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A FRICTIONLESS CONTACT PROBLEM FOR ELASTIC-VISCOPLASTIC MATERIALS WITH INTERNAL STATE VARIABLE

Abstract. We study a mathematical model for frictionless contact between an elastic-viscoplastic body and a foundation. We model the material with a general elastic-viscoplastic constitutive law with internal state variable and the contact with a normal compliance condition. We derive a variational formulation of the model. We establish existence and uniqueness of a weak solution, using general results on first order nonlinear evolution equations with monotone operators and fixed point arguments. Finally, we study the dependence of the solution on perturbations of contact conditions and prove a convergence result.

1. Introduction. In this paper we study a mathematical model describing contact between deformable bodies. We model the material behavior with a general elastic-viscoplastic constitutive law with internal state variable of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)\big)\, ds, \tag{1.1}$$

$$\dot{\mathbf{k}}(t) = \phi\big(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\big), \tag{1.2}$$

where $\mathbf{u}$ denotes the displacement field, and $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ represent the stress tensor and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively; $\mathcal{G}$ is a nonlinear constitutive function which describes

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the viscoplastic behavior of the material and depends on the internal state variable $\mathbf{k}$; and ϕ is also a nonlinear constitutive function which depends on $\mathbf{k}$.

We suppose that $\mathbf{k}$ is a vector-valued function whose evolution is governed by the differential equation (1.2); the set of admissible internal state variables is given by

$$Y = \{\boldsymbol{\alpha} = (\alpha_i) \mid \alpha_i \in L^2(\Omega),\, 1 \le i \le m\}.$$

In (1.1) and throughout, the dot above a variable represents the derivative with respect to the time variable t.

It follows from (1.1) that, at each time moment t, the stress tensor $\boldsymbol{\sigma}(t)$ is split into two parts: $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t)$, where $\boldsymbol{\sigma}^V(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))$ represents the purely viscous part of the stress whereas $\boldsymbol{\sigma}^R(t)$ satisfies a rate-type elastic-viscoplastic relation with internal state variable

$$\boldsymbol{\sigma}^R(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}^R(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)\big)\, ds, \tag{1.3}$$

$$\dot{\mathbf{k}}(t) = \phi\big(\boldsymbol{\sigma}^R(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\big). \tag{1.4}$$

When $\mathcal{G} = 0$ the constitutive law (1.1) reduces to the Kelvin–Voigt viscoelastic constitutive relation given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)). \tag{1.5}$$

Examples and mechanical interpretation of elastic-viscoplastic materials of the form (1.3) in which the function $\mathcal{G}$ does not depend on the internal variable $\mathbf{k}$ were considered by many authors: see for instance [4, 11] and the references therein. Contact problems for materials of the form (1.1), (1.3) without internal variable and (1.5) are the topic of numerous papers, e.g. [7, 8, 9, 15, 16, 21] and the recent references [1, 10]. Contact problems for elastic-viscoplastic materials of the form (1.3)–(1.4) were studied in [3, 6, 17]. Dynamic frictionless contact problems for materials of the form (1.1), in which the internal state variable represents the damage field whose evolution is described by a differential inclusion, are investigated in [18, 19].

In the present paper we consider a mathematical model for frictionless contact between an elastic-viscoplastic body and a deformable foundation. The contact is modeled with the normal compliance condition (see, e.g., [16]). We derive the variational formulation and prove existence and uniqueness of a weak solution of the model. Finally, we study the dependence of the solution on perturbations of contact conditions and prove a convergence result.

The paper is organized as follows. In Section 2 we present the notation and some preliminaries. In Section 3 we present the mechanical problem, we list the assumptions on the data and give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result based on first order evolution equations with monotone operators and fixed point arguments. In Section 5 we study the dependence of the solution on perturbations of contact conditions and prove a convergence result.

2. Notation and preliminaries. In this section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [5].

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere, the indices i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable.

