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ON FULLY COUPLED CONTINUOUS TIME RANDOM WALKS

Abstract. Continuous time random walks with jump sizes equal to the corresponding waiting times for jumps are considered. Sufficient conditions for the weak convergence of such processes are established and the limiting processes are identified. Furthermore one-dimensional distributions of the limiting processes are given under an additional assumption.

1. Introduction. A *continuous time random walk* (for short CTRW) is a random walk in which jumps of random sizes are separated by random intervals of time. Such a process is generated by a sequence of random vectors (Y_k, J_k) with values in $\mathbb{R}^d \times \mathbb{R}_+$, where the sequence $\{Y_k\}$ represents the successive jump sizes with values in $\mathbb{R}^d$ and the sequence of positive random variables J_k gives the waiting times between the successive jumps. A CTRW is called *coupled* if Y_k and J_k are dependent for all $k \geq 1$ (see [1]). Let $N(t) \stackrel{df}{=} \max\{k : J_1 + \cdots + J_k \leq t\}$ be the number of jumps up to time t. There are the following two types of CTRW (see [5, 11]):

$$Z(t) \stackrel{df}{=} \sum_{k=1}^{N(t)} Y_k \quad \text{and} \quad \tilde{Z}(t) \stackrel{df}{=} \sum_{k=1}^{N(t)+1} Y_k.$$

$Z(t)$ represents the position of the random walk after the last jump before time t and $\tilde{Z}(t)$ is its position at the first jump after t. The CTRWs have numerous applications in physics, financial mathematics and many other fields. For an overview of applications see [9, 4, 8, 6].

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A CTRW in case $Y_k = J_k$, $k \geq 1$, is called here *fully coupled* and in this case the processes Z and $\tilde{Z}$ are denoted

$$X(t) \stackrel{df}{=} \sum_{k=1}^{N(t)} J_k \quad \text{and} \quad \tilde{X}(t) \stackrel{df}{=} \sum_{k=1}^{N(t)+1} J_k.$$

Such processes have various applications, for example in renewal theory and reliability theory. Namely, consider a situation in which components of a machine break down at random times and are instantly replaced by new ones. Then $N(t)$ is the number of failures up to time t, while $X(t)$ is the time of the last failure before t and $\tilde{X}(t)$ is the time of the first failure after t. Hence $\gamma(t) \stackrel{df}{=} t - X(t)$ can be interpreted as the age of the component currently working and $\tilde{\gamma}(t) \stackrel{df}{=} \tilde{X}(t) - t$ as the residual life time of the component working at time t. Well known results on asymptotic behavior of these characteristics are gathered in Proposition 2.8 of [4].

The asymptotics at infinity of the processes Z and $\tilde{Z}$ is usually investigated via weak convergence in the Skorokhod J_1 topology of sequences of scaled CTRWs $b_n^{-1}Z(n\,\cdot)$ and $b_n^{-1}\tilde{Z}(n\,\cdot)$, where $\{b_n\}$ with $b_n \to 0$ is an appropriate scaling sequence (see e.g. [1, 5]). This problem is usually investigated in a general setting via weak convergence of processes Z_n and $\tilde{Z}_n$ defined by the rows of the array $\{(Y_{n,k}, J_{n,k}), n, k \geq 1\}$ as follows:

$$Z_n(t) \stackrel{df}{=} \sum_{k=1}^{N_n(t)} Y_{n,k} \quad \text{and} \quad \tilde{Z}_n(t) \stackrel{df}{=} \sum_{k=1}^{N_n(t)+1} Y_{n,k},$$

where $N_n(t) = \max\{k \geq 0 : J_{n,1} + \cdots + J_{n,k} \leq t\}$. Recently, nice conditions for weak convergence of the processes Z_n and $\tilde{Z}_n$ were given by Theorem 3.6 in [11]. It states that if for each $n \geq 1$, $\{(Y_{n,k}, J_{n,k}), k \geq 1\}$ is a sequence of i.i.d. random vectors, and the processes

$$T_n(t) \stackrel{df}{=} J_{n,1} + \cdots + J_{n,[nt]}$$

weakly converge to a strictly increasing process D in the Skorokhod J_1 topology, then the processes Z_n, $\tilde{Z}_n$, and related processes of age and remaining lifetime

$$\gamma_n(t) = t - \sum_{i=1}^{N_n(t)} J_{n,i} \quad \text{and} \quad \tilde{\gamma}_n(t) = \sum_{i=1}^{N_n(t)+1} J_{n,i} - t, \tag{1}$$

weakly converge in the J_1 topology. The proof of this theorem in [11] is based on a continuous mapping approach in the Skorokhod J_1 topology.

The aim of the paper is to strengthen the general results given in [11, Theorem 3.6] in the case of fully coupled CTRW processes, i.e.

$$X_n(t) \stackrel{df}{=} \sum_{k=1}^{N_n(t)} J_{n,k} \quad \text{and} \quad \tilde{X}_n(t) \stackrel{df}{=} \sum_{k=1}^{N_n(t)+1} J_{n,k},$$

by relaxing the assumption that the process D is strictly increasing. We emphasize that elements of the array $\{J_{n,k}, n, k \geq 1\}$ may be dependent. In Theorem 1 we show that the processes X_n and $\tilde{X}_n$ weakly converge in the J_1 topology whenever the processes T_n weakly converge to a nondecreasing process D (note that D is always nondecreasing since the processes T_n are nondecreasing). Lemmas 1 and 2 play a key role in proving Theorem 1 and are similar to Proposition 2.3 in [11]; they are stronger in a sense, although less general. Theorem 2 characterizes the one-dimensional distributions of the limiting processes for the sequences $\{X_n\}$ and $\{\tilde{X}_n\}$ when D is a Lévy process with strictly increasing sample paths. The formulas given in Theorem 2 may be obtained from Theorem 4.9 in [11] but we present an alternative proof, more elementary, based on the invariance principle.

2. Main results. In this paper $\mathbb{D}[0,\infty)$ denotes the space of *càdlàg* functions on $[0,\infty)$, i.e. of all right-continuous functions having left limits and taking real values; $\mathbb{D}_{u,\uparrow}$ denotes the space of all nonnegative, nondecreasing and unbounded *càdlàg* functions; $x \circ y$ denotes composition of functions x and y defined as $x \circ y(t) = x(y(t))$, where $x, y \in \mathbb{D}_{u,\uparrow}[0,\infty)$; y^{-1} denotes the inverse function to y defined as $y^{-1}(t) = \inf\{s > 0 : y(s) > t\}$; and $\mathrm{Disc}(x)$ denotes the set of discontinuity points of x. Furthermore, $\xi_n \Rightarrow \xi$ denotes convergence in distribution of the random elements ξ_n to ξ, which we also call *weak convergence*.

