GENERAL METHOD OF REGULARIZATION.
III: THE UNILATERAL CONTACT PROBLEM

Abstract. The aim of this paper is to prove that the relaxation of the elastic-perfectly plastic energy (of a solid made of a Hencky material with the Signorini constraints on the boundary) is the weak* lower semicontinuous regularization of the plastic energy. We consider an elastic-plastic solid endowed with the von Mises (or Tresca) yield condition. Moreover, we show that the set of solutions of the relaxed problem is equal to the set of solutions of the relaxed problem proposed by Suquet. We deduce that the relaxation of the energy of plastic soil (with the Signorini condition on the boundary) is the weak* lower semicontinuous regularization of the energy.

1. Introduction. In this paper we investigate the convex functional

\[ BD \times \mathbb{M}_b \ni (u, \mu) \mapsto S(\varepsilon(u), \mu) = \int_{\Omega} h(x, \varepsilon(u), \mu) \]

with constraints on the boundary of \( \Omega \), where \( \varepsilon(u) \) is the symmetrized gradient of \( u \), \( BD(\Omega) \) is the space of bounded deformations and \( \mu \) is a regular measure on \( \Omega \), \( \mu \in \mathbb{M}_b(\Omega, \mathbb{R}^n) \). Moreover we assume that \( S(\varepsilon(u), \mu) = \infty \) if \( \varepsilon(u) \notin L^1 \). In [6] we find the lower semicontinuous (l.s.c.) relaxation of \( S \), and we show that the relaxation is a l.s.c. function (in the weak* \( BD \times \mathbb{M}_b \) topology), not greater than \( S \). Now we prove that the above mentioned relaxation is the largest l.s.c. minorant less than \( S \), in the case of a material with the von Mises (or Tresca) plastic yield condition (i.e. the l.s.c. regularization of \( S \), in this case). Moreover, we consider the Signorini problem in soil mechanics (i.e. a unilateral contact condition on the boundary of a piece of soil).

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The l.s.c. regularization (in \( L^1_{\text{loc}} \) topology) of functionals defined on the space \( BV(\Omega) \) was investigated in many papers ([1]–[3], [18]), but their authors did not consider problems with constraints on the boundary of \( \Omega \).

In [12], the global method of relaxation (cf. [11]) is applied to l.s.c. regularization of quasiconvex functionals with constraints (Dirichlet condition). These functionals are defined on \( BV(\Omega) \). The relevant constraints do not describe the relaxation proposed by Suquet (see [22] and [9]) or unilateral contact conditions (cf. condition \((H4)\) from [12]). In [12] the essential assumption of the method is the local coercivity of the density of elastic-plastic energy.

Kohn and Temam [20] solved the existence problem for an elastic-perfectly plastic solid made of a homogeneous Hencky material. To prove that the functional of the total potential energy is weak* l.s.c. in the space \( BD(\Omega) \), they used the method of relaxation of the kinematic boundary condition. They did not show that the relaxed problem was the l.s.c. regularization of the original problem.

Unilateral problems in plasticity were considered by Baiocchi, Buttazzo, Gastaldi and Tomarelli [4]. In [4] they solved the Signorini-like Problem, but they were not able to solve the classical Signorini Problem (see [15]) in Hencky plasticity. Tomarelli [25] solved the Signorini problem in the space \( \mathbb{U}f \subset BD(\Omega) \), \( \text{div}u \in L^2(\Omega) \), for an isotropic and homogeneous body made of a Hencky material.

A l.s.c. relaxation of the Signorini problem for an anisotropic and non-homogeneous Hencky material has been found in [6]. Now we prove that the relaxation (of [6]) is the l.s.c. regularization in the case of a material with the von Mises (or Tresca) plastic yield condition. This generalizes Tomarelli’s [25] result to the case of a non-homogeneous Hencky material with non-linear elastic law. Moreover, we show that an analogous result holds in soil mechanics (where the set of admissible stresses is a convex cone; see Remark 2).

2. Some basic definitions and theorems. Let \( \Omega \) be a bounded, open, connected set of class \( C^1 \) in \( \mathbb{R}^n \). The space of continuous functions with compact support is denoted by \( C_c \). Let \( C^\infty(\Omega, \mathbb{R}^m) \) be the space of \( \mathbb{R}^m \)-valued, infinitely differentiable functions. Moreover, the space of infinitely differentiable functions equal to 0 at the boundary \( \text{Fr} \Omega \) of \( \Omega \) is denoted by \( C^\infty_0(\Omega) \). Finally, \( \mathbb{M}_b(\Omega, \mathbb{R}^m) \) is the space of \( \mathbb{R}^m \)-valued, Radon, bounded, regular measures on \( \Omega \), with the norm \( \| \cdot \|_{\mathbb{M}_b(\Omega, \mathbb{R}^m)} \).

We will use the duality pairs \((\mathbb{M}_r, C_c)\) or \((\mathbb{M}_b, C_0)\), where \( \mathbb{M}_r \) is the space of regular measures. Duality pairings will be denoted by \( \langle \cdot, \cdot \rangle \), and the scalar product of \( z, z^* \in \mathbb{R}^n \) by \( z \cdot z^* \) or \( zz^* \). The scalar product of \( w, w^* \in \mathbb{R}^{n \times n} \) is denoted by \( w : w^* = w_{ij}w_{ij}^* \). Let \( g = (g_1, \ldots, g_m) \in C(\overline{\Omega}, \mathbb{R}^m) \) and
\( \mu = (\mu_1, \ldots, \mu_m) \in M_b(\Omega, \mathbb{R}^m) \). Then \( \int_\Omega g \cdot \mu = \int_\Omega g \mu \equiv \sum_{i=1}^m \int_\Omega g_i \mu_i \).

If \( F : Y \to \mathbb{R} \cup \{\infty\} \), then \( F^* \) denotes its polar function \( F^*(y^*) = \sup\{\langle y^*; y \rangle - F(y) \mid y \in Y\} \), and \( \text{dom } F = \{y \in Y \mid F(y) < \infty\} \) (see [16]). If \( Q \) is a subset of \( Y \), then \( I_Q(\cdot) \) is its indicator function (taking the value 0 in \( Q \) and \( \infty \) outside), and \( I_Q^*(\cdot) \) stands for its support function.

Finally, we need the following notations. \( B_V(\Xi, r) \) is the closed ball in a metric space \( V \) with center \( \Xi \) and radius \( r \). Furthermore, \( \text{cl}_V(Z) \) stands for the closure of \( Z \subset V \) in the topology of the space \( V \); analogously, \( \text{cl}_{\| \cdot \|}(Z) \) is the closure of the set \( Z \) in the norm \( \| \cdot \| \). Similarly, \( \text{int } Z \) denotes the interior of \( Z \). \( \mathcal{L}^0(\Omega, \mathbb{R}^m)_\mu \) is the set of \( \mu \)-measurable functions from \( \Omega \) into \( \mathbb{R}^m \). We denote by \( E^n_s \) the space of symmetric \( n \times n \) matrices. We set \( \|e_{ij}\|_{E^n_s} = \sum_{i,j=1}^n |e_{ij}| \). We denote by \( \otimes \) (resp. \( \otimes_s \)) the tensor product (resp. symmetric tensor product). If \( \tau \subset 2^X \) is a linear topology in a vector space \( X \), then \( [X, \tau] \) denotes the topological space and \( [X, \tau]^* \) its dual. We define the following Banach spaces (see [20], [23]):

\[
\text{(2.1) } LD(\Omega) \equiv \left\{ u \in L^1(\Omega, \mathbb{R}^n) \right\} \\
\epsilon_{ij}(u) \equiv \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \in L^1(\Omega), i, j = 1, \ldots, n, \right\},
\]

\[
\text{(2.2) } BD(\Omega) \equiv \{ u \in L^1(\Omega, \mathbb{R}^n) \mid \epsilon_{ij}(u) \in M_b(\Omega), i, j = 1, \ldots, n \},
\]

with the natural norms

\[
\|u\|_{LD(\Omega)} = \|u\|_{L^1} + \sum_{i,j=1}^n \|\epsilon_{ij}(u)\|_{L^1},
\]

\[
\|u\|_{BD(\Omega)} = \|u\|_{L^1} + \sum_{i,j=1}^n \|\epsilon_{ij}(u)\|_{M_b}.
\]

Moreover, \( \mathcal{R}_0 \equiv \{ u \in BD(\Omega) \mid \epsilon(u) = 0 \} \) denotes the space of rigid motions in \( \mathbb{R}^n \).

There exists a continuous surjective linear trace \( \gamma_B : [BD(\Omega), \| \cdot \|_{BD}] \to [L^1(\text{Fr } \Omega, \mathbb{R}^n), \| \cdot \|_{L^1}] \) such that \( \gamma_B(u) = u|_{\text{Fr } \Omega} \) for all \( u \in BD(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^n) \) (see [23]). We endow the spaces

\[
\text{(2.4) } X \equiv C_c(\Omega, \mathbb{R}^n) \times C_c(\Omega, E^n_s), \quad X_0 \equiv \{ (g, h) \in X \mid g = \text{div } h \},
\]

with the natural norm

\[
\|g\|_{C(\Omega, \mathbb{R}^n)} + \|h\|_{C(\Omega, E^n_s)} \equiv \sup\{ \|g(x)\|_{\mathbb{R}^n} \mid x \in \Omega \} + \sup\{ \|h(x)\|_{E^n_s} \mid x \in \Omega \}
\]

for \( g \in C(\Omega, \mathbb{R}^n) \) and \( h \in C(\Omega, E^n_s) \). Then \( BD(\Omega) \) is isomorphic to the dual of \( [(X/X_0), \| \cdot \|_{C(\Omega, \mathbb{R}^n)} + \| \cdot \|_{C(\Omega, E^n_s)}] \) (see [24]). The topology \( \sigma((X/X_0)^*, X) = \sigma(BD(\Omega), C_c(\Omega, \mathbb{R}^n) \times C_c(\Omega, E^n_s)) \) is called the \textit{weak* BD topology}. A net
\{u_\delta\}_{\delta \in D} \subset BD(\Omega) is convergent to \(u_0 \in BD(\Omega)\) in this topology if and only if
\begin{equation}
\int_\Omega g \cdot (u_0 - u_\delta) \, dx + \int_\Omega h : \varepsilon(u_0 - u_\delta) \to 0
\end{equation}
for all \((g, h) \in X\) (see [17, pp. 73–81]). For every \(\varphi \in L^1(Fr \Omega, \mathbb{R}^n)\), the set \(\{u \in BD(\Omega) \mid \gamma_B(u) = \varphi\}\) is dense in \([BD(\Omega), \text{weak}^* \text{ topology}]\) (see [6, Proposition 2.5]). Then the trace operator \(\gamma_B\) is not continuous on \([BD(\Omega), \text{weak}^* \text{ topology}]\) if the space \(L^1(Fr \Omega, \mathbb{R}^n)\) is endowed with a Hausdorff topology.