We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces:

$$\begin{aligned} H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}&, \\ \mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}&, \\ H_1 = \{\mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\}&, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}&. \end{aligned}$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \varepsilon_{ij}(\mathbf{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i v_i \, dx && \forall \mathbf{u}, \mathbf{v} \in H, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \tau_{ij} \, dx && \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}, \\ (\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} && \forall \mathbf{u}, \mathbf{v} \in H_1, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H && \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1. \end{aligned}$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively. For every $\mathbf{v} \in H_1$ we also use write $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v}\, da \qquad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm

$$|\mathbf{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$|\mathbf{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X + \max_{t\in[0,T]} |\dot{\mathbf{f}}(t)|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

We recall the following standard result for first order evolution equations (see [2] or [21]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V'$$

and we say that the inclusions above define a *Gelfand triple*. We denote by $|\cdot|_V, |\cdot|_H$ and $|\cdot|_{V'}$ the norms on the spaces V, H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V.

Note that if $\mathbf{f} \in H$ then

$$(\mathbf{f}, \mathbf{v})_{V'\times V} = (\mathbf{f}, \mathbf{v})_H \qquad \forall \mathbf{v} \in V.$$

THEOREM 2.1. *Let V, H be as above, and let $A : V \to V'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\mathbf{u}, \mathbf{u})_{V'\times V} \geq \omega |\mathbf{u}|_V^2 + \lambda \qquad \forall \mathbf{u} \in V, \tag{2.4}$$

$$|A\mathbf{u}|_{V'} \leq C(|\mathbf{u}|_V + 1) \qquad \forall \mathbf{u} \in V, \tag{2.5}$$

for some constants $\omega > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0,T;V')$, there exists a unique function $\mathbf{u}$ which satisfies

$$\begin{aligned} &\mathbf{u} \in L^2(0,T;V) \cap C(0,T;H), \qquad \dot{\mathbf{u}} \in L^2(0,T;V'),\\ &\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{f}(t), \qquad a.e.\ t \in (0,T),\\ &\mathbf{u}(0) = \mathbf{u}_0. \end{aligned}$$

3. Mechanical and variational formulations. An elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\mathrm{meas}(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ is applied in $\Omega \times (0, T)$. The body is in frictionless contact with a deformable foundation over the potential contact surface $\Gamma_3 \times (0, T)$. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. Then, the classical formulation of the mechanical contact problem with normal compliance of an elastic-viscoplastic material with internal state variable is as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$ *and an internal state variable field* $\mathbf{k} : \Omega \times [0, T] \to \mathbb{R}^m$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \mathbf{k}(s)\big)\, ds \quad \text{in } \Omega \times (0, T), \tag{3.1}$$

$$\dot{\mathbf{k}} = \phi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\mathbf{u}), \mathbf{k}) \quad \text{in } \Omega \times (0, T), \tag{3.2}$$

$$\rho\ddot{\mathbf{u}} = \operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega \times (0, T), \tag{3.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{3.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.5}$$

$$\sigma_\nu = p(u_\nu - g), \quad \boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3 \times (0, T), \tag{3.6}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \mathbf{k}(0) = \mathbf{k}_0 \quad \text{in } \Omega. \tag{3.7}$$

Here equations (3.1)–(3.2) represent the elastic-viscoplastic constitutive law with internal state variable introduced in the first section. Equation (3.3) represents the equation of motion where ρ is the mass density. Equations (3.4)–(3.5) are the displacement-traction conditions.

Let us give some remarks on the contact conditions (3.6) in which u_ν, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the normal displacement, the normal stress and the tangential stress, respectively. The first equality in (3.6) describes the *normal compliance contact condition*, where p is a prescribed function such that $p(r) = 0$ for $r \leq 0$, and g represents the gap between the potential contact surface Γ_3 and the foundation, measured along the outward normal vector. When $u_\nu - g$ is positive, it represents the penetration of the body into the foundation. So penetration is allowed but penalized.

Normal compliance contact conditions were first introduced in [14] in the study of dynamic problems for linearly elastic materials and have been used later in the literature; see for example [3, 7, 13, 16] and the references therein. An example of a normal compliance function p is

$$p(r) = c_\nu r_+, \tag{3.8}$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$. Another example of the normal compliance function is

$$p(r) = \begin{cases} c_\nu r_+ & \text{if } r \le \alpha, \\ c_\nu \alpha & \text{if } r > \alpha, \end{cases} \tag{3.9}$$

where α is a positive constant related to the wear and hardness of the surface. In this case the last contact condition means that when the penetration is large enough, i.e. it exceeds α, the obstacle desintegrates and offers no additional resistance to the penetration.

The second relation in (3.6) indicates that the friction force on the contact surface vanishes, i.e. the contact is frictionless.

