APPLICATIONES MATHEMATICAE
32,4 (2005), pp. 477–488

Karel Kolman (Praha)

SUPERCONVERGENCE BY STEKLOV AVERAGING IN THE FINITE ELEMENT METHOD

Abstract. The Steklov postprocessing operator for the linear finite element method is studied. Superconvergence of order $\mathcal{O}(h^2)$ is proved for a class of second order differential equations with zero Dirichlet boundary conditions for arbitrary space dimensions. Relations to other postprocessing and averaging schemes are discussed.

1. Introduction. Most finite element postprocessing schemes for second order problems discretized on simplicial partitions of domains appear in the form of gradient postprocessing, i.e., an improved approximation of the gradient of the true solution is obtained, but the resulting vector function is not a potential field.

Steklov averaging based postprocessing proposed by Oganesyan and Rukhovets in [6] does not suffer from this disadvantage.

The monograph [6] (pp. 94–101, 189) presents the Steklov operator S_h (which will be defined in Section 2) as a tool for postprocessing of finite element solutions of second order elliptic boundary value problems. Fixing a subdomain Ω_0 with $\overline{\Omega}_0 \subset \Omega$, and taking the finite element approximate solution u_h of the Poisson problem $-\Delta u = f$ with zero Dirichlet boundary condition, Oganesyan and Rukhovets proved the error bound

$$\|u - S_h u_h\|_{1,\Omega_0} = \mathcal{O}(h^{3/2}) \tag{1}$$

for 2-dimensional uniform meshes consisting of right isosceles triangles, whereas the piecewise linear finite element solution u_h approximates the exact solution u with order

$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h).$$

2000 *Mathematics Subject Classification*: Primary 65N30.

Key words and phrases: superconvergence, finite elements, postprocessing, averaging.

Numerical experiments, however, suggest that bound (1) is not optimal, i.e., it may be improved.

In this paper we show that, in fact, for $u \in H^3(\Omega)$ and uniform meshes we have

$$\|u - S_h u_h\|_{1,\Omega_0} \le Ch^2 \|u\|_{3,\Omega}$$

for some $C > 0$.

2. Steklov averaging operator. Let Ω_0 with $\overline{\Omega}_0 \subset \Omega \subset \mathbb{R}^d$, $d \ge 1$, be a subdomain of a bounded domain Ω. The d-dimensional *Steklov averaging operator* acting on a function $v \in L^1(\Omega)$ is defined by

$$(S_h v)(x) = \frac{1}{(2h)^d} \int\limits_{\Pi_h} v(x+y)\,dy, \tag{2}$$

where $\Pi_h = \{y = (y_1, \ldots, y_d) \in \mathbb{R}^d \mid |y_i| \le h,\ i = 1, \ldots, d\}$, $x \in \Omega_0$, and h is small enough so that $x + \Pi_h \subset \Omega$.

LEMMA 2.1. *For each integer $l \ge 0$ and a fixed subdomain Ω_0 with $\overline{\Omega}_0 \subset \Omega$ there exist constants $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\|S_h v\|_{l,\Omega_0} \le C\|v\|_{l,\Omega} \quad \forall v \in H^l(\Omega) \tag{3}$$

and

$$\|v - S_h v\|_{1,\Omega_0} \le Ch^2 \|v\|_{3,\Omega} \quad \forall v \in H^3(\Omega). \tag{4}$$

Proof. Oganesyan and Rukhovets' monograph [6] contains an elementary proof of (3) and the bound

$$\|v - S_h v\|_{0,\Omega_0} \le Ch^2 \|v\|_{2,\Omega} \tag{5}$$

for $d = 2$ (pp. 94–96). Bramble and Schatz's paper [1] includes the bounds (3) and (5) in Lemmas 5.2 and 5.3 for arbitrary space dimensions (take $S_h v = K_{h,l}^{2t} * v$ with $t = 1$ and $l = 1$).