Define the inverse of the process T_n as $T_n^{-1}(t) \stackrel{df}{=} \inf\{s : T_n(s) > t\}$, $t \geq 0$. Then $T_n^{-1}(t) = N_n(t)/n + 1/n$. Hence $\tilde{X}_n(t) = T_n(T_n^{-1}(t))$ and $X_n(t) = T_n(T_n^{-1}(t) - 1/n)$. Trajectories of the above processes are in $\mathbb{D}_{u,\uparrow}$. Auxiliary results formulated below in Lemmas 1–3 are the basis for our main results stated in Theorem 1 and dealing with weak convergence of X_n and $\tilde{X}_n$.

LEMMA 1. *Let x_n, $n \geq 1$, be elements of $\mathbb{D}_{u,\uparrow}[0,\infty)$ converging in the J_1 topology to $x \in \mathbb{D}_{u,\uparrow}[0,\infty)$. Then $x_n \circ x_n^{-1} \to x \circ x^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology.*

LEMMA 2. *Let x_n, $n \geq 1$, be elements of $\mathbb{D}_{u,\uparrow}[0,\infty)$ converging in the J_1 topology to $x \in \mathbb{D}_{u,\uparrow}[0,\infty)$. Then $(x_n \circ x_n^{-1})^{-1} \to (x \circ x^{-1})^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology.*

LEMMA 3. *Let x_n, $n \geq 1$, be elements of $\mathbb{D}_{u,\uparrow}[0,\infty)$ starting from zero with* $\mathrm{Disc}(x_n) = \{i/n : i \in \mathbb{N}\}$ *and such that each x_n is constant on the intervals $[i/n, (i+1)/n)$, $i = 0, 1, \ldots$. Then*

$$x_n \circ (x_n^{-1} - 1/n) = (x_n \circ x_n^{-1})^{-1}.$$

Theorem 1. *If $T_n \Rightarrow D$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology, then $\tilde{X}_n \Rightarrow D \circ D^{-1}$ and $X_n \Rightarrow (D \circ D^{-1})^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology.*

Let f_- denote the left-continuous version of the right-continuous function f, i.e. $f_-(t) = \lim_{h\downarrow 0} f(t-h)$, and let g_+ denote the right-continuous version of the left-continuous function g, i.e. $g_+(t) = \lim_{h\downarrow 0} g(t+h)$.

Remark 1. Using formula (18) and the right-continuity of the trajectories of D it is easy to check that

$$(D \circ D^{-1})^{-1} = (D_- \circ (D^{-1})_-)_+.$$

This agrees with the general form of the limiting process for the sequence $\{Z_n\}$ given in Theorem 3.6 of [11] (see also Remark 3.3 of [5]).

The following theorem gives the one-dimensional distributions of the processes M and $\tilde{M}$ defined by $M(t) = (D \circ D^{-1})^{-1}(t)$ and $\tilde{M}(t) = D \circ D^{-1}(t)$, $t \geq 0$, where D is a Lévy process with strictly increasing sample paths.

Theorem 2. *Let D be a Lévy process with Lévy measure ν_D such that $\nu_D(0,\infty) = \infty$, i.e. D has strictly increasing sample paths. Then*

$$P(M(t) < x) = \int_0^x \int_0^\infty \nu_D(t-u,\infty) P(D(s) \in du)\, ds \quad \text{for } x < t, \tag{2}$$

and

$$P(\tilde{M}(t) - t \geq x) = \int_0^t \int_0^\infty \nu_D(x+t-u,\infty) P(D(s) \in du)\, ds \quad \text{for } x > 0. \tag{3}$$

Remark 2. If T_n, $n \geq 1$, converge weakly to a strictly increasing Lévy process D, then by Theorem 1 we have $X_n \Rightarrow M$ and $\tilde{X}_n \Rightarrow \tilde{M}$ and the distributions of the limiting processes are given by (2) and (3). Notice that mutual independence of elements in the array $\{J_{n,k}, n, k \geq 1\}$ is not necessary. Sufficient conditions for weak convergence of processes of partial sums of dependent random variables to a Lévy process are given in [3] (see also [12]).

Using the relation (1) we get the following illustration of Theorems 1–2 in terms of age and remaining life time processes.

Corollary 1. *Let $\{J_{n,k}, n, k \geq 1\}$ be an array such that T_n converges weakly to an α-stable subordinator D with $0 < \alpha < 1$. Then the following convergences hold:*

$$P(\gamma_n(t) \leq x) \to \frac{\sin(\pi\alpha)}{\pi} \int_{1-x/t}^{1} (1-u)^{-\alpha} u^{\alpha-1}\, du \quad \text{for all } x \leq t, \tag{4}$$

$$P(\tilde{\gamma}_n(t) \leq x) \to \frac{\sin(\pi\alpha)}{\pi} \int_{1}^{1+x/t} (v-1)^{-\alpha} v^{-1}\, dv \quad \text{for all } x \geq 0. \tag{5}$$

REMARK 3. Convergence (4) generalizes the result of Kotulski [7, p. 790]. All necessary calculations needed to prove (4) and (5) are given in Example 5.2 in [5], so we omit the proof of the above corollary.

3. Proofs of the results

Proof of Lemma 1. The first part of the proof gives an explicit formula for the function $y \circ y^{-1}$ and characterizes its discontinuity points, i.e. the set $\mathrm{Disc}(y \circ y^{-1})$ for any $y \in \mathbb{D}_{u,\uparrow}[0,\infty)$. This is used in the second part of the proof to show that $x_n \circ x_n^{-1} \to x \circ x^{-1}$ if $x_n \to x$ in the J_1 topology.

Observe that for any $y \in \mathbb{D}_{u,\uparrow}[0,\infty)$ the composition $y \circ y^{-1}$ is also an element of $\mathbb{D}_{u,\uparrow}[0,\infty)$ and is of the form

$$\begin{aligned}(6)\qquad (y \circ y^{-1})(t) &= y(y^{-1}(t)) = y(\inf\{s > 0 : y(s) > t\}) \\ &= \inf\{y(s) : y(s) > t\}.\end{aligned}$$

Functions in $\mathbb{D}_{u,\uparrow}[0,\infty)$ have countably many discontinuity points and are continuous at 0. For technical reasons, to avoid considering too many cases in the proof, we add 0 to $\mathrm{Disc}(y)$ and put $y(0-) = 0$. Furthermore we allow $\mathrm{Disc}(y)$ to be dense in $[0,\infty)$. The domain of $y \circ y^{-1}$ can be written as $B_1 \cup B_2$ where

$$B_1 = \bigcup_{\tau_y \in \mathrm{Disc}(y)} [y(\tau_y-), y(\tau_y)) \quad \text{and} \quad B_2 = \mathbb{R}_+ \setminus B_1.$$