In (3.7), $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ is the initial velocity and $\mathbf{k}_0$ is the initial internal state variable.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. To obtain a variational formulation of the problem (3.1)–(3.7) we need additional notation.

Let V be the closed subspace of H_1 given by

$$V = \{\mathbf{v} \in H_1 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Then the following Korn inequality holds:

$$|\boldsymbol{\varepsilon}(\mathbf{v})|_{\mathcal{H}} \ge C_k |\mathbf{v}|_{H_1} \quad \forall \mathbf{v} \in V,$$

where $C_k > 0$ is a constant depending only on Ω and Γ_1. On the space V we consider the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \tag{3.10}$$

and let $|\cdot|_V$ be the associated norm. It follows from Korn's inequality that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \le C_0 |\mathbf{v}|_V \quad \forall \mathbf{v} \in V. \tag{3.11}$$

In the study of the mechanical problem (3.1)–(3.7) we make the following assumptions. The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

(3.12)

$$\begin{cases} \text{(a) There exist constants } C^1_{\mathcal{A}}, C^2_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon})| \le C^1_{\mathcal{A}} |\boldsymbol{\varepsilon}| + C^2_{\mathcal{A}} \quad \forall \boldsymbol{\varepsilon} \in S_d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \\ \qquad \ge m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(c) } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in S_d. \\ \text{(d) } \boldsymbol{\varepsilon} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is continuous on } S_d, \ \text{a.e. } \mathbf{x} \in \Omega. \end{cases}$$

The elasticity operator $\mathcal{E} : \Omega \times S_d \to S_d$ satisfies

(3.13)

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{E}} > 0 \text{ such that} \\ \quad |\mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}_2)| \le L_{\mathcal{E}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) For any } \boldsymbol{\varepsilon} \in S_d, \mathbf{x} \mapsto \mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega. \\ \text{(c) } \mathbf{x} \mapsto \mathcal{E}(\mathbf{x}, \mathbf{0}) \text{ is in } \mathcal{H}. \end{cases}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R}^m \to S_d$ satisfies

(3.14)

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \mathbf{k}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \mathbf{k}_2)| \\ \qquad \le L_{\mathcal{G}} (|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\mathbf{k}_1 - \mathbf{k}_2|) \\ \quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^m, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \mathbf{k} \in \mathbb{R}^m, \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{k}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ \text{(c) } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ is in } \mathcal{H}. \end{cases}$$

The function $\phi : \Omega \times S_d \times S_d \times \mathbb{R}^m \to \mathbb{R}$ satisfies

(3.15)

$$\begin{cases} \text{(a) There exists a constant } L_{\phi} > 0 \text{ such that} \\ \quad |\phi(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \mathbf{k}_1) - \phi(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \mathbf{k}_2)| \\ \qquad \le L_{\phi} \big(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\mathbf{k}_1 - \mathbf{k}_2| \big) \\ \quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^m, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \mathbf{k} \in \mathbb{R}^m, \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{k}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ \text{(c) } \mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ is in } \mathcal{H}. \end{cases}$$

The normal compliance function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(3.16)\quad \begin{cases} \text{(a) There exists a constant } L_p > 0 \text{ such that} \\ \qquad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \le L_p |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b) } r \mapsto p(\cdot, r) \text{ is Lebesgue measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \text{(c) } p(\cdot, r) = 0 \text{ for all } r \le 0. \end{cases}$$

We notice that the assumption (3.16) is satisfied by functions defined in (3.8) and (3.9) and therefore our results apply to the corresponding frictionless contact. The mass density satisfies

$$(3.17)\quad \rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \ge \rho^*, \text{ a.e. } \mathbf{x} \in \Omega.$$

The gap function g is such that

$$(3.18)\qquad g \in L^2(\Gamma_3), \qquad g \ge 0 \text{ a.e. on } \Gamma_3.$$

The body forces and surface tractions have the regularity

$$(3.19)\qquad \mathbf{f}_0 \in L^2(0, T; H), \qquad \mathbf{f}_2 \in L^2(0, T; L^2(\Gamma_2)^d).$$

Finally we assume that the initial data satisfy

$$(3.20)\qquad \mathbf{u}_0 \in V, \qquad \mathbf{v}_0 \in H, \qquad \mathbf{k}_0 \in Y.$$