If we make use of the equality $S_h \nabla v = \nabla S_h v$ proved in Lemma 2.2 below, where the Steklov operator for vector functions is naturally defined by

$$(S_h \nabla v)_i = S_h \frac{\partial}{\partial x_i} v, \qquad i = 1, \ldots, d,$$

inequality (5) immediately implies (4). ■

LEMMA 2.2. *If $v \in H^1(\Omega)$, then*

$$S_h \nabla v = \nabla S_h v \tag{6}$$

in Ω_0 and

$$S_h v \in C^1(\overline{\Omega}_0). \tag{7}$$

Proof. For a function $\varphi \in C_0^\infty(\Omega_0)$, by the Fubini theorem and the definition of the distributional derivative we have

$$\begin{aligned}(S_h\partial_i v, \varphi)_{\Omega_0} &= \frac{1}{(2h)^d}\int\limits_{\Omega_0}\Big(\int\limits_{\Pi_h}\partial_i v(x+y)\,dy\Big)\varphi(x)\,dx\\ &= \frac{1}{(2h)^d}\int\limits_{\Pi_h}\Big(\int\limits_{\Omega_0}\partial_i v(x+y)\varphi(x)\,dx\Big)\,dy\\ &= -\frac{1}{(2h)^d}\int\limits_{\Pi_h}\Big(\int\limits_{\Omega_0}v(x+y)\partial_i\varphi(x)\,dx\Big)\,dy\\ &= -\frac{1}{(2h)^d}\int\limits_{\Omega_0}\Big(\int\limits_{\Pi_h}v(x+y)\,dy\Big)\partial_i\varphi(x)\,dx = -(S_h v, \partial_i\varphi)_{\Omega_0}\end{aligned}$$

and the distributional derivative $(\partial/\partial x_i)S_h v$ is, therefore, a function from $L^2(\Omega_0)$ and equals $S_h(\partial/\partial x_i)v$.

For a function $v \in L^2(\Omega)$ we get $S_h v \in C(\overline{\Omega}_0)$, since

$$(S_h v)(x) - (S_h v)(y) = \frac{1}{(2h)^d}\int\limits_{\Upsilon_h(x,y)} v(\xi)\,d\xi \le \frac{1}{(2h)^d}\,\|1\|_{L^2(\Upsilon_h(x,y))}\|v\|_{L^2(\Omega)},$$

where the set $\Upsilon_h(x,y)$ is defined by

$$\Upsilon_h(x,y) = ((x+\Pi_h)\cup(y+\Pi_h))\backslash((x+\Pi_h)\cap(y+\Pi_h))$$

(see Figure 1) and apparently

$$\lim_{x\to y}\|1\|_{L^2(\Upsilon_h(x,y))} = 0$$

for a fixed h.

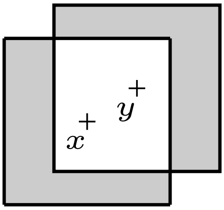

Fig. 1. $\Upsilon_h(x,y)$ for $d=2$ (shaded area)

Applying the above continuity property to $S_h\nabla u$, we get $S_h u \in C^1(\overline{\Omega}_0)$. ∎

REMARK 2.3. Clearly, (2) can be rewritten as

$$S_h v = \frac{1}{(2h)^d} v * \chi_h,$$

where χ_h is the characteristic function of the set Π_h and we extend the function v in some way to $L^1(\mathbb{R}^d)$.

EXAMPLE 2.4. For $v(x) = \sin(\pi x)$, $x \in \mathbb{R}^1$, we obtain

$$(S_h v)(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} \sin \pi y \, dy = \frac{\sin \pi h}{\pi h} \sin \pi x$$

and

$$(S_h v)^{(n)}(x) = (S_h v^{(n)})(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} \sin^{(n)} \pi y \, dy = \frac{\sin \pi h}{\pi h} \sin^{(n)} \pi x$$

for the nth derivative.

3. Steklov averaging in FEM. We shall consider second order elliptic boundary value problems of the type

$$\begin{aligned} -\operatorname{div}(\mathbb{A}(x)\nabla u(x)) &= f(x) && \text{in } \Omega, \\ u(x) &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{8}$$

where the matrix $\mathbb{A}(x)$ is uniformly positive definite and has a bounded total derivative, and the right-hand side $f(x)$ and Ω are such that the conditions of the so-called *supercloseness* (see [2] and Section 3.1 below) of the piecewise linear finite element solution u_h to the piecewise linear nodal Lagrange interpolant $L_h u$ are satisfied.