Let $t \in [y(\tau_y^*-), y(\tau_y^*))$ for some $\tau_y^* \in \mathrm{Disc}(y)$. Then $y^{-1}(t) = \inf\{s > 0 : y(s) > t\} = \tau_y^*$, so

$$(7)\qquad (y \circ y^{-1})(t) = y(\tau_y^*) \quad \text{for } t \in [y(\tau_y^*-), y(\tau_y^*)) \subset B_1.$$

Let $t \in B_2$. Then there exists s such that $t = y(s)$. Because y is right-continuous and nondecreasing we have

$$y^{-1}(t) = \inf\{s > 0 : y(s) > t\} = \sup\{s : t = y(s)\} = s_0.$$

We show that $y(s_0) = t$. This is obvious if there is only a single $s > 0$ such that $y(s) = t$. If $y(s) = t$ for more than one s, then $y(s_0-) = t$. Since $t \in B_2$, the equality $y(s_0) = t$ must also hold. Indeed, if $y(s_0) > t$, then $s_0 \in \mathrm{Disc}(y)$ and it would mean that $t \in [y(s_0-), y(s_0))$, so $t \in B_1$, which contradicts the assumption that $t \in B_2$. Hence

$$(8)\qquad (y \circ y^{-1})(t) = y(\sup\{s : t = y(s)\}) = t \quad \text{for } t \in B_2.$$

Define

$$(9)\qquad \tau_y(t) := \sup\{\tau_y \in \mathrm{Disc}(y) : t \ge y(\tau_y-)\}.$$

Observe that if $t \in B_1$, then there exists $\tau_y^* \in \mathrm{Disc}(y)$ such that $\tau_y^* = \tau_y(t)$.

Observe also that the following equivalences hold:

$$(10)\qquad t < y(\tau_y(t)) \;\Leftrightarrow\; \exists \tau_y^* \in \mathrm{Disc}(y) : t \in [y(\tau_y^*-), y(\tau_y^*)) \;\Leftrightarrow\; t \in B_1.$$

To show $\Rightarrow$ notice that by the definition of $\tau_y(t)$ there exists a sequence $\{\tau_{y,n}\}$ of discontinuity points of y such that $\tau_{y,n} \uparrow \tau_y(t)$ and $t \ge y(\tau_{y,n}-)$. Since y has limits from the left we get $t \ge \lim_n y(\tau_{y,n}-) = y(\tau_y(t)-)$. This and the assumption $t < y(\tau_y(t))$ imply that $y(\tau_y(t)-) \le t < y(\tau_y(t))$. Hence $\tau_y(t) \in \mathrm{Disc}(y)$, which proves the first implication. The other implications are obvious.

Combining (7), (8) and (10) we get

$$(y \circ y^{-1})(t) = \max\{t, y(\tau_y(t))\}. \tag{11}$$

Now we characterize $\mathrm{Disc}(y \circ y^{-1})$. Notice that by (7) for any $\tau^* \in \mathrm{Disc}(y)$ we have

$$(y \circ y^{-1})(y(\tau^*-)) = y(\tau^*).$$

Moreover, by (11) we get

$$\begin{aligned}(y \circ y^{-1})(y(\tau^*-)-) &= \lim_{h \searrow 0} (y \circ y^{-1})(y(\tau^*-) - h) \\ &= \lim_{h \searrow 0} \max\{y(\tau^*-) - h,\ y(\sup\{\tau_y \in \mathrm{Disc}(y) : y(\tau^*-) - h \ge y(\tau_y-)\})\} \\ &= \max\{y(\tau^*-),\ y(\sup\{\tau_y \in \mathrm{Disc}(y) : y(\tau^*-) > y(\tau_y-)\})\} = y(\tau^*-).\end{aligned}$$

Hence

$$(y \circ y^{-1})(y(\tau^*-)) - (y \circ y^{-1})(y(\tau^*-)-) = y(\tau^*) - y(\tau^*-) > 0$$

and $y(\tau^*-) \in \mathrm{Disc}(y \circ y^{-1})$. By (11) it follows that the only discontinuity points of $y \circ y^{-1}$ are $y(\tau_y-)$, $\tau_y \in \mathrm{Disc}(y)$. Therefore

$$\mathrm{Disc}(y \circ y^{-1}) = \{y(\tau_y-) : \tau_y \in \mathrm{Disc}(y)\}.$$

Since for all $\tau_y \in \mathrm{Disc}(y)$ we have $(y \circ y^{-1})(y(\tau_y-)-) = y(\tau_y-)$ and $(y \circ y^{-1})(y(\tau_y-)) = y(\tau_y)$ it follows that the sets of values y and $y \circ y^{-1}$ are the same, i.e. $\{y(t) : t \ge 0\} = \{y \circ y^{-1}(t) : t \ge 0\}$ and the corresponding jump sizes of y and $y \circ y^{-1}$ are the same.

Now we prove that if $x, x_n \in \mathbb{D}_{u,\uparrow}[0,\infty)$ and $x_n \to x$ in the J_1 topology, then $x_n \circ x_n^{-1} \to x \circ x^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ with the J_1 topology. It is sufficient to show that $x_n \circ x_n^{-1} \to x \circ x^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,T]$ in the J_1 topology for any $0 < T < \infty$ which is a continuity point of $x \circ x^{-1}$. Choose S such that $x(S) = T$ and $S \notin \mathrm{Disc}(x)$. For the remaining part of the proof, $x_n \circ x_n^{-1}$ and $x \circ x^{-1}$ denote the functions restricted to $[0,T]$ and x_n, x denote the functions restricted to $[0,S]$.

Let $\varepsilon > 0$ be an arbitrarily small number. Then x has finitely many, say K_ε, jumps greater than or equal to ε. Let

$$G(\varepsilon) := \{\tau \in \mathrm{Disc}(x) : x(\tau) - x(\tau-) \ge \varepsilon\} = \{\tau^{(1)} < \cdots < \tau^{(K_\varepsilon)}\}.$$

Arguing as in [13, p. 79], for each n one can choose a set $G_n(\varepsilon) =$

$\{\tau_n^{(1)} < \cdots < \tau_n^{(K_\varepsilon)}\}$ such that

$$\max_{i \le K_\varepsilon} |\tau_n^{(i)} - \tau^{(i)}| \to 0,$$
$$\max_{i \le K_\varepsilon} |x_n(\tau_n^{(i)}) - x(\tau^{(i)})| \to 0,$$
$$\max_{i \le K_\varepsilon} |x_n(\tau_n^{(i)}-) - x(\tau^{(i)}-)| \to 0.$$

