APPLICATIONES MATHEMATICAE
31,4 (2004), pp. 369–395

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SHARP BOUNDS FOR EXPECTATIONS OF SPACINGS FROM DECREASING DENSITY AND FAILURE RATE FAMILIES

Abstract. We apply the method of projecting functions onto convex cones in Hilbert spaces to derive sharp upper bounds for the expectations of spacings from i.i.d. samples coming from restricted families of distributions. Two families are considered: distributions with decreasing density and with decreasing failure rate. We also characterize the distributions for which the bounds are attained.

1. Introduction. Let $X_1, \ldots, X_n$ be i.i.d. random variables with common cumulative distribution function (cdf) F, mean μ, finite variance σ^2 and quantile function given by

$$F^{-1}(u) = \sup\{x : F(x) \le u\}, \quad 0 \le u < 1.$$

We write $X_{1:n}, \ldots, X_{n:n}$ for order statistics and consider *spacings*, that is, differences of consecutive order statistics, $R_{j:n} = X_{j+1:n} - X_{j:n}$, $1 \le j \le n-1$. Spacings are widely used in goodness-of-fit tests, quality control problems and characterizations of distributions. For a deeper discussion of their properties and applications we refer the reader to Pyke [10].

Moriguti [8] presented sharp upper bounds for spacings in the class of distributions with finite variance, expressed in σ units. López-Blázquez [6] derived bounds for the expectations of $X_{j+k:n} - X_{j:n}$ in $\sigma_{j:n-k} = (\operatorname{Var} X_{j:n-k})^{1/2}$ units for general distributions with finite second moments and for discrete distributions of N points [7]. Danielak and Rychlik [3] obtained bounds in the classes of distributions with decreasing density on

2000 *Mathematics Subject Classification*: Primary 60E15, 62G30; Secondary 62N05.

Key words and phrases: sharp bound, convex cone, decreasing density, decreasing failure rate, order statistics, projection, spacings.

The second author was supported by the KBN (Polish State Committee for Scientific Research) Grant No. 5 P03A 012 20.

the average (DDA for brevity) and decreasing failure rate on the average (DFRA). Bounds for arbitrary differences $X_{k:n} - X_{j:n}$, $1 \le j < k \le n$, expressed in different scale units generated by various central absolute moments of the parent distribution of a single observation are presented in Danielak [2].

In this paper we present sharp upper bounds for the expectations of spacings, in σ units, when the parent distribution F belongs to the class of distributions with decreasing density (DD) or with decreasing failure rate (DFR). Let U and V denote the distribution function of standard uniform distribution and standard exponential distribution, respectively. We say that F belongs to the class DD if $F^{-1}U = F^{-1}$ is convex on $(0,1)$. Similarly F belongs to the class DFR if $F^{-1}V$ is convex in $(0,\infty)$. These two classes can be treated together as a family of distributions F such that $F^{-1}W$ is convex on the support of W, where $W = U, V$. We then say that F *succeeds* W *in convex order* ($F \succeq_{\mathrm{c}} W$), a notion introduced for continuous life distributions by van Zwet [14].

Denote the density function and the cdf of the ith order statistic from the standard uniform sample of size n by

$$f_{i:n}(x) = nB_{i-1,n-1}(x), \qquad F_{i:n}(x) = \sum_{k=i}^{n} B_{k,n}(x),$$

respectively, where

$$B_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}, \qquad 0 \le x \le 1,\ i = 0,\dots,k,\ k = 0,1,\dots,$$

are Bernstein polynomials. Using the representation

$$E_F X_{i:n} = \int_0^1 F^{-1}(x) f_{i:n}(x)\, dx,$$

and setting $r_{j:n} = f_{j+1:n} - f_{j:n}$, we obtain

$$E_F R_{j:n} = \int_0^1 [F^{-1}(x) - \mu] r_{j:n}(x)\, dx. \tag{1.1}$$

Changing variables in (1.1) we get

$$E_F R_{j:n} = \int_0^d [F^{-1}W(x) - \mu] r_{j:n}W(x) w(x)\, dx, \tag{1.2}$$

where W is an absolutely continuous cdf with density w, support $[0,d) = [0,d_W)$ and a finite variance. The last integral can be treated as the inner product in the real Hilbert space $\mathcal{H} = L^2([0,d), w(x)dx)$ of square integrable functions on $[0,d)$ with respect to the weight function w. Applying the Schwarz inequality to (1.2) and noting that

$$\|F^{-1}W - \mu\|_W = \sigma,$$

we obtain the bound

$$E_F R_{j:n} \le \sigma \|r_{j:n} W\|_W, \tag{1.3}$$

which is attained iff the two factors of the integrand in (1.2) are proportional. If F is an arbitrary cdf with finite variance and $F \succeq_{\mathrm{c}} W$, then the transformation $F^{-1}W - \mu$ defines a family of functions

$$\mathcal{C}_W^0 = \Big\{ g \in \mathcal{C}_W : \int_0^d g(x)w(x)\,dx = 0 \Big\}, \tag{1.4}$$

where

$$\mathcal{C}_W = \{ g \in \mathcal{H} : g \text{ is nondecreasing and convex} \}. \tag{1.5}$$

In general, the functions $r_{j:n}W$ are neither nondecreasing nor convex. In order to derive sharp bounds for (1.3) we apply the projection method presented in Gajek and Rychlik [4]. For a thorough justification and numerous applications we refer the reader to Rychlik [12]. Below we only briefly sketch some basic ideas. Observe that (1.4) is a convex cone in the Hilbert space $\mathcal{H}$. We need to replace a fixed function $r_{j:n}W$ by its projection onto (1.4) denoted by $P_W^0 r_{j:n} W$. The norm of the projection is the optimal bound in σ units, which is achieved by F such that $F^{-1}W - \mu$ is proportional to $P_W^0 r_{j:n} W$. Note that (1.5) is a translation invariant convex cone: $g \in \mathcal{C}_W$ implies that $g + c \in \mathcal{C}_W$ for any real c. Due to the following lemma (cf. Rychlik [11]) we can replace the original projection problem by a simpler one of finding the projection $P_W r_{j:n} W$ of the function $r_{j:n}W$ onto (1.5).

LEMMA 1. *Let $\mathcal{H} = L^2([0,d), w(x)dx)$ with $\int_0^d w(x)\,dx = 1$ and $\mathcal{C}$ be a translation invariant convex cone in $\mathcal{H}$. If the projection Ph of an arbitrary $h \in \mathcal{H}$ onto $\mathcal{C}$ exists*, *then*

$$\int_0^d Ph(x)w(x)\,dx = \int_0^d h(x)w(x)\,dx. \tag{1.6}$$

As $\int_0^d r_{j:n}W(x)w(x)\,dx = 0$, we have $P_W^0 r_{j:n}W = P_W r_{j:n} W$, and finally the bound

$$E_F R_{j:n} \le \sigma \|P_W r_{j:n} W\|_W$$

is sharp and is attained by a unique F satisfying

$$\frac{F^{-1}W(x) - \mu}{\sigma} = \frac{P_W r_{j:n} W(x)}{\|P_W r_{j:n} W\|_W}. \tag{1.7}$$

In Section 2 we describe the shape of the projection in terms of three parameters and determine them. Section 3 contains the main results of the paper. The proofs (quite long) are given in Section 4.

2. The projection problem. We present assumptions on the projected functions $h = r_{j:n}W$ chosen so as to cover the cases $W = U$, V.

(A) Let h be a bounded, twice differentiable function on $[0,d)$ such that $h(0) = 0$, $\lim_{x\nearrow d} h(d) \ge 0$ and $\int_0^d h(x)w(x)\,dx = 0$, where w is a positive weight function satisfying $\int_0^d w(x)\,dx = 1$. Moreover, we assume that h is decreasing on $(0,a)$, convex increasing on (a,b), concave increasing on (b,c) and decreasing on (c,d) for some $0 < a < b < c \le d$.

The lemma below describes the behavior of the functions $r_{j:n}W$ in $[0,d)$ for $W = U$, V.

LEMMA 2.

(a) *Let $W = U$. The function $r_{1:2}$ is linear increasing. If $n \ge 3$, then $r_{1:n}$ is first concave increasing, then decreasing; for $2 \le j \le n-2$ the function $r_{j:n}$ is decreasing, convex increasing, concave increasing and decreasing; and $r_{n-1:n}$ is first decreasing, then convex increasing. Moreover, $r_{j:n}$ has a unique zero in $(0,1)$ at $\theta = j/n$.*

(b) *Let $W = V$. The function $r_{1:2}V$ is concave increasing. If $n \ge 3$, then $r_{1:n}V$ is first concave increasing, then decreasing; for $2 \le j \le n-2$ the function $r_{j:n}V$ is decreasing, convex increasing, concave increasing and decreasing; and $r_{n-1;n}V$ is first decreasing, then convex increasing and ultimately concave increasing. The function $r_{j:n}V$ has a unique zero in $(0,\infty)$ at $\theta = -\ln(1-j/n)$.*

It follows that $r_{j:n}W$ satisfies (A) for $W = U$, $2 \le j \le n-2$ and $W = V$, $2 \le j \le n-1$. From now on we assume that h satisfies (A). It follows that h has exactly one zero $\theta \in (a,c)$ and the sign of h at the inflection point b may be arbitrary.

The following lemma describes the shape of the projection of an arbitrary function h satisfying (A) onto the convex cone (1.5).