3.1. *Uniform simplicial partitions.* In the following we suppose that the finite element solution u_h is computed on a d-simplicial partition $\mathcal{T}_h = \{\tau_{h,i}\}_i$ of $\overline{\Omega}$ (which is a bounded polytopic domain). We further assume that the partition is constructed by translation of a partition of the d-hypercube K_h^d with edge size h into $d!$ d-simplices,

$$S_{\sigma,h} = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(d)} \le h\},$$

where σ denotes a permutation of $1, \ldots, d$ and $\sigma(j)$ its jth component.

The details of the construction and properties of the partition are presented in [2]. The partition constructed has internal vertices and midpoints of internal edges of the d-simplices as symmetry points (see Theorem 2.5 in [2]). We call a partition *uniform* if for each internal edge the union of simplices sharing the edge is a point-symmetric set with respect to the midpoint of the edge.

An important property of the Galerkin solution u_h computed on partitions of this type is its so-called *supercloseness* to $L_h u$. It is described by the bound

$$\|L_h u - u_h\|_{1,\Omega} \le C h^2 \|u\|_{3,\Omega} \tag{9}$$

(see [2]). Here $L_h u$ is the piecewise linear Lagrange interpolation of u at the vertices of the simplices of the partition.

3.2. *Superconvergence bound.* The main goal of this paper is to prove the following theorem.

THEOREM 3.1. *Let Ω_0 with $\overline{\Omega}_0 \subset \Omega \subset \mathbb{R}^d$, $d \geq 1$, be a subdomain of a bounded domain Ω. Suppose that a regular family of uniform partitions is used in the discretization and that $u \in H^3(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|u - S_h u_h\|_{1,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega} \tag{10}$$

for h small enough.

In the following paragraphs we will be working with the nodal linear Lagrange interpolant $L_h u$ of the true solution. To guarantee that $L_h u$ is well defined we shall suppose that $u \in H^s(\Omega)$ with $s = 3$ when $d \leq 5$ and $s > d/2$ when $d \geq 6$. This assumption is not an obstacle to proving the superconvergence property for $u \in H^3(\Omega)$, which is obtained by a density argument.

By the triangle inequality

$$\begin{aligned} \|u - S_h u_h\|_{1,\Omega_0} &\leq \|u - S_h L_h u\|_{1,\Omega_0} + \|S_h(L_h u - u_h)\|_{1,\Omega_0} \\ &\leq \|u - S_h u\|_{1,\Omega_0} + \|S_h(u - L_h u)\|_{1,\Omega_0} \\ &\quad + \|S_h(L_h u - u_h)\|_{1,\Omega_0}. \end{aligned} \tag{11}$$

We can use (4) to bound the first term on the right-hand side of (11),

$$\|u - S_h u\|_{1,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega}. \tag{12}$$

Using the supercloseness property (9) of the finite element solution u_h and boundedness (3) of S_h, we get

$$\|S_h(L_h u - u_h)\|_{1,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega}. \tag{13}$$

The boundedness of S_h and approximation theory (see [3]) imply the L^2-bound

$$\|S_h(u - L_h u)\|_{0,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega},$$

and therefore, it remains to estimate the term

$$|S_h(u - L_h u)|_{1,\Omega_0}.$$

3.3. *One-dimensional case.* In the one-dimensional case, we have

$$|S_h(u - L_h u)|_{1,\Omega_0} = \|(S_h(u - L_h u))'\|_{0,\Omega_0} = \|S_h(u' - (L_h u)')\|_{0,\Omega_0}.$$

For any node N_i of a given uniform partition of the interval Ω we obtain

$$(S_h u')(N_i) = \frac{1}{2h} \int\limits_{N_{i-1}}^{N_{i+1}} u'(x)\, dx = \frac{u(N_{i+1}) - u(N_{i-1})}{2h}$$

