$, are independent of the random variables forming the modulation series u_t, $t = 0, 1, \ldots, n-1$. We can prove the following lemmas and corollaries.

LEMMA 4.1. *Under the assumptions of Lemma* 2.1 *the DFT of the finite time series* $r_t = Z_t y_t$, $t = 0, 1, \ldots, n-1$, *satisfies, for* $\nu = 0, 1, \ldots, n-1$,

$$E_\eta E_z \tilde{r}_\nu = m_z \tilde{o}_\nu, \qquad E_\eta E_z |\tilde{r}_\nu - E_\eta E_z \tilde{r}_\nu|^2 = \frac{\sigma_z^2}{n^2} \sum_{t=0}^{n-1} |o_t|^2 + \frac{(\sigma_z^2 + |m_z|^2)\sigma_\eta^2}{n}.$$

Proof. Since the assertions of Lemmas 2.1 hold, Lemma 3.1 ensures that $E_z \tilde{r}_\nu = m_z \tilde{y}_\nu$ and by (8) and Lemma 2.1 applied to the observation errors η_t, $t = 0, 1, \ldots, n-1$, we have $E_\eta E_z \tilde{r}_\nu = m_z E_\eta(\tilde{o}_\nu + \tilde{\eta}_\nu) = m_z \tilde{o}_\nu$ for $\nu = 0, 1, \ldots, n-1$. Furthermore,

$$E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 = \frac{\sigma_z^2}{n^2} \sum_{t=0}^{n-1} |y_t|^2$$

for $\nu = 0, 1, \ldots, n-1$, and simple calculation shows that

$$\begin{aligned} E_z |\tilde{r}_\nu - E_\eta E_z \tilde{r}_\nu|^2 &= E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu + E_z \tilde{r}_\nu - E_\eta E_z \tilde{r}_\nu|^2 \\ &= E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 + |E_z \tilde{r}_\nu - E_\eta E_z \tilde{r}_\nu|^2 \\ &= E_z |\tilde{r}_\nu - E_z \tilde{r}_\nu|^2 + |m_z(\tilde{y}_\nu - \tilde{o}_\nu)|^2. \end{aligned}$$

Hence, the above equalities together with (8) and Lemma 2.1 imply

$$\begin{aligned} E_\eta E_z |\tilde{r}_\nu - E_\eta E_z \tilde{r}_\nu|^2 &= \frac{\sigma_z^2}{n^2} \sum_{t=0}^{n-1} E_\eta |o_t + \eta_t|^2 + E_\eta |m_z \tilde{\eta}_\nu|^2 \\ &= \frac{\sigma_z^2}{n^2} \sum_{t=0}^{n-1} [|o_t|^2 + E_\eta |\eta_t|^2] + \frac{|m_z|^2 \sigma_\eta^2}{n} \end{aligned}$$

for $\nu = 0, 1, \ldots, n-1$, which completes the proof. ∎

LEMMA 4.2. *Under the assumptions of Lemma* 2.2 *the DFT of the finite time series* $s_t = z_{\sigma(t)} y_t$, $t = 0, 1, \ldots, n-1$, *satisfies, for* $\nu = 0, 1, \ldots, n-1$,

$$E_\eta E_\sigma \tilde{s}_\nu = m_n \tilde{o}_\nu, \quad \textit{where} \quad m_n = \frac{1}{n} \sum_{j=0}^{n-1} z_j,$$

$$E_\eta E_\sigma |\tilde{s}_\nu - E_\eta E_\sigma \tilde{s}_\nu|^2 \le \frac{V_n}{n-1} \frac{1}{n} \sum_{t=0}^{n-1} |o_t|^2 + \sigma_\eta^2 \left(\frac{V_n}{n-1} + \frac{|m_n|^2}{n} \right),$$

where

$$V_n = \frac{1}{n}\sum_{j=0}^{n-1} |z_j - m_n|^2.$$

Proof. The proof is analogous to the proof of Lemma 4.1 except that it is now based on Lemmas 2.2 and 3.2. ■

From Lemmas 4.1 and 4.2 we can easily deduce corollaries analogous to 3.1–3.3 for the case of bounded signals o_t, $t = 0, 1, \ldots, n-1$, and bounded modulation sequences z_j, $j = 0, 1, \ldots, n-1$, respectively. Thus, we see that the presence of zero-mean uncorrelated errors corrupting the original signal values does not change the character of the DFT spectrum of the modulated series. Indeed, the formulae for the spectrum mean values remain the same as in the case of errorless signal modulation, and the spectrum variances in Lemmas 4.1 and 4.2 differ from those of Lemmas 3.1 and 3.2 by the relevant additive terms which occur because of non-zero second moment of the observation errors. This means that our earlier assertions concerning the behaviour of the modulated signal spectrum hold also in the case of a bounded signal corrupted by uncorrelated random errors which have zero mean and identical finite second moment.

5. Conclusions. The properties of the DFT spectrum examined in this work are helpful in understanding the possible changes such a spectrum undergoes in the case of random amplitude and/or phase modulation of the original signal. Our modulation model includes distortions of stochastic nature in the amplitudes and/or phases of the original signal values at observation times. For bounded signals of deterministic character (like a sum of monochromatic oscillations with constant amplitudes and phases) it is proved that occurrence of random or pseudo-random amplitude and/or phase modulation of the signal can completely change the character of its DFT spectrum. Namely, the amplitude and/or phase modulated signal spectrum may have a purely stochastic character. On the other hand it is also shown that in certain cases the modulated bounded signal spectrum can still resemble the spectrum of the original signal, although small peaks can be significantly smoothed. Similar conclusions are deduced also in the case of a deterministic signal which is corrupted at the times of observation by uncorrelated random errors with zero mean and finite second moment. Since the DFT is linear, the results obtained also help to understand the influence of modulating a particular component on the spectrum of a signal which is a sum of several modulated components (e.g. modulated monochromatic oscillations).

It is worth remarking that our conclusions complete the observations of Ni and Huo [13], concerning importance of phase and amplitude information in signal and image reconstruction. The concept of phase randomization used to obtain multivariate surrogate time series [12] with distribution similar to the series being observed is also related to the subject considered here.

Hinich [10] used a similar approach to amplitude modulation modeling to derive statistics for detecting randomly modulated pulses in noise. Detection of random amplitude modulation is also the subject of [14]. The Singular Spectrum Analysis (SSA) method has an important property, first noted by Vautard and Ghil [19], that it may be used directly to identify modulated oscillations in the presence of noise. Allen and Robertson [1] proposed a generalization of the "Monte Carlo SSA" algorithm which allows for objective testing for the presence of modulated oscillations at low signal-to-noise ratios in multivariate data.

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