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QUANTILE HEDGING FOR BASKET DERIVATIVES

Abstract. The problem of quantile hedging for basket derivatives in the Black–Scholes model with correlation is considered. Explicit formulas for the probability maximizing function and the cost reduction function are derived. Applicability of the results to the widely traded derivatives like digital, quantos, outperformance and spread options is shown.

1. Introduction. As recent events on the market have shown, the risk in pricing financial contracts should be more thoroughly surveyed. Although the problem of minimizing risk is widely studied in the literature, the great majority of the results do not meet the expectations of practitioners who are interested in straightforward applications. This paper is concerned with risk analysis for the basket derivatives and provides explicit computing methods for the risk parameters.

The risk is measured by the possibility of a partial hedging of the payoff. Thus our approach is based on the idea of quantile hedging which was introduced in [6] and later developed in various directions (see for instance [4], [10], [2], [1]). Let us briefly sketch the general concept. Denote by H a contingent claim and assume that the arbitrage free pricing method indicates its price $p(H)$. This means that if the investor has an initial endowment $x \geq p(H)$ then he is able to follow some trading strategy such that his portfolio hedges H with probability 1. If this is the case, then x carries no risk and *the probability maximizing function* Φ_1 equals 1, i.e. $\Phi_1(x) = 1$. On the other hand, if $x < p(H)$ then the shortfall probability is strictly greater than zero for each trading strategy and then $\Phi_1(x) < 1$. The greater the probability of shortfall, the smaller the value $\Phi_1(x)$. Thus the function Φ_1 can be viewed as a measure of risk sensitivity to the price reduction of the option. There

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is also another aspect of the problem. Assume that the hedger is willing to accept some risk measured by the shortfall probability in order to reduce initial cost. He chooses a number $\alpha \in [0,1]$ and searches for a minimal initial capital $\Phi_2(\alpha)$ which allows him to find a strategy such that the probability of the shortfall is smaller than $1-\alpha$. Thus if the hedger accepts no risk, i.e. $\alpha = 0$, then the minimal cost required to replicate H is just $p(H)$. In this case *the cost reduction function* satisfies $\Phi_2(0) = p(H)$. However, if $\alpha > 0$ then $\Phi_2(\alpha) < p(H)$ and the function Φ_2 enables us to view the effect of how the risk acceptance affects the cost reduction of the option. Recall the numerical example from [6, p. 261] which shows that $\Phi_2(0,05) = 0,59 \cdot p(H)$ for a call option with certain parameters. This means that the acceptance of a 5% margin of risk reduces the hedging cost by 41%. This shows that quantile hedging is an attractive tool for risk analysis and should be taken into account by traders.

The basic problem, however, is to determine the functions Φ_1 and Φ_2 for specific derivatives. There are only a few examples in the literature where they are explicitly found. In [6] explicit formulas are given for the most important case of a call option in a classical Black–Scholes model. The method can be mimicked to obtain formulas for the put option. The idea is based on reducing the original dynamic problem to a static one which can be solved with methods used in the theory of statistical tests. Since the market considered in [6] was complete, the solution of the static problem could be obtained, via the Neyman–Pearson lemma, by indicating a non-randomized test for appropriate probability measures. The Neyman–Pearson lemma can be generalized to the case of composite hypotheses, i.e. when measures are replaced by families of measures (see [3] where the solution in the abstract form is presented). However, straightforward applicability of this result to incomplete markets seems to be questionable.

This paper is devoted to determining the functions Φ_1 and Φ_2 for the basket derivatives in the Black–Scholes framework with correlation. As the market is complete, we follow the same general method as in [6], but we find the solutions explicitly using specific features of the model. More precisely, we show that the original problem can be reduced to that of finding another two deterministic functions Ψ_1, Ψ_2 depending on H, which turn out to be regular, i.e. continuous and strictly monotone if H is of a reasonable form (see Propositions 3.4 and 3.5). Then, roughly speaking, $\Phi_1 = \Psi_1 \circ \Psi_2^{-1}$ and $\Phi_2 = \Psi_2 \circ \Psi_1^{-1}$; for a precise formulation see Theorem 3.6. In the one-dimensional case when H is a call option the result covers the above mentioned example from [6]. We also determine explicit forms of Ψ_1 and Ψ_2 for commonly traded derivatives (see Section 4). As Ψ_1, Ψ_2 are rather complicated, the inverse functions cannot be given by analytic formulas but can be determined with the use of numerical methods. Thus a great advantage of our results is that they can be used in practice.

The paper is organized as follows. In Section 2 we briefly recall the multidimensional Black–Scholes model and formulate the problem precisely. Section 3 contains the main result, Theorem 3.6, which is preceded by a general discussion on the results from [6] and the Neyman–Pearson technique. The method established in Theorem 3.6 is used in Section 4 for calculating the functions Ψ_1, Ψ_2 for two-asset derivatives which are widely traded: digital option, quanto domestic, quanto foreign, outperformance and spread options.

2. The model. Let $(\Omega, \mathcal{F}_t, t \in [0,T], P)$ be a fixed probability space with filtration. The prices of d shares are given by the Black–Scholes equations

$$dS_t^i = S_t^i(\alpha_i dt + \sigma_i dW_t^i), \qquad i = 1, \ldots, d,\ t \in [0,T],$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1, \ldots, d$ and $W_t = (W_t^1, \ldots, W_t^d), t \in [0,T]$, is a sequence of standard Wiener processes adapted to $\{\mathcal{F}_t; t \in [0,T]\}$ with the correlation matrix Q of the form

$$Q = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,d} \\ \rho_{2,1} & 1 & \rho_{2,3} & \cdots & \rho_{2,d} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{d,1} & \rho_{d,2} & \rho_{d,3} & \cdots & 1 \end{bmatrix},$$

where

$$\rho_{i,j} = \mathrm{cor}\{W_1^i, W_1^j\}, \qquad i,j = 1, \ldots, d.$$

We assume that Q is positive definite. The process given above will be called a *Q-Wiener process*. The trader can invest his money in stocks as well as put it on a savings account whose dynamics is given by

$$dB_t = rB_t dt, \qquad t \in [0,T],$$

with r standing for a constant short rate.

REMARK 2.1. The most common approach to the description of the market is based on a sequence of independent Wiener processes (see for instance the classical textbook [9]). It can be shown that the model described above is equivalent to the model with d independent Wiener processes and the $d \times d$ diffusion matrix with constant coefficients. We work with a correlated Wiener process because it is more convenient for later calculations. Let us also mention that parameters in such a model can be easily estimated from data (see [7, p. 104]).

Let us now briefly characterize a martingale measure of the model, i.e. a measure $\tilde{P}$ which is equivalent to P such that the discounted price processes

$\hat{S}_t^i := e^{-rt} S_t^i$, $i = 1, \ldots, d$, are martingales. The following is a version of Theorem 10.14 in [5] adapted to our finite-dimensional setting.

THEOREM 2.2. *Let φ be a predictable process taking values in $\mathbb{R}^d$ satisfying*

$$\mathbf{E}(e^{\int_0^T (Q^{-1/2}\varphi_t, dW_t) - \frac{1}{2}\int_0^T |\varphi_t|^2 \, dt}) = 1.$$

Then the process

$$\widetilde{W}_t = W_t - \int_0^t Q^{1/2}\varphi_s \, ds, \qquad t \in [0,T],$$

is a Q-Wiener process with respect to the measure $\widetilde{P}$ with density

$$\frac{d\widetilde{P}}{dP} = e^{\int_0^T (Q^{-1/2}\varphi_t, dW_t) - \frac{1}{2}\int_0^T |\varphi_t|^2 \, dt}.$$

It can be shown that each measure equivalent to P can be characterized by a density process

$$Z_t := e^{\int_0^t (Q^{-1/2}\varphi_s, dW_s) - \frac{1}{2}\int_0^t |\varphi_s|^2 \, ds}, \qquad t \in [0,T], \tag{2.1}$$

for some predictable $\mathbb{R}^d$-valued process φ. The process $\hat{S}^i$ is a $\widetilde{P}$-martingale if and only if $\hat{S}^i Z$ is a P-martingale. Thus the measure $\widetilde{P}$ can be determined by finding a process φ in (2.1) such that $\hat{S}^i Z$, $i = 1, \ldots, d$, are P-martingales. Simple calculations based on the Itô formula yield

$$\varphi_t = -Q^{-1/2}\left[\frac{\alpha - r\mathbf{1}_d}{\sigma}\right] := -Q^{-1/2}\begin{bmatrix} \frac{\alpha_1 - r}{\sigma_1} \\ \vdots \\ \frac{\alpha_d - r}{\sigma_d} \end{bmatrix}, \qquad t \in [0,T].$$

The martingale measure $\widetilde{P}$ is thus unique and given by the density process

$$\tilde{Z}_t := e^{-(Q^{-1}[\frac{\alpha - r\mathbf{1}_d}{\sigma}], W_t) - \frac{1}{2}|Q^{-1/2}[\frac{\alpha - r\mathbf{1}_d}{\sigma}]|^2 t}, \qquad t \in [0,T]. \tag{2.2}$$

Moreover, it follows from Theorem 2.2 that the process

$$\widetilde{W}_t := W_t + \frac{\alpha - r\mathbf{1}_d}{\sigma} t, \qquad t \in [0,T],$$

is a Q-Wiener process under $\widetilde{P}$. The dynamics of the prices under the measure $\widetilde{P}$ can be written as

$$dS_t^i = S_t^i (r dt + \sigma_i d\widetilde{W}_t^i), \qquad i = 1, \ldots, d.$$