LEMMA 3. *Let $\mathcal{C}^* \subset \mathcal{C}_W$ be the class of functions of the form*

$$g^*(x) = \begin{cases} h(\alpha), & 0 \le x < \alpha, \\ h(x), & \alpha \le x < \beta, \\ \lambda(x-\beta) + h(\beta), & \beta \le x < d, \end{cases} \tag{2.1}$$

for some $a \le \alpha < \beta \le b$ and $\lambda \ge h'(\beta)$, or

$$g^*(x) = \begin{cases} \gamma, & 0 \le x < \beta, \\ \lambda(x-\beta) + \gamma, & \beta \le x < d. \end{cases} \tag{2.2}$$

for $\lambda \ge 0$ and $\gamma \in \mathbb{R}$. Then for any $g \in \mathcal{C}_W$ there exists a function $g^* \in \mathcal{C}^*$ such that

$$\|h - g^*\| \le \|h - g\|.$$

Lemma 4. *Let* $\overline{h} : (0, d] \to \mathbb{R}$ *be given by*

$$\overline{h}(\beta) = \frac{\int_0^\beta h(x)w(x)\,dx}{\int_0^\beta w(x)\,dx}.$$

Then

(i) $\overline{h}(d) = 0$ *and* $\overline{h}(\beta) < 0$ *for any* $\beta \in (0, d)$,

(ii) *there exists a unique* $\alpha \in (a, \theta)$ *such that*

$$\overline{h}(\alpha) = h(\alpha), \tag{2.3}$$

the function $\overline{h}$ *is decreasing with* $\overline{h} > h$ *in* $(0, \alpha]$, *and* $\overline{h}$ *is increasing with* $\overline{h} < h$ *in* (α, d).

We introduce the following notations:

$$\begin{aligned} \lambda_1(\beta) &= \frac{\int_\beta^d (x-\beta)[h(x)-h(\beta)]w(x)\,dx}{\int_\beta^d (x-\beta)^2 w(x)\,dx}, \\ Y(\beta) &= \lambda_1(\beta) - h'(\beta), \\ Z(\beta) &= \int_\beta^d [h(x) - \lambda_1(\beta)(x-\beta) - h(\beta)]w(x)\,dx. \end{aligned} \tag{2.4}$$

Proposition 1. *Assume that* α *satisfies* (2.3). *If the set*

$$\mathcal{Y} = \{\beta \in (\alpha, b) : Y(\beta) \geq 0 \text{ and } Z(\beta) = 0\}$$

is not empty, then

$$P_W h(x) = \begin{cases} h(\alpha), & 0 \leq x \leq \alpha, \\ h(x), & \alpha < x \leq \beta, \\ h(\beta) + \lambda(x-\beta), & \beta < x < d, \end{cases} \tag{2.5}$$

for $\beta = \beta^* = \sup\{\beta \in \mathcal{Y}\}$ *and* $\lambda = \lambda_1(\beta^*)$. *Otherwise*,

$$P_W h(x) = -\overline{h}(\beta)\left[\frac{(x-\beta)\mathbb{I}_{[\beta,d)}(x)}{\int_\beta^d (x-\beta)w(x)\,dx} - 1\right] \tag{2.6}$$

for the greatest $0 < \beta \leq \alpha$ *satisfying*

$$\begin{aligned} &\int_\beta^d h(x)w(x)\,dx \left[\int_\beta^d (x-\beta)^2 w(x)\,dx - \left(\int_\beta^d (x-\beta)w(x)\,dx\right)^2\right] \\ &\qquad = \int_0^\beta w(x)\,dx \int_\beta^d (x-\beta)w(x)\,dx \int_\beta^d (x-\beta)h(x)w(x)\,dx. \end{aligned} \tag{2.7}$$

3. Main results. Sharp upper bounds for $E_F R_{j:n}$, $2 \le j \le n-2$, and F belonging to the class DD are presented in the following

PROPOSITION 2. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with decreasing density, cdf F, finite $E_F X_1 = \mu$ and $\operatorname{Var}_F X_1 = \sigma^2$. Put*

$$\begin{aligned}(3.1)\quad Y_1(x) = \sum_{m=1}^{j-2} f_{m:n+2}(x) &+ \left[1 + \frac{(n-j+3)!}{3(n-j)!}\right] f_{j-1:n+2}(x)\\ &+ \left[1 + (n-j+2)(n-j+1)\left(\tfrac{1}{2} - \tfrac{2}{3}(n-j)\right)\right] f_{j:n+2}(x)\\ &+ \left[1 + (n-j+1)(n-j)\left(-\tfrac{1}{2} + \tfrac{1}{3}(n-j-1)\right)\right] f_{j+1:n+2}(x),\end{aligned}$$

$$\begin{aligned}(3.2)\quad Z_1(x) = &-\sum_{m=1}^{j-1} f_{m:n+2}(x) + \left[\tfrac{1}{6}(n-j+2)(n-j+1) - 1\right] f_{j:n+2}(x)\\ &- \left[\tfrac{1}{6}(n-j+1)(n-j-4) + 1\right] f_{j+1:n+2}(x).\end{aligned}$$

If

$$(3.3)\qquad Y_1(\alpha) > 0, \qquad Z_1(\alpha) < 0 < Z_1(y),$$

where $\alpha = (j-1)/(n-1)$ and y is the smallest positive zero of (3.1), *then*

$$(3.4)\qquad \frac{E_F R_{j:n}}{\sigma} \le B = B(j,n)$$

for

$$\begin{aligned}(3.5)\quad B^2 = {} & \alpha[f_{j+1:n}(\alpha) - f_{j:n}(\alpha)]^2\\ &+ \frac{(n!)^2}{(2n-1)!}\left\{\binom{2j}{j}\binom{2n-2j-2}{n-j-1}[F_{2j+1:2n-1}(\beta) - F_{2j+1:2n-1}(\alpha)]\right.\\ &- 2\binom{2j-1}{j}\binom{2n-2j-1}{n-j-1}[F_{2j:2n-1}(\beta) - F_{2j:2n-1}(\alpha)]\\ &+ \left.\binom{2j-2}{j-1}\binom{2n-2j}{n-j}[F_{2j-1:2n-1}(\beta) - F_{2j-1:2n-1}(\alpha)]\right\}\\ &+ (1-\beta)\left\{\tfrac{1}{3}\lambda^2(1-\beta)^2 + \lambda(1-\beta)[f_{j+1:n}(\beta) - f_{j:n}(\beta)]\right.\\ &+ \left.[f_{j+1:n}(\beta) - f_{j:n}(\beta)]^2\right\},\end{aligned}$$

with β being the smallest positive zero of (3.2) *and*

$$\begin{aligned}\lambda &= \lambda_1(\beta)\\ &= \frac{1 - F_{j+1:n+1}(\beta) - \frac{n-j+1}{2(n+2)}[(n-j)f_{j+1:n+2}(\beta) - (n-j+2)f_{j:n+2}(\beta)]}{\frac{1}{3}(1-\beta)^3(n+1)}.\end{aligned}$$

Equality holds in (3.4) *for*

$$
(3.6)\quad F(x) = \begin{cases} 0, & \frac{x-\mu}{\sigma} < \frac{r_{j:n}(\alpha)}{B}, \\ r_{j:n}^{-1}\left(\frac{x-\mu}{\sigma}B\right), & \frac{r_{j:n}(\alpha)}{B} \le \frac{x-\mu}{\sigma} < \frac{r_{j:n}(\beta)}{B}, \\ \frac{x-\mu}{\sigma\lambda}B - \frac{r_{j:n}(\beta)}{\lambda} + \beta, & \frac{r_{j:n}(\beta)}{B} \le \frac{x-\mu}{\sigma} < \frac{\lambda(1-\beta)+r_{j:n}(\beta)}{B}, \\ 1, & \frac{x-\mu}{\sigma} \ge \frac{\lambda(1-\beta)+r_{j:n}(\beta)}{B}. \end{cases}
$$

If (3.3) *fails, then*

$$
(3.7)\qquad \frac{E_F R_{j:n}}{\sigma} \le \frac{f_{j+1:n+1}(\beta)}{\beta(n+1)}\sqrt{\frac{1+3\beta}{3(1-\beta)}}
$$

for β *being the smallest positive solution to*

$$
(3.8)\qquad \frac{n-j+1}{6}\left[4(j+1)f_{j+2:n+3}(x) + (n-j+2)f_{j+1:n+3}(x)\right] = \sum_{m=1}^{j+1} m f_{m+1:n+3}(x).
$$

Equality holds in (3.7) *for*

$$
(3.9)\qquad F(x) = \begin{cases} 0, & \frac{x-\mu}{\sigma} < a_1, \\ \beta+\frac{1}{2}(1-\beta)^2\left(1+\frac{x-\mu}{\sigma}\sqrt{\frac{1+3\beta}{3(1-\beta)}}\right), & a_1 \le \frac{x-\mu}{\sigma} < a_2, \\ 1, & \frac{x-\mu}{\sigma} \ge a_2, \end{cases}
$$

with

$$
a_1 = -\sqrt{\frac{3(1-\beta)}{1+3\beta}}, \qquad a_2 = \frac{1+\beta}{1-\beta}\sqrt{\frac{3(1-\beta)}{1+3\beta}}.
$$

Distributions (3.6) and (3.9) are not absolutely continuous. The former has a jump of size α at the left end of its support, then is the inverse function of a nondecreasing polynomial, and has a right uniform tail. The cdf (3.9) is a mixture of an atom and a cdf of uniform distribution. However, it is easy to find sequences of absolutely continuous $F_k \succeq_{\mathrm{c}} U$, $k \to \infty$, which attain the bounds asymptotically.

For $j = 1$, $n = 2$ the bound derived by Plackett [9] is optimal and is attained by a uniform distribution belonging to the class DD. If $j = 1$, $n \ge 3$, then the projection is a linear function (cf. Danielak [1]) of the form $P_U r_{1:n}(x) = 2(2x-1)/(n+1)$ and $E_F R_{1:n}/\sigma \le 2\sqrt{3}/(n+1)$. The bound is attained for the uniform distribution on $\left[\mu-\sqrt{3}\sigma, \mu+\sqrt{3}\sigma\right]$. If $j = n-1$, then the optimal bound coincides with that obtained in the class of arbitrary distributions with finite variance (see Danielak [2]). In this case the bound (3.4) is sharp and becomes an equality for the cdf (3.6) with $\beta = 1$.