Hence we can take m_ε so large that for all $n > m_\varepsilon$ the expressions on the left-hand sides above are smaller than ε, $G_n(\varepsilon) \subset \mathrm{Disc}(x_n)$ and

$$\sup\{|x_n(\tau) - x_n(\tau-)| : \tau \in \mathrm{Disc}(x_n) \setminus G_n(\varepsilon)\} < \varepsilon.$$

Let $\lambda_n, n \ge 1$, be continuous, strictly increasing mappings of $[0,T]$ onto $[0,T]$ such that

$$\lambda_n(x(\tau^{(i)}-)) = x_n(\tau_n^{(i)}-), \qquad \lambda_n(x(\tau^{(i)})) = x_n(\tau_n^{(i)}), \tag{12}$$

for $\tau_n^{(i)} \in G_n(\varepsilon)$, $\tau^{(i)} \in G(\varepsilon)$, $1 \le i \le K_\varepsilon$, and suppose they are linear between the points

$$\{(x(\tau^{(i)}-), x_n(\tau_n^{(i)}-)), (x(\tau^{(i)}), x_n(\tau_n^{(i)})) : 1 \le i \le K_\varepsilon\}.$$

Then for all $n > m_\varepsilon$ we have

$$\begin{aligned}\|\lambda_n - e\| &= \max_{i \le K_\varepsilon}\{|\lambda_n(x(\tau^{(i)}-)) - x(\tau^{(i)}-)| \vee |\lambda_n(x(\tau^{(i)})) - x(\tau^{(i)})|\} \\ &= \max_{i \le K_\varepsilon}\{|x_n(\tau_n^{(i)}-) - x(\tau^{(i)}-)| \vee |x_n(\tau_n^{(i)}) - x(\tau^{(i)})|\} < \varepsilon,\end{aligned}$$

where $a \vee b = \max\{a,b\}$. Moreover $\|\lambda_n^{-1} - e\| = \|\lambda_n - e\| < \varepsilon$ for all $n > m_\varepsilon$.

Now, using (11), we will estimate

$$\|x_n \circ x_n^{-1} \circ \lambda_n - x \circ x^{-1}\| = \sup_{t \in [0,T]} R_n(t)$$

for arbitrary $n > m_\varepsilon$, where

$$R_n(t) \overset{df}{=} |\max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} - \max\{t, x(\tau_x(t))\}|.$$

Assuming first that $\max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} \le \max\{t, x(\tau_x(t))\}$ we get

$$R_n(t) = \max\{t, x(\tau_x(t))\} - \max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\}.$$

Of course we have either (1°) $t \ge x(\tau_x(t))$ or (2°) $t < x(\tau_x(t))$.

In case (1°) we have

$$R_n(t) = t - \max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} \le t - \lambda_n(t) \le \|\lambda_n - e\| < \varepsilon.$$

In case (2°), we use (10) to find that $t \in [x(\tau_x^*-), x(\tau_x^*))$ for some $\tau_x^* \in \mathrm{Disc}(x)$ and $\tau_x(t) = \tau_x^*$. Then we have two further possibilities.

(i): τ_x^* is a jump time of x with jump size greater than ε, i.e. $\tau_x^* = \tau^{(i_0)} \in G(\varepsilon)$ for some $i_0 \le K_\varepsilon$. Then $t \in [x(\tau^{(i_0)}-), x(\tau^{(i_0)}))$ and using (12) we get

$$\lambda_n(t) \in [\lambda_n(x(\tau^{(i_0)}-)), \lambda_n(x(\tau^{(i_0)}))) = [x_n(\tau_n^{(i_0)}-), x_n(\tau_n^{(i_0)})).$$

Notice that by (9) we have $\tau_{x_n}(\lambda_n(t)) = \tau_n^{(i_0)}$ and $x_n(\tau_{x_n}(\lambda_n(t))) = x_n(\tau_n^{(i_0)})$, which gives

$$R_n(t) = x(\tau^{(i_0)}) - \max\{\lambda_n(t), x_n(\tau_n^{(i_0)})\} \le x(\tau^{(i_0)}) - x_n(\tau_n^{(i_0)}) \le \varepsilon.$$

(ii): τ_x^* is a jump time of x with jump size smaller than ε, i.e. $\tau_x^* \notin G(\varepsilon)$. Then $t \in [x(\tau_x^*-), x(\tau_x^*)) \subset [x(\tau_x^*) - \varepsilon, x(\tau_x^*))$ and $\lambda_n(t) \in [\lambda_n(x(\tau_x^*) - \varepsilon), \lambda_n(x(\tau_x^*)))$. Hence

$$\begin{aligned} R_n(t) &= x(\tau_x(t)) - \max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} \le x(\tau_x(t)) - \lambda_n(t) \\ &\le x(\tau_x^*) - \lambda_n(x(\tau_x^*) - \varepsilon) \\ &= x(\tau_x^*) - (x(\tau_x^*) - \varepsilon) + (x(\tau_x^*) - \varepsilon) - \lambda_n(x(\tau_x^*) - \varepsilon) \\ &\le \varepsilon - (\lambda_n(x(\tau_x^*) - \varepsilon) + (x(\tau_x^*) - \varepsilon)) \le \|\lambda_n - e\| + \varepsilon < 2\varepsilon. \end{aligned}$$

Now assuming that $\max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} > \max\{t, x(\tau_x(t))\}$ we get

$$R_n(t) = \max\{\lambda_n(t), x_n(\tau_{x_n}(\lambda_n(t)))\} - \max\{t, x(\tau_x(t))\}.$$

Again we have either (1°) $\lambda_n(t) \ge x_n(\tau_{x_n}(\lambda_n(t)))$ or (2°) $\lambda_n(t) < x_n(\tau_{x_n}(\lambda_n(t)))$. In case (1°) we have

$$R_n(t) = \lambda_n(t) - \max\{t, x(\tau_x(t))\} \le \lambda_n(t) - t \le \|\lambda_n - e\| < \varepsilon.$$

In case (2°) we use (10) to find that $\lambda_n(t) \in [x_n(\tau_{x_n}^*-), x_n(\tau_{x_n}^*))$ for some $\tau_{x_n}^* \in \mathrm{Disc}(x_n)$ and $\tau_{x_n}(\lambda_n(t)) = \tau_{x_n}^*$. Then we need to consider two further possibilities.