A net \(\{u_\delta\}_{\delta \in D}\) converges to \(u_0\) (in the topology (2.7)–(2.8)) if
\begin{equation}
u_\delta \to \nu_0 \quad \text{in} \quad \|\nu\|_{L^p(\Omega, \mathbb{R}^n)} \quad \forall p \text{ such that } 1 \leq p < q = n/(n-1)
\end{equation}
and weakly in \(L^q(\Omega, \mathbb{R}^n)\) \((q = \infty \text{ if } n = 1)\),
\begin{equation}
\varepsilon(u_\delta) \to \varepsilon(u_0) \quad \text{weak}^* \text{ in } M_b(\Omega, E^n_s).
\end{equation}

The weak* \(BD(\Omega)\) topology and the topology (2.7)–(2.8) are equivalent on bounded subsets of \(BD(\Omega)\) (cf. Proposition 2 of [8]). Therefore the injection of \([BD(\Omega), \text{weak}^* \text{ topology}]\) into \([L^p(\Omega, \mathbb{R}^n), \text{weak topology}]\) is continuous on bounded subsets of \(BD(\Omega)\), where \(1 \leq p \leq q = n/(n-1)\) \((q = \infty \text{ if } n = 1)\).

We define the Banach space of measurable functions
\begin{equation}
W^n(\Omega, \text{div}) \equiv \{\sigma \in L^\infty(\Omega, E^n_s) \mid \text{div } \sigma \in L^n(\Omega, \mathbb{R}^n)\}
\end{equation}
endowed with the natural norm \(\|\sigma\|_{W^n(\Omega, \text{div})} = \|\sigma\|_{L^\infty(\Omega, E^n_s)} + \|\text{div } \sigma\|_{L^n(\Omega, \mathbb{R}^n)}\) (cf. [23, Chap. 2, Sec. 7] and [6]). The distribution \(\sigma : \varepsilon(u)\), where \(\sigma \in W^n(\Omega, \text{div}), u \in BD(\Omega)\), defined (for \(\varphi_1 \in C^\infty_0(\Omega)\)) by
\begin{equation}
\langle \sigma : \varepsilon(u); \varphi_1 \rangle_{D' \times D} = -\int_\Omega (\text{div } \sigma) \cdot u \varphi_1 \, dx - \int_\Omega \sigma : (u \otimes \nabla \varphi_1) \, dx,
\end{equation}
is a bounded measure on \(\Omega\), absolutely continuous with respect to \(|\varepsilon(u)|\) (see [23]).

**Assumption 1.** \(\Omega\) and \(\Omega_1\) are bounded open connected sets of class \(C^1\) in \(\mathbb{R}^n \text{ with } \Omega \subset \subset \Omega_1\). 

**Theorem 1** (cf. [23]). There exists a continuous, linear, surjective, open map \(\beta_B\) from \([W^n(\Omega, \text{div}), \|\cdot\|_{W^n(\Omega, \text{div})}]\) onto \([L^\infty(Fr \Omega, \mathbb{R}^n), \|\cdot\|_{L^\infty}]\) such that \(\beta_B(\sigma) = \sigma|_{Fr \Omega} \cdot \nu\) for every \(\sigma \in C(\Omega, E^n_s)\), where \(\nu\) denotes the exterior unit vector normal to \(Fr \Omega\). Furthermore, for all \(u \in BD(\Omega)\) and all \(\sigma \in W^n(\Omega, \text{div})\), the following Green formula holds:
\begin{equation}
\int_\Omega \sigma : \varepsilon(u) + \int_\Omega (\text{div } \sigma) \cdot u \, dx = \int_{Fr \Omega} \beta_B(\sigma) \cdot \gamma_B(u) \, ds.
\end{equation}

**3. Auxiliary theorems and spaces.** In this paper, the Lebesgue and Hausdorff measures on \(\Omega\) and \(Fr \Omega\) are denoted by \(dx\) and \(ds\), respectively.
ASSUMPTION 2 (cf. [8] and [6]). $\mathcal{K} : \overline{\Omega} \to 2^{E^n_s}$ is a multifunction such that $\mathcal{K}(x)$ is a convex closed subset of $E^n_s$ for all $x \in \overline{\Omega}$; there exists $z_0 \in C^1(\overline{\Omega}, E^n_s)$ such that $z_0(x) \in \mathcal{K}(x)$ for every $x \in \overline{\Omega}$; and the following conditions hold:

(i) if $z(x) \in \mathcal{K}(x)$ for $dx$-almost every (dx-a.e.) $x \in \Omega$, $z \in C(\overline{\Omega}, E^n_s)$ and $z_{\text{int}\Omega} \in W^n(\Omega, \text{div})$, then $z(y) \in \mathcal{K}(y)$ for every $y \in \overline{\Omega}$;

(ii) for every $y \in \overline{\Omega}$ and every $w \in \mathcal{K}(y)$ there exists $z \in C(\overline{\Omega}, E^n_s)$ such that $z_{\text{int}\Omega} \in W^n(\Omega, \text{div})$, $z(y) = w$ and $z(x) \in \mathcal{K}(x)$ for every $x \in \overline{\Omega}$.

Conditions (i) and (ii) are equivalent to the condition that for every $y \in \overline{\Omega}$,

\[
(3.1) \quad \mathcal{K}(y) = \{ z(y) \mid z \in C(\overline{\Omega}, E^n_s), z_{\text{int}\Omega} \in W^n(\Omega, \text{div}), z(x) \in \mathcal{K}(x) \text{ for } dx\text{-a.e. } x \in \Omega \}. 
\]

DEFINITION 1. Let $j^* : \Omega \times E^n_s \to \mathbb{R} \cup \{ \infty \}$ be a convex normal integrand (cf. [16, Chapter 8, p. 232] and [8, Definition 2]) such that

\[
(3.2) \quad \{ w^* \in E^n_s \mid j^*(x, w^*) < \infty \} = \mathcal{K}(x) \quad \text{for } dx\text{-a.e. } x \in \Omega.
\]

ASSUMPTION 3. For every $\widehat{r} > 0$ there exists $c_{\widehat{r}}$ such that

\[
(3.3) \quad \sup \left\{ \int_{\Omega} j^*(x, z^*) \, dx \mid z^* \in L^\infty(\Omega, E^n_s), \| z^* \|_{L^\infty} < \widehat{r} \right\} < c_{\widehat{r}} < \infty.
\]

ASSUMPTION 4. There exist $u^c \in LD(\Omega)$ and $q \in L^1(\Omega, \mathbb{R})$ such that

\[
(3.4) \quad j(x, w) \equiv j^{**}(x, w) \equiv \sup \{ w : w^* - j^*(x, w^*) \mid w^* \in E^n_s \}
\]

for $dx$-a.e. $x \in \Omega$ and all $w \in E^n_s$. Then $j$ is a convex normal integrand. Define $j_\infty : \overline{\Omega} \times E^n_s \to \mathbb{R} \cup \{ \infty \}$ by

\[
(3.5) \quad j_\infty(x, w) \equiv \sup \{ w : w^* - I_{\mathcal{K}(x)}(w^*) \mid w^* \in E^n_s \}
\]

for $x \in \overline{\Omega}$ and $w \in E^n_s$.

ASSUMPTION 5. $I_0$, $I_1$ and $I_s$ are Borel subsets of Fr $\Omega$ such that $I_0 \cap I_1 = \emptyset$, $I_0 \cap I_s = \emptyset$, $I_s \cap I_1 = \emptyset$, $I_1 = \overline{I}_1$, $\text{ds}(\text{int} I_s) = \text{ds}(\overline{I}_s)$ and $\text{ds}(\text{Fr} \Omega \setminus (I_0 \cup I_1 \cup I_s)) = 0$. Moreover, $I_1 = \text{Fr} \Omega \cap \mathcal{C}$, where $\mathcal{C} = \text{cl} \text{int} \mathcal{C} \subset \Omega_1$ is a closed Caccioppoli set and $\text{ds}(\text{Fr} \Omega \cap \text{Fr} \mathcal{C}) = 0$ (see [19], [8, Definition 3]). Furthermore, $\gamma_B(u)|_{I_0} = 0$ on $I_0$ and an $(n-1)$-dimensional density force $g \in L^\infty(I_1, \mathbb{R}^n)$ is prescribed on $I_1$. Moreover, $\gamma_B(u)(x) \cdot v(x) \leq 0$ for $\text{ds}$-a.e. $x \in I_s$. This means that we have a potential function $h_s : I_s \times E^n_s \to \mathbb{R} \cup \{ \infty \}$, defined by
\[ h_s(x, y) = \begin{cases} \infty & \text{if } \sum_{i=1}^n y_{ii} < 0, \\ 0 & \text{otherwise}, \end{cases} \]

where \( y = -\gamma_B(u) \otimes_s \nu. \)