The wealth process with the initial endowment x and the trading strategy π is defined by

$$X_t^{x,\pi} := \pi_t^0 B_t + \sum_{i=1}^d \pi_t^i S_t^i, \qquad t \in [0,T],$$

and assumed to satisfy $X_0^{x,\pi} = x$. All strategies are assumed to be admissible, i.e. $X_t^{x,\pi} \geq 0$ for each $t \in [0,T]$ almost surely, and self-financing, i.e.

$$dX_t^{x,\pi} = \pi_t^0 dB_t + \sum_{i=1}^{d} \pi_t^i dS_t^i, \qquad t \in [0,T].$$

A *contingent claim*, representing future random payoff, is a random variable $H \geq 0$ measurable with respect to $\mathcal{F}_T$. A *hedging strategy* against H is a pair (x,π) such that

$$P(X_T^{x,\pi} \geq H) = 1.$$

A *replicating strategy* is a pair (x,π) such that

$$P(X_T^{x,\pi} = H) = 1.$$

A *price* of H is defined by

$$p(H) := \inf\{x : \exists \pi \text{ such that } P(X_T^{x,\pi} \geq H) = 1\}$$

and, due to the fact that the market is complete, it follows from the general theory that $p(H) = \tilde{\mathbf{E}}[e^{-rT}H]$, where the expectation is calculated under the measure $\widetilde{P}$.

If $x < p(H)$ then $P(X_T^{x,\pi} \geq H) < 1$ for all π and the question under consideration is to find a strategy maximizing the probability of successful hedge, i.e.

$$P(X_T^{x,\pi} \geq H) \underset{\pi}{\longrightarrow} \max. \tag{2.3}$$

We will refer to the corresponding function $\varPhi_1 : [0,+\infty) \to [0,1]$ given by

$$\varPhi_1(x) := \max_{\pi} P(X_T^{x,\pi} \geq H)$$

as the *maximal probability function*. If there exists $\hat{\pi}$ such that $P(X_T^{x,\hat{\pi}} \geq H) = \varPhi_1(x)$ then it will be called the *probability maximizing strategy for* x.

We also consider the problem of cost reduction. Let $\alpha \in [0,1]$ be a fixed number describing the level of shortfall risk accepted by the trader. Then we are searching for a minimal initial cost such that there exists a strategy with the probability of successful hedge exceeding $1-\alpha$, i.e.

$$x \to \min; \quad \exists \pi \text{ such that } P(X_T^{x,\pi} \geq H) \geq 1-\alpha. \tag{2.4}$$

The *cost reduction function* $\varPhi_2 : [0,1] \to [0,p(H)]$ is thus defined by

$$\varPhi_2(\alpha) := \min\{x : \exists \pi \text{ such that } P(X_T^{x,\pi} \geq H) \geq 1-\alpha\}.$$

If there exists $\hat{\pi}$ such that $P(X_T^{\varPhi_2(\alpha),\hat{\pi}} \geq H) \geq 1-\alpha$ then it will be called the *cost minimizing strategy for* α.

In what follows we study the problem of determining the functions $\varPhi_1$ and $\varPhi_2$ for the contingent claim H of a general form. Then in Section 4 specific payoffs are examined.

3. Main results. In this section we present a general method of determining Φ_1 and Φ_2. Let us start with the auxiliary problems which can be solved via the Neyman–Pearson lemma.

Assume that we are given two probability measures P_1, P_2 with strictly positive density $\frac{dP_1}{dP_2}$ and consider two types of optimizing problems

$$\begin{cases} P_1[A] \to \max, \\ P_2[A] \le x, \end{cases} \tag{3.5}$$

$$\begin{cases} P_1[B] \ge 1-\alpha, \\ P_2[B] \to \min, \end{cases} \tag{3.6}$$

where $\alpha, x \in [0,1]$ are fixed constants. Problem (3.5) is a classical one appearing in the statistical hypotheses testing. Recall that if there exists a constant $c \ge 0$ such that $P_2\big(\frac{dP_1}{dP_2} \ge c\big) = x$ then the set

$$\tilde{A} := \left\{ \frac{dP_1}{dP_2} \ge c \right\}$$

is a solution of (3.5). It is not surprising that the solution of the problem (3.6) is of a similar form. For the convenience of the reader we prove the following.

PROPOSITION 3.1. *If there exists a constant $c \ge 0$ satisfying $P_1\big(\frac{dP_2}{dP_1} \le c\big) = 1-\alpha$ then the set*

$$\tilde{B} := \left\{ \frac{dP_2}{dP_1} \le c \right\}$$

is a solution of the problem (3.6).

Proof. Let B be an arbitrary set satisfying $P_1(B) \ge 1-\alpha$. We will show that $P_2(B) \ge P_2(\tilde{B})$. The following estimate holds:

$$\begin{aligned} P_2(B) - P_2(\tilde{B}) &= \int_{\Omega} (\mathbf{1}_B - \mathbf{1}_{\tilde{B}})\, dP_2 \\ &= \int_{\{\frac{dP_2}{dP_1} \le c\}} (\mathbf{1}_B - \mathbf{1}_{\tilde{B}})\, dP_2 + \int_{\{\frac{dP_2}{dP_1} > c\}} (\mathbf{1}_B - \mathbf{1}_{\tilde{B}})\, dP_2 \\ &\ge c \int_{\{\frac{dP_2}{dP_1} \le c\}} (\mathbf{1}_B - \mathbf{1}_{\tilde{B}})\, dP_1 + c \int_{\{\frac{dP_2}{dP_1} > c\}} \mathbf{1}_B\, dP_1 \\ &= c\Big(\int_{\Omega} \mathbf{1}_B\, dP_1 - \int_{\Omega} \mathbf{1}_{\tilde{B}}\, dP_1 \Big) = c(P_1(B) - P_1(\tilde{B})) \\ &\ge c(P_1(B) - (1-\alpha)) \ge 0. \ \blacksquare \end{aligned}$$

Let us notice that both optimal sets $\tilde{A}$, $\tilde{B}$ have a similar form

$$\left\{\frac{dP_1}{dP_2} \geq c\right\}, \tag{3.7}$$

with suitable constants $c \geq 0$. More precisely, for $\tilde{A}$ the constant c is such that

$$P_2\left(\frac{dP_1}{dP_2} \geq c\right) = x, \tag{3.8}$$

and for $\tilde{B}$ it is such that

$$P_1\left(\frac{dP_1}{dP_2} \geq c\right) = 1 - \alpha. \tag{3.9}$$

Now, let us come back to the initial problem of determining Φ_1, Φ_2. Let us start by presenting two auxiliary results which are nonrandomized versions of Theorems 2.34 and 2.42 in [6].

THEOREM 3.2. *Let $x \geq 0$. If $\tilde{A}$ is a set solving the problem*

$$\begin{cases} P[A] \to \max, \\ \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_A] \leq x, \end{cases} \tag{3.10}$$

then $\Phi_1(x) = P(\tilde{A})$ and the probability maximizing strategy for x is the one replicating the payoff $H\mathbf{1}_{\tilde{A}}$.

Let us notice that if $x \geq p(H)$ then $\tilde{A} = \Omega$ and thus $\Phi_1(x) = 1$. Moreover, if (3.10) has a solution for every $x \geq 0$, then Φ_1 is increasing.

THEOREM 3.3. *Let $\alpha \in [0, 1]$ be fixed. If $\tilde{B}$ is a set solving the problem*

$$\begin{cases} P[B] \geq 1 - \alpha, \\ \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_B] \to \min, \end{cases} \tag{3.11}$$

then $\Phi_2(\alpha) = \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\tilde{B}}]$ and the cost minimizing strategy for α is the one replicating the payoff $H\mathbf{1}_{\tilde{B}}$.

Notice that $\Phi_2(0) = p(H)$ and if (3.11) has a solution for each $\alpha \in [0, 1]$ then Φ_2 is decreasing.

Now we apply the method of solving the problems (3.5) and (3.6) to (3.10) and (3.11). Notice that (3.10) and (3.11) can be reformulated

$$\begin{cases} P[A] \to \max, \\ P^*(A) \leq x/\tilde{\mathbf{E}}[e^{-rT} H], \end{cases} \tag{3.12}$$

and

$$\begin{cases} P[B] \geq 1 - \alpha, \\ P^*(B) \to \min, \end{cases} \tag{3.13}$$

where P^* is the probability measure given by the density

$$\frac{dP^*}{d\widetilde{P}} = \frac{H}{\tilde{\mathbf{E}}[H]}.$$

In view of (3.7) we are searching for solutions $\tilde{A}$, $\tilde{B}$ to (3.12), (3.13) in the family of sets

$$\left\{\frac{dP}{dP^*} \geq c\right\} = \left\{\frac{dP}{d\widetilde{P}}\,\frac{d\widetilde{P}}{dP^*} \geq c\right\} = \left\{\tilde{Z}_T^{-1} \geq c\frac{H}{\tilde{\mathbf{E}}[H]}\right\}, \quad c \geq 0,$$

where $\tilde{Z}_T$ is given by (2.2). Denoting, for simplicity, the constant $c/\tilde{\mathbf{E}}[H]$ by c we see that the optimal sets $\tilde{A}$, $\tilde{B}$ are of the form

$$A_c := \{\tilde{Z}_T^{-1} \geq cH\}, \tag{3.14}$$

where, by (3.8) and (3.9), c is such that

$$P^*(A_c) = \frac{x}{\tilde{\mathbf{E}}[e^{-rT}H]} \quad \text{for } \tilde{A}, \tag{3.15}$$

$$P(A_c) = 1 - \alpha \quad \text{for } \tilde{B}. \tag{3.16}$$

Now define $\Psi_1 : [0, +\infty) \to [0, 1]$ and $\Psi_2 : [0, +\infty) \to [0, p(H)]$ by

$$\Psi_1(c) := P(A_c), \tag{3.17}$$

$$\Psi_2(c) := P^*(A_c) \cdot \tilde{\mathbf{E}}[e^{-rT}H] = \tilde{\mathbf{E}}[e^{-rT}H\mathbf{1}_{A_c}]. \tag{3.18}$$