(the central difference derivative approximation at N_i) and

$$
\begin{aligned}
(S_h(L_h u)')(N_i) &= \frac{1}{2h}\left[u(N_{i+1}) - u(N_i) + u(N_i) - u(N_{i-1})\right] \\
&= \frac{u(N_{i+1}) - u(N_{i-1})}{2h}.
\end{aligned}
$$

Since $S_h(L_h u)'$ is piecewise linear (on the same partition of Ω), we get

$$S_h(L_h u)' = L_h S_h u',$$

and consequently,

$$|S_h(u - L_h u)|_{1,\Omega_0} = \|(I - L_h)S_h u'\|_{0,\Omega_0} \le Ch^2\|S_h u'\|_{2,\Omega_0} \le Ch^2\|u\|_{3,\Omega},$$

which is the required bound.

3.4. *Properties of* $S_h L_h u$ *and* $S_h \nabla L_h u$. The situation for $d > 1$ is more complicated. We define

$$V_h^k(\Omega) = \{v \in C(\overline{\Omega}) \mid v|_{\tau_h} \in P^k(\tau_h)\ \forall \tau_h \in \mathcal{T}_h\},$$

where $P^k(\tau_h)$ is the space of polynomials of degree k on a d-simplex τ_h.

LEMMA 3.2. *If* $v_h \in V_h^1(\Omega)$, *then*

$$S_h \nabla v_h \in (V_h^d(\Omega_0))^d, \tag{14}$$

$$S_h v_h \in V_h^{1+d}(\Omega_0). \tag{15}$$

Proof. The function $\partial v_h/\partial x_i$ is constant on each $\tau_h \in \mathcal{T}_h$ for $i = 1, \ldots, d$. From the structure of the simplicial mesh defined in Section 3.1 (see the definition of $S_{\sigma,h}$ in Section 3.1 and the definition of Π_h in Section 2), we clearly see that for a fixed simplex τ_h, the function $f_{\tau_h}(x) = \operatorname{meas}((x + \Pi_h) \cap \tau_h)$ is a piecewise polynomial function in $V_h^d(\Omega_0)$. The inclusion (14) is thus proved. Inclusion (15) then follows from the equality $S_h \nabla v_h = \nabla S_h v_h$. ■

For $d = 1$ the smoothed function $S_h L_h u$ is thus piecewise quadratic ($S_h \nabla L_h u$ is piecewise linear).

For $d = 2$ the smoothed function $S_h L_h u$ is piecewise cubic ($S_h \nabla L_h u$ is piecewise quadratic).

For $d = 3$ the smoothed function $S_h L_h u$ is piecewise quartic ($S_h \nabla L_h u$ is piecewise cubic).

3.5. *Quadratic polynomial gradient recovery at symmetry points.* For $N \in \overline{\Omega}_0$ define

$$F_{i,N}(u) = S_h \frac{\partial}{\partial x_i}(u - L_h u)(N) = \frac{1}{(2h)^d} \int\limits_{\Pi_h} \frac{\partial}{\partial x_i}(u - L_h u)(N + y)\, dy.$$

Let us now take N to be a local symmetry point of the partition, by which we mean that for any continuous function v_{even} that is even with respect to N, the linear interpolation $L_h v_{\text{even}}$ is even on $N + \varPi_h$ (e.g., N being a vertex or a midpoint of an edge of the uniform partition considered).

We now prove that for a quadratic polynomial q we have $F_{i,N}(q) = 0$ for $i = 1, \dots, d$. Decompose

$$q(x) = q_{\text{even}}(x) + q_{\text{odd}}(x)$$

into even and odd (with respect to the point N) polynomial parts; clearly q_{odd} is linear, thus $q_{\text{odd}} = L_h q_{\text{odd}}$ and $F_{i,N}(q_{\text{odd}}) = 0$. The local symmetry condition implies that $L_h q_{\text{even}}$ is an even function. Then both $(\partial/\partial x_i) q_{\text{even}}$ and $(\partial/\partial x_i) L_h q_{\text{even}}$ are odd functions and have zero mean over $N + \varPi_h$ (a set that is point-symmetric with respect to N). Therefore, $F_{i,N}(q) = 0$ for all quadratic functions and $i = 1, \dots, d$ (for all $d \geq 1$).