The displacement formulation of the equilibrium problem for the elastic-perfectly plastic body made of a Hencky material with the Signorini contact condition, reads (see [6]):

\[ (SN_{\lambda,j}) \quad \text{Find } \inf \{ \lambda F_1(u) + G_j(\varepsilon(u)) \mid u \in LD(\Omega) \}, \]

where

\begin{align*}
\lambda F_1(u) & \equiv -\lambda \left( \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx + \int_{\Gamma_1} \mathbf{g} \cdot \gamma_B(u) \, ds \right) \\
& \quad + \int_{\Gamma_0} 1_{\{\gamma_B(u)(x) = 0\}} (-\gamma_B(u) \otimes_s \nu) \, ds + \int_{\Gamma_s} h_s(x, -\gamma_B(u) \otimes_s \nu) \, ds
\end{align*}

for \( u \in BD(\Omega), \) and the functional \( G_j(\varepsilon(\cdot)) : BD(\Omega) \to \mathbb{R} \cup \{ \infty \} \) is given by

\[ G_j(\varepsilon(u)) \equiv \begin{cases} \int_{\Omega} j(x, \varepsilon(u)) \, dx & \text{if } u \in LD(\Omega), \\ \infty & \text{if } u \in BD(\Omega) - LD(\Omega). \end{cases} \]

The quantity in (3.7) describes the total elastic-perfectly plastic energy of a body occupying the given subset \( \Omega \) of \( \mathbb{R}^n. \) This body is subjected to volume forces \( \mathbf{f} \in L^n(\Omega, \mathbb{R}^n) \) and boundary forces \( \mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n). \) The constant \( \lambda \geq 0, \lambda < \infty, \) is the load multiplier (see [23, Chap. 1, Sec. 4]). The body is clamped on \( \Gamma_0. \) We assume the Signorini contact condition on \( \Gamma_s. \)

Similarly, we define the Suquet unilateral contact problem (cf. [22]):

\[ (SS_{\lambda,j}) \quad \text{Find } \inf \{ \lambda F_2(u, \mu) + G_j(\varepsilon(u)) \mid (u, \mu) \in LD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n) \}, \]

where

\begin{align*}
\lambda F_2(u, \mu) & \equiv -\lambda \left( \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx + \int_{\Gamma_1} \mathbf{g} \cdot \mu \right) \\
& \quad + \int_{\Gamma_0} 1_{\{\gamma_B(u)(x) = 0\}} (-\gamma_B(u) \otimes_s \nu) \, ds + \int_{\Gamma_s} h_s(x, -\gamma_B(u) \otimes_s \nu) \, ds
\end{align*}

if \( u \in LD(\Omega), \mu = \gamma_B(u) \, ds \) for \( |\mu - \gamma_B(u) \, ds| \)-a.e. \( x \in \Gamma_1, \) and \( \lambda F_2(u, \mu) \equiv \infty \) otherwise. Unfortunately, \((SN_{\lambda,j})\) and \((SS_{\lambda,j})\) are not l.s.c. in the weak* \( BD(\Omega) \) and weak* \( BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n) \) topologies, respectively (cf. [9]). Below we obtain explicitly the relaxed form of \((SS_{\lambda,j})\) and show the relaxed form of \((SN_{\lambda,j})\).

Let \( \mu \in M_b(\Omega, E^*_s). \) We recall that \( |\mu| \) is the total variation measure associated with \( \mu. \) The density of \( \mu \) with respect to \( |\mu| \) will be denoted by \( d\mu/d|\mu|. \) Let \( \mu = \mu_a(x) \, dx + \mu_s \) be the Lebesgue decomposition of \( \mu \) into the absolutely continuous and singular parts with respect to \( dx. \)
We consider the spaces $Y^1(\overline{\Omega})$ and $C_{\text{div}}(\overline{\Omega}, E^n_a)$, given by
\begin{equation}
Y^1(\overline{\Omega}) \equiv \{ M \in M_b(\overline{\Omega}, E^n_a) \mid \exists u_1 \in BD(\Omega_1), \\
\varepsilon(u_1)|_{\overline{\Omega}} = M, \ u_1|_{\Omega_1 - \overline{\Omega}} = 0 \}, \quad (3.12)
\end{equation}
\begin{equation}
C_{\text{div}}(\overline{\Omega}, E^n_a) \equiv \{ \sigma \in C(\overline{\Omega}, E^n_a) \mid \sigma|_{\Omega} \in W^n(\Omega, \text{div}) \}. \quad (3.13)
\end{equation}
These are topological vector spaces placed in duality by the bilinear pairing
\begin{equation}
\langle M; \sigma \rangle_{Y^1 \times C(\overline{\Omega}, E^n_a)} = \int_{\overline{\Omega}} \sigma : M = \sum_{i,j=1}^{n} \int_{\overline{\Omega}} \sigma_{ij} M^{ij}. \quad (3.14)
\end{equation}
We say that a sequence $\{M_m\}_{m \in \mathbb{N}} \subset Y^1(\overline{\Omega})$ converges to $M_0$ in the topology $\sigma(Y^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, E^n_a))$ if $\langle (M_m - M_0); \sigma \rangle_{Y^1 \times C(\overline{\Omega}, E^n_a)} \to 0$ for every $\sigma \in C_{\text{div}}(\overline{\Omega}, E^n_a)$ as $m \to \infty$. By [14, Theorem V.3.9] we have $[Y^1(\overline{\Omega}), \sigma(Y^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, E^n_a))]^* = C_{\text{div}}(\overline{\Omega}, E^n_a)$.

Assumption 6. There exists $\sigma_0 \in C_{\text{div}}(\overline{\Omega}, E^n_a)$ for which $\beta_B(\sigma_0) = \lambda g$ on $\Gamma_1$ and $\sigma_0(x) \in K(x)$ for $dx$-a.e. $x \in \Omega$. ■

The space $BD(\Omega)$ is isomorphic to $A_{BD} \equiv \{ u \in BD(\Omega_1) \mid u|_{\Omega_1 - \overline{\Omega}} = 0 \}$ (cf. Assumption 1). Moreover, $A_{BD}$ is isomorphic to $Y^1(\overline{\Omega})$, and the isomorphism is given by $A_{BD} \ni u \mapsto \varepsilon(u)|_{\overline{\Omega}} \in Y^1(\overline{\Omega})$. The Banach spaces $[BD(\Omega), \| \cdot \|_{BD}]$ and $[Y^1(\overline{\Omega}), \| \cdot \|_{M_b(\overline{\Omega})}]$ are isomorphic (cf. [6, Proposition 4.24]). Each closed ball $cl_{\| \cdot \|}(B(0, r))$ (in $Y^1$) is compact in the topology $\sigma(Y^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, E^n_a))$, where $cl_{\| \cdot \|}$ denotes the closure in the norm of $BD(\Omega)$ (see [6, Proposition 4.23]). The space $[cl_{\| \cdot \|_{BD}}(B_{BD}(0, r)), \text{weak}^* BD(\Omega)$ topology] is isomorphic to $[cl_{\| \cdot \|_{BD}}(B_{BD}(0, r)), \sigma(Y^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, E^n_a))]$ (cf. [6, Proposition 4.25]). Let
\begin{equation}
M^1(\Gamma_1) \equiv \{ \mu \otimes s \nu \in M_b(\Gamma_1, E^n_s) \mid \mu \in M_b(\Gamma_1, \mathbb{R}^n) \}. \quad (3.15)
\end{equation}
Consider the vector spaces
\begin{equation}
[C(\Gamma_1, E^n_s), \sigma(C(\Gamma_1, E^n_s)), M^1(\Gamma_1)], \quad (3.16)
\end{equation}
where $cl_{\| \cdot \|}$ denotes the closure in the norm of $BD(\Omega)$ (see [23]).

The inside (resp. outside) trace of $BD(\Omega)$ into $L^1(\text{Fr} \Omega, \mathbb{R}^n)$ (resp. $BD(\Omega_1 - \overline{\Omega})$ into $L^1(\text{Fr} \Omega, \mathbb{R}^n)$) is denoted by $\gamma^I_B$ (resp. $\gamma^O_B$) (see [23]).

The functional $S^I_B : Y^1(\overline{\Omega}) \times M^1(\Gamma_1) \to \mathbb{R} \cup \{ \infty \}$ is defined by
\begin{equation}
S^I_B(\varepsilon(u)|_{\overline{\Omega}}, \mu \otimes s \nu) \equiv - \int_{\Gamma_1} \lambda g \cdot \mu + \int_{\Omega} j(x, \varepsilon(u)_a) \, dx \\
+ \int_{\Gamma} I_{\{ (\gamma^I_B(\mu \otimes s \nu)(x) = 0) \}} (\gamma^I_B(\mu) \otimes s \nu) \, ds + \int_{\Gamma} h_s(x, -\gamma^I_B(\mu) \otimes s \nu) \, ds \quad (3.17)
\end{equation}
if $u|_{\Omega} \in LD(\Omega)$ and $\mu = \gamma_{B}^{I}(u)\,ds$ for $|\mu - \gamma_{B}^{I}(u)\,ds|$-a.e. $x \in \Gamma_1$, and $S_{\lambda}^{j}(\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu) = \infty$ otherwise, where $\varepsilon(u)|_{\Gamma} \in Y^{1}(\overline{\Omega})$ and $\mu \in M_{b}(\Gamma_1, \mathbb{R}^{n})$ (cf. [6]). We assume that there exist $\tilde{u} \in BD(\Omega)$ and $\tilde{\mu} \in M_{b}(\Gamma_1, \mathbb{R}^{n})$ such that $\tilde{u}|_{\Omega} \in LD(\Omega)$ and $S_{\lambda}^{j}(\varepsilon(\tilde{u})|_{\Gamma}, \tilde{\mu} \otimes s, \nu) < \infty$.

4. Lower semicontinuous regularization. In this section the l.s.c. regularization of the functional $S_{\lambda}^{j}$ is found, where the spaces $BD(\Omega)$ and $M^{1}(\Gamma_1)$ are endowed with the weak* $BD$ topology and $\sigma(M^{1}(\Gamma_1), C(\Gamma_1, E_{s}^{n}))$ topology, respectively. We apply this result to the problem of relaxation in Hencky plasticity.