Notice that both Ψ_1, Ψ_2 are decreasing and $\Psi_1(0) = 1$, $\Psi_2(0) = p(H)$. Thus $\Psi_2(0)$ provides the arbitrage free price of the continent claim H. Below we list some properties of Ψ_1, Ψ_2 needed later. First let us introduce two conditions concerning a real function $f : \mathbb{R}^d \to [0, +\infty)$:

(C1) $\lambda_d\big(\{z : f(z) = c\}\big) = 0$ for each $c > 0$,
(C2) $\lambda_d\big(\{z : f(z) \in (a, b]\}\big) > 0$ for each $0 < a < b$.

Here λ_d stands for the Lebesgue measure on $\mathbb{R}^d$.

PROPOSITION 3.4.

(a) *The function Ψ_1 is left continuous with right hand limits at each point of the domain.*
(b) $\lim_{c\to+\infty} \Psi_1(c) = P(H = 0)$.

Assume that $\tilde{Z}_T H = f(W_T)$ where $f : \mathbb{R}^d \to [0, +\infty)$. Then Ψ_1 is

(c) *continuous if and only if* (C1) *is satisfied,*
(d) *strictly decreasing if and only if* (C2) *is satisfied.*

Proof. (a) The function Ψ_1 can be written in the form

$$\Psi_1(c) = P\left(\tilde{Z}_T H \leq \frac{1}{c}\right) = F_{\tilde{Z}_T H}\left(\frac{1}{c}\right), \quad c > 0, \tag{3.19}$$

where $F_{\tilde{Z}_T H}$ stands for the distribution function of the random variable $\tilde{Z}_T H$. Thus Ψ_1 has one-sided limits for any $c > 0$ and the left continuity follows from the right continuity of $F_{\tilde{Z}_T H}$ for any $c > 0$. Left continuity at $c = 0$ follows from monotonicity.

(b) The assertion follows from the formula

$$\Psi_1(c) = P(\tilde{Z}_T^{-1} \geq cH \mid H > 0)P(H > 0) + P(\tilde{Z}_T^{-1} \geq cH \mid H = 0)P(H = 0)$$

and from

$$\lim_{c \to +\infty} P(\tilde{Z}_T^{-1} \geq cH \mid H > 0) = 0.$$

(c) First we show continuity at zero. If $c_n \downarrow 0$ then $\{\tilde{Z}_T^{-1} \geq c_n H\}_n$ is an increasing family of sets and by the continuity of probability we have

$$\begin{aligned} \lim_{n \to +\infty} \Psi_1(c_n) &= \lim_{n \to +\infty} P(\tilde{Z}_T^{-1} \geq c_n H) = P\Big(\bigcup_n \{\tilde{Z}_T^{-1} \geq c_n H\}\Big) \\ &= P(\tilde{Z}_T^{-1} > 0) = 1 = \Psi_1(0). \end{aligned}$$

Taking into account (3.19) we see that Ψ_1 is continuous for each $c > 0$ if and only if the random variable $\tilde{Z}_T H = f(W_T)$ has no positive atoms. In view of the equality

$$P(\tilde{Z}_T H = c) = P(f(W_T) = c) = \mathcal{L}_{W_T}(\{z : f(z) = c\}), \quad c > 0,$$

and the fact that the distribution of W_T is nondegenerate we see that the continuity of Ψ_1 is equivalent to (C1). $\mathcal{L}_{W_T}$ above stands for the distribution of W_T.

(d) For $0 < c_1 < c_2$ we have

$$\begin{aligned} \Psi_1(c_1) - \Psi_1(c_2) &= P(\tilde{Z}_T H \leq 1/c_1) - P(\tilde{Z}_T H \leq 1/c_2) \\ &= P(f(W_T) \in (1/c_2, 1/c_1]) \\ &= \mathcal{L}_{W_T}(\{z : f(z) \in (1/c_2, 1/c_1]\}), \end{aligned}$$

and it follows from the nondegeneracy of the distribution of W_T that the strict monotonicity of Ψ_1 is equivalent to (C2). ∎

PROPOSITION 3.5.

(a) *The function Ψ_2 is left continuous with right hand limits at each point of the domain.*
(b) $\lim_{c \to +\infty} \Psi_2(c) = 0$.

Assume that $\tilde{Z}_T H = f(W_T)$ where $f : \mathbb{R}^d \to [0, +\infty)$. Then Ψ_2 is

(c) *continuous if and only if* (C1) *is satisfied,*
(d) *strictly decreasing if and only if* (C2) *is satisfied.*

Proof. (a) It follows from monotonicity that one-sided limits exist. We show left continuity for any $c > 0$. For $c_n \uparrow c$ the family $\{\tilde{Z}_T^{-1} \geq c_n H\}_n$ is decreasing and

$$\bigcap_n \{\tilde{Z}_T^{-1} \geq c_n H\} = \{H = 0\} \cup \{\tilde{Z}_T^{-1} \geq cH\} = \{\tilde{Z}_T^{-1} \geq cH\}.$$

Thus by dominated convergence we have

$$\begin{aligned}\lim_{n\to+\infty} \Psi_2(c_n) &= \lim_{n\to+\infty} \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c_n H\}}] \\ &= \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq cH\}}] = \Psi_2(c).\end{aligned}$$

(b) For $c_n \uparrow +\infty$ we have

$$\{\tilde{Z}_T^{-1} \geq c_n H\}_n \downarrow \bigcap_n \{\tilde{Z}_T^{-1} \geq c_n H\} = \{H = 0\} \cup \{\tilde{Z}_T^{-1} = +\infty\} = \{H = 0\},$$

and thus

$$\lim_{n\to+\infty} \Psi_2(c_n) = \lim_{n\to+\infty} \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c_n H\}}] = \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{H=0\}}] = 0.$$

(c) We show that the right continuity of Ψ_2 is equivalent to (C1). Then continuity follows from (a). For $c_n \downarrow c \geq 0$ we have

$$\begin{aligned}\{\tilde{Z}_T^{-1} \geq c_n H\} \uparrow \bigcup_n \{\tilde{Z}_T^{-1} \geq c_n H\} &= \{H = 0\} \cup \{H > 0,\, \tilde{Z}_T^{-1} > cH\} \\ &= \{\tilde{Z}_T^{-1} > cH\} = \{1 > cf(W_T)\},\end{aligned}$$

and thus

$$\lim_{n\to+\infty} \Psi_2(c_n) = \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{1 > cf(W_T)\}}].$$

The condition $\lim_{n\to+\infty} \Psi_2(c) = \Psi_2(c)$ holds if and only if $\tilde{P}(1 \geq cf(W_T)) = \tilde{P}(1 > cf(W_T))$. The last condition holds for $c = 0$, and for $c > 0$ it is equivalent to (C1).

(d) Fix $0 < c_1 < c_2$. The inequality

$$\Psi_2(c_1) - \Psi_2(c_2) = \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{1/c_1 < f(W_T) \leq 1/c_2\}}] > 0$$

holds if and only if $\tilde{P}(1/c_1 < f(W_T) \leq 1/c_2) > 0$. The last condition is equivalent to (C2). ■

Now assume that $\tilde{Z}_T H = f(W_T)$ for some $f : \mathbb{R}^d \to [0, +\infty)$. Fix $\alpha \in [0, 1]$, $x > 0$ and consider the problem of existence of solutions to the equation

$$\Psi_1(c) = 1 - \alpha, \tag{3.20}$$

as well as to

$$\Psi_2(c) = x. \tag{3.21}$$

In view of Propositions 3.4 and 3.5, if (C1) is satisfied then Ψ_1, Ψ_2 are continuous decreasing functions with images $(P(H = 0), 1]$ and $(0, p(H)]$ respectively. Thus for $\alpha \in [0, P(H \neq 0))$ and $x \in (0, p(H)]$ the equations (3.20) and (3.21) do have solutions. Moreover, if (C2) is satisfied then the solutions are unique.

The description of Φ_1 and Φ_2 is provided by the following theorem, which is the main result of the paper.

THEOREM 3.6. *Assume that* $\tilde{Z}_T H = f(W_T)$ *for some* $f : \mathbb{R}^d \to [0, +\infty)$ *satisfying* (C1).

(a) *Let* $c = c(x) \in [0, +\infty)$ *be a solution of the equation*

$$\Psi_2(c) = x, \qquad x \in (0, p(H)). \tag{3.22}$$

Then the maximal probability function is given by

$$\Phi_1(x) = \begin{cases} P(H = 0) & \text{for } x = 0, \\ \Psi_1(c(x)) & \text{for } x \in (0, p(H)), \\ 1 & \text{for } x \geq p(H). \end{cases}$$

Moreover, for any $x \in (0, p(H))$ *the probability maximizing strategy for* x *is the one replicating the payoff* $H\mathbf{1}_{A_{c(x)}}$.