We now turn to the case when F belongs to the class DFR. Assume that $2 \le j \le n-1$.

PROPOSITION 3. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with decreasing failure rate, cdf F, finite $E_F X_1 = \mu$ and $\operatorname{Var}_F X_1 = \sigma^2$. Put*

$$\begin{aligned} Y_2(x) = {} & \frac{1}{n-j}\sum_{m=1}^{j-2} f_{m:n+2}(x) + \left[\frac{1}{n-j} + \frac{2(n-j+2)!}{(n-j)!}\right] f_{j-1:n+2}(x) \\ & + \left[\frac{1}{n-j} + (n-j+1)[1-4(n-j)]\right] f_{j:n+2}(x) \\ & + \left[\frac{1}{n-j} + (n-j)[2(n-j)-3]\right] f_{j+1:n+2}(x), \end{aligned} \tag{3.10}$$

$$\begin{aligned} Z_2(x) = {} & -\sum_{m=1}^{j-1} f_{m:n+1}(x) + [(n-j)(n-j+1)-1]\, f_{j:n+1}(x) \\ & + [(n-j)(2-n+j)-1] f_{j+1:n+1}(x). \end{aligned} \tag{3.11}$$

If

$$Y_2(\alpha_0) > 0, \qquad Z_2(\alpha_0) < 0 < Z_2(y), \tag{3.12}$$

where $\alpha_0 = (j-1)/(n-1)$ and y is the smallest positive zero of (3.10), *then*

$$\frac{E_F R_{j,j+1:n}}{\sigma} \le B = B(j,n) \tag{3.13}$$

for

$$\begin{aligned} B^2 = {} & \beta_0 r^2_{j:n}(\alpha_0) + (1-\beta_0)[2\lambda^2 - 2\lambda r_{j:n}(\beta_0) + r^2_{j:n}(\beta_0)] \\ & + \frac{(n!)^2}{(2n-1)!}\left\{\binom{2j}{j}\binom{2n-2j-2}{n-j-1}[F_{2j+1:2n-1}(\beta_0) - F_{2j+1:2n-1}(\alpha_0)]\right. \\ & - 2\binom{2j-1}{j}\binom{2n-2j-1}{n-j-1}[F_{2j:2n-1}(\beta_0) - F_{2j:2n-1}(\alpha_0)] \\ & \left. + \binom{2j-2}{j-1}\binom{2n-2j}{n-j}[F_{2j-1:2n-1}(\beta_0) - F_{2j-1:2n-1}(\alpha_0)]\right\}, \end{aligned} \tag{3.14}$$

where β_0 is the smallest positive zero of (3.11) *and*

$$\begin{aligned} \lambda = \lambda_1(V^{-1}(\beta_0)) = \frac{1}{2(1-\beta_0)} & \left\{\frac{1}{n-j}\sum_{m=1}^{j+1} f_{m:n+1}(\beta_0)\right. \\ & \left. -(n-j) f_{j+1:n+1}(\beta_0) + (n-j+1) f_{j:n+1}(\beta_0)\right\}. \end{aligned}$$

The bound (3.13) *is achieved for*

$$F(x)=\begin{cases}0, & \dfrac{x-\mu}{\sigma}<a_1,\\ r_{j:n}^{-1}\Big(\dfrac{x-\mu}{\sigma}\,B\Big), & a_1\le\dfrac{x-\mu}{\sigma}<a_2,\\ V\Big(\dfrac{x-\mu}{\sigma\lambda}\,B-\dfrac{r_{j:n}(\beta_0)}{\lambda}+V^{-1}(\beta_0)\Big), & \dfrac{x-\mu}{\sigma}\ge a_2,\end{cases}\tag{3.15}$$

with $a_1=r_{j:n}(\alpha_0)/B$, $a_2=r_{j:n}(\beta_0)/B$.

If (3.12) *does not hold*, *then*

$$\frac{E_F R_{j,j+1:n}}{\sigma}\le\frac{f_{j+1:n+1}(\varrho)}{\varrho(n+1)}\sqrt{\frac{1+\varrho}{1-\varrho}},\tag{3.16}$$

where ϱ *is the smallest positive solution to*

$$\sum_{m=1}^{j+1}\frac{m}{n-j}\,f_{m+1:n+2}(x)=(n-j+1)f_{j+1:n+2}(x)+2(j+1)f_{j+2:n+2}(x).$$

The bound (3.16) *is attained for*

$$F(x)=\begin{cases}0, & \dfrac{x-\mu}{\sigma}\le-\sqrt{\dfrac{1-\varrho}{1+\varrho}},\\ V\Big((1-\varrho)\Big[\dfrac{x-\mu}{\sigma}\sqrt{\dfrac{1+\varrho}{1-\varrho}}+1\Big]+V^{-1}(\varrho)\Big), & \dfrac{x-\mu}{\sigma}>-\sqrt{\dfrac{1-\varrho}{1+\varrho}}.\end{cases}\tag{3.17}$$

The cdf (3.15) has a jump of size $\alpha_0=(j-1)/(n-1)$ at the left end of its support, then is the inverse function of a nondecreasing polynomial, and finally has an exponential tail. The cdf (3.17) is a mixture of an atom and an exponential distribution. If $j=1$, $n\ge 2$, then $P_V r_{1:n}V(x)=(x-1)/(n-1)$ and $E_F R_{1:n}/\sigma\le 1/(n-1)$. The bound is attained for the exponential distribution with location and scale parameters equal to $\mu-\sigma$ and σ, respectively.

4. Proofs. We shall frequently apply the following lemma:

LEMMA 5. *The number of zeros of a linear combination of Bernstein polynomials*

$$W(x)=\sum_{k=0}^{m}a_k B_{k,m}(x),\qquad x\in(0,1),\tag{4.1}$$

does not exceed the number of sign changes of the sequence $a_0,\ldots,a_m$. *The initial and final signs of* (4.1) *in* $(0,1)$ *are identical with the signs of the first and last nonzero elements of* $a_0,\ldots,a_m$, *respectively.*

The proof of the former statement, known as variation diminishing property of Bernstein polynomials, can be found in Schoenberg [13], and of the latter was presented in Gajek and Rychlik [5].

We also use the formulae below (with the convention that $B_{l,m}(x) = 0$ for $l > m$ or $l < 0$):

$$
\begin{aligned}
xB_{l,m}(x) &= \frac{l+1}{m+1}\, B_{l+1,m+1}(x),\\
(1-x)^s B_{l,m}(x) &= \frac{(m-l+s)!m!}{(m-l)!(m+s)!}\, B_{l,m+s}(x),\\
B'_{l,m}(x) &= m[B_{l-1,m-1}(x) - B_{l,m-1}(x)],\\
\int_y^1 B_{l,m}(x)\,dx &= \frac{1}{m+1}\sum_{s=0}^{l} B_{s,m+1}(y).
\end{aligned}
\tag{4.2}
$$

Proof of Lemma 2. (a) Let $W = U$. We have $r_{1:2}(x) = 4x - 2$. Assume that $n \ge 3$. Using (4.2) we get

$$
\begin{aligned}
r'_{j:n}(x) &= n(n-1)[-B_{j-2,n-2}(x) + 2B_{j-1,n-2}(x) - B_{j,n-2}(x)],\\
r''_{j:n}(x) &= n(n-1)(n-2)\\
&\quad\times [-B_{j-3,n-3}(x) + 3B_{j-2,n-3}(x) - 3B_{j-1,n-3}(x) + B_{j,n-3}(x)].
\end{aligned}
$$

If $2 \le j \le n-2$, then, by Lemma 5, $r'_{j:n}$ is either first positive, then negative and ultimately positive ($+-+$, for brevity) or negative everywhere in $[0,1)$. The latter is impossible, because $r_{j:n}$ integrates to 0 in $(0,1)$ and vanishes at 0 and 1. Thus, $r_{j:n}$ has first a minimum, then a maximum, and it is convex and concave about the minimum and maximum, respectively. This combined with Lemma 5 implies that $r''_{j:n}$ is $-+-+$, and our claim follows. Similar considerations apply to the remaining cases.

(b) Assume that $W = V$. The function $r_{1:2}V(x) = 2(1-2e^{-x})$ is concave increasing on $[0,\infty)$. Take $n \ge 3$. Defining $C_{j,m}(x) = B_{j,m}V(x)$ we get

$$
r_{j:n}V(x) = n\,[-C_{j-1,n-1}(x) + C_{j,n-1}(x)]
$$

and

$$
\begin{aligned}
(r_{j:n}V)'(x) &= n(n-1)e^{-x}[-C_{j-2,n-2}(x) + 2C_{j-1,n-2}(x) - C_{j,n-2}(x)],\\
(r_{j:n}V)''(x) &= \frac{n(n-1)}{n-2}\, e^{-x}\{-(n-j+1)C_{j-3,n-2}(x)\\
&\quad + (3n-3j+1)C_{j-2,n-2}(x) - (3n-3j-1)C_{j-1,n-2}(x)\\
&\quad + (n-j-1)C_{j,n-2}(x)\}.
\end{aligned}
$$

Since each $C_{l,m}$ is a superposition of an increasing function V and a Bernstein polynomial, the statement of Lemma 5 holds for linear combinations of $C_{l,m}$ as well. Analyzing the signs of $(r_{j:n}V)'$ and $(r_{j:n}V)''$, analogously to the proof of part (a) we easily obtain the desired conclusions. ∎

Proof of Lemma 3. We show that for any $g \in \mathcal{C}_W$ we can find a function $g^* \in \mathcal{C}^*$ which is closer to h than g. Our proof starts with the observation that it suffices to consider functions g satisfying $g(0) < 0$. Monotonicity of g and the fact that g integrates to 0 imply that either $g(0) < 0 < \lim_{x \nearrow d} g(x)$ or $g(x) = 0$ for $x \in [0, d)$. We exclude the latter case since there exists a function that vanishes in $[0, \theta]$, is linear increasing in (θ, d) and is a better approximation to h than the constant 0 (see Gajek and Rychlik [5]). As $\max\{g, h(a)\}$ is nondecreasing convex and is closer to h than g, it suffices to restrict our attention to functions g satisfying $0 > g(0) \geq h(a)$. Since $h - g$ is continuous, the set $\{x \in [0, d) : h(x) = g(x)\}$ is closed. It follows that there exist at most countably many closed intervals (possibly degenerate) where $h = g$. Note that it suffices to consider those g for which the set $\{h = g\}$ contains at most one nondegenerate interval. Indeed, suppose that there are at least two such intervals. They must be subsets of $[a, b]$, because g is nondecreasing convex. If $h = g$ in some $[\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$ with $\beta_1 < \alpha_2$ and $h \neq g$ in (β_1, α_2) then

$$\widetilde{g}(x) = \begin{cases} h(x), & x \in (\beta_1, \alpha_2), \\ g(x), & x \notin (\beta_1, \alpha_2), \end{cases}$$

is nondecreasing convex and $\|h - g\| \geq \|h - \widetilde{g}\|$, a contradiction. Now we need to consider two cases:

(I) the set $\{g = h\}$ contains a nondegenerate interval,
(II) the set $\{g = h\}$ does not contain any interval.