(i): $\tau_{x_n}^*$ is a jump time of x_n with jump size greater than ε, i.e. $\tau_{x_n}^* = \tau_n^{(i_0)} \in G_n(\varepsilon)$ for some $i_0 \le K_\varepsilon$. Then

$$\lambda_n(t) \in [x_n(\tau_n^{(i_0)}-), x_n(\tau_n^{(i_0)})) = [\lambda_n(x(\tau^{(i_0)}-)), \lambda_n(x(\tau^{(i_0)}))),$$

which gives $t \in [x(\tau^{(i_0)}-), x(\tau^{(i_0)}))$. Hence $\tau_x(t) = \tau^{(i_0)}$ and $x(\tau_x(t)) = x(\tau^{(i_0)})$ and we have

$$\begin{aligned} R_n(t) &= x_n(\tau_{x_n}(\lambda_n(t))) - \max\{t, x(\tau_x(t))\} \\ &= x_n(\tau_n^{(i_0)}) - \max\{t, x(\tau^{(i_0)})\} \le x_n(\tau_n^{(i_0)}) - x(\tau^{(i_0)}) \le \varepsilon. \end{aligned}$$

(ii): $\tau_{x_n}^*$ is a jump time of x_n with jump size smaller than ε, i.e. $\tau_{x_n}^* \notin G_n(\varepsilon)$. Then $\lambda_n(t) \in [x_n(\tau_{x_n}^*-), x_n(\tau_{x_n}^*)) \subset [x_n(\tau_{x_n}^*) - \varepsilon, x_n(\tau_{x_n}^*))$ and $t \in [\lambda_n^{-1}(x_n(\tau_{x_n}^*) - \varepsilon), \lambda_n^{-1}(x_n(\tau_{x_n}^*)))$. Hence

$$\begin{aligned} R_n(t) &= x_n(\tau_{x_n}(\lambda_n(t))) - \max\{t, x(\tau_x(t))\} \\ &= x_n(\tau_{x_n}^*) - \max\{t, x(\tau_x(t)\} \le x_n(\tau_{x_n}^*) - t \le x_n(\tau_{x_n}^*) - \lambda_n^{-1}(x_n(\tau_{x_n}^*) - \varepsilon) \\ &= x_n(\tau_{x_n}^*) - (x_n(\tau_{x_n}^*) - \varepsilon) + (x_n(\tau_{x_n}^*) - \varepsilon) - \lambda_n^{-1}(x_n(\tau_{x_n}^*) - \varepsilon) \\ &\le \varepsilon - \lambda_n^{-1}(x_n(\tau_{x_n}^*) - \varepsilon) + (x_n(\tau_{x_n}^*) - \varepsilon)) \\ &\le \|\lambda_n^{-1} - e\| + \varepsilon = \|\lambda_n - e\| + \varepsilon < 2\varepsilon. \end{aligned}$$

Summarizing, we get $\|x_n \circ x_n^{-1} \circ \lambda_n - x \circ x^{-1}\| = \sup_{t\in[0,T]} R_n(t) < 2\varepsilon$ for any $n > m_\varepsilon$.

This implies that for arbitrary ε there exists m_ε such that for all $n > m_\varepsilon$ we have

$$d_{J_1}(x_n \circ x_n^{-1}, x \circ x^{-1}) \le \max\{\|\lambda_n - e\|, \|x_n \circ x_n^{-1} \circ \lambda_n - x \circ x^{-1}\|\} < 2\varepsilon,$$

which completes the proof of the lemma. ∎

Proof of Lemma 2. First we give an explicit formula for $(y \circ y^{-1})^{-1}(t)$. For that choose an arbitrary $t \ge 0$ and consider the case when $t \in B_1$. Then there exists $\tau_y^* \in \mathrm{Disc}(y)$ such that $t \in [y(\tau_y^*-), y(\tau_y^*))$. Observe that

$$\begin{aligned} y(\tau_y^*) &= \inf\{y(v) : y(v) > y(\tau_y^*-)\} \\ &= \inf\{y(v) : y(v) > s\} \quad \text{if } s \in [y(\tau_y^*-), y(\tau_y^*)). \end{aligned}$$

Using this and formula (6) we get

$$\begin{aligned} (13)\quad & (y \circ y^{-1})^{-1}(t) \\ &= \inf\{s > 0 : (y \circ y^{-1})(s) > t\} = \inf\{s > 0 : \inf\{y(v) : y(v) > s\} > t\} \\ &= \inf\{s > 0 : \inf\{y(v) : y(v) > s\} > y(\tau_y^*-)\} \\ &= \inf\{s \in [y(\tau_y^*-), y(\tau_y^*)) : \inf\{y(v) : y(v) > s\} > y(\tau_y^*-)\} = y(\tau_y^*-). \end{aligned}$$

Now fix $t \in B_2$. Then there exists $s > 0$ such that $y(s) = t$. Let $s_0 = \sup\{s : y(s) = t\}$. We will show that $y(s_0) = t$. This is obvious if there is only a single $s > 0$ such that $y(s) = t$. If $y(s) = t$ for more than one s then $y(s_0-) = t$ and from $t \in B_2$ it also follows that $y(s_0) = t$. Now we will show that

$$(14)\qquad (y \circ y^{-1})^{-1}(t) = t \quad \text{for } t \in B_2.$$

First notice that by the right continuity of y it follows that for every $\{s_n\}$ such that $s_n \searrow s_0$ we also have $y(s_n) \searrow y(s_0) = t$. Fix such a sequence $\{s_n\}$. Since

$$\begin{aligned} (15)\qquad (y \circ y^{-1})^{-1}(t) &= \inf\{z > 0 : (y \circ y^{-1})(z) > t\} \\ &= \inf\{z : \inf\{y(v) : y(v) > z\} > t\}, \end{aligned}$$

it is sufficient to find $z_n \searrow t$ such that $\inf\{y(v) : y(v) > z_n\} > t$. The sequence $\{z_n = y(s_n)\}$ is one we are looking for. Indeed, $z_n \searrow t$ and $\inf\{y(v) : y(v) > z_n\} \ge z_n = y(s_n) > y(s_0) = t$, which proves (14). Now define

$$(16)\qquad \eta_y(t) := \inf\{\tau_y \in \mathrm{Disc}(y) : t \le y(\tau_y)\}.$$

Notice that if $t \in B_1$, then $\eta_y(t) \in \mathrm{Disc}(y)$. Observe also that the following equivalence holds:

$$(17)\qquad t > y(\eta_y(t)-) \ \Leftrightarrow\ \eta_y(t) \in \mathrm{Disc}(y) \text{ and } t \in (y(\eta_y(t)-), y(\eta_y(t))].$$

To prove $\Rightarrow$ notice that by the definition of $\eta_y(t)$, there exist $\eta_{y,n} \in \mathrm{Disc}(y)$, $n \geq 1$, such that $\eta_{y,n} \downarrow \eta_y(t)$ and $y(\eta_{y,n}) \geq t$. Then by right continuity of y we have $t \leq y(\eta_{y,n}) \downarrow y(\eta_y(t))$, so $t \leq y(\eta_y(t))$. Moreover, if $t > y(\eta_y(t)-)$, we get $y(\eta_y(t)-) < t \leq y(\eta_y(t))$, which implies that $\eta_y(t) \in \mathrm{Disc}(y)$, so $\Rightarrow$ holds. Notice that $\Leftarrow$ is obvious.