(b) *Let* $c = c(\alpha) \in [0, +\infty)$ *be a solution of the equation*

$$\Psi_1(c) = 1 - \alpha, \qquad \alpha \in [0, P(H \neq 0)). \tag{3.23}$$

Then the cost reduction function is given by

$$\Phi_2(\alpha) = \begin{cases} \Psi_2(c(\alpha)) & \text{for } \alpha \in [0, P(H \neq 0)), \\ 0 & \text{for } \alpha \in [P(H \neq 0), 1]. \end{cases}$$

Moreover, for any $\alpha \in [0, P(H \neq 0))$ *the cost reduction strategy for* α *is the one replicating the payoff* $H\mathbf{1}_{A_{c(\alpha)}}$.

Proof. The proof is based on the considerations preceding the formulation of the theorem.

(a) If $x \geq p(H)$ then the hedging strategy is the probability maximizing strategy and then clearly $\Phi_1(x) = 1$. Consider the case $x \in (0, p(H))$. By Theorem 3.2 we know that $\Phi_1(x) = P(\tilde{A})$, where $\tilde{A}$ is a solution of (3.10). The solution of (3.12), which is equivalent to (3.10), is of the form (3.14) with c satisfying (3.15). But (3.15) is equivalent to (3.22). Thus we have

$$\Phi_1(x) = P(A_c) = \Psi_1(c),$$

where c is given by the condition $\Psi_2(c) = x$. For $x = 0$ consider the trivial strategy $\pi = 0$. Then $P(X_T^{x,\pi} \geq H) = P(H = 0)$. On the other hand, due to the monotonicity of Φ_1, we have $\Phi_1(0) \leq \lim_{x\downarrow 0} \Phi_1(x) = \lim_{x\downarrow 0} \Psi_1(c(x)) = \lim_{z\uparrow +\infty} \Psi_1(z) = P(H = 0)$. As a consequence, $\Phi_1(0) = P(H = 0)$. The second part of the assertion follows from Theorem 3.2.

(b) If $\alpha \in [P(H \neq 0), 1]$ then consider a trivial strategy $\pi = 0$ with zero initial endowment $x = 0$. Then $X_T^{x,\pi} = 0$ and thus $P(X_T^{x,\pi} \geq H) = P(H = 0) \geq 1 - \alpha$. As a consequence, $\Phi_2(\alpha) = 0$. Now consider the case

$\alpha \in [0, P(H \neq 0)]$. It follows from Theorem 3.3 that $\Phi_2(\alpha) = \tilde{\mathbf{E}}[e^{-rT}H\mathbf{1}_{\tilde{B}}]$, where $\tilde{B}$ is a solution to (3.11). The optimal solution of (3.11) is the same as for (3.13) and has the form (3.14) with c satisfying (3.16). The condition (3.16) can be written as $\Psi_1(c) = 1 - \alpha$. Thus we have

$$\Phi_2(\alpha) = \tilde{\mathbf{E}}[e^{-rT}H\mathbf{1}_{A_c}] = \Psi_2(c).$$

The second part of the assertion follows from Theorem 3.3. ■

In virtue of Theorem 3.6, to determine Φ_1, Φ_2 one has to find Ψ_1, Ψ_2 and solve the equations (3.22), (3.23). In general, as Ψ_1, Ψ_2 have a rather sophisticated form, one should not expect to find analytic formulas for the constants in (3.22), (3.23). However, the equations (3.22), (3.23) can be solved with the use of numerical methods. In the following we solve the problem of determining Ψ_1, Ψ_2 for the most common basket derivatives.

4. Quantile hedging in a two-dimensional model. In this section we determine explicit formulas for the functions Ψ_1, Ψ_2 for a few examples of popular options. Since our derivatives depend on two underlying assets we first simplify the general formulas from Section 3.

For $d = 2$ we denote the correlation matrix by

$$Q = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

Consequently,

$$Q^{-1} = \frac{1}{\rho^2 - 1}\begin{bmatrix} -1 & \rho \\ \rho & -1 \end{bmatrix}, \qquad Q^{-1/2} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \end{bmatrix}.$$

Hence the density of the martingale measure (2.2) can be written as

$$\tilde{Z}_T = e^{-A_1 W_T^1 - A_2 W_T^2 - BT}, \tag{4.24}$$

where

$$A_1 := \frac{1}{\rho^2 - 1}\left(-\frac{\alpha_1 - r}{\sigma_1} + \rho\frac{\alpha_2 - r}{\sigma_2}\right),$$

$$A_2 := \frac{1}{\rho^2 - 1}\left(\rho\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2}\right),$$

$$B := \frac{1}{8}\left(\left(\left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_1 - r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_2 - r}{\sigma_2}\right)^2 + \left(\left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_1 - r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_2 - r}{\sigma_2}\right)^2\right).$$

The formula (3.14) for the set A_c simplifies to

$$A_c = \{\tilde{Z}_T^{-1} \geq cH\} = \{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq cH\},$$

and consequently formulas (3.17), (3.18) become

$$\Psi_1(c) = P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq cH),$$
$$\Psi_2(c) = \tilde{\mathbf{E}}[e^{-rT} H \mathbf{1}_{\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq cH\}}].$$

Now we set the notation concerning the multidimensional normal distribution and recall its basic properties, which can be found in standard textbooks on probability theory or statistics (see for instance [8]). A random vector X taking values in $\mathbb{R}^d$ has a multidimensional normal distribution if its density is of the form

$$f_X(x) = \frac{1}{(2\pi)^{d/2} (\det \Sigma)^{1/2}} \cdot e^{-\frac{1}{2}(x-m)^T \Sigma^{-1} (x-m)}, \qquad x \in \mathbb{R}^d, \tag{4.25}$$

where $m \in \mathbb{R}^d$ is the mean of X and Σ is a symmetric positive definite $d \times d$ covariance matrix of X. The fact that X has a density (4.25) will be denoted by $X \sim N_d(m, \Sigma)$ or $\mathcal{L}(X) = N_d(m, \Sigma)$. If $d = 1$ then the subscript is omitted and $N(m, \sigma)$ denotes the normal distribution with mean m and variance σ. If $X \sim N_d(m, \Sigma)$ and A is a $k \times d$ matrix, then

$$AX \sim N_k(Am, A\Sigma A^T); \tag{4.26}$$

in particular if $a \in \mathbb{R}^d$ then

$$a^T X \sim N(a^T m, a^T \Sigma a). \tag{4.27}$$

Let X be a random vector taking values in $\mathbb{R}^d$ and fix an integer $0 < k < d$. Let us divide X into two vectors $X^{(1)}$ and $X^{(2)}$ with lengths k, $d-k$ respectively, i.e.

$$X^{(1)} = (X_1, \ldots, X_k)^T, \qquad X^{(2)} = (X_{k+1}, \ldots, X_d)^T.$$

Analogously, divide the mean vector m and the covariance matrix Σ,

$$m = \begin{pmatrix} m^{(1)} \\ m^{(2)} \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma^{(11)} & \Sigma^{(12)} \\ \Sigma^{(21)} & \Sigma^{(22)} \end{bmatrix},$$

so that $\mathbf{E}X^{(1)} = m^{(1)}$, $\mathbf{E}X^{(2)} = m^{(2)}$, $\operatorname{Cov} X^{(1)} = \Sigma^{(11)}$, $\operatorname{Cov} X^{(2)} = \Sigma^{(22)}$, $\operatorname{Cov}(X^{(1)}, X^{(2)}) = \Sigma^{(12)} = {\Sigma^{(21)}}^T$. Denote by $\mathcal{L}(X^{(1)} \mid X^{(2)} = x^{(2)})$ the conditional distribution of $X^{(1)}$ given $X^{(2)} = x^{(2)} \in \mathbb{R}^{d-k}$. If $\Sigma^{(22)}$ is nonsingular then

$$\mathcal{L}(X^{(1)} \mid X^{(2)} = x^{(2)}) = N_k(m^{(1)}(x^{(2)}), \Sigma^{(11)}(x^{(2)})), \tag{4.28}$$

where

$$m^{(1)}(x^{(2)}) = m^{(1)} + \Sigma^{(12)}{\Sigma^{(22)}}^{-1}(x^{(2)} - m^{(2)}),$$
$$\Sigma^{(11)}(x^{(2)}) = \Sigma^{(11)} - \Sigma^{(12)}{\Sigma^{(22)}}^{-1}\Sigma^{(21)}. \tag{4.29}$$

Actually the conditional variance $\Sigma^{(11)}(x^{(2)})$ does not depend on $x^{(2)}$ but we keep the notation for the sake of consistency. The conditional density will be denoted by $f_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$, where $x^{(1)} \in \mathbb{R}^k$. In particular if (X,Y) is a two-dimensional normal vector with parameters

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

then

$$\mathcal{L}(X \mid Y = y) = N(m_1(y), \sigma_1(y)),$$

where

$$m_1(y) := m_1 + \frac{\sigma_{12}}{\sigma_{22}}(y - m_2), \qquad \sigma_1(y) := \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}. \tag{4.30}$$

If X is a random vector then its distribution with respect to the measure $\widetilde{P}$ will be denoted by $\tilde{\mathcal{L}}(X)$ and its density by $\tilde{f}_X$. Analogously, the notation $\tilde{f}_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$ stands for the conditional density with respect to $\widetilde{P}$.