(I) Suppose that $h = g$ on some $[\alpha, \beta] \subset [a, b]$, $\alpha < \beta$. We are going to show that there exists a function $\widetilde{g}$ of the form (2.1), closer to h than g. Take an arbitrary $\xi \in [\alpha, \beta]$ and denote by h_1 the nondecreasing function closest to $h_{|[0,\xi]}$ taking value $h(\xi)$ at ξ, and by h_2 the nondecreasing convex function closest to $h_{|[\xi,d)}$ such that $h_2(\xi) = h(\xi)$. We are now in a position to show that h_1 is either constantly $h(\xi)$, or for some $a \leq \eta < \xi$, constantly $h(\eta)$ on $[0, \eta]$ and equal to h on $[\eta, \xi]$, and h_2 is continuous and equal to h on $[\xi, \nu]$ and increasing linear on $[\nu, d)$ for some $\nu \in [\xi, d]$. Note that h_1 is convex, and so it is the best approximation of $h_{|[0,\xi]}$ in the class of nondecreasing convex functions. Furthermore, the function

$$\widetilde{g}(x) = \begin{cases} h_1(x), & x \in [0, \xi], \\ h_2(x), & x \in (\xi, d), \end{cases}$$

is nondecreasing and convex, satisfies (2.1) and is closer to h than g.

Now, our goal is to find the nondecreasing function $P_1 h$ closest to $h_{|[0,\xi]}$. Applying the modification of the Moriguti method of obtaining greatest convex minorants, presented in Rychlik [12, Example 3, pp. 14–16], we observe that either $P_1 h(x) = \zeta$ and $\zeta > h(\xi)$, or $P_1 h$ is constantly $h(\eta)$ on $[0, \eta]$ for some $a \leq \eta < \xi$, and equal to h on $[\eta, \xi]$. Only in the latter case the projection has the required form. We proceed to show that if the former holds, then the nondecreasing function closest to $h_{|[0,\xi]}$ taking value $h(\xi)$ at ξ is constantly $h(\xi)$ on $[0,\xi]$. Lemma 1 yields

$$\int_0^\xi h(x)w(x)\,dx = \int_0^\xi P_1h(x)w(x)\,dx = \zeta\int_0^\xi w(x)\,dx.$$

Any nondecreasing function g such that $g(\xi)=h(\xi)$ can be represented as

$$g(x) = g_0(x) + h(\xi) - g_0(\xi),$$

where g_0 is nondecreasing and

$$\int_0^\xi g_0(x)w(x)\,dx = \int_0^\xi h(x)w(x)\,dx = \zeta\int_0^\xi w(x)\,dx,$$

and $g_0(0)\le\zeta\le g_0(\xi)$. Therefore

$$\begin{aligned}\|h-g\|^2 &= \|h-g_0-[h(\xi)-g_0(\xi)]\mathbf{1}\|^2\\ &= \|h-g_0\|^2 - 2[h(\xi)-g_0(\xi)](h-g_0,\mathbf{1}) + [h(\xi)-g_0(\xi)]^2(\mathbf{1},\mathbf{1}).\end{aligned}$$

Combining $\|P_1h-h\|\le\|g_0-h\|$, $(h-g_0,\mathbf{1})=0$, and $(h-P_1h,\mathbf{1})=0$ with $h(\xi)\le\zeta\le g_0(\xi)$ we deduce that

$$\begin{aligned}\|h-g\|^2 &\ge \|h-P_1h\|^2 + [h(\xi)-P_1h(\xi)]^2(\mathbf{1},\mathbf{1})\\ &= \|h-P_1h-[h(\xi)-P_1h(\xi)]\mathbf{1}\|^2 = \|h-h(\xi)\mathbf{1}\|^2.\end{aligned}$$

It follows that the nondecreasing function closest to $h_{|[0,\xi]}$ taking value $h(\xi)$ at ξ is either constantly $h(\xi)$ on $[0,\xi]$ or constantly $h(\eta)$ on $[0,\eta]$ for some $\eta<\xi$, and equal to h on $[\eta,\xi]$.

It remains to observe that h_2 is of the form described above. This was proved in Lemma 1 of Gajek and Rychlik [5].

(II) Assume now that the set $\{g=h\}$ does not contain any interval. We will prove that there exists a function $\widetilde{g}$ of the form (2.1) or (2.2) such that $\|h-\widetilde{g}\|\le\|h-g\|$.

Suppose that there are some $a\le\alpha<\beta\le b$ such that $g(\alpha)=h(\alpha)$, $g(\beta)=h(\beta)$ and $g(x)<h(x)$ for $x\in(\alpha,\beta)$. Set

$$g_1(x) = \begin{cases} h(x), & x\in(\alpha,\beta),\\ g(x), & x\notin(\alpha,\beta).\end{cases}$$

Obviously, $\|h-g_1\|\le\|h-g\|$ and g_1 is nondecreasing. If $g(\alpha-)<h(\alpha-)$, then $g'(\alpha)=h'(\alpha)$. If $g(\alpha-)>h(\alpha-)$, then $g'(\alpha)<h'(\alpha)$. Similarly, the inequalities $g(\beta+)<h(\beta+)$ and $g(\beta+)>h(\beta+)$ imply $g'(\beta)=h'(\beta)$ and $g'(\beta)>h'(\beta)$, respectively. Consequently, g_1 has nondecreasing derivative. Thus g_1 is convex. Note that the set $\{g_1=h\}$ contains an interval, so we are in case (I) and can find an approximation of the form (2.1).

• Suppose now that $g(a+)\le h(a+)$. Our claim is that there exists a function $\widetilde{g}$ of the form (2.1) which is closer to h than g. As we have assumed that $g(a)\ge g(0)\ge h(a)$, it suffices to consider the case $g(a)=h(a)$. If

$g(a+) = h(a+)$, which means that $g = h$ on some interval, then we proceed as in case (I).

Assume then that $g(a+) < h(a+)$. If there exists $\beta \in (a, b]$ such that $g(\beta) = h(\beta)$, then g can be replaced by a function equal to h on some interval, which leads to case (I). Otherwise, two subcases are possible: either g crosses h at $\delta \in (b, d)$, or g runs beneath h in the whole (a, d). But then $d < \infty$ and we set $\delta = d$.

Let l_t denote the tangent line to h at t. The slope of l_t continuously increases in $[a, b)$ and so does the value $l_t(\delta)$ ranging from $l_a(\delta) < h(\delta)$ to $l_b(\delta) > h(\delta)$. It follows that there exists $\phi \in (a, b)$ such that $l_\phi(\delta) = h(\delta)$. Set

$$\widetilde{g}(x) = \begin{cases} h(a), & 0 \le x < a, \\ h(x), & a \le x < \phi, \\ l(x), & \phi \le x < d. \end{cases}$$

Obviously $\widetilde{g}$ is of the form (2.1), it is nondecreasing and convex, and closer to h than g.

• Consider the case $g(a+) > h(a+)$. Since $g(0) < 0$, it follows that there exists $\alpha \in [0, a]$ such that $h(\alpha) = g(\alpha)$. Clearly, there also exists $\alpha' \in [a, b]$ such that $h(\alpha') = h(\alpha)$. Note that the constant $h(\alpha)$ is closer to h than g on $[0, \alpha']$.

Our next objective is to show that if $g(a+) > h(a+)$ and the set of degenerate intervals $\{g = h\} \cap (a, b]$ contains at least two points then one of the following two cases holds:

1) there exists a function $\widetilde{g}$ of the form (2.1) closer to h than g;
2) there are only two such points $\beta < \phi$, and then $g > h$ on (β, ϕ) and $g(\phi+) < h(\phi+)$.

If $h(x) = g(x)$ for $x \in \{\beta, \phi\}$, $\beta < \phi$ and $g < h$ on (β, ϕ), then, as already shown, g can be replaced by some function of the form (2.1). If $h(x) = g(x)$ for $x \in \{\beta, \phi\}$, $h(x) \neq g(x)$ for $x \in (a, \beta)$ and $g(\beta-) > h(\beta-)$, $g > h$ on (β, ϕ) and $g(\phi+) > h(\phi+)$, then the function

$$g_1(x) = \begin{cases} h(x), & x \in (\beta, \phi), \\ g(x), & x \notin (\beta, \phi), \end{cases}$$

is closer to h than g, belongs to (1.5) and can be replaced by a function of the form (2.1), because $\{g_1 = h\}$ contains an interval. Therefore the inequality $g(a+) > h(a+)$, combined with the assumption that $\{g = h\}$ does not contain an interval, implies four cases:

(i) $\{g = h\} \cap (a, b) = \emptyset$, i.e. $g > h$ on (a, b),
(ii) $\{g = h\} \cap (a, b) = \{\phi\}$ and $g(\phi+) > h(\phi+)$,
(iii) $\{g = h\} \cap (a, b) = \{\phi\}$ and $g(\phi+) < h(\phi+)$,
(iv) $\{g = h\} \cap (a, b) = \{\beta, \phi\}$ and $g > h$ on (β, ϕ) and $g(\phi+) < h(\phi+)$.