Combining (13), (15) and (17) we get

$$(y \circ y^{-1})^{-1}(t) = \min\{t, y(\eta_y(t)-)\}. \tag{18}$$

Fix $\tau_y^* \in \mathrm{Disc}(y)$. Using the same argument as in the proof of (13) we show that

$$\begin{aligned}(y \circ y^{-1})^{-1}(y(\tau_y^*)-) &= \inf\{s > 0 : (y \circ y^{-1})(s) > y(\tau_y^*)-\} \\ &= \lim_{h \searrow 0} \inf\{s > 0 : \inf\{y(v) : y(v) > s\} > y(\tau_y^*) - h\} \\ &= \inf\{s > 0 : \inf\{y(v) : y(v) > s\} > y(\tau_y^*-)\} \\ &= \inf\{s \in [y(\tau_y^*-), y(\tau_y^*)) : \inf\{y(v) : y(v) > s\} > y(\tau_y^*-)\} = y(\tau_y^*-).\end{aligned}$$

Moreover

$$(y \circ y^{-1})^{-1}(y(\tau_y^*)) = \inf\{s > 0 : \inf\{y(v) : y(v) > s\} > y(\tau_y^*)\} \geq y(\tau_y^*).$$

Hence $y(\tau_y^*) \in \mathrm{Disc}((y \circ y^{-1})^{-1})$. By (18) and (17) we easily check that

$$\mathrm{Disc}((y \circ y^{-1})^{-1}) = \{y(\tau_y) : \tau_y \in \mathrm{Disc}(y)\}.$$

Now we show that $(x_n \circ x_n^{-1})^{-1} \to (x \circ x^{-1})^{-1}$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology. We use the same method as in the proof of Lemma 1. Let $T > 0$ be an arbitrary continuity point of $(x \circ x^{-1})^{-1}$ and consider the restrictions of $(x_n \circ x_n^{-1})^{-1}$ and $(x \circ x^{-1})^{-1}$ to $[0, T]$. Choose $S \notin \mathrm{Disc}(x)$ such that $x(S) = T$ and restrict x_n and x to $[0, S]$. Then choose an arbitrarily small $\varepsilon > 0$ and take sets $G_n(\varepsilon)$, $G(\varepsilon)$, numbers K_ε and m_ε and functions λ_n as in the proof of Lemma 1.

Now, using (18), we estimate

$$\|(x_n \circ x_n^{-1})^{-1} \circ \lambda_n - (x \circ x^{-1})^{-1}\| = \sup_{t \in [0,T]} R_n(t)$$

for $n > m_\varepsilon$, where

$$R_n(t) := |\min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} - \min\{t, x(\eta_x(t)-)\}|.$$

Assuming first that $\min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} \geq \min\{t, x(\eta_x(t)-)\}$ we get

$$R_n(t) = \min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} - \min\{t, x(\eta_x(t)-)\}.$$

Of course we have either (1°) $t \leq x(\eta_x(t)-)$ or (2°) $t > x(\eta_x(t)-)$.

In case (1°) we have

$$R_n(t) = \min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} - t \leq \lambda_n(t) - t \leq \|\lambda_n - e\| < \varepsilon.$$

In case (2°) we use (17) to get $\eta_x(t) = \tau_x^*$ for some $\tau_x^* \in \mathrm{Disc}(x)$ and $t \in (x(\tau_x^*-), x(\tau_x^*)]$. In this case we need to consider two further possibilities.

(i): τ_x^* is a jump time of x with jump size greater than ε, i.e. $\tau_x^* = \tau^{(i_0)} \in G(\varepsilon)$ for some $i_0 \le K_\varepsilon$. Then $t \in (x(\tau^{(i_0)}-), x(\tau^{(i_0)})]$ and

$$\lambda_n(t) \in (\lambda_n(x(\tau^{(i_0)}-)), \lambda_n(x(\tau^{(i_0)}))] = (x_n(\tau_n^{(i_0)}-), x_n(\tau_n^{(i_0)})].$$

Observe that $\eta_{x_n}(\lambda_n(t)) = \inf\{\tau_{x_n} \in \mathrm{Disc}(x_n) : \lambda_n(t) \le x_n(\tau_{x_n})\} = \tau_n^{(i_0)}$. Hence $x_n(\eta_{x_n}(\lambda_n(t))-) = x_n(\tau_n^{(i_0)}-)$ and

$$\begin{aligned} R_n(t) &= \min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} - x(\eta_x(t)-) \\ &= \min\{\lambda_n(t), x_n(\tau_n^{i_0}-)\} - x(\tau^{i_0}-) \le x_n(\tau_n^{(i_0)}-) - x(\tau^{(i_0)}-) \le \varepsilon. \end{aligned}$$

(ii): τ_x^* is a jump time of x with jump size smaller than ε, i.e. $\tau_x^* \notin G(\varepsilon)$. Then $t \in (x(\tau_x^*-), x(\tau_x^*)] \subset (x(\tau_x^*-), x(\tau_x^*-)+\varepsilon]$ and $\lambda_n(t) \in (\lambda_n(x(\tau_x^*-)), \lambda_n(x(\tau_x^*-)+\varepsilon)]$. Hence

$$\begin{aligned} R_n(t) &= \min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} - x(\eta_x(t)-) \le \lambda_n(t) - x(\tau_x^*-) \\ &\le \lambda_n(x(\tau_x^*-)+\varepsilon) - x(\tau_x^*-) \\ &= \lambda_n(x(\tau_x^*-)+\varepsilon) - (x(\tau_x^*-)+\varepsilon) + (x(\tau_x^*-)+\varepsilon) - x(\tau_x^*-) \\ &\le \lambda_n(x(\tau_x^*-)+\varepsilon) - (x(\tau_x^*-)+\varepsilon) + \varepsilon \le \|\lambda_n - e\| + \varepsilon < 2\varepsilon. \end{aligned}$$

To estimate $R_n(t)$ in case $\min\{\lambda_n(t), x_n(\eta_{x_n}(\lambda_n(t))-)\} < \min\{t, x(\tau_x(t)-)\}$ we interchange the roles of x and x_n, i.e. we consider the cases $\lambda_n(t) \le x_n(\eta_{x_n}(\lambda_n(t))-)$ and $\lambda_n(t) > x_n(\eta_{x_n}(\lambda_n(t))-)$ instead of the cases $t \le x(\eta_x(t)-)$ and $t > x(\eta_x(t)-)$, respectively, as in the proof of Lemma 1. Hence for all $n > m_\varepsilon$ we have $\|(x_n \circ x_n^{-1})^{-1} \circ \lambda_n - (x \circ x^{-1})^{-1}\| = \sup_{t\in[0,T]} R_n(t) < 2\varepsilon$.