3.6. *Dimension* $d = 2$. For $d = 2$ the postprocessed function $S_h \nabla L_h u$ is piecewise quadratic (see Lemma 3.2). Since $S_h \nabla q = \nabla q$ is linear for a quadratic polynomial q ($S_h \nabla q$ being the average of a linear function over an area symmetric with respect to its center of gravity) and $S_h \nabla q = S_h \nabla L_h q$ at the vertices and midpoints of edges of triangles, we have

$$\nabla q = S_h \nabla q = S_h \nabla L_h q. \tag{16}$$

We shall now follow a Bramble–Hilbert lemma like sequence of arguments. From (16) we see that

$$\begin{aligned} \|S_h(\nabla u - \nabla L_h u)\|_{0,\tau_h} &= \|S_h(\nabla(u - q) - \nabla L_h(u - q))\|_{0,\tau_h} \\ &\leq C\|\nabla(u - q) - \nabla L_h(u - q)\|_{0,\tau_h + \varPi_h} \\ &\leq C' h |u - q|_{2,\tau_h + \varPi_h}. \end{aligned}$$

Making use of results of approximation theory (see [3]), we get

$$\inf_q |u - q|_{2,\tau_h + \varPi_h} \leq C h |u|_{3,\tau_h + \varPi_h}$$

and consequently,

$$\|S_h(\nabla u - \nabla L_h u)\|_{0,\tau_h} \leq C h^2 |u|_{3,\tau_h + \varPi_h}.$$

Summing over all triangles τ_h, we get

$$\begin{aligned} |S_h(u - L_h u)|_{1,\varOmega_0} &= \sqrt{\sum_{\tau_h} \|S_h(\nabla u - \nabla L_h u)\|^2_{0,\tau_h}} \\ &\leq C h^2 \sqrt{\sum_{\tau_h} |u|^2_{3,\tau_h + \varPi_h}} \leq C' h^2 |u|_{3,\varOmega}. \end{aligned} \tag{17}$$

Putting (12), (13), and (17) together, we finally prove the bound

$$\|u - S_h u_h\|_{1,\varOmega_0} \leq C h^2 \|u\|_{3,\varOmega} \tag{18}$$

for $u \in H^3(\varOmega)$.

3.7. *Superconvergence for the diagonal directional derivative* $\sum_{i=1}^{d} \partial/\partial x_i$. Before we look into the general case of arbitrary space dimension, we shall take a closer look at the case of the diagonal derivative.

The postprocessed function $S_h\big(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} L_h u\big)$ has a simpler structure than the functions $S_h\big(\frac{\partial}{\partial x_i} L_h u\big)$, $i = 1, \dots, d$, have, allowing a simple proof of a superconvergence result for the diagonal derivative.

For an edge e of the simplicial partition in the direction $(1, \dots, 1) \in \mathbb{R}^d$, the patch of simplices sharing this edge is a d-hypercube with e being its diagonal (see [2]). Furthermore, the derivative along e is constant on the d-hypercube. Now clearly $S_h\big(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} L_h u\big)$ is piecewise linear in each direction $e_i \in \mathbb{R}^d$ with jth component $e_{ij} = \delta_{ij}$, so it is a piecewise linear tensor product function (e.g., piecewise bilinear for $d = 2$).

The bound

$$\left\| S_h \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(u - L_h u) \right\|_{0,\Omega_0} \le C h^2 \|u\|_{3,\Omega}$$

is now apparent, since similarly to (16) we have

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} q = S_h \sum_{i=1}^{d} \frac{\partial}{\partial x_i} q = S_h \sum_{i=1}^{d} \frac{\partial}{\partial x_i} L_h q$$

for any quadratic polynomial q.

3.8. *Dimension* $d \ge 3$. We shall now use the results for $d = 2$ to prove the quadratic polynomial gradient recovery property (16) for general $d \ge 3$.

For q linear, we clearly have $L_h q = q$, and therefore, (16) is true.