If either (i) or (ii) hold, then there are two possibilities:

(a) $g \geq h$ on the whole (b, d) (g and h may be tangent at some $\delta \in (b, c)$),
(b) there exist $\delta \in (b, c)$ and $\Delta \in (\delta, d)$ such that $g > h$ on (b, δ) and (Δ, d) and $g < h$ on (δ, Δ).

If (a) holds, then there exists a nondecreasing linear function l such that $h \leq l \leq g$ on (b, c), where g is convex and h concave. Moreover, $h \leq l \leq g$ on (c, d), and $l \leq g$ on $(0, b)$. In case (i) we have

$$l(0) \leq g(0) \leq g(\alpha) = h(\alpha) = h(\alpha'),$$

so l crosses $h_1 = \max\{h(\alpha), h\}$ at some α''. The function

$$g_2(x) = \begin{cases} h(\alpha), & 0 \leq x \leq \alpha'', \\ l(x), & \alpha'' < x < d, \end{cases}$$

belongs to (1.5) and is of the form (2.1) provided $\alpha'' > \alpha'$. Otherwise it is of the form (2.2). The constantly $h(\alpha)$ function is closer to h than g on $(0, \alpha'')$, and $h \leq l \leq g$ on (α'', d). If case (ii) holds, then $g(\phi) = h(\phi)$. It follows that l crosses h at $\alpha'' \geq \phi$. Hence g_2 is of the form (2.1) and is closer to h than g.

Assume now that (b) holds. Take the line l_2 passing through $(\delta, h(\delta))$ and $(\Delta, h(\Delta))$. Then $l_2(x) \leq g(x)$ for $x \in [0, \delta]$, $g(x) \leq l_2(x) \leq h(x)$ for $x \in [\delta, \Delta]$ and $h(x) \leq l_2(x) \leq g(x)$ for $x > \Delta$ and $h(\delta-) < l_2(\delta-)$. Applying the same arguments as in (a) we conclude that l_2 crosses $h_1 = \max\{h(\alpha), h\}$ at some α''. Therefore, we improve the approximation if we replace g by

$$g_3(x) = \begin{cases} h(\alpha), & 0 \leq x \leq \alpha'', \\ l_2(x), & \alpha'' < x < d, \end{cases}$$

which is of the form (2.1) or (2.2).

It remains to consider cases (iii) and (iv). Then g runs beneath h on $[\phi, b]$ where h is convex and either $g < h$ on (b, d) or g crosses h at a unique $\Delta \in (b, d)$. If the former holds, then we set $\Delta = d < \infty$. Let l_3 be the line passing through $(\phi, h(\phi))$ and $(\Delta, h(\Delta))$, with $h(\Delta) \geq g(\Delta)$. Then $l_3(x) \leq g(x)$ for $x \in [0, \phi]$, $h(\phi-) < l_3(\phi-)$ and $g(x) \leq l_3(x) \leq h(x)$ for $x \in [\phi, \Delta]$ and $h(x) \leq l_3(x) \leq g(x)$ for $x > \Delta$. Since $l_3(x) \leq g(x)$ for $x < \phi$ and $g(0) < h(0)$, we see that $l_3(x) = h(x)$ for some $x > \phi$. Moreover l_3 crosses $h_1 = \max\{h(\alpha), h\}$ at some $\alpha'' \in (\alpha, \phi]$. Let

$$g_4(x) = \begin{cases} h(\alpha), & 0 \leq x \leq \alpha'', \\ l_3(x), & \alpha'' < x < d. \end{cases}$$

If (iv) holds, then g_4 is of the form (2.1). If (iii) holds, then g_4 is either of the form (2.1) or (2.2). Obviously, $\|g_4 - h\| \leq \|g - h\|$. This ends the proof. ■

Proof of Lemma 4. (i) From (A) we have $\int_0^d h(x)w(x)\,dx = 0$ and clearly $\overline{h}(d) = 0$. Since $h(\beta) \leq 0$ for any $\beta \leq \theta$, we see that $\int_0^\beta h(x)w(x)\,dx < 0$, which implies $\overline{h}(\beta) < 0$ for $\beta \le \theta$. Likewise, $h(\beta) \ge 0$ for $\beta > \theta$, which gives $0 < \int_\beta^d h(x)w(x)\,dx = -\int_0^\beta h(x)w(x)\,dx$. Thus $\overline{h}(\beta) < 0$ for $\beta > \theta$.

(ii) We have

$$\begin{aligned}\overline{h}'(\beta) &= \frac{h(\beta)w(\beta)\int_0^\beta w(x)\,dx - w(\beta)\int_0^\beta h(x)w(x)\,dx}{[\int_0^\beta w(x)\,dx]^2}\\ &= \frac{w(\beta)[h(\beta)-\overline{h}(\beta)]}{\int_0^\beta w(x)\,dx}.\end{aligned}$$

It suffices to show that $\overline{h} > h$ in $[0,\alpha]$ and $\overline{h} < h$ in (α,d). Let $W(\beta) = \int_0^\beta w(x)\,dx$. Then W is strictly increasing and $W(d) = 1$. With the notation

$$H_W(\gamma) = \int_0^\gamma hW^{-1}(x)\,dx$$

for $\gamma \in [0,1]$ we have

$$H_W(W(\beta)) = \int_0^{W(\beta)} hW^{-1}(x)\,dx = \int_0^\beta h(x)w(x)\,dx. \tag{4.3}$$

Note that $H_W'(\gamma) = hW^{-1}(\gamma)$ and $H_W'(W(\beta)) = h(\beta)$. Thus $H_W'(0) = 0$, H_W' decreases in $(0,W(a))$ and in $(W(c),1)$, increases in $(W(a),W(c))$, and $H_W'(W(\theta)) = 0$. Hence H_W is concave decreasing in $(0,W(a))$, convex decreasing in $(W(a),W(\theta))$, convex increasing in $(W(\theta),W(c))$, concave increasing in $(W(c),1)$ and attains its local minimum at $\gamma = W(\theta)$. Moreover $H_W(0) = 0$ and

$$H_W(1) = \int_0^1 hW^{-1}(x)\,dx = \int_0^d h(x)w(x)\,dx = 0.$$

By definition of W and (4.3), we have $\overline{h}(\beta) = H_W(W(\beta))/W(\beta)$. It follows that $\overline{h}(\beta)$ is the slope of the linear function l_β passing through $(0,0)$ and $(W(\beta),H_W(W(\beta)))$. By concavity of H_W in $[0,W(a)]$, l_β lies beneath H_W for $W(\beta) \in (0,W(a))$, that is, for $\beta \in [0,a]$. Every line tangent to H_W at $W(\beta) \in (0,W(a))$ lies over H_W in that interval. As $H_W'(W(\beta)) = h(\beta)$, we see that $\overline{h}(\beta) > h(\beta)$ for $\beta \in [0,a]$.

It follows from the convexity of H_W in $[W(a),W(\theta)]$ that there is a unique $\alpha \in [a,\theta]$ such that the difference $\overline{h} - h$ changes its sign from positive to negative. Precisely, α is the point such that the line through $(0,0)$ and $(W(\alpha),H_W(W(\alpha)))$ is tangent to H_W at $W(\alpha)$. Moreover, for $\beta \in (\theta,d)$ we have $\overline{h}(\beta) < 0 < h(\beta)$. ∎

LEMMA 6. *Assume that α satisfies* (2.3). *Then for any $\beta \in (0,c]$ the function*

$$g(x) = \begin{cases} \overline{h}(\beta), & 0 \le \beta \le \alpha,\\ h(\max\{x,\alpha\}), & \alpha < \beta \le c,\end{cases} \tag{4.4}$$

is the projection of $h_{|[0,\beta]}$ *onto the convex cone of nondecreasing functions in the Hilbert space* $L^2([0,\beta), w(x)dx)$.

Proof. Rychlik ([12, Example 3, pp. 14–16]) presented the solution to the problem of projecting functions onto the convex cone of nondecreasing functions in spaces of the type L^2 with nonuniform weight function. In particular, the projection of $h_{|[0,\beta)} \in L^2([0,\beta), w(x)dx)$ is $(\overline{H}_W^\beta)'W$, where $\overline{H}_W^\beta$ is the greatest convex minorant of H_W on $[0,\beta)$. From the properties of H_W and $H_W W$, given above, we conclude that $(\overline{H}_W^\beta)'W(x)$ has the form (4.4), which completes the proof. ∎

Clearly, for any $\beta \le \max\{b,\alpha\}$ the function (4.4) is the projection of $h_{|[0,\beta]}$ onto the convex cone of nondecreasing and convex functions in the space $L^2([0,\beta), w(x)dx)$.

Proof of Proposition 1. Suppose first that the assumptions of the first claim are satisfied. Take an arbitrary $\xi \in (\alpha,\beta)$ and g defined by (2.5). By Lemma 6, $g_{|[0,\xi]}$ is the projection of $h_{|[0,\xi]}$ onto the cone of nondecreasing functions in $L^2([0,\xi), w(x)dx)$. Proposition 1 in Danielak [1] implies that $g_{|[\xi,d]}$ is the projection of $h_{|[\xi,d]}$ onto the convex cone of nondecreasing convex functions in $L^2([\xi,d), w(x)dx)$. Therefore, for any $f \in \mathcal{C}_W$ (see (1.5)) we have

$$\int_0^d [f(x)-h(x)]^2 w(x)\,dx \ge \int_0^d [g(x)-h(x)]^2 w(x)\,dx$$

because the inequality holds for the integrals over both $[0,\xi]$ and $[\xi,d]$. It follows that g is the projection of h, because it is closer to h than any other function f from the cone.