In the following subsections we will use the universal constants: A_1, A_2, B defined in (4.24) as well as $a_1, a_2, b, \tilde{a}_1, \tilde{a}_2, \tilde{b}$ introduced below.

Fix a number $K > 0$. One can check the following:

$$\{S_T^1 \geq K\} = \{W_T^1 \geq a_1\} = \{\widetilde{W}_T^1 \geq \tilde{a}_1\}, \tag{4.31}$$

$$\{S_T^2 \geq K\} = \{W_T^2 \geq a_2\} = \{\widetilde{W}_T^2 \geq \tilde{a}_2\}, \tag{4.32}$$

$$\{S_T^1 \geq S_T^2\} = \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\} = \{\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \geq \tilde{b}\}, \tag{4.33}$$

where

$$a_1 := \frac{1}{\sigma_1}\left(\ln\frac{K}{S_0^1} - \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)T\right), \quad \tilde{a}_1 := \frac{1}{\sigma_1}\left(\ln\frac{K}{S_0^1} - \left(r - \frac{1}{2}\sigma_1^2\right)T\right),$$
$$a_2 := \frac{1}{\sigma_2}\left(\ln\frac{K}{S_0^2} - \left(\alpha_2 - \frac{1}{2}\sigma_2^2\right)T\right), \quad \tilde{a}_2 := \frac{1}{\sigma_2}\left(\ln\frac{K}{S_0^2} - \left(r - \frac{1}{2}\sigma_2^2\right)T\right),$$
$$b := \ln\frac{S_0^2}{S_0^1} + \left(\alpha_2 - \alpha_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\right)T, \quad \tilde{b} := \ln\frac{S_0^2}{S_0^1} - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T.$$

In all the formulas below it is understood that $\ln 0 = -\infty$ and Φ stands for the cumulative distribution function of $N(0,1)$.

4.1. Digital option. In this section we determine Ψ_1, Ψ_2 for the payoff

$$H = K \cdot \mathbf{1}_{\{S_T^1 \geq S_T^2\}}, \quad \text{where } K > 0. \tag{4.34}$$

By (4.33) we have

$$\begin{aligned}
\Psi_1(c) &= P(A_c) = P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq cK\mathbf{1}_{\{S_T^1 \geq S_T^2\}}) \\
&= P(A_1 W_T^1 + A_2 W_T^2 + BT \geq \ln(cK), S_T^1 \geq S_T^2) \\
&\quad + P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0, S_T^1 < S_T^2) \\
&= P(A_1 W_T^1 + A_2 W_T^2 + BT \geq \ln(cK) \mid \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad \cdot P(\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) + P(\sigma_1 W_T^1 - \sigma_2 W_T^2 < b).
\end{aligned} \tag{4.35}$$

Let us notice that

$$X := \begin{bmatrix} A_1 W_T^1 + A_2 W_T^2 \\ \sigma_1 W_T^1 - \sigma_2 W_T^2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \sigma_1 & -\sigma_2 \end{bmatrix} \begin{bmatrix} W_T^1 \\ W_T^2 \end{bmatrix},$$

so in view of (4.26) we have $X \sim N_2(0, \Sigma)$, where

$$\Sigma = \begin{bmatrix} (A_1 + A_2 r)TA_1 + (A_1 r + A_2)TA_2 & (\sigma_1 - \sigma_2 r)TA_1 + (\sigma_1 r - \sigma_2)TA_2 \\ (\sigma_1 - \sigma_2 r)TA_1 + (\sigma_1 r - \sigma_2)TA_2 & (\sigma_1 - \sigma_2 r)T\sigma_1 - (\sigma_1 r - \sigma_2)T\sigma_2 \end{bmatrix}.$$

In virtue of (4.30) we have

$$\mathcal{L}(A_1 W_T^1 + A_2 W_T^2 \mid \sigma_1 W_T^1 - \sigma_2 W_T^2 = y) = N(m(y), \sigma(y)),$$

where

$$\begin{aligned}
m(y) &= y \frac{(\sigma_1 - \sigma_2 r)A_1 + (\sigma_1 r - \sigma_2)A_2}{(\sigma_1 - \sigma_2 r)\sigma_1 - (\sigma_1 r - \sigma_2)\sigma_2}; \\
\sigma(y) &= \frac{T(A_1\sigma_2 + A_2\sigma_1)^2(\rho^2 - 1)}{-\sigma_1^2 + 2\rho\sigma_1\sigma_2 - \sigma_2^2}.
\end{aligned}$$

By (4.27) we have $\sigma_1 W_T^1 - \sigma_2 W_T^2 \sim N(0, T(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2))$. Going back to (4.35) we obtain

$$\begin{aligned}
\Psi_1(c) = &\int_b^{+\infty} P(A_1 W_T^1 + A_2 W_T^2 \geq \ln(cK) - BT \mid \sigma_1 W_T^1 - \sigma_2 W_T^2 = y) \\
&\qquad \cdot f_{\sigma_1 W_T^1 - \sigma_2 W_T^2}(y)\, dy + P(\sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\
= &\int_b^{+\infty} \Phi\bigg(\frac{m(y) + BT - \ln(cK)}{\sqrt{\sigma(y)}}\bigg) f_{\sigma_1 W_T^1 - \sigma_2 W_T^2}(y)\, dy \\
&+ \Phi\bigg(\frac{b}{\sqrt{T(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}}\bigg).
\end{aligned}$$

Now let us determine Ψ_2. In virtue of (4.33) we have

$$\begin{aligned}\Psi_2(c) &= e^{-rT}\tilde{\mathbf{E}}[H\mathbf{1}_{A_c}] = e^{-rT}\tilde{\mathbf{E}}[K\mathbf{1}_{\{S_T^1\ge S_T^2\}}\cdot \mathbf{1}_{\{\tilde{Z}_T^{-1}\ge cK\mathbf{1}_{\{S_T^1\ge S_T^2\}}\}}] \\ &= e^{-rT}K\widetilde{P}(S_T^1\ge S_T^2, \tilde{Z}_T^{-1}\ge cK\mathbf{1}_{\{S_T^1\ge S_T^2\}}) \\ &= e^{-rT}K\widetilde{P}(\tilde{Z}_T^{-1}\ge cK \mid S_T^1\ge S_T^2)\widetilde{P}(S_T^1\ge S_T^2) \\ &= e^{-rT}K\widetilde{P}(e^{A_1W_T^1+A_2W_T^2+BT} > cK \mid \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2\ge \tilde{b}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot \widetilde{P}(\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2\ge\tilde{b}) \\ &= e^{-rT}K\int_{\tilde{b}}^{+\infty}\widetilde{P}(e^{A_1W_T^1+A_2W_T^2+BT} > cK \mid \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2 = y) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot \tilde{f}_{\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2}(y)\,dy \\ &= e^{-rT}K\int_{\tilde{b}}^{+\infty}\widetilde{P}\Big(A_1\widetilde{W}_T^1+A_2\widetilde{W}_T^2 > \ln(cK)+A_1\frac{\alpha_1-r}{\sigma_1}T \\ &\qquad + A_2\frac{\alpha_2-r}{\sigma_2}T - BT \,\Big|\, \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2 = y\Big)\cdot \tilde{f}_{\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2}(y)\,dy \\ &= e^{-rT}K\int_{\tilde{b}}^{+\infty}\Phi\bigg(\frac{m(y)-\ln(cK)-A_1\frac{\alpha_1-r}{\sigma_1}T-A_2\frac{\alpha_2-r}{\sigma_2}T+BT}{\sqrt{\sigma(y)}}\bigg) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot \tilde{f}_{\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2}(y)\,dy.\end{aligned}$$

4.2. Quantos

4.2.1. *Quanto domestic.* The contingent claim is of the form

$$H = S_T^2(S_T^1-K)^+,\quad K>0. \tag{4.36}$$

First, notice that

$$\begin{aligned}A_c &= \{e^{A_1W_T^1+A_2W_T^2+BT}\ge cS_T^2(S_T^1-K)\} \\ &= \{(A_2-\sigma_2)W_T^2\ge v(c,W_T^1)\} = \{(A_2-\sigma_2)\widetilde{W}_T^2\ge w(c,\widetilde{W}_T^1)\},\end{aligned} \tag{4.37}$$

where

$$v(c,x) := \ln\big(cS_0^2e^{(\alpha_2-\frac{1}{2}\sigma_2^2-B)T-A_1x}(S_0^1e^{(\alpha_1-\frac{1}{2}\sigma_1^2)T+\sigma_1x}-K)\big),$$

$$w(c,x) := \ln\big[cS_0^2e^{(r-\frac{1}{2}\sigma_2^2-B+A_1\frac{\alpha_1-r}{\sigma_1}+A_2\frac{\alpha_2-r}{\sigma_2})T-A_1x}(S_0^1e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1x)}-K)\big].$$

By (4.31) and (4.37) we have

$$\Psi_1(c) = P(A_c\mid S_T^1\ge K)P(S_T^1\ge K)+P(A_c\mid S_T^1<K)P(S_T^1<K)$$

$$= P\big((A_2-\sigma_2)W_T^2 \ge \ln(cS_0^2 e^{(\alpha_2-\frac{1}{2}\sigma_2^2-B)T-A_1W_T^1}(S_T^1-K)) \;\big|\; W_T^1 \ge a_1\big)$$
$$\cdot P(W_T^1 \ge a_1) + P(W_T^1 < a_1)$$
$$= \int_{a_1}^{+\infty} P\big((A_2-\sigma_2)W_T^2 \ge v(c,W_T^1) \,\big|\, W_T^1 = x\big) f_{W_T^1}(x)\,dx + \Phi\bigg(\frac{a_1}{\sqrt{T}}\bigg).$$