The bilinear form between $Y^{1}(\overline{\Omega}) \times M^{1}(\Gamma_1)$ and $C_{\text{div}}(\overline{\Omega}, E_{s}^{n}) \times C(\Gamma_1, E_{s}^{n})$ is given by

$$
\langle (\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu); (\sigma, \kappa) \rangle_{2} \equiv \int_{\Omega} \sigma : \varepsilon(u)|_{\Omega} + \int_{\Gamma_1} \sigma : (-\gamma_{B}^{I}(u) \otimes s, \nu)\,ds + \int_{\Gamma_1} \kappa : [\mu \otimes s, \nu]
$$

for $\varepsilon(u)|_{\Gamma} = (\varepsilon(u)|_{\Omega}, (-\gamma_{B}^{I}(u)\,ds \otimes s, \nu)) \in Y^{1}(\overline{\Omega})$, $\mu \otimes s, \nu \in M^{1}(\Gamma_1)$, $\sigma \in C_{\text{div}}(\overline{\Omega}, E_{s}^{n})$ and $\kappa \in C(\Gamma_1, E_{s}^{n})$ (cf. Proposition 6 of [7]). Because of the duality between $Y^{1}(\overline{\Omega}) \times M^{1}(\Gamma_1)$ and $C_{\text{div}}(\overline{\Omega}, E_{s}^{n}) \times C(\Gamma_1, E_{s}^{n})$ we define a functional $(S_{\lambda}^{j})^{*} : C_{\text{div}}(\overline{\Omega}, E_{s}^{n}) \times C(\Gamma_1, E_{s}^{n}) \to \mathbb{R} \cup \{\infty\}$ by

$$
(S_{\lambda}^{j})^{*}(\sigma, \kappa) \equiv \sup \{ \langle (\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu); (\sigma, \kappa) \rangle_{2} - S_{\lambda}^{j}(\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu) : (\varepsilon(u)|_{\Gamma} \in Y^{1}(\overline{\Omega}), \mu \otimes s, \nu \in M^{1}(\Gamma_1) \}
$$

for $\sigma \in C_{\text{div}}(\overline{\Omega}, E_{s}^{n})$ and $\kappa \in C(\Gamma_1, E_{s}^{n})$. The bidual functional $(S_{\lambda}^{j})^{**} : Y^{1}(\overline{\Omega}) \times M^{1}(\Gamma_1) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$
(S_{\lambda}^{j})^{**}(\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu) = \sup \{ \langle (\varepsilon(u)|_{\Gamma}, \mu \otimes s, \nu); (\sigma, \kappa) \rangle_{2} - (S_{\lambda}^{j})^{*}(\sigma, \kappa) : (\sigma, \kappa) \in C_{\text{div}}(\overline{\Omega}, E_{s}^{n}), \kappa \in C(\Gamma_1, E_{s}^{n}) \}
$$

for $\varepsilon(u)|_{\Gamma} \in Y^{1}(\overline{\Omega})$ and $\mu \in M_{b}(\Gamma_1, \mathbb{R}^{n})$.

The bilinear form between $M_{b}(\Omega, E_{s}^{n}) \times Y^{1}(\overline{\Omega})|_{\Gamma_{1}} \times M^{1}(\Gamma_1)$ and $C_{\text{div}}(\overline{\Omega}, E_{s}^{n}) \times C(\Gamma_1, E_{s}^{n})$ is

$$
\langle (w, -\gamma_{B}^{I}(u) \otimes s, \nu, \mu \otimes s, \nu); (\sigma, \kappa) \rangle_{3} \equiv \int_{\Omega} \sigma : w + \int_{\Gamma_1} \sigma : (-\gamma_{B}^{I}(u) \otimes s, \nu)\,ds + \int_{\Gamma_1} \kappa : [u \otimes s, \nu]
$$

for $w \in M_{b}(\Omega, E_{s}^{n})$, $\gamma_{B}^{I}(u)\,ds \otimes s, \nu \in Y^{1}(\overline{\Omega})|_{\Gamma_{1}}$, $\mu \otimes s, \nu \in M^{1}(\Gamma_1)$, $\sigma \in C_{\text{div}}(\overline{\Omega}, E_{s}^{n})$ and $\kappa \in C(\Gamma_1, E_{s}^{n})$. The extension of $S_{\lambda}^{j}$ onto $M_{b}(\Omega, E_{s}^{n}) \times Y^{1}(\overline{\Omega})|_{\Gamma_{1}} \times M^{1}(\Gamma_1)$ (denoted by $\tilde{S}_{\lambda}^{j}$) is given by
(4.5) \[ \tilde{S}^j_\lambda(w, -\gamma^j_B(u)) ds \otimes_s \nu, \mu \otimes_s \nu) \equiv -\int_{\Gamma_1} \lambda g \cdot \mu + \int_{\Gamma_1} j(x, w) \, dx \]
\[ + \int_{\Gamma_0} I_{\{\gamma^j_B(u) \otimes_s \nu = 0\}} (\gamma^j_B(u) \otimes_s \nu) ds + \int_{\Gamma_s} h_s(x, -\gamma^j_B(u) \otimes_s \nu) \, ds \]
if \( w \in L^1(\Omega, E^n_s) \) and \( \mu = \gamma^j_B(u) \) ds for \( |\mu - \gamma^j_B(u) \) ds\)-a.e. \( x \in \Gamma_1 \), and \( \tilde{S}^j_\lambda(w, -\gamma^j_B(u) \otimes_s \nu, \mu \otimes_s \nu) \equiv \infty \) otherwise.

By duality between \( M_b(\Omega, E^n_s) \times Y^1(\Omega)_{BR} \times M^1(\Gamma_1) \) and \( C_{div}(\tilde{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s) \), we define the functional \( (\tilde{S}^j_\lambda)^* : C_{div}(\tilde{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s) \to \mathbb{R} \cup \{\infty\} \). It is given by

(4.6) \[ (\tilde{S}^j_\lambda)^*(\sigma, \kappa) \equiv \sup\{(w, -\gamma^j_B(u) \otimes_s \nu, \mu \otimes_s \nu); (\sigma, \kappa)\} + \tilde{S}^j_\lambda(w, -\gamma^j_B(u)) ds \otimes_s \nu, \mu \otimes_s \nu) \mid w \in L^1(\Omega, E^n_s), u \in BD(\Omega) \]
and \( \mu \otimes_s \nu \in M^1(\Gamma_1) \}

for \( \sigma \in C_{div}(\tilde{\Omega}, E^n_s) \) and \( \kappa \in C(\Gamma_1, E^n_s) \). The bidual functional \( (\tilde{S}^j_\lambda)^{**} : Y^1(\tilde{\Omega}) \times M^1(\Gamma_1) \to \mathbb{R} \cup \{\infty\} \) is defined by

(4.7) \[ (\tilde{S}^j_\lambda)^{**}(w, -\gamma^j_B(u) ds \otimes_s \nu, \mu \otimes_s \nu) \equiv \sup\{(w, -\gamma^j_B(u) \otimes_s \nu, \mu \otimes_s \nu); (\sigma, \kappa)\} - (\tilde{S}^j_\lambda)^*(\sigma, \kappa) \mid \sigma \in C_{div}(\tilde{\Omega}, E^n_s), \kappa \in C(\Gamma_1, E^n_s) \}
for \( (w, -\gamma^j_B(u) ds \otimes_s \nu) \in Y^1(\tilde{\Omega}) \) and \( \mu \otimes_s \nu \in M^1(\Gamma_1) \).

**Lemma 2** (see [6] and [8, Lemma 6]). For every \( \sigma \in C_{div}(\tilde{\Omega}, E^n_s) \) and \( \kappa \in C(\Gamma_1, E^n_s) \) we have \( (\tilde{S}^j_\lambda)^*(\sigma, \kappa) \geq (S^j_\lambda)^*(\sigma, \kappa) \). Moreover, \( (\tilde{S}^j_\lambda)^{**}(M, \mu \otimes_s \nu) \leq (S^j_\lambda)^{**}(M, \mu \otimes_s \nu) \) for every \( M \in Y^1(\tilde{\Omega}) \) and \( \mu \in M_b(\Gamma_1, \mathbb{R}^n) \).

**Definition 2** (cf. [13]). A subset \( H_0 \) of \( \mathcal{L}^0(\tilde{\Omega}, \mathbb{R}^m) \) is said to be PCU-stable if for any continuous partition of unity \( (\alpha_0, \ldots, \alpha_d) \) with \( \alpha_0, \ldots, \alpha_d \in C^\infty(\tilde{\Omega}, \mathbb{R}) \), and any \( z_0, \ldots, z_d \in H_0 \), the sum \( \sum_{i=0}^d \alpha_i z_i \) is in \( H_0 \).

**Proposition 3.** The functional \( (\tilde{S}^j_\lambda)^{**} \) defined by (4.5)–(4.7) satisfies

(4.8) \[ (\tilde{S}^j_\lambda)^{**}(\varepsilon(u)|_{\Omega}, -\gamma^j_B(u) ds \otimes_s \nu, \mu \otimes_s \nu) = -\int_{\Gamma_1} \lambda g \cdot \mu \]
\[ + \int_{\Gamma_0} j(x, -\gamma^j_B(u) \otimes_s \nu) ds + \int_{\Gamma_s} (h^*_s + I_K)^*(x, -\gamma^j_B(u) \otimes_s \nu) \, ds \]
\[ + \int_{\Gamma_1} j(x, d((\mu - \gamma^j_B(u) ds \otimes_s \nu) \otimes_s \nu) \, dx + \int_{\Omega} j(x, (\mu - \gamma^j_B(u) ds \otimes_s \nu) \otimes_s \nu) d|\mu - \gamma^j_B(u) ds \otimes_s \nu| \]
\[ + \int_{\Omega} j(x, d\varepsilon(u)_s) dx + \int_{\Omega} j(x, d\varepsilon(u)_s/d\varepsilon(u)_s) |d\varepsilon(u)_s| \]
for every \((e(u)|_{\Omega}, -\gamma_B^j(u) \, ds \otimes_s \nu) \in Y^1(\Omega)\) and \(\mu \otimes_s \nu \in M^1(\Gamma_1)\), where
\[
(4.9) \quad h^*_s(x, \sigma) = \begin{cases} 
0 & \text{if } \sigma(x) \in \mathcal{A}(x), \\
\infty & \text{otherwise},
\end{cases}
\]
and
\[
(4.10) \quad \mathcal{A}(x) = \left\{ \sigma \in E^n_s \mid \sum_{i,j=1}^n \sigma_{ij}(\gamma_B^j(u) \otimes_s \nu)_{ij} \geq 0 \text{ for all } u \text{ such that} \right. \\
\left. \sum_{i=1}^n (\gamma_B^i(u) \otimes_s \nu)_{ii} = (\gamma_B^i(u) \cdot \nu) \leq 0 \right\} \quad \text{for } ds\text{-a.e. } x \in \Gamma_s.
\]