Suppose now that $P_W h$ is of the form (2.5), but the parameters α, β do not satisfy the assumptions of the proposition. We will show that we can find better approximations of h than g, which gives a contradiction. First, assume that $\alpha \in (a,\beta)$ does not satisfy (2.3). Consider g of the form (2.5) restricted to $[0,\beta]$ with β fixed and $\alpha \in [a,\beta]$. Let

$$D(\alpha) = \int_0^\beta [g(x)-h(x)]^2 w(x)\,dx.$$

Then

$$D'(\alpha) = 2h'(\alpha)[h(\alpha)-\overline{h}(\alpha)] \int_0^\alpha w(x)\,dx.$$

If $D'(\alpha) < 0$, then $D(\alpha)$ decreases if α increases. If $D'(\alpha) > 0$, then $D(\alpha)$ decreases if α decreases. In both cases approximation of h can be improved, a contradiction.

Suppose now that g is of the form (2.5), α satisfies condition (2.3), but $\lambda \neq \lambda_1(\beta)$. Set

$$D(\beta,\lambda) = \int_{\alpha}^{d} [g(x) - h(x)]^2 w(x)\,dx = \int_{\beta}^{d} [h(\beta) + \lambda(x-\beta) - h(x)]^2 w(x)\,dx.$$

We fix β and look for $\lambda \geq h'(\beta)$ for which $D(\beta,\lambda)$ is minimized. The function $D(\beta,\lambda)$ is quadratic in λ and attains its minimum for $\lambda_1(\beta)$ defined by (2.4). Hence, we minimize $D(\beta,\lambda)$ by taking $\lambda^*(\beta) = \max\{h'(\beta), \lambda_1(\beta)\}$. Thus we can exclude all $\lambda \neq \lambda^*(\beta)$. Furthermore, if $\lambda^*(\beta) = h'(\beta)$, then we improve the approximation by decreasing β (cf. Gajek and Rychlik [5, pp. 170–171]). So, we can also exclude $\lambda = \lambda(\beta) \neq \lambda_1(\beta)$.

Suppose then that g is of the form (2.5), α satisfies (2.3) and $\lambda = \lambda_1(\beta)$. If $Y(\beta) < 0$, then g is not convex. If $Z(\beta) \neq 0$, then the necessary condition (1.6) fails and g is not a projection. Therefore, conditions (2.3), $Y(\beta) \geq 0$ and $Z(\beta) = 0$ are necessary for g of the form (2.5) with $\lambda = \lambda_1(\beta)$ to be the projection of h onto $\mathcal{C}_W$. If there exist $\alpha < \beta_1 < \beta_2$ satisfying these conditions, then (2.5) with β_2 and $\lambda_1(\beta_2)$ approximates h better on $[\alpha, d)$ than (2.5) with β_1 and $\lambda_1(\beta_1)$ (cf. Danielak [1, proof of Proposition 1]). Therefore we take the greatest β^* satisfying the above conditions.

Summing up, we have proved that the assumptions of the first part of the proposition are necessary and sufficient. If they are not satisfied, then by Lemma 3, the projection is of the form (2.2). We now show that the function satisfying the assumptions of the second claim is the best approximation of h of this form.

Consider the function

$$D(\gamma,\lambda,\beta) = \|g-h\|^2 = \int_{0}^{\beta} [h(x) - \gamma]^2 w(x)\,dx + \int_{\beta}^{d} [h(x) - \lambda(x-\beta) - \gamma]^2 w(x)\,dx.$$

Defining

$$\widetilde{g}_{\lambda,\beta}(x) = h(x) - \lambda(x-\beta)\mathbb{I}_{[\beta,d)}(x),$$

we can write

$$D(\gamma,\lambda,\beta) = \int_{0}^{d} [\widetilde{g}_{\lambda,\beta}(x) - \gamma]^2 w(x)\,dx.$$

The function $D(\gamma,\lambda,\beta)$, with λ and β fixed, is quadratic convex and attains its minimum at

$$\gamma^* = \gamma^*(\lambda,\beta) = \int_0^d \widetilde{g}_{\lambda,\beta}(x)w(x)\,dx = -\lambda\int_\beta^d (x-\beta)w(x)\,dx. \tag{4.5}$$

Set $D(\lambda,\beta) = D(\gamma^*,\lambda,\beta)$ and

$$\widetilde{g}_\beta(x) = (x-\beta)\mathbb{I}_{[\beta,d)}(x) - \int_\beta^d (x-\beta)w(x)\,dx.$$

Then the quantity

$$D(\lambda,\beta) = \int_0^d [h(x)-\lambda\widetilde{g}_\beta(x)]^2 w(x)\,dx$$

for any fixed $\beta \in (0,d)$ is minimized at

$$\lambda^* = \lambda^*(\beta) = \frac{\int_0^d h(x)\widetilde{g}_\beta(x)w(x)\,dx}{\int_0^d \widetilde{g}_\beta(x)^2 w(x)\,dx} = \frac{L(\beta)}{M(\beta)},$$

where

$$L(\beta) = \int_\beta^d (x-\beta)h(x)w(x)\,dx,$$

$$M(\beta) = \int_\beta^d (x-\beta)^2 w(x)\,dx - \Big[\int_\beta^d (x-\beta)w(x)\,dx\Big]^2.$$

We have

$$L'(\beta) = -\int_\beta^d h(x)w(x)\,dx,$$

$$M'(\beta) = -2\int_\beta^d (x-\beta)w(x)\,dx\int_0^\beta w(x)\,dx.$$

Note that $L(d) = 0$ and $L'(\beta) < 0$, and the same holds for M. It follows that $L(\beta)$, $M(\beta)$ and $\lambda^*(\beta)$ are positive for any $\beta \in [0,d)$. Therefore, for arbitrary fixed β the function of the form (2.2) with optimal parameters γ^* and λ^* is nondecreasing and convex.

Let $D(\beta)$ stand for $D(\lambda^*(\beta),\beta)$. Then

$$D(\beta) = \int_0^d h^2(x)w(x)\,dx - 2\,\frac{L(\beta)}{M(\beta)}\int_0^d h(x)\widetilde{g}_\beta(x)w(x)\,dx$$
$$+ \left(\frac{L(\beta)}{M(\beta)}\right)^2\int_0^d \widetilde{g}^2_\beta(x)w(x)\,dx = \int_0^d h^2(x)w(x)\,dx - \frac{L^2(\beta)}{M(\beta)}$$

and

$$D'(\beta) = 2\,\frac{L(\beta)}{M(\beta)}\left[\lambda^*(\beta)\,\frac{M'(\beta)}{2} - L'(\beta)\right].$$

Since $D'(\beta)$ is continuous and $D(\beta)$ does not attain its minimum at $\beta = 0$ or $\beta = d$, we see that the condition $D'(\beta) = 0$ is necessary for $\beta \in (0, d)$ to be optimal. Thus we need

$$\lambda^*(\beta) = 2\,\frac{L'(\beta)}{M'(\beta)} = -\frac{\overline{h}(\beta)}{\int_\beta^d (x-\beta)w(x)\,dx}. \tag{4.6}$$

Substituting (4.6) into (4.5), we conclude that β has to satisfy

$$\gamma^*(\beta) = \frac{\int_0^\beta h(x)w(x)\,dx}{\int_0^\beta w(x)\,dx} = \overline{h}(\beta).$$

Plugging the optimal $\gamma^*(\beta)$ and $\lambda^*(\beta)$ into (2.2) we obtain (2.6). The function g is the best constant approximation of h in $[0, \beta]$ because

$$\int_0^\beta g(x)w(x)\,dx = \int_0^\beta h(x)w(x)\,dx.$$

It is also the best linear approximation of h in $[\beta, d]$ since it is of the form $\lambda^*(\beta)(x-\beta) + \gamma^*$. It remains to find $\beta \in (0, d)$ satisfying $D'(\beta) = 0$, for which the function of the form (2.6) is the projection of h onto (1.5). We have stated that the function (2.6) is the projection if the following conditions are satisfied:

1) a constant approximation in $[0, \beta]$ is optimal in this interval,
2) the point $(\beta, \gamma(\beta))$ lies on the curve $(\beta, \overline{h}(\beta))$,
3) the increasing linear part of (2.6), say l, is the optimal linear approximation of h on $[\beta, d)$.

Assume that $D'(\beta) = 0$ for some $\beta > \alpha$. Then either g runs beneath h on (β, c), or g and h have at least one common point over (β, c). In both cases we can find a better approximation of the form (2.5).

If conditions 1)–3) hold for $\beta = \alpha$, then the constant and linear parts are the optimal nondecreasing approximations of h on $[0, \alpha]$ and (α, d), respectively, and they together define the desired projection of h on $[0, d)$.

Assume that $l(\alpha) < h(\alpha) = \overline{h}(\alpha)$ for $\beta = \alpha$, where $l(\alpha)$ denotes the value of the optimal linear approximation of h on $[\alpha, d)$ at α. Then g can be replaced by a function of the form (2.5), passing through $(\alpha, h(\alpha))$. So we have a contradiction. If the function (2.6) with $\beta = \alpha$ is not the projection, then the best linear approximation of h on (α, d) is the best nondecreasing and convex approximation on this interval and $l(\alpha) > h(\alpha)$. Now we decrease β starting from $\beta = \alpha$. Linear functions are still the best nondecreasing convex approximations of h on (β, d), the values $l(\beta)$ change continuously until they reach the level $\overline{h}(\beta)$, earlier than $h(\beta)$, because $\overline{h}(\beta) > h(\beta)$ for $\beta < \alpha$. Then the function that is constantly $\overline{h}(\beta)$ on $[0, \beta]$ and equal to l on (β, d) is the projection of h onto (1.5), because $\overline{h}(\beta)$ and l are the projections of h on $[0, \beta)$ and (β, d), respectively. ■

The following lemma is a simplified version of Lemma 4 in Gajek and Rychlik [5].