Thus we have shown that for every ε there exists m_ε such that for all $n > m_\varepsilon$ we have

$$\begin{aligned} &d_{J_1}((x_n \circ x_n^{-1})^{-1}, (x \circ x^{-1})^{-1}) \\ &\qquad \le \max\{\|\lambda_n - e\|, \|(x_n \circ x_n^{-1})^{-1} \circ \lambda_n - (x \circ x^{-1})^{-1}\|\} < 2\varepsilon, \end{aligned}$$

which completes the proof. ∎

Proof of Lemma 3. By (11) we have $(x_n \circ x_n^{-1})(t) = x_n(i/n)$ for $t \in [x_n(i/n-), x_n(i/n)) = [x_n((i-1)/n), x_n(i/n))$. For any t there exists i_0 such that $t \in [x_n((i_0-1)/n), x_n(i_0/n))$. For this t we have

$$\begin{aligned} (x_n \circ x_n^{-1})^{-1}(t) &= \inf\{s > 0 : (x_n \circ x_n^{-1})(s) > t\} \\ &= \inf\{s > 0 : (x_n \circ x_n^{-1})(s) > x_n((i_0-1)/n)\} \\ &= \inf\{s > 0, s \in [x_n((i-1)/n), x_n(i/n)) :\ x_n(i/n) > x_n((i_0-1)/n)\} \\ &= x_n((i_0-1)/n). \end{aligned}$$

On the other hand,

$$x_n^{-1}(t) = \inf\{s > 0 : x_n(s) > t\} = \inf\{i/n : x_n(i/n) > t\}$$
$$= \inf\{i/n : x_n(i/n) > x_n((i_0 - 1)/n)\} = i_0/n.$$

Hence $x_n(x_n^{-1}(t) - 1/n) = x_n(i_0/n - 1/n) = x_n((i_0 - 1)/n) = (x_n \circ x_n^{-1})^{-1}(t)$, which completes the proof. ∎

Proof of Theorem 1. By Lemma 3 we have $X_n = T_n(T_n^{-1} - 1/n) = (T_n \circ T_n^{-1})^{-1}$. Now Theorem 1 immediately follows from Lemmas 1–2 and Theorem 5.5 of [2]. ∎

Proof of Theorem 2. We apply Theorem 1 to a special type of distribution of $J_{n,k}$. Assume that for each $n \geq 1$ the random variables $J_{n,1}, J_{n,2}, \ldots$ are iid with distribution such that for any nonnegative number x,

$$P(J_{n,1} > x) = \begin{cases} 1 & \text{for } x < x_n, \\ \nu_D(x,\infty)/\nu_D(x_n,\infty) & \text{for } x > x_n, \end{cases}$$

where ν_D is a Lévy measure with subordinator D satisfying $\nu_D(0,\infty) = \infty$, and $\{x_n\}$ is a sequence of positive numbers such that $x_n \downarrow 0$ in such a way that $\nu_D(x_n,\infty)/n \to 1$. Then $J_{n,1} \xrightarrow{p} 0$ and $nP(J_{n,1} > x) \to \nu_D(x,\infty)$ for all $x > 0$ and $T_n \Rightarrow D$. Using Theorem 1 we get $X_n \Rightarrow M$ and $\tilde{X}_n \Rightarrow \tilde{M}$. Since D is stochastically continuous, we have $X_n(t) \Rightarrow M(t)$ and $\tilde{X}_n(t) \Rightarrow \tilde{M}(t)$ for any $t > 0$.

Now we show (2) for $x < t$. Choose a sufficiently small $\varepsilon > 0$. Since $\nu_D(0,\infty) = \infty$, by using Theorem 3.1 from [9] we find that the integral $\int_0^\infty \int_0^t \nu_D(t-u,\infty) P(D(s) < x, D(s) \in du)\, ds$ is finite, so there exists b_1 such that $\int_{b_1}^\infty \int_0^t \nu_D(t-u,\infty) P(D(s) < x, D(s) \in du)\, ds \leq \varepsilon$.

Note that the weak convergence $T_n \Rightarrow D$ implies the weak convergence of the sequence $\{N_n(t)/n\}$ for any fixed $t > 0$, so also its tightness. Therefore for the chosen ε and fixed $t > 0$ there exists a positive integer b_2 such that $P(N_n(t)/n > b_2) < \varepsilon$ for all $n \geq 1$. Take $b = \max\{b_1, b_2\}$ and define $R_n \stackrel{df}{=} \{r = k/n : k \leq bn\}$. Notice that

$$\begin{aligned} (19) \qquad P(X_n(t) < x) &= P(T_n(N_n(t)/n) < x,\, N_n(t)/n > b) \\ &\quad + P(T_n(N_n(t)/n) < x,\, N_n(t)/n \leq b) \\ &\leq \varepsilon + \sum_{r \in R_n} P(T_n(N_n(t)/n) < x,\, N_n(t) = nr) \\ &= \varepsilon + \sum_{r \in R_n} P(T_n(r) < x,\, N_n(t) = nr). \end{aligned}$$

Observe that

$$\begin{aligned} (20) \qquad & P(T_n(r) < x,\, N_n(t) = nr) \\ &= P(T_n(r) < x,\, N_n(t) \geq nr) - P(T_n(r) < x,\, N_n(t) \geq nr + 1) \end{aligned}$$

$$= P(T_n(r) < x,\, T_n(r) \le t) - P(T_n(r) < x,\, T_n(r) + J_{n,nr+1} \le t)$$

$$= \int_0^t (1 - P(J_{n,1} \le t - u))P(T_n(r) < x,\, T_n(r) \in du).$$

Define

$$H_n(x) \stackrel{df}{=} \sum_{r \in R_n} P(T_n(r) < x,\, N_n(t) = nr).$$

Since sample paths of T_n are step functions, for any nonnegative integer k we have

$$\int_{k/n}^{(k+1)/n} P(T_n(s) < x,\, T_n(s) \in du)\, ds = \frac{1}{n} P(T_n(k/n) < x,\, T_n(k/n) \in du).$$

Hence

$$H_n(x) = \int_0^t nP(J_{n,1} > t - u) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\, ds$$
$$= \int_0^t nP(J_{n,1} > t - u)\, \kappa_n(du),$$

where

$$\kappa_n(G) \equiv \int_0^b P(T_n(s) < x,\, T_n(s) \in G)\, ds, \quad G \in \mathcal{B}(\mathbb{R}_+).$$