**Proof.** Let \(p \equiv \mu - \gamma_B^j(u) \, ds \) on \(\Gamma_1\). By Assumption 6 we have \(\beta_B(\sigma_0) = 2^j g\) on \(\Gamma_1\). Hence \(\int_{\Gamma_1} \lambda g \cdot \mu = \int_{\Gamma_1} \sigma_0 : (\mu \otimes_s \nu)\) for every \(\mu \in M_b(\Gamma_1, \mathbb{R}^n)\). Note that \(\gamma_B^j(LD(\Omega)) = L^1(\Gamma_1, \mathbb{R}^n)\) (cf. [23, Chapter 2, Theorem 1.1]). By [21, Theorem 3A], for every \(\sigma \in C_{div}(\Omega, E^n_s)\) and \(\kappa \in C(\Gamma_1, E^n_s)\) we get
\[
(4.11) \quad (\tilde{S}^j_\lambda)^* (\sigma, \kappa) \equiv \sup \left\{ \int_{\Gamma_1} (\kappa - \sigma) : (\mu \otimes_s \nu) + \int_{\Gamma_1} \sigma : (p \otimes_s \nu) \\
+ \int_{\Gamma_0} \sigma_0 : (\mu \otimes_s \nu) - \int_{\Gamma_1} I_{\{d(p \otimes_s \nu) = 0\}} \left( \frac{d(p \otimes_s \nu)}{|d(p \otimes_s \nu)|} \right) d|p \otimes_s \nu| \\
+ \int_{\Gamma_0} \sigma : (-\gamma_B^j(u) \otimes_s \nu) \, ds - \int_{\Gamma_0} I_{\{\gamma_B^j(u) = 0\}} (-\gamma_B^j(u) \otimes_s \nu) \, ds \\
+ \int_{\Gamma_0} \sigma : (-\gamma_B^j(u) \otimes_s \nu) \, ds - \int_{\Gamma_0} h_s(x, -\gamma_B^j(u) \otimes_s \nu) \, ds \\
+ \int_{\Omega} \sigma : w \, dx - \int_{\Omega} j(x, w) \, dx \mid u \in BD(\Omega_1) \text{ such that } u|_{\Omega} \in LD(\Omega), \right. \\
\left. u|_{\Omega_1 - \Omega} = 0, w \in L^1(\Omega, E^n_s) \text{ and } \mu \in M_b(\Gamma_1, \mathbb{R}^n) \right\} \\
= \int_{\Omega} j^*(x, \sigma) \, dx + \sup \left\{ \int_{\Gamma_1} (\kappa - \sigma) \cdot \nu + \int_{\Gamma_1} (\sigma_0 \cdot \nu) \cdot \mu \right. \\
+ \int_{\Gamma_1} \sigma : (p \otimes_s \nu) \mid \mu \in M_b(\Gamma_1, \mathbb{R}^n) \text{ and } p \otimes_s \nu = 0 \right. \\
\left. + \int_{\Gamma_1} h^*_s(x, \sigma) \, ds \right\} \\
= \int_{\Omega} j^*(x, \sigma) \, dx + \int_{\Gamma_1} I_{\{(\kappa - \sigma) \cdot \nu(x) = -\lambda g(x)\}} (\kappa - \sigma) \, ds + \int_{\Gamma_1} h^*_s(x, \sigma) \, ds,
\]
because \(\sigma_0 \cdot \nu = 2^j g\) on \(\Gamma_1\) (see also [8, (4.9)] and [7, (3.18)]).

By the duality between \(Y^1(\Omega) \times M^1(\Gamma_1)\) and \(C_{div}(\Omega, E^n_s) \times C(\Gamma_1, E^n_s)\) we obtain the functional \((\tilde{S}^j_\lambda)^{**}\). The spaces \(C_{div}(\Omega, E^n_s)\) and \(C(\Gamma_1, E^n_s)\) are
PCU-stable, so by [13, Theorem 1] and the proof of Theorem 4 of [13] we get

\[
(4.12) \quad (\ldots)^* \varepsilon(u)_{\mathcal{H}, \mu \otimes \nu} = \sup \left\{ \int_{\Omega} \sigma : \varepsilon(u)_{\Omega} - \int_{\Omega} j^*(x, \sigma) \, dx \right. \\
+ \int_{\Gamma_1} (|\kappa - \sigma| \cdot \nu) \cdot \mu - \int \rho \{((\kappa - \sigma) \cdot \nu)(x) = -\lambda g(x)\} \left((\kappa - \sigma) \cdot \nu\right) \, ds \\
+ \int_{\Gamma_1} \sigma : [((\mu - \gamma_B^j(u) \, ds) \otimes s \, \nu] - \int_{\Gamma_1} j^*_\infty(x, \sigma) \, ds |(\mu - \gamma_B^j(u) \, ds) \otimes s \, \nu| \\
+ \int_{\Gamma_0} \sigma : (-\gamma_B^j(u) \otimes s \, \nu) \, ds - \int_{\Gamma_0} j^*_\infty(x, \sigma) \, ds \\
+ \int_{\Gamma_s} \sigma : (-\gamma_B^j(u) \otimes s \, \nu) \, ds - \int_{\Gamma_s} (h^*_s + j^*_\infty)(x, \sigma) \, ds \left| \sigma \in C_{\text{div}}(\bar{\Omega}^n, E^s_n), \kappa \in C(\Gamma_1, E^s_n) \right. \\
= \sup \left\{ \int_{\Omega} [\sigma : (\varepsilon(u)_a) - j^*(x, \sigma)] \, dx \\
+ \int_{\Omega} [\sigma : (d\varepsilon(u)_s/d|\varepsilon(u)_s|) - j^*_\infty(x, \sigma)] \, d|\varepsilon(u)_s| \\
- \int \lambda g \cdot \mu + \int \rho \{((\mu \otimes s \, \nu) = d|p \otimes s \, \nu|) - j^*_\infty(x, \sigma)] \, d|p \otimes s \, \nu| \\
+ \int_{\Gamma_s} \sigma : (-\gamma_B^j(u) \otimes s \, \nu) - (h^*_s + j^*_\infty)(x, \sigma) \, ds \\
+ \int_{\Gamma_0} \sigma : (-\gamma_B^j(u) \otimes s \, \nu) - j^*_\infty(x, \sigma) \, ds \left| \sigma \in C_{\text{div}}(\bar{\Omega}, \bar{E}_n^s) \right. \\
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\text{for every } \varepsilon(u)_{\mathcal{H}} = (\varepsilon(u)_{\mathcal{H}, \mu \otimes \nu}, -\gamma_B^j(u) \, ds \otimes s \, \nu) \in Y^1(\bar{\Omega}) \text{ and } (\mu \otimes s \, \nu) \in M_1(\Gamma_1) \text{ (where } p \equiv \mu - \gamma_B^j(u) \, ds \text{ on } \Gamma_1), \text{ which is (4.8). In the above calculations we use the equality } j^*_\infty(x, \sigma) = I_{K(x)}(\sigma) = I_{K(x)}(x, \sigma), \text{ which holds for every } \sigma \in E^s_n \text{ and } x \in \bar{\Omega}. \text{ Moreover, by (3.2) we can assume that } \sigma(x) \in K(x) \text{ for every } x \in \bar{\Omega}. \text{ Thus the proof is complete.} \quad \blacksquare

Let \( w \in E_n^s \). Then \( w_{ij}^D \equiv w_{ij} - n^{-1} \delta_{ij} \tr w \), where \( \delta_{ij} \) is the Kronecker delta and \( w^D \) denotes the matrix \( [w_{ij}^D] \).

\text{Assumption 7 (cf. [10]). There exist functionals } j^D : \Omega \times (E^s_n)^D \rightarrow \mathbb{R} \cup \{\infty\} \text{ and } j^{tr} : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\} \text{ such that there exist } k_0 > 0 \text{ and } k_1 > 0 \text{ such that for every } z \in \mathbb{R} \text{ and } dx\text{-a.e. } x \in \Omega, j^{tr}(x, z) \geq k_1 z^2 - k_0, \text{ and the normal integrand } j^*( \text{given by Definition 1 and Assumption 3) satisfies}
for every $\varepsilon \in \mathbb{E}_s^n$ and $dx$-a.e. $x \in \Omega$. ■

We say that a net $\{(u_t, \mu_t)\}_{t \in T} \subset BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ converges to $(u_0, \mu_0)$ in the weak* $BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ topology if $u_t \rightharpoonup u_0$ in the weak* $BD(\Omega)$ topology and $\mu_t \rightharpoonup \mu_0$ in the weak* $\mathbb{M}_b(I_1, \mathbb{R}^n)$ topology.

Now we prove that $(\mathcal{S}_h^j)^{**}$ is the l.s.c. regularization of $\mathcal{S}_h^j$ in the weak* $BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ topology, for the elastic-perfectly plastic potential $j$ given by (4.13). Recall that the space $U(\Omega)$ is given by

\begin{equation}
U(\Omega) \equiv \{u \in BD(\Omega) \mid \text{div } u \in L^2(\Omega)\}. 
\end{equation}

**Proposition 4** (cf. [10]). The functionals $\mathcal{S}_h^j, \mathcal{S}_h^{j^*}$ and $(\mathcal{S}_h^j)^{**}$ are equal to $\infty$ for every $(\varepsilon(u)|_{\Omega}, -\gamma_B(u) ds \otimes_s \nu) \in Y^1(\Omega)$ and $\mu \otimes_s \nu \in M^1(I_1)$ such that $u|_{\Omega} \in BD(\Omega) - U(\Omega)$.