The conditional distribution is given by

$$\mathcal{L}((A_2-\sigma_2)W_T^2 \mid W_T^1 = x) \sim N(m(x),\sigma(x)),$$

where $m(x),\sigma(x)$ are given by (4.29). Hence

$$\Psi_1(c) = \int_{a_1}^{+\infty} \Phi\bigg(\frac{m(x)-v(c,x)}{\sqrt{\sigma(x)}}\bigg) f_{W_T^1}(x)\,dx + \Phi\bigg(\frac{a_1}{\sqrt{T}}\bigg).$$

To avoid technicalities assume that $A_2 \ne \sigma_2$. We have

$$\begin{aligned}\Psi_2(c) &= e^{-rT}\tilde{\mathbf{E}}[S_T^2(S_T^1-K)^+\mathbf{1}_{A_c}]\\ &= e^{-rT}\tilde{\mathbf{E}}[S_T^2(S_T^1-K)^+\mathbf{1}_{A_c} \mid S_T^1 \le K]\widetilde{P}(S_T^1 \le K)\\ &\quad + e^{-rT}\tilde{\mathbf{E}}[S_T^2(S_T^1-K)^+\mathbf{1}_{A_c} \mid S_T^1 > K]\widetilde{P}(S_T^1 > K)\\ &= e^{-rT}\tilde{\mathbf{E}}[S_T^2(S_T^1-K)\mathbf{1}_{A_c} \mid S_T^1 > K]\widetilde{P}(S_T^1 > K).\end{aligned}$$

By (4.31) and (4.37) we have

$$\Psi_2(c) = e^{-rT}\tilde{\mathbf{E}}[S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2\widetilde{W}_T^2}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1\widetilde{W}_T^1}-K)$$
$$\cdot \mathbf{1}_{\{(A_2-\sigma_2)\widetilde{W}_T^2 \ge w(c,\widetilde{W}_T^1)\}} \mid \widetilde{W}_T^1 > \tilde{a}_1]\widetilde{P}(\widetilde{W}_T^1 > \tilde{a}_1)$$

$$= e^{-rT}\int_{\tilde{a}_1}^{+\infty} \tilde{\mathbf{E}}[S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2\widetilde{W}_T^2}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1\widetilde{W}_T^1}-K)$$
$$\cdot \mathbf{1}_{\{(A_2-\sigma_2)\widetilde{W}_T^2 \ge w(c,\widetilde{W}_T^1)\}} \mid \widetilde{W}_T^1 = x]\tilde{f}_{\widetilde{W}_T^1}(x)\,dx$$

$$= C_1\int_{\tilde{a}_1}^{+\infty} e^{\sigma_1 x}\int_{w(c,x)}^{+\infty} e^{\frac{\sigma_2}{A_2-\sigma_2}y}\tilde{f}_{(A_2-\sigma_2)\widetilde{W}_T^2|\widetilde{W}_T^1=x}(y)\,dy\,\tilde{f}_{\widetilde{W}_T^1}(x)\,dx$$
$$- C_2\int_{\tilde{a}_1}^{+\infty}\int_{w(c,x)}^{+\infty} e^{\frac{\sigma_2}{A_2-\sigma_2}y}\tilde{f}_{(A_2-\sigma_2)\widetilde{W}_T^2|\widetilde{W}_T^1=x}(y)\,dy\,\tilde{f}_{\widetilde{W}_T^1}(x)\,dx.$$

with $C_1 := e^{-rT}S_0^1S_0^2e^{(2r-\frac{1}{2}\sigma_1^2-\frac{1}{2}\sigma_2^2)T}$, $C_2 := e^{-rT}KS_0^2e^{(r-\frac{1}{2}\sigma_2^2)T}$. From (4.30) we have $\tilde{\mathcal{L}}((A_2-\sigma_2)\widetilde{W}_T^2 \mid \widetilde{W}_T^1 = x) = N((A_2-\sigma_2)\rho x, T(1-\rho^2)(A_2-\sigma_2)^2)$

and hence

$$\Psi_2(c) = C_1 \frac{\int_{\tilde{a}_1}^{+\infty} e^{\sigma_1 x} \int_{w(c,x)}^{+\infty} e^{\frac{\sigma_2}{A_2-\sigma_2}y + \frac{(y-(A_2-\sigma_2)\rho x)^2}{2T(1-\rho^2)(A_2-\sigma_2)^2}}\, dy\, \tilde{f}_{\widetilde{W}_T^1}(x)\, dx}{\sqrt{2\pi T(1-\rho^2)(A_2-\sigma_2)^2}}$$
$$- C_2 \frac{\int_{\tilde{a}_1}^{+\infty} \int_{w(c,x)}^{+\infty} e^{\frac{\sigma_2}{A_2-\sigma_2}y + \frac{(y-(A_2-\sigma_2)\rho x)^2}{2T(1-\rho^2)(A_2-\sigma_2)^2}}\, dy\, \tilde{f}_{\widetilde{W}_T^1}(x) dx}{\sqrt{2\pi T(1-\rho^2)(A_2-\sigma_2)^2}}.$$

4.2.2. *Quanto foreign.* The payoff is of the form

$$H = \left(S_T^1 - \frac{K}{S_T^2}\right)^+, \qquad K > 0.$$

First, notice that

$$\left\{S_T^1 - \frac{K}{S_T^2} \geq 0\right\} = \{\sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d\} \tag{4.38}$$
$$= \{\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d}\} =: \Omega_0,$$

where

$$d := \ln \frac{K}{S_0^1 S_0^2} - \left(\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right)T, \quad \tilde{d} := d + (\alpha_1 + \alpha_2 - 2r)T,$$

and

$$A_c = \left\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c\left(S_T^1 - \frac{K}{S_T^2}\right)\right\} \tag{4.39}$$
$$= \{A_1 W_T^1 + (A_2 + \sigma_2) W_T^2 \geq v(c, \sigma_1 W_T^1 + \sigma_2 W_T^2)\}$$
$$= \{A_1 \widetilde{W}_T^1 + (A_2 + \sigma_2) \widetilde{W}_T^2 \geq w(c, \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2)\},$$

where

$$v(c,z) := \ln\left(\frac{c}{S_0^2} e^{(\frac{1}{2}\sigma_2^2 - \alpha_2 - B)T} (S_0^1 S_0^2 e^{\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)T + z} - K)\right),$$
$$w(c,z) := \ln\left[\frac{c}{S_0^2} e^{T(\frac{1}{2}\sigma_2^2 - r + A_1 \frac{\alpha_1 - r}{\sigma_1} + A_2 \frac{\alpha_2 - r}{\sigma_2} - B)} (S_0^1 S_0^2 e^{(2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))T + z} - K)\right].$$

By (4.38) we have

$$\Psi_1(c) = P\left(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c\left(S_T^1 - \frac{K}{S_T^2}\right) \;\middle|\; \Omega_0\right) P(\Omega_0)$$
$$+ P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0 \mid \Omega_0^c) P(\Omega_0^c)$$
$$= P(S_T^2 e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c(S_T^1 S_T^2 - K) \mid \Omega_0) P(\Omega_0) + P(\Omega_0^c),$$

As a consequence of (4.39) we obtain

$$\begin{aligned}\Psi_1(c) &= P(A_1 W_T^1 + (A_2+\sigma_2)W_T^2 \geq v(c, \sigma_1 W_T^1 + \sigma_2 W_T^2) \mid \Omega_0) P(\Omega_0) \\ &\quad + P(\Omega_0^c) \\ &= \int_d^{+\infty} P(A_1 W_T^1 + (A_2+\sigma_2)W_T^2 \geq v(c,z) \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 = z) \\ &\qquad\qquad \cdot f_{\sigma_1 W_T^1 + \sigma_2 W_T^2}(z)\, dz + P(\Omega_0^c).\end{aligned}$$

By (4.30) we have

$$\mathcal{L}(A_1 W_T^1 + (A_2+\sigma_2)W_T^2 \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 = z) = N(m(z), \sigma(z)),$$

where

$$\begin{aligned}m(z) &:= \frac{(A_1 + (A_2+\sigma_2)\rho)\sigma_1 + (A_1\rho + A_2 + \sigma_2)\sigma_2}{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \\ \sigma(z) &:= T\bigg\{(A_1 + (A_2+\sigma_2)\rho)A_1 + (A_1\rho + (A_2+\sigma_2))(A_2+\sigma_2) \\ &\qquad - \frac{\big((A_1 + (A_2+\sigma_2)\rho)\sigma_1 + (A_1\rho + (A_2+\sigma_2))\sigma_2\big)^2}{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\bigg\},\end{aligned}$$

and thus

$$\begin{aligned}\Psi_1(c) = \int_d^{+\infty} &\Phi\bigg(\frac{m(z) - v(c,z)}{\sqrt{\sigma(z)}}\bigg) f_{\sigma_1 W_T^1 + \sigma_2 W_T^2}(z)\, dz \\ &+ \Phi\bigg(\frac{d}{\sqrt{T(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)}}\bigg).\end{aligned}$$