LEMMA 7. *If* $\{y \in (0, b] : Y(y) > 0\} = (0, v)$ *and* Z *has a finite number of zeros*, *then* Z *is either positive or negative or changes its sign once from* $-$ *to* $+$ *in* $(0, v)$.

Proof of Proposition 2. We begin by finding α defined by (2.3). Since $h(x) = s_{j:n}(x) = n[B_{j,n-1}(x) - B_{j-1,n-1}(x)]$, α is the unique solution to

$$\int_0^\alpha n[B_{j,n-1}(x) - B_{j-1,n-1}(x)]\,dx = n\alpha[B_{j,n-1}(\alpha) - B_{j-1,n-1}(\alpha)],$$

which is equivalent to

$$-B_{j,n}(\alpha) = (j+1)B_{j+1,n}(\alpha) - jB_{j,n}(\alpha), \tag{4.7}$$

and finally $\alpha = (j-1)/(n-1)$. Note that, by Lemma 4(ii), $\alpha > a$. The next step is to evaluate

$$\lambda_1(\beta) = \frac{n\int_\beta^1 (x-\beta)[B_{j,n-1}(x) - B_{j-1,n-1}(x) - B_{j,n-1}(\beta) + B_{j-1,n-1}(\beta)]\,dx}{\int_\beta^1 (x-\beta)^2\,dx}.$$

Since

$$n\int_\beta^1 x[B_{j,n-1}(x) - B_{j-1,n-1}(x)]\,dx = \frac{\sum_{m=0}^{j} B_{m,n+1}(\beta) + (j+1)B_{j+1,n+1}(\beta)}{n+1},$$

$$n\beta\int_\beta^1 [B_{j,n-1}(x) - B_{j-1,n-1}(x)]\,dx = \frac{j+1}{n+1}\,B_{j+1,n+1}(\beta),$$

$$n[B_{j,n-1}(\beta) - B_{j-1,n-1}(\beta)]\int_\beta^1 (x-\beta)\,dx = \frac{(n-j)(n-j+1)B_{j,n+1}(\beta) - (n-j+1)(n-j+2)B_{j-1,n+1}(\beta)}{2(n+1)},$$

we finally obtain

$$\text{(4.8)}\quad \lambda_1(\beta) = \frac{\sum_{m=0}^{j} B_{m,n+1}(\beta) - \frac{(n-j+1)!}{2(n-j-1)!} B_{j,n+1}(\beta) + \frac{(n-j+2)!}{2(n-j)!} B_{j-1,n+1}(\beta)}{\frac{1}{3}(n+1)(1-\beta)^3}.$$

We next evaluate $Y(\beta) = \lambda_1(\beta) - h'(\beta)$, where

$$h'(\beta) = n(n-1)[-B_{j-2,n-2}(\beta) + 2B_{j-1,n-2}(\beta) - B_{j,n-2}(\beta)].$$

Multiplying $h'(\beta)$ by the denominator of (4.8) we get

$$\frac{h'(\beta)(n+1)(1-\beta)^3}{3} = -\frac{(n-j+3)!}{3(n-j)!} B_{j-2,n+1}(\beta) + \frac{2(n-j+2)!}{3(n-j-1)!} B_{j-1,n+1}(\beta) - \frac{(n-j+1)!}{3(n-j-2)!} B_{j,n+1}(\beta).$$

Therefore

$$\text{(4.9)}\qquad Y(\beta) = \frac{\sum_{m=0}^{j} a_m B_{m,n+1}(\beta)}{\frac{1}{3}(n+1)(1-\beta)^3},$$

where

$$\begin{aligned} a_m &= 1, \qquad m = 0, \dots, j-3, \\ a_{j-2} &= 1 + \frac{(n-j+3)!}{3(n-j)!}, \\ a_{j-1} &= 1 + (n-j+2)(n-j+1)\left[\tfrac{1}{2} - \tfrac{2}{3}(n-j)\right], \\ a_j &= 1 + (n-j)(n-j+1)\left[\tfrac{1}{3}(n-j-1) - \tfrac{1}{2}\right]. \end{aligned}$$

Since the denominator in (4.9) is positive, Y has the same sign as the polynomial (3.1) in the numerator. The coefficients a_m are positive for $m = 0, \dots, j-2$, and a_{j-1} is negative, because such is the expression in square brackets. Furthermore, $a_j = 0$ for $j = n-2$, and $a_j > 0$ for $j \le n-3$. Thus, by Lemma 5, (3.1) is positive near 0 and negative near 1, provided $j = n-2$. If $j \le n-3$, then (3.1) is either $+-+$ or positive in $(0,1)$. Since the line tangent to h at b lies over the graph of h for $\beta > b$, we have $h'(b) > \lambda_1(b)$, which implies $Y_1(b) < 0$. It follows that (3.1) is positive near 0 and has exactly one zero $y \in (0, b)$. The condition $y > \alpha$, which is equivalent to $Y_1(\alpha) > 0$, is necessary for the existence of a projection of the form (2.5). If $y \le \alpha$, then the projection is of the form (2.6).

Next, we find the exact form of the polynomial

$$Z(\beta) = n\int_{\beta}^{1} [B_{j,n-1}(x) - B_{j-1,n-1}(x)]\,dx - n(1-\beta)[B_{j,n-1}(\beta) - B_{j-1,n-1}(\beta)] - \tfrac{1}{2}\lambda_1(\beta)(1-\beta)^2.$$

Since

$$n\int_\beta^1 [B_{j,n-1}(x) - B_{j-1,n-1}(x)]\,dx = B_{j,n}(\beta),$$

$$n(1-\beta)[B_{j,n-1}(\beta) - B_{j-1,n-1}(\beta)] = (n-j)B_{j,n}(\beta) - (n-j+1)B_{j-1,n}(\beta),$$

$$\lambda_1(\beta) = \frac{2\sum_{m=0}^{j} B_{m,n+1}(\beta) - \frac{(n-j+1)!}{(n-j-1)!}B_{j,n+1}(\beta) + \frac{(n-j+2)!}{(n-j)!}B_{j-1,n+1}(\beta)}{\frac{2}{3}(n+1)(1-\beta)^3},$$

with the notation $Z_1(\beta) = \frac{2}{3}(n+1)(1-\beta)Z(\beta)$, we obtain

$$\begin{aligned}Z_1(\beta) = &-\frac{2}{3}(n-j-1)(n-j+1)B_{j,n+1}(\beta) + \frac{2(n-j+2)!}{3(n-j)!}B_{j-1,n+1}(\beta)\\ &-\sum_{m=0}^{j} B_{m,n+1}(\beta) + \frac{(n-j+1)!}{2(n-j-1)!}B_{j,n+1}(\beta) - \frac{(n-j+2)!}{2(n-j)!}B_{j-1,n+1}(\beta).\end{aligned}$$

The polynomial $Z_1(\beta)$ can be represented as $Z_1(\beta) = \sum_{m=0}^{j} b_m B_{m,n+1}(\beta)$, where

$$\begin{aligned} b_m &= -1, \qquad m = 0,\ldots,j-2,\\ b_{j-1} &= \tfrac{1}{6}(n-j+1)(n-j+2) - 1,\\ b_j &= -\tfrac{1}{6}(n-j+1)(n-j-4) - 1\end{aligned}$$

(cf. (3.2)). The coefficient b_{j-1} is positive, because $j < n-1$. If $j = n-2$, then $b_j = 0$, and $b_j < 0$ otherwise. It follows that for $j = n-2$ the polynomial Z_1 (and so Z) changes its sign once from $-$ to $+$ at some $z \in (0,1)$. If $j < n-2$, then (3.2) is either negative in the whole $(0,1)$ and then the projection is of the form (2.5), or it is $-+-$. By Lemma 7, (3.2) changes its sign in $(0,y)$ at most once and only from $-$ to $+$. Therefore, for $2 \le j \le n-2$ the polynomial (3.2) is either negative on $(0,y]$, or there exists a unique $z \in (0,y]$ such that (3.2) changes its sign at z from $-$ to $+$. If the former holds, then the necessary condition (1.6) fails and the projection is not of the form (2.5). In the latter case, the projection is of the form (2.5) if $z > \alpha$. Summing up, (3.3) are necessary and sufficient conditions for (2.5) to be the projection, with $\alpha^* = \alpha = (j-1)/(n-1)$ and β^* the smallest positive zero of the polynomial (3.2). Then

$$\|P_U h\|_U^2 = \int_0^{\alpha^*} h^2(\alpha^*)\,dx + \int_{\alpha^*}^{\beta^*} h^2(x)\,dx + \int_{\beta^*}^1 [h(\beta^*) + \lambda(x-\beta^*)]^2\,dx,$$

where $\lambda = \lambda_1(\beta^*)$, and we finally get (3.5). Using (1.7), we find the cdf (3.6) for which the bound (3.4) is achieved.