**Proof.** By (4.13) we get

\begin{equation}
j^*(x, w^*) = \sup\{w : w^* - j^D(x, w^D) - j^{tr}(x,\text{tr } w) \mid w \in \mathbb{E}_s^n \}
= \sup\{w^D : (w^*)^D + n^{-1}(\text{tr } w)(\text{tr } w^*)
- j^D(x, w^D) - j^{tr}(x,\text{tr } w) \mid w \in \mathbb{E}_s^n \}
= \sup\{w^D : (w^*)^D - j^D(x, w^D) \mid w \in \mathbb{E}_s^n \}
+ \sup\{z(\text{tr } w^*)^n - j^{tr}(x, z) \mid z \in \mathbb{R} \}
= (j^D)^*(x, (w^*)^D) + (j^{tr})^*(x,\text{tr } w^*)n^{-1}.
\end{equation}

Then by Assumption 7 there exist $k_2 > 0$ and $k_3 > 0$ such that for every $z \in \mathbb{R}$ and for $dx$-a.e. $x \in \Omega$, $(j^{tr})^*(x, z) \leq k_2 z^2 + k_3$. Thus by (3.2) there exists a multifunction $K^D$ such that

\begin{equation}
K(x) = K^D(x) \oplus \{zI \in \mathbb{E}_s^n \mid I_{ij} = \delta_{ij} \text{ and } z \in \mathbb{R}\},
\end{equation}

where $K^D : \Omega \rightarrow 2(\mathbb{E}_s^n)^D$ is defined for every $x \in \Omega$ (see (3.1)). By (3.5), we have $j_\infty(x, w) = \infty$ for every $x \in \Omega$ and $w \in \mathbb{E}_s^n$ such that $\text{tr } w \neq 0$. By (3.17), (4.5) and (4.8) we get $S_h^j(\varepsilon(u)|_{\Omega}, \mu \otimes_s \nu) = \mathcal{S}_h^j(\varepsilon(u)|_{\Omega}, -\gamma_B(u) ds \otimes_s \nu, \mu \otimes_s \nu) = (\mathcal{S}_h^j)^{**}(\varepsilon(u)|_{\Omega}, -\gamma_B(u) ds \otimes_s \nu, \mu \otimes_s \nu) = \infty$ for every $(\varepsilon(u)|_{\Omega}, \mu \otimes_s \nu) \in Y^1(\Omega) \times M^1(I_1)$ such that $\text{tr } \varepsilon(u)|_{\Omega} = \text{div } (u|_{\Omega}) \in M_b(\Omega, \mathbb{R}) - L^1(\Omega, \mathbb{R})$. ■

**Corollary 5.** For every $u \in U(\Omega)$ and for $ds$-a.e. $x \in \text{Fr } \Omega$,

\begin{equation}
j_\infty(x, (-\gamma_B(u) \otimes_s \nu)) = \infty \quad \text{if } -\gamma_B(u) \cdot \nu \neq 0
\end{equation}
on $\Gamma_0$, and
\begin{equation}
(h^*_s + I_\kappa)^*(x, -\gamma_B(u) \otimes s) = h^*_s(x, -\gamma_B(u) \otimes s)
= h_s(x, -\gamma_B(u) \otimes s)
\end{equation}
on \Gamma_s \text{ (cf. Remark 2).}\]

Now we turn to the direct method of regularization. The study of the difference between \((S^j_\lambda)^*\) and \((\tilde{S}^j_\lambda)^*\) plays an essential role in this section.

**Lemma 6.** For every \(u \in BD(\Omega_1)\) and \(\mu \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)\) such that \(u|_{\Omega_1} \in LD(\Omega), u|_{\Omega_1 - \overline{\Omega}} = 0, \gamma^l_B(u) = 0\) on \(\Gamma_0\) and \(\gamma^l_B(u) ds = \mu\) on \(\Gamma\) we have

\begin{equation}
(S^j_\lambda)^*(\varepsilon(u)|_{\overline{\Omega}}, -\gamma^l_B(u) ds \otimes s, \mu \otimes s) = (S^j_\lambda)^*(\varepsilon(u)|_{\overline{\Omega}}, \mu \otimes s).
\end{equation}

**Proof.** By Proposition 3 and by (4.18), we obtain (4.19) as in \([8, Lemma 8]\) (see also \([9, Lemma 1]\)).

**Lemma 7.** Let \(\kappa_s \in C(\Gamma_1, E^n_s), \sigma_s \in C_{\text{div}}(\overline{\Omega}, E^n_s)\), where \(\text{div} \sigma_s = 0\) on \(\Omega\) and \(\int_{\Gamma_1} \kappa_s : [t \otimes s] ds = 0\) for every \(t \in L^1(\Gamma_1, \mathbb{R}^n)\). Then

\begin{equation}
(S^j_\lambda)^*(\sigma, \kappa) = (S^j_\lambda)^*(\sigma + \sigma_s, \kappa + \kappa_s)
\end{equation}

for every \(\sigma \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) and \(\kappa \in C(\Gamma_1, E^n_s)\).

**Proof.** By (4.2) and the Green formula (2.11) we have

\begin{equation}
(S^j_\lambda)^*(\sigma, \kappa)
= \sup \left\{ -\int_{\Omega} (\text{div} \sigma) \cdot u dx + \int_{\Gamma_1} \kappa : (\mu \otimes s) + S^j_\lambda(\varepsilon(u)|_{\overline{\Omega}}, \mu \otimes s) \bigg|_{u|_{\Gamma_1} \in LD(\Omega), u|_{\Omega_1 - \overline{\Omega}} = 0, \mu \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)} \right\}
= \sup \left\{ -\int_{\Omega} [\text{div}(\sigma + \sigma_s)] \cdot u dx + \int_{\Gamma_1} (\kappa + \kappa_s) : (\gamma_B(u) \otimes s) ds \bigg|_{u|_{\Gamma_1} \in LD(\Omega), u|_{\Omega_1 - \overline{\Omega}} = 0} - S^j_\lambda(\varepsilon(u)|_{\overline{\Omega}}, \gamma_B(u) ds \otimes s) \right\} = (S^j_\lambda)^*(\sigma + \sigma_s, \kappa + \kappa_s)
\end{equation}

(cf. \([7, Lemma 18]\) and (3.17)). Moreover, the trace \(\gamma_B\) is a surjection from \(LD(\Omega)\) onto \(L^1(\text{Fr} \Omega, \mathbb{R}^n)\) (see Theorem 1.1 of \([23, Chapter 2]\)).

We say that a net \(\{(\sigma_t, \kappa_t)\}_{t \in T} \subset C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s)\) converges to \((\tilde{\sigma}, \tilde{\kappa})\) in the topology

\begin{equation}
\sigma(\text{div}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s), L^1(\Omega, E^n_s))
\times \{(\varphi, \mu \otimes s) \in Y^1(\overline{\Omega}|_{\text{Fr} \Omega} \times M^1(\Gamma_1) | \varphi|_{\Gamma_0} = 0, \varphi|_{\Gamma_1} ds = \mu \otimes s)\}
\end{equation}
if \((w \, dx, \hat{p} \otimes_s \nu, \hat{p} \, ds \otimes_s \nu); ((\sigma_t, \kappa_t) - (\hat{\sigma}, \hat{\kappa}))_3\) \(\to 0\) for all \(w \in L^1(\Omega, E^n_s)\) and \(\hat{p} \in L^1(\Gamma_1, \mathbb{R}^n)\).

**Lemma 8** (see [9, Lemma 3]). Let
\[(4.23) \quad \tilde{f} : C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s) \to \mathbb{R}\]
be a linear functional, continuous in the topology (4.22), such that for some \(\kappa_0 \in C(\Gamma_1, E^n_s)\) and \(\sigma_1 \in C_{\text{div}}(\overline{\Omega}, E^n_s)\), and every \(\sigma_s \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) with \(\text{div} \sigma_s = 0\) (in \(L^1(\Omega, \mathbb{R}^n)\)), we have
\[(4.24) \quad \tilde{f}(\sigma_1 + \sigma_s, \kappa_0) = \tilde{f}(\sigma_1, \kappa_0)\).

Moreover, assume that for every \(\sigma \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) and \(\kappa, \kappa_s \in C(\Gamma_1, E^n_s)\) such that \(\int_{\Gamma_1} \kappa_s : [t \otimes_s \nu] \, ds \to 0\) for every \(t \in L^1(\Gamma_1, \mathbb{R}^n)\), we have \(\tilde{f}(\sigma, \kappa) = \tilde{f}(\sigma, \kappa + \kappa_s)\). Then there exists \(\hat{\nu}_1 \in LD(\Omega)\) such that \(\gamma_B(\hat{\nu}_1) = 0\) on \(\Gamma_0\)
and for every \((\sigma, \kappa) \in C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s)\) we have
\[(4.25) \quad \tilde{f}(\sigma, \kappa) = \int_{\Omega} \sigma : \varepsilon(\hat{u}_1) \, dx - \int_{\text{Fr} \, \Omega} \sigma : (\gamma_B(\hat{u}_1) \otimes_s \nu) \, ds + \int_{\Gamma_1} \kappa : (\gamma_B(\hat{u}_1) \otimes_s \nu) \, ds.\]

Let \(Q_1 : C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s) \to \mathbb{R} \cup \{\infty\}\) be defined by
\[(4.26) \quad Q_1(\sigma, \kappa) = \inf_{\sigma_s} \{ (S_{\chi}^i)^*(\sigma + \sigma_s, \kappa) \mid \sigma_s \in C(\overline{\Omega}, E^n_s) \text{ with } \text{div} \sigma_s = 0 \}\]
for \(\sigma \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) and \(\kappa \in C(\Gamma_1, E^n_s)\).

**Proposition 9** (see [9, Proposition 4]). For every \(\sigma \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) and \(\kappa \in C(\Gamma_1, E^n_s)\) we have
\[(4.27) \quad (S_{\chi}^i)^*(\sigma, \kappa) = \text{cl}_{(4.22)} Q_1(\sigma, \kappa),\]
where \(\text{cl}_{(4.22)} Q_1\) denotes the largest minorant which is less than \(Q_1\) and l.s.c. in the topology (4.22) (i.e. \(\text{cl}_{(4.22)} Q_1\) the l.s.c. regularization of \(Q_1\) in (4.22)).