It remains to prove (16) for quadratic monomials

$$q(x) = x_i x_j, \qquad i, j \in \{1, \dots, d\}.$$

Fix i, j. For an integer $k \notin \{i, j\}$, the monomial q does not depend on x_k and so $\partial q/\partial x_k$ is zero. The derivative $(\partial/\partial x_k) L_h q$ is constant on each d-simplex and is determined by the values of $L_h q$ at the vertices of the edge in the canonical direction e_k. Each d-simplex of the partition considered has such an edge (see [2]). Therefore,

$$\frac{\partial}{\partial x_k} q = \frac{\partial}{\partial x_k} L_h q = 0 \qquad \text{for } k \notin \{i, j\},$$

and thus,

$$S_h \frac{\partial}{\partial x_k} L_h q = 0 \qquad \text{for } k \notin \{i, j\}.$$

We shall now investigate the case of the $\partial/\partial x_i$ derivative. Setting $\Pi_h^m = (-h, h)^m$ and taking a permutation σ of $1, \dots, d$ such that $\sigma(d-1) = i$ and

$\sigma(d) = j$, we have

$$S_h \frac{\partial}{\partial x_i} L_h q = \frac{1}{(2h)^{d-2}} \int\limits_{\Pi_h^{d-2}} \left(\frac{1}{(2h)^2} \int\limits_{\Pi_h^2} \frac{\partial}{\partial x_i} L_h q \; dx_i dx_j \right) dx_{\sigma(1)} \cdots dx_{\sigma(d-2)}$$
$$= \frac{1}{(2h)^2} \int\limits_{\Pi_h^2} \frac{\partial}{\partial x_i} L_h q \, dx_i \, dx_j,$$

since $L_h q$ does not depend on $x_{\sigma(1)}, \ldots, x_{\sigma(d-2)}$.

It now follows from (16) for the two-dimensional case that $S_h(\partial/\partial x_i)L_h q$ is a linear function (depending on x_i and x_j only).

Therefore, (16) is proven for $d \geq 3$. Following an identical sequence of arguments as for the two-dimensional case, we come to the bound (10) of Theorem 3.1.

Having worked so far with functions $u \in H^s(\Omega)$ (with $s = 3$ when $d \leq 5$ and $s > d/2$ when $d \geq 6$), we shall now complete the proof for functions $u \in H^3(\Omega)$ by a density argument.

Let $u \in H^3(\Omega)$ and $\varepsilon > 0$ be given. By density of $H^s(\Omega)$ in $H^3(\Omega)$ there exists a function $w \in H^s(\Omega)$ such that $\|u - w\|_{3,\Omega} \leq \varepsilon$. By the triangle inequality

$$\|u - S_h u_h\|_{1,\Omega_0} \leq \|(u - w) - S_h(u_h - w_h)\|_{1,\Omega_0} + \|w - S_h w_h\|_{1,\Omega_0}$$
$$\leq C\varepsilon + Ch^2\|w\|_{3,\Omega} \leq C'\varepsilon + Ch^2\|u\|_{3,\Omega},$$

where w_h is the Galerkin projection of w (corresponding to the boundary value problem considered and the finite element space used). Since $\varepsilon > 0$ was arbitrary, we arrive at

$$\|u - S_h u_h\|_{1,\Omega_0} \leq Ch^2\|u\|_{3,\Omega}$$

for $u \in H^3(\Omega)$.

3.9. *Pointwise estimates at local symmetry points.* Estimates at a symmetry point N yield

$$|F_{i,N}(u)| \leq \frac{C}{(2h)^d} \left\| \frac{\partial}{\partial x_i}(u - L_h u) \right\|_{0,N+\Pi_h} \|1\|_{0,N+\Pi_h} \leq \frac{C'}{(2h)^{d/2}} \, h|u|_{2,N+\Pi_h}$$

and since $F_{i,N}(q) = 0$ for all quadratic polynomials q, we can write

$$|F_{i,N}(u)| \leq \frac{C}{(2h)^{d/2}} \, h|u - q|_{2,N+\Pi_h}.$$

The use of approximation theory (see [3]) then yields the following result, which we will employ later in Section 3.11.

LEMMA 3.3. *There exists a constant $C > 0$ such that if N is a symmetry point of the mesh, then*

$$|F_{i,N}(u)| \le \frac{C}{(2h)^{d/2}}\, h^2 |u|_{3,N+\Pi_h}, \quad i = 1, \dots, d, \tag{19}$$

for h small enough.