Since the Lévy process D is a subordinator with Lévy measure $\nu_D(0,\infty) = \infty$, Theorem 27.4 on p. 175 of [10] shows that $D(s)$ has continuous distribution for all s. By the convergence $T_n \Rightarrow D$ in $\mathbb{D}_{u,\uparrow}[0,\infty)$ in the J_1 topology and by continuity of distribution of $D(s)$ for all $s \ge 0$ we deduce that for any Borel set G which is a continuity set of the measure κ the following convergence holds:

$$\kappa_n(G) \to \int_0^b P(D(s) < x,\, D(s) \in G)\, ds \equiv \kappa(G). \tag{21}$$

Notice that

$$\int_0^{t-x_n} nP(J_{n,1} > t - u) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\, ds$$
$$= \frac{n}{\nu_D(x_n,\infty)} \int_0^{t-x_n} \nu_D(t-u,\infty) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\, ds$$
$$= \frac{n}{\nu_D(x_n,\infty)} \int_0^{t-x_n} \nu_D(t-u,\infty)\, \kappa_n(du).$$

Define probability measures $\tilde{\kappa}$ and $\tilde{\kappa}_n$, $n \geq 1$, on $[0,t]$ by $\tilde{\kappa}(G) \stackrel{df}{=} \kappa(G)/\kappa[0,t]$ and $\tilde{\kappa}_n(G) \stackrel{df}{=} \kappa_n(G)/\kappa_n[0,t]$ for Borel sets G in $[0,t]$. Then from (21) we get $\tilde{\kappa}_n \Rightarrow \tilde{\kappa}$. Since $\tilde{\kappa}$ is a continuous distribution, the set of discontinuities of the function $h(u) = \nu_D(t-u,\infty)$ has zero measure with respect to $\tilde{\kappa}$. Hence by Theorem 5.1 of [2],

$$\int_0^t \nu_D(t-u,\infty)\,\kappa_n(du) \to \int_0^t \nu_D(t-u,\infty)\,\kappa(du).$$

The assumptions $x_n \downarrow 0$ and $\nu_D(x_n,\infty)/n \to 1$ now give the convergence

$$\begin{aligned}\frac{n}{\nu_D(x_n,\infty)} \int_0^{t-x_n} \nu_D(t-u,\infty) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\,ds \\ \to \int_0^t \nu_D(t-u,\infty) \int_0^b P(D(s) < x,\, D(s) \in du)\,ds.\end{aligned}$$

Observe that for $x < t$ we get

$$\begin{aligned}\int_{t-x_n}^t nP(J_{n,1} > t-u) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\,ds \\ = n\int_0^b P(T_n(s) < x,\, T_n(s) \in (t-x_n,t))\,ds = 0,\end{aligned}$$

because $(0,x) \cap (t-x_n,t) = \emptyset$ for all sufficiently large n. Finally we get

$$\begin{aligned}H_n(x) &= \int_0^t nP(J_{n,1} > t-u) \int_0^b P(T_n(s) < x,\, T_n(s) \in du)\,ds \\ &\to \int_0^t \nu_D(t-u,\infty) \int_0^b P(D(s) < x,\, D(s) \in du)\,ds.\end{aligned} \tag{22}$$

Using $P(X_n(t) < x) \to P(M(t) < x)$, the inequality $0 \leq P(X_n(t) < x) - H_n(x) \leq \varepsilon$ and the convergence (22), we get

$$P(X_n(t) < x) \to \int_0^t \nu_D(t-u,\infty) \int_0^\infty P(D(s) < x,\, D(s) \in du)\,ds,$$

which completes the proof of the equality (2) in the theorem.

The proof of (3) is similar. Here we only consider the main differences. Now we set $B = (x,\infty)$ for any fixed $x > 0$ and we have

$$\begin{aligned}&P(\tilde{X}_n(t) - t \in B) \\ &\quad = P(\tilde{X}_n(t) \in B+t,\, N_n(t)/n > b) + P(\tilde{X}_n(t) \in B+t,\, N_n(t)/n \leq b) \\ &\quad \leq \varepsilon + P(\tilde{X}_n(t) \in B+t,\, N_n(t)/n \leq b)\end{aligned}$$

and

$$P(\tilde{X}_n(t) \in B+t,\ N_n(t)/n \le b) = \sum_{r\in R_n} P(\tilde{X}_n(t) \in B+t,\ N_n(t) = rn)$$
$$= \sum_{r\in R_n} P(T_n(r+1/n) \in B+t,\ N_n(t) \ge rn)$$
$$- \sum_{r\in R_n} P(T_n(r+1/n) \in B+t,\ N_n(t) \ge rn+1).$$

Observe that

$$\begin{aligned}P(T_n(r+1/n) &\in B+t,\ N_n(t) \ge rn) \\ &= P(T_n(r) + J_{n,rn+1} \in B+t,\ T_n(r) \le t) \\ &= \int_0^t P(J_{n,rn+1} \in B+t-u) P(T_n(r) \in du)\end{aligned}$$

and

$$\begin{aligned}P(T_n(r+1/n) &\in B+t,\ N_n(t) \ge rn+1) \\ &= P(T_n(r) + J_{n,rn+1} \in B+t,\ T_n(r) + J_{n,rn+1} \le t) = 0.\end{aligned}$$

Using the above we get

$$P(T_n(r+1/n) \in B+t,\ N_n(t) = rn) = \int_0^t P(J_{n,1} \in B+t-u) P(T_n(r) \in du).$$

Finally we obtain

$$\begin{aligned}P(\tilde{X}_n(t) &\in B+t) \\ &\le \varepsilon + \sum_{r\in R_n} \int_0^t P(J_{n,1} > x+t-u) P(T_n(r) \in du) = \varepsilon + \tilde{H}_n(B),\end{aligned}$$

where

$$\tilde{H}_n(B) = \int_0^t nP(J_{n,1} > x+t-u) \int_0^b P(T_n(s) \in du)\, ds.$$

But for sufficiently large n we have

$$\tilde{H}_n(B) = \frac{n}{\nu_D(x_n,\infty)} \int_0^t \nu_D(x+t-u,\infty) \int_0^b P(T_n(s) \in du)\, ds$$
$$\to \int_0^t \nu_D(x+t-u,\infty) \int_0^b P(D(s) \in du)\, ds.$$

Since $0 \le P(\tilde{X}_n(t) - t > x) - \tilde{H}_n(B) \le \varepsilon$, and

$$0 \le \int_0^t \nu_D(x+t-u,\infty) \int_0^\infty P(D(s)\in du)\,ds - \int_0^t \nu_D(x+t-u,\infty) \int_0^b P(D(s)\in du)\,ds \le \varepsilon,$$

where $\varepsilon > 0$ is any small number, we get (3), which completes the proof of the theorem. ■

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