We say that a net \(\{(\sigma_t, \kappa_t)\}_{t \in T} \subset C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s)\) converges to \((\hat{\sigma}, \hat{\kappa})\) in the topology
\[(4.28) \quad \sigma(C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s), Y^1(\overline{\Omega}) \times (L^1(\Gamma_1, \mathbb{R}^n) \otimes_s \nu)\]
if\[(4.29) \quad (\varepsilon(u)|_{\overline{\Omega}}, z \, ds \otimes_s \nu; ((\sigma_t, \kappa_t) - (\hat{\sigma}, \hat{\kappa}))_2) \to 0\]
for every \(\varepsilon(u)|_{\overline{\Omega}} \in Y^1(\overline{\Omega})\) and \(z \in L^1(\Gamma_1, \mathbb{R}^n)\).

**Proposition 10.** Let \(\tilde{A}_k \equiv \{(\sigma, \kappa) \in C_{\text{div}}(\overline{\Omega}, E^n_s) \times C(\Gamma_1, E^n_s) \mid \|\text{div} \sigma\|_{L^n} \leq k, \|\kappa\|_{C(\Gamma_1, E^n_s)} \leq k\}\). For every \(\hat{\sigma} \in C_{\text{div}}(\overline{\Omega}, E^n_s)\) and \(\hat{\kappa} \in C(\Gamma_1, E^n_s)\) there
exists \( k(\tilde{\sigma}, \tilde{\kappa}) > 0 \) such that for every \( k \geq k(\tilde{\sigma}, \tilde{\kappa}) \),

\[
(4.30) \quad (S_{\lambda}^J)^*(\tilde{\sigma}, \tilde{\kappa}) = \text{cl}_{\mathcal{A}_k} Q_1(\tilde{\sigma}, \tilde{\kappa}),
\]

where \( \text{cl}_{\mathcal{A}_k} Q_1(\cdot, \cdot) \) is the l.s.c. regularization of the function \((\sigma, \kappa) \mapsto Q_1(\sigma, \kappa) + I_{\mathcal{A}_k}(\sigma, \kappa)\) in the topology \((4.22)\) and \( I_{\mathcal{A}_k}(\cdot) \) is the indicator function of \( \mathcal{A}_k \).

**Proof.** The proof is similar to that of [8, Proposition 13] (see also [9, Proposition 6]). 🔮

**Theorem 11.** For every \( \varepsilon(u)|_{\overline{\Omega}} \in Y^1(\overline{\Omega}) \) and \( \mu \in M_b(\Gamma_1, \mathbb{R}^n) \) we have

\[
(4.31) \quad (S_{\lambda}^J)^*(\varepsilon(u)|_{\overline{\Omega}}, \mu \otimes_s \nu) = (S_{\lambda}^J)^*(\varepsilon(u)|_{\overline{\Omega}}, \mu \otimes_s \nu).
\]

**Proof.** The proof is similar to that of [9, Theorem 7] (see also [8, Theorem 14]). 🔮

The displacement formulation \((SN_{\lambda,j})\) of the equilibrium problem for the elastic-plastic body made of a Hencky material with the Signorini contact condition is given by \((3.7)\). The Suquet unilateral contact problem \((SS_{\lambda,j})\) is defined by \((3.10)\). Moreover, in [6] we studied the bidual relaxed Signorini problem

\[
(4.32) \quad (OP_{\lambda,j}^*) \quad \text{Find} \inf\{ (\lambda F_s)^*(u, \mu) + G_j^*(\varepsilon(u)) \mid u \in BD(\Omega), \mu \in M_b(\Gamma_1, \mathbb{R}^n) \},
\]

where for every \( u \in BD(\Omega) \) and \( \mu \in M_b(\Gamma_1, \mathbb{R}^n) \),

\[
(4.33) \quad (\lambda F_s)^*(u, \mu) \equiv -\lambda \left[ \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_1} g \cdot \mu \right] \\
+ \int_{\Gamma_0} j(x, -\gamma_B(u) \otimes_s \nu) \, ds + \int_{\Gamma_s} (h_s^+ + I_K)^*(x, -\gamma_B(u) \otimes_s \nu) \, ds \\
+ \int_{\Gamma_1} \left( x, \frac{d((\mu - \gamma_B(u) \, ds) \otimes_s \nu)}{d(\mu - \gamma_B(u) \, ds) \otimes_s \nu} \right) d||\mu - \gamma_B(u) || \, s
\]

and

\[
(4.34) \quad G_j^*(\varepsilon(u)) = \int_{\Omega} j(x, \varepsilon(u)|_{\Omega}) \, dx + \int_{\Omega} j(x, \frac{d\varepsilon(u)|_{\Omega}}{d\varepsilon(u)|_{\Omega}}) \, d||\varepsilon(u)|_{\Omega} ||.
\]

**Lemma 12** (cf. Lemma 15 of [8]). If \( f \in L^{n+\delta}(\Omega, \mathbb{R}^n) \), where \( \delta \geq 0 \), then the functional \( BD(\Omega) \ni u \mapsto \int_{\Omega} f \cdot u \, dx \in \mathbb{R} \) is continuous in the weak* \( BD(\Omega) \) and \( \sigma(Y^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, E_s^n)) \) topologies on bounded subsets of \( BD(\Omega) \).

**Assumption 8.** There exist \( k_b > 0 \) and \( r_1 > 0 \) such that \( j^*(x, w^*) \leq k_b \) for every \( w^* \in B_{E_s^n}(0, r_1) \) and \( dx\text{-a.e.} \ x \in \Omega \), where \( B_{E_s^n}(0, r_1) \) is the closed ball in \( E_s^n \) with center 0 and radius \( r_1 \). 🔮
Suppose the function
\[(4.35) \quad (u, \mu) \mapsto [SS_{\lambda,j}](u, \mu) \equiv \lambda F_2(u, \mu) + G_j(\varepsilon(u))\]
is coercive over $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$, where $F_2$ and $G_j$ are given by (3.11) and (3.9). Moreover, let $0 \leq \lambda_1 < \lambda$. Then $[SS_{\lambda_1,j}]$ is coercive on $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$. Similarly, if the function
\[(4.36) \quad (u, \mu) \mapsto [OP_{\lambda,j}^*](u, \mu) = (\lambda F_s)^*(u, \mu) + G_j^*(\varepsilon(u))\]
is coercive and $0 \leq \lambda_1 < \lambda$, then $[OP_{\lambda_1,j}^*]$ is coercive over $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$. Moreover,
\[(4.37) \quad [SS_{\lambda,j}](u, \mu) \geq [OP_{\lambda,j}^*](u, \mu)\]
for every $(u, \mu) \in BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$.

We say that a net $\{(u_t, \mu_t)\}_{t \in T} \subset BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$ converges to $(u_0, \mu_0)$ in the weak* $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$ topology if $u_t \rightharpoonup u_0$ in the weak* $BD(\Omega)$ topology and $\mu_t \rightharpoonup \mu_0$ in the weak* $M_b(\Gamma_1, \mathbb{R}^n)$ topology.

**Theorem 13.** Let $f \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, where $\delta \geq 0$. If the function $[SS_{\lambda,j}]$ is coercive over $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$, then the l.s.c. regularization of $[SS_{\lambda,j}]$ (in the weak* $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$ topology) is $[OP_{\lambda,j}^*]$.

*Proof.* The proof is similar to that of [8, Theorem 18] and [9, Theorem 8].

Thus for every solution $(u_0, \mu_0)$ of $(OP_{\lambda,j})^*$ there is a net $\{(u_m, \mu_m)\}_{m \in \mathcal{H}}$ which minimizes $[SS_{\lambda,j}]$ and $(u_m, \mu_m) \rightharpoonup (u_0, \mu_0)$ in the weak* $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$ topology.

**Corollary 14.** The functional $[SS_{\lambda,j}]$ is coercive over the space $BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)$ if and only if $[OP_{\lambda,j}^*]$ is coercive.

*Proof.* By Theorem 13, if $[SS_{\lambda,j}]$ is coercive, then so is $[OP_{\lambda,j}^*]$ (cf. Corollary 19 of [8]). By (4.37) the proof is complete.

The bidual relaxed Signorini problem (where the relaxation on $\Gamma_1$, proposed by Suquet in [22], is omitted) is defined by the formula
\[(4.38) \quad (NP_{\lambda,j}) \quad \text{Find} \ \inf\{(\lambda F_N)^*(u) + G_j^*(\varepsilon(u)) \mid u \in BD(\Omega)\},\]
where for every $u \in BD(\Omega)$,
\[(4.39) \quad (\lambda F_N)^*(u) \equiv -\lambda \left[ \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_1} g \cdot \gamma_B(u) \, ds \right] + \int_{\Gamma_0} j_{\infty}(x, -\gamma_B(u) \otimes s \nu) \, ds + \int_{\Gamma_s} (h_s^* + I_K)^*(x, -\gamma_B(u) \otimes s \nu) \, ds\]
and $G_j^*$ is defined by (4.34).

Similarly to Theorem 13 (cf. [9, Theorem 18]) we obtain the following result:
THEOREM 15. Let $f \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, where $\delta \geq 0$. If the function
\begin{equation}
BD(\Omega) \ni u \mapsto [SN_{\lambda,j}](u) = \lambda F_1(u) + G_j(\varepsilon(u))
\end{equation}
is coercive over $BD(\Omega)$, then the l.s.c. regularization of $[SN_{\lambda,j}]$ in the weak$^*$
$BD(\Omega)$ topology is the functional
\begin{equation}
BD(\Omega) \ni u \mapsto [NP_{\lambda,j}^*](u) = (\lambda F_N)^*(u) + G_j^*(\varepsilon(u)).
\end{equation}
Remark 1. Let $f \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, where $\delta \geq 0$, and suppose the functional
$[SN_{\lambda,j}]$ is coercive over $LD(\Omega)$. Since $[SN_{\lambda,j}](u) = \infty$ for $u \in
BD(\Omega) - LD(\Omega)$, the l.s.c. regularization of $[SN_{\lambda,j}]$ in the weak$^*$ $BD(\Omega)$
topology (over the space $SBD(\Omega)$) is
\begin{equation}
SBD(\Omega) \ni u \mapsto [NP_{\lambda,j}^*](u) = (\lambda F_N)^*(u) + G_j^*(\varepsilon(u))
\end{equation}
(cf. [5]). Indeed, (4.42) is the supremum over the set of affine mappings (continuous in the weak$^*$
$BD(\Omega)$ topology) less than the functional $[SN_{\lambda,j}]$.