If (3.3) fails, then $P_W h$ is of the form (2.6). Condition (2.7) takes on the form

$$B_{j,n}(\beta)\left[\tfrac{1}{3}(1-\beta)^3-\tfrac{1}{4}(1-\beta)^4\right]=\frac{\beta(1-\beta)^2}{2(n+1)}\sum_{m=0}^{j}B_{m,n+1}(\beta),$$

which can be rewritten as

$$\tfrac{2}{3}(j+1)(n-j+1)B_{j+1,n+2}(\beta)+\frac{(n-j+2)!}{6(n-j)!}\,B_{j,n+2}(\beta)=\sum_{m=1}^{j+1}mB_{m,n+2}(\beta).$$

This is equivalent to

$$K(\beta)=\sum_{m=1}^{j+1}c_mB_{m,n+2}(\beta)=0,$$

where

$$\begin{aligned}c_m&=m>0,\qquad m=1,\ldots,j-1,\\ c_j&=j-\tfrac{1}{6}(n-j+1)(n-j+2),\\ c_{j+1}&=(j+1)\left[1-\tfrac{2}{3}(n-j+1)\right]\end{aligned}$$

(cf. (3.8)). Since $n-j\ge 2$, we have $c_{j+1}<0$. This combined with Lemma 5 implies that K is positive near 0, negative near 1 and has a unique zero $\beta^*\in(0,1)$. It uniquely determines $P_U h$ of the form (2.6). Its norm gives the bound in (3.7). Applying (1.7), we obtain the cdf (3.9) attaining the bound (3.7). ∎

Proof of Proposition 3. We proceed analogously to the proof of Proposition 2. Set $h(x)=r_{j:n}V(x)=f_{j+1:n}V(x)-f_{j:n}V(x)$. Suppose first that $P_V h$ is of the form (2.5). The task is now to find α satisfying

$$\begin{aligned}&\int_0^\alpha[B_{j,n-1}(1-e^{-x})-B_{j-1,n-1}(1-e^{-x})]e^{-x}\,dx\\ &\qquad=(1-e^{-\alpha})[B_{j,n-1}(1-e^{-\alpha})-B_{j-1,n-1}(1-e^{-\alpha})]\end{aligned}$$

(cf. (2.3)) or equivalently

$$-B_{j,n}(1-e^{-\alpha})=(j+1)B_{j+1,n}(1-e^{-\alpha})-jB_{j,n}(1-e^{-\alpha}).$$

Writing $\alpha_0=1-e^{-\alpha}$, we obtain equation (4.7) and so $\alpha_0=(j-1)/(n-1)$. Next we determine

$$\lambda_1(\beta)=\frac{\int_\beta^\infty(x-\beta)[s_{j:n}V(x)-s_{j:n}V(\beta)]e^{-x}\,dx}{\int_\beta^\infty(x-\beta)^2e^{-x}\,dx}.\tag{4.10}$$

The denominator of (4.10) is equal to $2e^{-\beta}$, and its numerator can be rewritten as

$$A(\beta) = n\int_{\beta}^{\infty} (x-\beta)[C_{j,n-1}(x) - C_{j-1,n-1}(x)]e^{-x}\,dx - s_{j:n}V(\beta)e^{-\beta}.$$

Gajek and Rychlik [5, pp. 176–177]) showed that

$$\int_{\beta}^{\infty} (x-\beta)C_{i,m}(x)e^{-x}\,dx = \frac{1}{m+1}\sum_{k=0}^{i} S(i+1-k, m+1-k)C_{k,m+1}(\beta),$$

where

$$S(i,n) = E_V(X_{i:n}) = \sum_{m=1}^{i} \frac{1}{n+1-m}, \qquad 1 \le i \le n.$$

Therefore

$$\begin{aligned} A(\beta) = \sum_{m=0}^{j-1} &[S(j+1-m, n-m) - S(j-m, n-m)]C_{m,n}(\beta) \\ &+ S(1, n-j)C_{j,n}(\beta) - s_{j:n}V(\beta)e^{-\beta}. \end{aligned}$$

Since $S(j+1-m, n-m) - S(j-m, n-m) = S(1, n-j) = 1/(n-j)$ and

$$s_{j:n}V(\beta)e^{-\beta} = (n-j)C_{j,n}(\beta) - (n-j+1)C_{j-1,n}(\beta),$$

it follows that

$$A(\beta) = \sum_{m=0}^{j} \frac{C_{m,n}(\beta)}{n-j} - (n-j)C_{j,n}(\beta) + (n-j+1)C_{j-1,n}(\beta)$$

and finally we obtain

$$\lambda_1(\beta) = \frac{e^{\beta}}{2}\left\{ \sum_{m=0}^{j} \frac{C_{m,n}(\beta)}{n-j} - (n-j)C_{j,n}(\beta) + (n-j+1)C_{j-1,n}(\beta)\right\}.$$

Our next goal is to determine

$$Y(\beta) = \lambda_1(\beta) - h'(\beta) = \frac{e^{\beta}}{2}[A(\beta) - 2e^{-\beta}h'(\beta)].$$

As

$$\begin{aligned} \frac{h'(\beta)}{e^{\beta}} = &-\frac{(n-j+2)!}{(n-j)!}\,C_{j-2,n}(\beta) \\ &+ \frac{2(n-j+1)!}{(n-j-1)!}\,C_{j-1,n}(\beta) - \frac{(n-j)!}{(n-j-2)!}\,C_{j,n}(\beta), \end{aligned}$$

we have

$$Y(\beta) = \frac{e^{\beta}}{2}\sum_{m=0}^{j} a_m C_{m,n}(\beta),$$

where

$$a_m = \frac{1}{n-j}, \qquad m = 0, \dots, j-3,$$
$$a_{j-2} = \frac{1}{n-j} + 2\,\frac{(n-j+2)!}{(n-j)!},$$
$$a_{j-1} = \frac{1}{n-j} + (n-j+1)[1-4(n-j)],$$
$$a_j = \frac{1}{n-j} + (n-j)[2(n-j)-3].$$

We easily check that $a_m > 0$ for $m = 0, \dots, j-2$ and $a_{j-1} < 0$. Moreover $a_j = 0$ for $j = n-1$ and $a_j > 0$ for $j < n-1$. Hence Y is $+-$ for $j = n-1$, and it is either $+$ or $+-+$ for the remaining values of j. By the argument used in the proof of the previous proposition, $Y(b) < 0$. Therefore, for any $2 \le j \le n-1$, there exists $y \in (0, b)$ such that $Y(y) = 0$ and $Y(\beta) > 0$ for $\beta \in (0, y)$. Setting $Y_2V(\beta) = 2(n+2)e^{-\beta}Y(\beta)$, we get (3.10) for $x = 1-e^{-\beta}$.

Next we analyze the behavior of

$$Z(\beta) = \int_\beta^\infty s_{j:n}V(x)e^{-x}\,dx - s_{j:n}V(\beta)\int_\beta^\infty e^{-x}\,dx - \lambda_1(\beta)\int_\beta^\infty (x-\beta)e^{-x}\,dx,$$

where

$$\int_\beta^\infty s_{j:n}V(x)e^{-x}\,dx = n\int_{1-e^{-\beta}}^1 [B_{j,n-1}(y) - B_{j-1,n-1}(y)]\,dy = C_{j,n}(\beta),$$
$$s_{j:n}V(\beta)\int_\beta^\infty e^{-x}\,dx = (n-j)C_{j,n}(\beta) - (n-j+1)C_{j-1,n}(\beta).$$

Finally, we obtain $Z(\beta) = \sum_{m=0}^{j} b_m C_{m,n}(\beta)$, where

$$b_m = -\frac{1}{2(n-j)}, \qquad m = 0, \dots, j-2,$$
$$b_{j-1} = \frac{n-j+1}{2} - \frac{1}{2(n-j)},$$
$$b_j = 1 - \frac{1}{2(n-j)} - \frac{n-j}{2}.$$

We see that $b_m < 0$ for $m = 0, \dots, j-2$, $b_{j-1} > 0$, $b_j < 0$ for $j < n-1$ and $b_j = 0$ for $j = n-1$. Therefore, for $j = n-1$ the function Z has exactly one zero. If $j < n-1$, then Z is either $-+-$, or negative everywhere in $(0, \infty)$. Analysis similar to that in the proof of Proposition 2 shows that the latter is impossible. Set $Z_2V^{-1}(\beta) = 2(n+1)(n-j)Z(\beta)$. As in the previous proof we deduce that relations (3.12) are necessary and sufficient for the existence of the projection of the form (2.5). Then $\alpha^* = V^{-1}(\alpha_0)$ and $\beta^* = V^{-1}(\beta_0)$, where β_0 is the smallest positive zero of (3.11). Determining the projection of the form (2.5) with $\lambda = \lambda_2(\beta^*)$ enables us to calculate the bound (3.14) and the distribution function (3.15) attaining the bound.

Suppose now that (3.12) fails. Then the projection is of the form (2.6). The aim is to find β satisfying (2.7). The condition can we rewritten as

$$C_{j,n}(\beta)(2e^{-\beta} - e^{-2\beta}) = (1-e^{-\beta})e^{-\beta}\,\frac{1}{n-j}\sum_{m=0}^{j} C_{m,n}(\beta),$$

or equivalently

$$2(j+1)C_{j+1,n+1}(\beta) + (n-j+1)C_{j,n+1}(\beta) = \frac{1}{n-j}\sum_{m=1}^{j+1} mC_{m,n+1}(\beta).$$

Further calculations lead to

$$K(\beta) = \sum_{m=1}^{j+1} d_m C_{m,n+1}(\beta) = 0$$

with

$$\begin{aligned} d_m &= \frac{m}{n-j}, \qquad m = 1,\dots,j-1,\\ d_j &= \frac{j}{n-j} - (n-j+1),\\ d_{j+1} &= (j+1)\left(\frac{1}{n-j} - 2\right). \end{aligned}$$

Since $d_m > 0$ for $m = 1,\dots,j-1$ and $d_{j+1} < 0$, independently of the sign of d_j, the function $K(\beta)$ has exactly one zero, say z. Thus, the projection is of the form

$$P_V h(x) = -\overline{h}(z)[e^z(x-z)\mathbb{I}_{[z,\infty)}(x) - 1],$$

where

$$\overline{h}(z) = \frac{\int_0^z [f_{j+1:n}V(x) - f_{j:n}V(x)]e^{-x}\,dx}{\int_0^z e^{-x}\,dx} = \frac{-f_{j+1:n+1}(1-e^{-z})}{(1-e^{-z})(n+1)}.$$

Then

$$\|P_V h\|_V^2 = [\overline{h}(z)]^2(2e^z - 1) = \frac{f_{j+1:n+1}^2(1-e^{-z})}{(1-e^{-z})^2(n+1)^2}\,(2e^z-1)$$

and substituting $\varrho = 1 - e^{-z}$, we obtain the square of the bound (3.16). Applying (1.7) we determine the cdf (3.17), for which the bound (3.16) is achieved. ∎

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Received on 28.5.2004 (1744)