By (4.38) and (4.39) we have

$$\begin{aligned}\Psi_2(c) &= e^{-rT}\tilde{\mathbf{E}}\bigg[\bigg(S_T^1 - \frac{K}{S_T^2}\bigg)\mathbf{1}_{A_c} \,\bigg|\, \Omega_0\bigg]\widetilde{P}(\Omega_0) \\ &= e^{-rT}\tilde{\mathbf{E}}\bigg[\bigg(S_T^1 - \frac{K}{S_T^2}\bigg)\mathbf{1}_{A_c} \,\bigg|\, \Omega_0\bigg]\widetilde{P}(\Omega_0) \\ &= e^{-rT}\int_{\tilde{d}}^{+\infty} \tilde{\mathbf{E}}[S_T^1 \mathbf{1}_{A_c} \mid \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 = z]\tilde{f}_{\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2}(z)\, dz \\ &\quad - e^{-rT}K\int_{\tilde{d}}^{+\infty} \tilde{\mathbf{E}}\bigg[\frac{1}{S_T^2}\mathbf{1}_{A_c} \,\bigg|\, \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 = z\bigg]\tilde{f}_{\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2}(z)\, dz.\end{aligned}$$

Using (4.28) we find the conditional distributions

$$\tilde{\mathcal{L}}(\widetilde{W}_T^1, A_1\widetilde{W}_T^1 + (A_2+\sigma_2)\widetilde{W}_T^2 \mid \sigma_1\widetilde{W}_T^1 + \sigma_2\widetilde{W}_T^2 = z) = N_2(M^1(z), \Sigma^1(z)),$$

$$\tilde{\mathcal{L}}(\widetilde{W}_T^2, A_1\widetilde{W}_T^1 + (A_2+\sigma_2)\widetilde{W}_T^2 \mid \sigma_1\widetilde{W}_T^1 + \sigma_2\widetilde{W}_T^2 = z) = N_2(M^2(z), \Sigma^2(z)),$$

where $M^1(z), M^2(z), \Sigma^1(z), \Sigma^2(z)$ are determined by (4.29). As a consequence,

$$\begin{aligned}\Psi_2(c) &= e^{-rT}S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T}\int_{\tilde{d}}^{+\infty}\int_{-\infty}^{+\infty}\int_{w(c,z)}^{+\infty} e^{\sigma_1 x}F^1(x,y)\,dy\,dx\,\tilde{f}_{\sigma_1\widetilde{W}_T^1+\sigma_2\widetilde{W}_T^2}(z)\,dz\\ &\quad - e^{-rT}\frac{K}{S_0^2}e^{-(r-\frac{1}{2}\sigma_2^2)T}\int_{\tilde{d}}^{+\infty}\int_{-\infty}^{+\infty}\int_{w(c,z)}^{+\infty} e^{-\sigma_2 x}F^2(x,y)\,dy\,dx\,\tilde{f}_{\sigma_1\widetilde{W}_T^1+\sigma_2\widetilde{W}_T^2}(z)\,dz,\end{aligned}$$

where F^1, F^2 stand for the density functions of the two-dimensional normal distributions $N_2(M^1(z), \Sigma^1(z))$, $N_2(M^2(z), \Sigma^2(z))$ respectively.

4.3. Outperformance option. The problem is studied for

$$H = (\max\{S_T^1, S_T^2\} - K)^+, \qquad K > 0.$$

Notice that

$$\{e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^1 - K)\} = \{A_2W_T^2 \geq v_1(c, W_T^1)\} = \{A_2\widetilde{W}_T^2 \geq w_1(c, \widetilde{W}_T^1)\} \tag{4.40}$$

$$\{e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^2 - K)\} = \{A_1W_T^1 \geq v_2(c, W_T^2)\} = \{A_1\widetilde{W}_T^1 \geq w_2(c, \widetilde{W}_T^2)\}, \tag{4.41}$$

where

$$\begin{aligned}v_1(c,x) &:= \ln\big[cS_0^1 e^{(\alpha_1-\frac{1}{2}\sigma_1^2)T+\sigma_1 x}\big] - A_1x - BT,\\ v_2(c,y) &:= \ln\big[cS_0^2 e^{(\alpha_2-\frac{1}{2}\sigma_2^2)T+\sigma_2 y}\big] - A_2y - BT,\\ w_1(c,x) &:= \ln\big(ce^{-A_1x+(A_1\frac{\alpha_1-r}{\sigma_1}-B)T}(S_0^1e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1x} - K)\big) + \frac{\alpha_2-r}{\sigma_2}T,\\ w_2(c,y) &:= \ln\big(ce^{-A_2y+(A_2\frac{\alpha_2-r}{\sigma_2}-B)T}(S_0^2e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2y} - K)\big) + \frac{\alpha_1-r}{\sigma_1}T.\end{aligned}$$

By (4.31)–(4.33) we have

$$\Psi_1(c) = P(e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^1 \vee S_T^2 - K)^+)$$

$$
\begin{aligned}
&= P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c(S_T^1 - K) \mid W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad \cdot P(W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad + P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0 \mid W_T^1 < a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad \cdot P(W_T^1 < a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad + P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c(S_T^2 - K) \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\
&\quad \cdot P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\
&\quad + P(e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0 \mid W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\
&\quad \cdot P(W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b).
\end{aligned}
$$

By (4.40), (4.41) we have

$$
\begin{aligned}
\Psi_1(c) &= P(A_2 W_T^2 \geq v_1(c, W_T^1) \mid W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad \cdot P(W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) + P(W_T^1 < a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\
&\quad + P(A_1 W_T^1 \geq v_2(c, W_T^2) \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\
&\quad \cdot P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) + P(W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b).
\end{aligned}
$$

Let $m_1(y,z), m_2(x,z), \sigma_1(y,z), \sigma_2(x,z)$ be the means and variances of the conditional distributions

$$
\begin{aligned}
\mathcal{L}(A_1 W_T^1 \mid W_T^2 = y, \sigma_1 W_T^1 - \sigma_2 W_T^2 = z) &= N(m_1(y,z), \sigma_1(y,z)), \\
\mathcal{L}(A_2 W_T^2 \mid W_T^1 = x, \sigma_1 W_T^1 - \sigma_2 W_T^2 = z) &= N(m_2(x,z), \sigma_2(x,z)),
\end{aligned}
$$

given by (4.29). Then

$$
\begin{aligned}
\Psi_1(c) = &\int_{a_1}^{+\infty} \int_{b}^{+\infty} \Phi\biggl(\frac{m_2(x,z) - v_1(c,x)}{\sqrt{\sigma_2(x,z)}}\biggr) f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x,z)\, dz\, dx \\
&+ \int_{-\infty}^{a_1} \int_{b}^{+\infty} f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x,z)\, dz\, dx \\
&+ \int_{a_2}^{+\infty} \int_{-\infty}^{b} \Phi\biggl(\frac{m_1(y,z) - v_2(c,y)}{\sqrt{\sigma_1(y,z)}}\biggr) f_{W_T^2, \sigma_1 W_T^1 - \sigma_2 W_T^2}(y,z)\, dz\, dy \\
&+ \int_{-\infty}^{a_2} \int_{-\infty}^{b} f_{W_T^2, \sigma_1 W_T^1 - \sigma_2 W_T^2}(y,z)\, dz\, dy.
\end{aligned}
$$

By (4.31)–(4.33), (4.40), (4.41) we have

$$\begin{aligned}\Psi_2(c) &= e^{-rT}\tilde{\mathbf{E}}((S_T^1 \vee S_T^2 - K)^+ \mathbf{1}_{A_c})\\ &= e^{-rT}\tilde{\mathbf{E}}((S_T^1 - K)\mathbf{1}_{A_c} \mid S_T^1 \geq K, S_T^1 \geq S_T^2)\widetilde{P}(S_T^1 \geq K, S_T^1 \geq S_T^2)\\ &\quad + e^{-rT}\tilde{\mathbf{E}}((S_T^2 - K)\mathbf{1}_{A_c} \mid S_T^2 \geq K, S_T^1 < S_T^2)\widetilde{P}(S_T^2 \geq K, S_T^1 < S_T^2)\\ &= e^{-rT}\tilde{\mathbf{E}}((S_T^1 - K)\mathbf{1}_{\{A_2\widetilde{W}_T^2 \geq w_1(c,\widetilde{W}_T^1)\}} \mid \widetilde{W}_T^1 \geq \tilde{a}_1, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2 \geq \tilde{b})\\ &\quad \cdot \widetilde{P}(\widetilde{W}_T^1 \geq \tilde{a}_1, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2 \geq \tilde{b})\\ &\quad + e^{-rT}\tilde{\mathbf{E}}((S_T^2 - K)\mathbf{1}_{\{A_1\widetilde{W}_T^1 \geq w_2(c,\widetilde{W}_T^2)\}} \mid \widetilde{W}_T^2 \geq \tilde{a}_2, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2 < \tilde{b})\\ &\quad \cdot \widetilde{P}(\widetilde{W}_T^2 \geq \tilde{a}_2, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2 < \tilde{b}).\end{aligned}$$

Let $m_1(y,z), \sigma_1(y,z)$ and $m_2(x,z), \sigma_2(x,z)$ denote the means and variances of the conditional distributions

$$\tilde{\mathcal{L}}(A_1\widetilde{W}_T^1 \mid \widetilde{W}_T^2, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2) = N(m_1(y,z), \sigma_1(y,z)),$$
$$\tilde{\mathcal{L}}(A_2\widetilde{W}_T^2 \mid \widetilde{W}_T^1, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_T^2) = N(m_2(x,z), \sigma_2(x,z)).$$