3.10. *Modifications.* We can alter the above approach for $d = 2$ to perform the averaging over a different area symmetric with respect to its center of gravity, e.g., Ξ_h or Λ_h of Figure 2, and obtain the same postprocessing accuracy as for the area Π_h. The proof would be identical as in the case of Π_h as long as we choose to average over a point-symmetric area such that the results of Lemma 3.2 are valid.

Whereas results of Section 3.9 carry over to the case of a general point-symmetric averaging area for $d \ge 3$, the condition of symmetry itself will not allow a similar proof of the quadratic polynomial recovery property (16) as in Section 3.8. However, (19) can still be utilized for proving error bounds of various averaging schemes as in the following section.

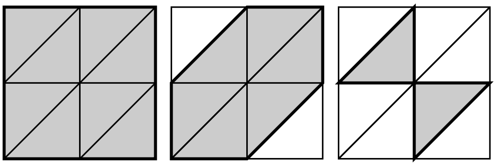

Fig. 2. Averaging areas Π_h, Ξ_h and Λ_h

3.11. *Gradient superconvergence.* For arbitrary space dimension d we easily obtain

$$\begin{aligned}
&\|\nabla u - L_h S_h \nabla u_h\|_0 \\
&\quad \le \|\nabla u - L_h S_h \nabla u\|_0 + \|L_h S_h(\nabla u - \nabla L_h u)\|_0 + \|L_h S_h(\nabla L_h u - \nabla u_h)\|_0 \\
&\quad \le C h^2 \|u\|_3 + \|L_h S_h(\nabla u - \nabla L_h u)\|_0,
\end{aligned}$$

and employing the bound (19) at vertices to bound the remaining linear term, we easily get

$$\|\nabla u - L_h S_h \nabla u_h\|_{0,\Omega_0} \le C h^2 \|u\|_{3,\Omega}.$$

Choosing to average over Ξ_h ($d = 2$, see Figure 2), one immediately gets the superconvergence result for the linear field (see [5])

$$L_h S_h \nabla u_h(N) = \frac{1}{6} \sum_{\tau \cap N \ne \emptyset} \nabla u_h|_\tau,$$

for $d = 3$ (see [4])

$$L_h S_h \nabla u_h(N) = \frac{1}{24} \sum_{\tau \cap N \neq \emptyset} \nabla u_h|_\tau,$$

or similarly for an arbitrary d with coefficient $1/(d+1)!$.

Averaging over Λ_h for $d = 2$ leads to the postprocessing scheme (see [2])

$$L_h S_h \partial_{x_1} u_h(N) = \frac{u_h(N + (h,0)) - u_h(N - (h,0))}{2h},$$
$$L_h S_h \partial_{x_2} u_h(N) = \frac{u_h(N + (0,h)) - u_h(N - (0,h))}{2h}.$$

Remark 3.4. The postprocessed solution $S_h u_h$ is not better in the L^2-sense, i.e., we only have

$$\|u - S_h u_h\|_{0,\Omega_0} = \mathcal{O}(h^2),$$

and this estimate cannot be improved.

4. Numerical experiments. We tested the superconvergence rate for the problem (8) with $A = I$ on the cube $\overline{\Omega} = [0,1]^3$. The first inequality in (11) allowed us to restrict ourselves to testing the second order of convergence of the interpolation $\|u - S_h L_h u\|_{1,\Omega}$ only. Results reported in Table 1 are for the function $u(x) = \prod_{i=1}^{3} \sin(\pi x_i)$, which is an eigenfunction of the problem considered. Note that in this case

$$\|\partial_1 u - S_h \partial_1 L_h u\|_0 > \Big\| \sum_{i=1}^{3} \partial_i u - S_h \sum_{i=1}^{3} \partial_i L_h u \Big\|_0,$$

as reported in Table 1. This observation is in agreement with Section 3.7.

The computation of the error in L^2-norms was performed with `HIntLib`, a `C++` package for high dimensional numerical integration.