COROLLARY 16. The functional $[SN_{\lambda,j}]$ is coercive over $BD(\Omega)$ if and
only if $[NP_{\lambda,j}^*]$ is coercive.

Proof. The proof is similar to that of Corollary 14. □

COROLLARY 17. The functional $[SN_{\lambda,j}]$ is coercive over $BD(\Omega)$ if and
only if $[OP_{\lambda,j}^*]$ is coercive over $BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$.

Proof. Indeed, by (3.7)-(3.11) the function $[SS_{\lambda,j}]$ is coercive over
$BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ if and only if $[SN_{\lambda,j}]$ is coercive over $BD(\Omega)$. By
Corollary 14 the proof is complete. □

THEOREM 18. Let $f \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, where $\delta \geq 0$. Moreover, let $j^* : \Omega \times
\mathbb{E}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a nonnegative normal integrand. If the function
$[SN_{\lambda,j}]$ is coercive over $BD(\Omega)$, then the couple $(\bar{u}, \bar{\mu}) \in BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ is
a solution of $(OP_{\lambda,j}^*)^*$ if and only if $\bar{\mu} = \gamma_B(\bar{u})|_{I_1} ds \in \mathbb{M}_b(I_1, \mathbb{R}^n)$ and $\bar{u}$ is a
minimum point of the function $[NP_{\lambda,j}^*]$.

Proof. Since $j^*$ is nonnegative, $(NP_{\lambda,j}^*)^*$ and $(OP_{\lambda,j}^*)^*$ have finite infima.
Indeed, $[NP_{\lambda,j}^*](0) < \infty$ and $[OP_{\lambda,j}^*](0,0) < \infty$ (cf. Assumptions 3 and 6).

First, let $\bar{u}$ be a solution of $(NP_{\lambda,j}^*)^*$. Then the couple $(\bar{u}, \gamma_B(\bar{u})|_{I_1} ds) \in
BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ is a solution of $(OP_{\lambda,j}^*)^*$, since
\begin{equation}
\inf(NP_{\lambda,j}^*) = \inf(SN_{\lambda,j}) = \inf(SS_{\lambda,j}) = \inf(OP_{\lambda,j}^*)
\end{equation}
(cf. Theorems 15 and 13 and formulae (3.7)-(3.11)).

Next, we show the converse implication. Let $(\bar{u}, \bar{\mu})$ be a solution of $(OP_{\lambda,j}^*)^*$. Then there exists a sequence \{$(u_m, \mu_m)$\}$_{m \in \mathbb{N}}$ in $BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ such that $\lim_{m \to \infty}[SS_{\lambda,j}](u_m, \mu_m) = \inf(OP_{\lambda,j}^*)$ and $(u_m, \mu_m) \rightharpoonup (\bar{u}, \bar{\mu})$ in
the weak$^*$ $BD(\Omega) \times \mathbb{M}_b(I_1, \mathbb{R}^n)$ topology. By (3.10) there exists $m_0 \in \mathbb{N}$ such that for every $m > m_0$ we obtain $\mu_m = \gamma_B(u_m) ds$ on $I_1$, because
inf(\(OP^{**}_{\lambda, j}\)) < \infty. Since inf(\(NP^{**}_{\lambda, j}\)) = inf(\(OP^{**}_{\lambda, j}\)), we have

\[
(4.44) \quad \lim_{m \to \infty} [SS_{\lambda, j}](u_m, \gamma_B(u_m)_{|\Gamma_1}) ds = \lim_{m \to \infty} [SN_{\lambda, j}](u_m) = inf(\(NP^{**}_{\lambda, j}\))
\]

(cf. (4.43)). Therefore, by Theorem 15, we get

\[
(4.45) \quad [NP^{**}_{\lambda, j}](\tilde{\mu}) = inf(\(NP^{**}_{\lambda, j}\)) = inf(\(OP^{**}_{\lambda, j}\)).
\]

We now prove that \(\tilde{\mu} = \gamma_B(\tilde{\mu}) ds\) on \(\Gamma_1\). Suppose otherwise. By (4.38), (4.32) and (4.45) we have \([OP^{**}_{\lambda, j}](\tilde{\mu}, \gamma_B(\tilde{\mu})_{|\Gamma_1}) ds = [OP^{**}_{\lambda, j}](\tilde{\mu}, \tilde{\mu})\). The function

\[
[0, \infty) \ni t \mapsto \int_{\Gamma_1} \left( x, \frac{d(t(\tilde{\mu} - \gamma_B(\tilde{\mu}) ds) \otimes_s \nu)}{d(\tilde{\mu} - \gamma_B(\tilde{\mu}) ds) \otimes_s \nu} \right) d(\tilde{\mu} - \gamma_B(\tilde{\mu}) ds) \otimes_s \nu + (1 - t) \int_{\Gamma_1} g \cdot \gamma_B(\tilde{\mu}) ds + t \int_{\Gamma_1} g \cdot \tilde{\mu}
\]

is positively homogeneous. Then

\[
[OP^{**}_{\lambda, j}](\tilde{\mu}, \tilde{\mu}) = [OP^{**}_{\lambda, j}](\tilde{\mu}, t(\tilde{\mu} - \gamma_B(\tilde{\mu}) ds)_{|\Gamma_1}) ds + \gamma_B(\tilde{\mu})_{|\Gamma_1} ds
\]

for every \(t > 0\). This contradicts with the coercivity of \([OP^{**}_{\lambda, j}]\) (cf. Corollary 17).

**Remark 2.** In the case of soil mechanics, the set of admissible stresses is a convex cone which contains the half-line \(A_L \equiv \{ \sigma \in E_s^n \mid tr \sigma \leq 0, \sigma^D = 0 \}\), where \(\sigma^D = \sigma - n^{-1} \delta tr \sigma\) and \(\delta_{ij}\) is the Kronecker delta. Moreover, by Assumption 8, the set of admissible stresses contains a neighborhood of \(0 \in E_s^n\) (the origin of the cone \(\neq 0\)). Then by direct calculation we obtain

\[
(4.46) \quad (h^*_s + I_{K_1}^*(x, -\gamma_B(u) \otimes_s \nu) = h^*_s(x, -\gamma_B(u) \otimes_s \nu)
\]

\[
= h_s(x, -\gamma_B(u) \otimes_s \nu),
\]

where \(K_1\) is the elastic domain (of the soil) at the point \(x\). Indeed,

\[
(4.47) \quad \sum_{i,j=1}^n \sigma_{ij}(\gamma^f_B(u) \otimes_s \nu)_{ij} = \frac{1}{n} tr \sigma tr(\gamma^f_B(u) \otimes_s \nu) + \sum_{i,j=1}^n \sigma_{ij}^d(\gamma^f_B(u) \otimes_s \nu)_{ij}^d
\]

and sup\(\{ \sigma : (-\gamma^f_B(u)(x) \otimes_s \nu) - I_{A_L}(\sigma) \mid \sigma \in E_s^n\} = h_s(x, -\gamma^f_B(u) \otimes_s \nu)\) (cf. (4.15)). Thus we obtain a generalization of Lemma 6 to the case of a soil material.

Let \(f \in L^{n+\delta}(\Omega, \mathbb{R}^n)\), where \(\delta \geq 0\). Moreover, suppose the functions

\([SN_{\lambda, j}], [NP^{**}_{\lambda, j}], [SS_{\lambda, j}]\) and \([OP^{**}_{\lambda, j}]\) describe a piece of elastic-plastic soil, i.e. \(j, j_\infty, K (K = K_1)\) describe the properties of the soil and Assumption 7 does not hold. If the function \([SS_{\lambda, j}]\) is coercive over \(BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)\), then its l.s.c. regularization in the weak* \(BD(\Omega) \times M_b(\Gamma_1, \mathbb{R}^n)\) topology
is $[\mathcal{OP}_{\lambda,j}^{**}]$. Moreover, Theorems 13, 15, 18, Corollaries 14, 16 and 17 hold for the case of a soil material.

Suppose $ds(I_0) = 0$ and $\int_{\Omega} f \cdot \widehat{u} \, dx + \int_{I_1} g \cdot \gamma_B(\widehat{u}) \, ds = 0$ for every $\widehat{u} \in \mathcal{R}_0$. Moreover, suppose that for every sequence $(\{u_m, \mu_m\})_{m \in \mathbb{N}} \subset BD(\Omega) \times \mathcal{M}_b(I_1, \mathbb{R}^n)$ such that $\inf \{ \|u_m + r\|_{BD} + \|\mu_m + \gamma_B(r)\|_{\mathcal{M}_b(I_1, \mathbb{R}^n)} : r \in \mathcal{R}_0 \} \to \infty$ as $m \to \infty$, we have $[\mathcal{OP}_{\lambda,j}^{**}](u_m, \mu_m) \to \infty$. Then there exists a solution $(u_s, \mu_s)$ of the problem $(\mathcal{OP}_{\lambda,j}^{**})$ in $BD(I_1)$. For every $u^1 \in \mathcal{R}_0$ the field $(u_s + u^1, \mu_s + \gamma_B(u^1)I_1 ds) \in BD(\Omega) \times \mathcal{M}_b(I_1, \mathbb{R}^n)$ is a solution of $[\mathcal{OP}_{\lambda,j}^{**}]$ if and only if $\int_{I_s}(h_s^* + I_K)^*(x, -\gamma_B(u_s) \otimes_s \nu) \, ds = \int_{I_s}(h_s^* + I_K)^*(x, -\gamma_B(u_s + u^1) \otimes_s \nu) \, ds$ (cf. Theorems 4.34 and 4.31 of [6]).

References


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