Finally we obtain

$$\begin{aligned}\Psi_2(c) = e^{-rT}\int_{\tilde{a}_1}^{+\infty}\int_{\tilde{b}}^{+\infty}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 x} - K)\Phi\left(\frac{m_2(x,z) - w_1(c,x)}{\sqrt{\sigma_2(x,z)}}\right)&\\ \cdot \tilde{f}_{\widetilde{W}_T^1, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_t^2}(x,z)\,dz\,dx&\\ + e^{-rT}\int_{\tilde{a}_2}^{+\infty}\int_{-\infty}^{\tilde{b}}(S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_1 y} - K)\Phi\left(\frac{m_1(y,z) - w_2(c,y)}{\sqrt{\sigma_1(y,z)}}\right)&\\ \cdot \tilde{f}_{\widetilde{W}_T^2, \sigma_1\widetilde{W}_T^1 - \sigma_2\widetilde{W}_t^2}(y,z)\,dz\,dy.&\end{aligned}$$

4.4. Spread option. The payoff is of the form

$$H = (S_T^1 - S_T^2 - K)^+, \qquad K > 0.$$

For any $y \in \mathbb{R}$,

$$\{S_T^1 - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T+\sigma_2 y} - K \geq 0\} = \{W_T^1 \geq e(y)\}, \tag{4.42}$$

where

$$e(y) := \frac{1}{\sigma_1}\left(\ln\left[\frac{1}{S_0^1}(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T+\sigma_2 y})\right] - \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)T\right).$$

Then

$$\begin{aligned}\Psi_1(c) &= P(e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^1 - S_T^2 - K)^+) \\ &= \int_{-\infty}^{+\infty} P(e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^1 - S_T^2 - K)^+ \mid W_T^2 = y) f_{W_T^2}(y)\, dy \\ &= \int_{-\infty}^{+\infty} P(e^{A_1W_T^1+A_2W_T^2+BT} \geq c(S_T^1 - S_T^2 - K), W_T^1 \geq e(y) \mid W_T^2 = y) \\ &\qquad\qquad \cdot f_{W_T^2}(y)\, dy \\ &\quad + \int_{-\infty}^{+\infty} P(e^{A_1W_T^1+A_2W_T^2+BT} \geq 0, W_T^1 < e(y) \mid W_T^2 = y) f_{W_T^2}(y)\, dy \\ &= \int_{-\infty}^{+\infty} P(e^{A_2y+BT}e^{A_1W_T^1} - cS_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T} e^{\sigma_1 W_T^1} \geq -c(S_T^2 + K), \\ &\qquad\qquad W_T^1 \geq e(y) \mid W_T^2 = y) f_{W_T^2}(y)\, dy \\ &\quad + \int_{-\infty}^{+\infty} P(W_T^1 < e(y) \mid W_T^2 = y) f_{W_T^2}(y)\, dy.\end{aligned}$$

Let

$$\mathcal{L}(W_T^1 \mid W_T^2 = y) = N(m(y), \sigma(y)).$$

Then

$$\begin{aligned}\Psi_1(c) = &\int_{-\infty}^{+\infty} \int_{S(c,y)\cap(e(y),+\infty)} f_{W_T^1 \mid W_T^2 = y}(x)\, dx\, f_{W_T^2}(y)\, dy \\ &+ \int_{-\infty}^{+\infty} \Phi\bigg(\frac{e(y) - m(y)}{\sqrt{\sigma(y)}}\bigg) f_{W_T^2}(y)\, dy,\end{aligned}$$

where for $y \in \mathbb{R}$,

$$\begin{aligned}S(c,y) := \{x : e^{A_2y+BT}e^{A_1x} - cS_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T} e^{\sigma_1 x} \\ \geq -c(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K)\}.\end{aligned}$$

For practical applications it is necessary to find a closed form of the set $S(c,y)$. In the formulation of the next result we will use the solutions of the equation

$$g(x) = -c(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K), \tag{4.43}$$

where $g(x) := e^{A_2y+BT}e^{A_1x} - cS_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T} e^{\sigma_1 x}$. These solutions can be found numerically.

PROPOSITION 4.1. *The set $S(c,y)$ is of the following form:*

(a) *If $A_1 > \sigma_1$ and*

(i) $g(\hat{x}) \geq -c(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K)$ *then* $S(c,y) = (-\infty, +\infty)$,

(ii) $g(\hat{x}) \geq -c(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K)$ *then* $S(c,y) = (-\infty, x_1) \cup (x_2, +\infty)$, *where $x_1 < x_2$ are the unique solutions of* (4.43).

Above, $\hat{x}$ stands for $\frac{1}{\sigma_1 - A_1} \ln\Big(\frac{A_1 e^{A_2 y + BT}}{\sigma_1 c S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}}\Big)$.

(b) *If $A_1 = \sigma_1$ and*

(i) $e^{A_2 y + BT} \geq c S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}$ *then* $S(c,y) = (-\infty, +\infty)$,

(ii) $e^{A_2 y + BT} < c S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}$ *then* $S(c,y) = (-\infty, x_0)$, *where x_0 is a unique solution of* (4.43).

(c) *If $A_1 < \sigma_1$ then $S(c,y) = (-\infty, x_0)$, where x_0 is a unique solution of* (4.43).

Proof. (a) One can check that g has a minimum at the point $\hat{x}$ and is decreasing on $(-\infty, \hat{x})$ and increasing on $(\hat{x}, +\infty)$. Hence (i) and (ii) follow.

(b) The formulas for $S(c,y)$ follow from the simplified form of the function $g(x) = (e^{A_2 y + BT} - c S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}) e^{A_1 x}$.

(c) It can be checked that g is strictly increasing on $\{x : g(x) < 0\}$ and $\lim_{x \to +\infty} g(x) = -\infty$. Thus (4.43) has a unique solution and the form of the set $S(c,y)$ follows. ■

Now let us determine Ψ_2. One can check that for $y \in \mathbb{R}$,

$$\{S_T^1 - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \geq 0\} = \{\widetilde{W}_T^1 \geq f(y)\}, \tag{4.44}$$

where

$$f(y) := \frac{1}{\sigma_1}\bigg(\ln\bigg[\frac{1}{S_0^1}(S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y})\bigg] - \Big(r - \frac{1}{2}\sigma_1^2\Big)T\bigg).$$

For $y \in \mathbb{R}$ define

$$\tilde{S}(c,y) := \{x : e^{A_1(x - \frac{\alpha_1 - r}{\sigma_1}T) + A_2(y - \frac{\alpha_2 - r}{\sigma_2}T) + BT} \geq c(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K)\}.$$

Then

$$\Psi_2(c) = e^{-rT} \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{A_c}] = e^{-rT} \int_{-\infty}^{+\infty} \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{A_c} \mid \widetilde{W}_T^2 = y] \tilde{f}_{\widetilde{W}_T^2}(y)\, dy$$

$$
\begin{aligned}
&= e^{-rT} \int_{-\infty}^{+\infty} \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)\mathbf{1}_{A_c}\mathbf{1}_{\{\widetilde{W}_T^1 \geq f(y)\}} \mid \widetilde{W}_T^2 = y]\tilde{f}_{\widetilde{W}_T^2}(y)\,dy \\
&\quad + e^{-rT} \int_{-\infty}^{+\infty} \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)^+\mathbf{1}_{A_c}\mathbf{1}_{\{\widetilde{W}_T^1 < f(y)\}} \mid \widetilde{W}_T^2 = y]\tilde{f}_{\widetilde{W}_T^2}(y)\,dy \\
&= e^{-rT} \int_{-\infty}^{+\infty} \int_{\tilde{S}(c,y)\cap(f(y),+\infty)} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 x} - S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 y} - K) \\
&\qquad\qquad \cdot \tilde{f}_{\widetilde{W}_T^1 \mid \widetilde{W}_T^2 = y}(x)\,dx\,\tilde{f}_{\widetilde{W}_T^2}(y)\,dy.
\end{aligned}
$$

The explicit form of the set $\tilde{S}(c,y)$ can be established in the same way as for $S(c,y)$ in Proposition 4.1.

References

[1] M. Baran, *Large losses—probability minimizing approach*, Appl. Math. (Warsaw) 31 (2004), 243–257.

[2] —, *Quantile hedging markets with proportional transaction costs*, ibid. 30 (2003), 193–208.

[3] J. Cvitanić and I. Karatzas, *Generalized Neyman–Pearson lemma via convex duality*, Bernoulli 7 (2001), 79–97.

[4] J. Cvitanić and G. Spivak, *Maximizing the probability of a perfect hedge*, Ann. Appl. Probab. 9 (1999), 1303–1328.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, 1992.

[6] H. Föllmer and P. Leukert, *Quantile hedging*, Finance Stoch. 3 (1999), 251–273.

[7] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2003.

[8] R. A. Johnson and D. W. Wichern, *Applied Multivariate Statistical Analysis*, 6th ed., Prentice-Hall, 2007.

[9] I. Karatzas, *Lectures on the Mathematics of Finance*, CRM Monogr. Ser. 8, Amer. Math. Soc., 1997.

[10] R. N. Krutchenko and A. V. Melnikov, *Quantile hedging for a jump-diffusion financial market model*, in: Mathematical Finance (Konstanz, 2000), Trends in Mathematics, Birkhäuser, Basel, 2001.

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