The smoothening effect of the Steklov operator applied to two-dimensional continuous piecewise linear functions is shown in Figures 3 and 4.

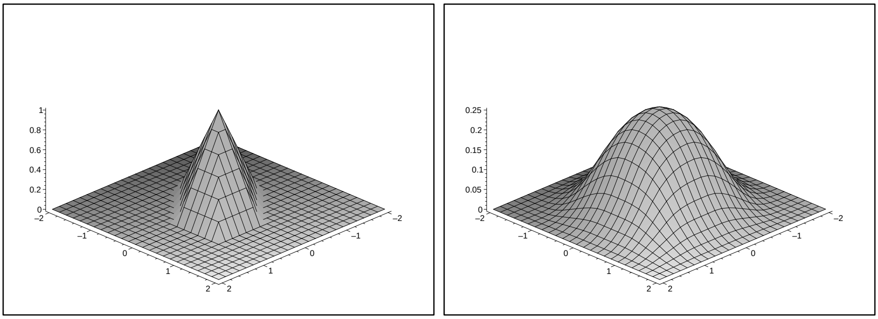

Fig. 3. Basis function for piecewise linear elements and its postprocessing

Table 1. Reduction factors for $u = \prod_{i=1}^{3} \sin(\pi x_i)$

$1/h$	$\|\partial_1 u - S_h \partial_1 L_h u\|_0$	reduct.	$\|\sum_{i=1}^{3} \partial_i u - S_h \sum_{i=1}^{3} \partial_i L_h u\|_0$	reduct.
4	0.290806	–	0.1440776	–
8	0.0997422	2.90	0.0485776	2.91
16	0.029411	3.19	0.0152295	3.39
32	0.00785763	3.32	0.00458811	3.74

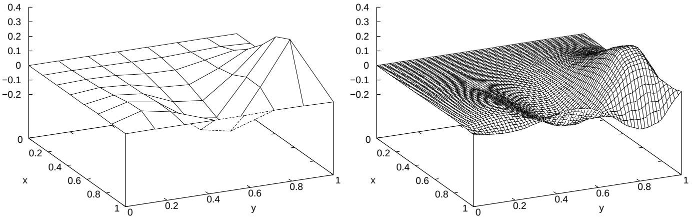

Fig. 4. Graphs of $L_h u$ and $S_h L_h u$ for $u(x_1, x_2) = e^{-2/\sqrt{\vartheta}} \exp((b_1 x_1 + b_2 x_2)/2\vartheta) \cdot \sin(k\pi x_1) \sin(l\pi x_2)$ with $\vartheta = 1/10$, $b_1 = 1$, $b_2 = 0$, $k = 1$, $l = 3$ for $h = 1/8$ (graph of $S_h L_h u$ fitted to an h^2 mesh)

Acknowledgements. This paper was supported by Grant No. A 1019201 of the Grant Agency of the Academy of Sciences of the Czech Republic.

References

[1] J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element method*, Math. Comp. 31 (1977), 94–111.

[2] J. Brandts and M. Křížek, *Gradient superconvergence on uniform simplicial partitions of polytopes*, IMA J. Numer. Anal. 23 (2003), 489–505.

[3] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 2002.

[4] V. Kantchev and R. D. Lazarov, *Superconvergence of the gradient of linear finite elements for 3D Poisson equation*, in: Proc. Internat. Conf. Optimal Algorithms, B. Sendov (ed.), Blagoevgrad, Izd. Bulg. Akad. Nauk, Sofia, 1986, 172–182.

[5] M. Křížek and P. Neittaanmäki, *Superconvergence phenomenon in the finite element method arising from averaging gradients*, Numer. Math. 45 (1984), 105–116.

[6] L. A. Oganesyan and L. A. Rukhovets, *Variational-Difference Methods for the Solution of Elliptic Equations*, Izd. Akad. Nauk Armyanskoĭ SSR, Erevan, 1979.

[7] R. Schürer, `HIntLib`, *High-dimensional Integration Library*, http://www.cosy.sbg.ac.at/˜rschuer/hintlib/

Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: kolman@math.cas.cz

Received on 12.4.2005;
revised version on 10.10.2005
(1781)