We will use a modified inner product on the Hilbert space $H = L^2(\Omega)^d$ given by

$$((\mathbf{u}, \mathbf{v}))_H = (\rho \mathbf{u}, \mathbf{v})_H \qquad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, it is weighted with ρ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\mathbf{v}\|_H = (\rho \mathbf{v}, \mathbf{v})_H^{1/2} \qquad \forall \mathbf{v} \in H.$$

It follows from assumptions (3.17) that $\|\cdot\|_H$ and $|\cdot|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we obtain the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(\cdot, \cdot)_{V' \times V}$ for the duality pairing between V' and V and recall that

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H \qquad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Assumptions (3.19) allow us, for a.e. $t \in (0, T)$, to define $\mathbf{f}(t) \in V'$ by

$$(3.21)\qquad (\mathbf{f}(t), \mathbf{v})_{V' \times V} = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da \qquad \forall \mathbf{v} \in V,$$

and note that

$$(3.22)\qquad \mathbf{f} \in L^2(0, T; V').$$

Finally we consider the functional $j : V \times V \to \mathbb{R}$ defined by

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} p(u_\nu - g)v_\nu\, da \qquad \forall \mathbf{u},\mathbf{v} \in V. \tag{3.23}$$

Using standard arguments we obtain the following variational formulation of the mechanical problem (3.1)–(3.7).

Problem PV. *Find a displacement field* $\mathbf{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ *and an internal state variable field* $\mathbf{k} : [0,T] \to Y$ *such that for a.e.* $t \in (0,T)$,

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)\big)\, ds, \end{aligned} \tag{3.24}$$

$$(\ddot{\mathbf{u}}(t),\mathbf{v})_{V'\times V} + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, \tag{3.25}$$

$$\dot{\mathbf{k}}(t) = \phi\big(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\big), \tag{3.26}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \qquad \mathbf{k}(0) = \mathbf{k}_0. \tag{3.27}$$

4. An existence and uniqueness result. Our main existence and uniqueness result is the following.

Theorem 4.1. *Let the assumptions* (3.12)–(3.20) *hold. Then Problem* PV *has a unique solution* $\{\mathbf{u},\boldsymbol{\sigma},\mathbf{k}\}$ *satisfying*

$$\mathbf{u} \in H^1(0,T;V)) \cap C^1(0,T;H), \qquad \ddot{\mathbf{u}} \in L^2(0,T;V'), \tag{4.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \qquad \operatorname{Div}\boldsymbol{\sigma} \in L^2(0,T;V'), \tag{4.2}$$

$$\mathbf{k} \in W^{1,2}(0,T;Y). \tag{4.3}$$

We conclude that under the assumptions (3.12)–(3.20) the mechanical problem (3.1)–(3.7) has a unique weak solution with regularity (4.1)–(4.3). The proof of this theorem will be carried out in several steps. It is based on arguments of nonlinear evolution equations with monotone operators (see [2] or [21]) and fixed point arguments.

In the first step we let $\boldsymbol{\eta} = (\boldsymbol{\eta}^1,\boldsymbol{\eta}^2) \in L^2(0,T;V' \times Y)$ be given, and prove that there exists a unique solution $\mathbf{u}_\eta$ of the following intermediate problem.

Problem PV$_\eta$. *Find a displacement field* $\mathbf{u}_\eta : [0,T] \to V$ *such that for a.e.* $t \in (0,T)$,

$$\begin{aligned} (\ddot{\mathbf{u}}_\eta(t),\mathbf{v})_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\eta}^1(t),\mathbf{v})_{V'\times V} \\ = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, \end{aligned} \tag{4.4}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{4.5}$$

Concerning Problem PV$_\eta$, we have the following result.

Lemma 4.2. *There exists a unique solution to Problem* PV_η *with regularity* (4.1).

Proof. We use the abstract existence and uniqueness result given by Theorem 2.1. We define $A : V \to V'$ by

$$(\mathrm{4.6})\qquad (A\mathbf{u}, \mathbf{v})_{V'\times V} = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

It follows from (4.6), (3.12)(a) and (3.10) that

$$(\mathrm{4.7})\qquad |A\mathbf{u} - A\mathbf{v}|_{V'} \le C|\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v})|_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Keeping in mind (3.12) and the Krasnosel'skiĭ theorem (see for example [12, p. 60]) we deduce that $A : V \to V'$ is continuous, and so hemicontinuous. Now, by (3.12)(b) and (3.10), we find

$$(\mathrm{4.8})\qquad (A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_{V'\times V} \ge m_{\mathcal{A}}|\mathbf{u} - \mathbf{v}|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

i.e., $A : V \to V'$ is monotone. Choosing $\mathbf{v} = \mathbf{0}_V$ in (4.8) we obtain

$$\begin{aligned}(A\mathbf{u}, \mathbf{u})_{V'\times V} &\ge m_{\mathcal{A}}|\mathbf{u}|_V^2 - |A\mathbf{0}_V|_{V'}|\mathbf{u}|_V \\ &\ge \tfrac{1}{2}m_{\mathcal{A}}|\mathbf{u}|_V^2 - \frac{1}{2m_{\mathcal{A}}}|A\mathbf{0}_V|_{V'}^2 \quad \forall \mathbf{u} \in V.\end{aligned}$$

Thus, A satisfies condition (2.4) with $\omega = m_{\mathcal{A}}/2$ and $\lambda = -|A\mathbf{0}_V|_{V'}^2/(2m_{\mathcal{A}})$. Next, by (4.6), (3.12)(a) and (3.10) we deduce that

$$|A\mathbf{u}|_{V'} \le C(|\mathbf{u}|_V + 1) \quad \forall \mathbf{u} \in V,$$

where C is a positive constant. This implies that A satisfies condition (2.5). Finally, we recall that by (3.20) and (3.22) we have $\mathbf{f} - \boldsymbol{\eta}^1 \in L^2(0, T; V')$ and $\mathbf{v}_0 \in H$.

Now Theorem 2.1 implies that there exists a unique function $\mathbf{v}_\eta$ which satisfies

$$(\mathrm{4.9})\qquad \mathbf{v}_\eta \in L^2(0, T; V) \cap C(0, T; H), \quad \dot{\mathbf{v}}_\eta \in L^2(0, T; V'),$$

$$(\mathrm{4.10})\qquad \dot{\mathbf{v}}_\eta(t) + A\mathbf{v}_\eta(t) + \boldsymbol{\eta}^1(t) = \mathbf{f}(t), \quad \text{a.e. } t \in (0, T),$$

$$(\mathrm{4.11})\qquad \mathbf{v}_\eta(0) = \mathbf{v}_0.$$

Let $\mathbf{u}_\eta : [0, T] \to V$ be defined by

$$(\mathrm{4.12})\qquad \mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\, ds + \mathbf{u}_0 \quad \forall t \in [0, T].$$

It follows from (4.6) and (4.9)–(4.12) that $\mathbf{u}_\eta$ is a solution of Problem PV_η with regularity (4.1). This concludes the existence part of Lemma 4.2. The uniqueness of the solution follows from the uniqueness of the solution to problem (4.10)–(4.11), guaranteed by Theorem 2.1. ∎

Define $\mathbf{k}_\eta \in W^{1,2}(0,T;Y)$ by

$$\mathbf{k}_\eta(t) = \mathbf{k}_0 + \int_0^t \boldsymbol{\eta}^2(s)\,ds. \tag{4.13}$$

In the second step we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and $\mathbf{k}_\eta$ defined in (4.13) to pose the following Cauchy problem for the stress field.

PROBLEM QV_η. *Find a stress field* $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))\,ds \qquad \forall t \in [0,T]. \tag{4.14}$$

For Problem QV_η we have the following result.

LEMMA 4.3. *There exists a unique solution of Problem* QV_η *and it satisfies* $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T,\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i$ *and* $\mathbf{u}_i$ *represent the solutions of Problem* QV_{η_i}, PV_{η_i}, *respectively, and* $\mathbf{k}_i$ *is defined in* (4.13) *for* $\boldsymbol{\eta}_i \in L^2(0,T;V' \times Y)$, $i = 1,2$, *then there exists* $C > 0$ *such that*

$$\begin{aligned} |\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2 \le C\Big(&|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \\ &+ \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2\,ds + \int_0^t |\mathbf{k}_1(s) - \mathbf{k}_2(s)|_Y^2\,ds\Big) \qquad \forall t \in [0,T]. \end{aligned} \tag{4.15}$$

Proof. Let $\Lambda_\eta : L^2(0,T,\mathcal{H}) \to L^2(0,T,\mathcal{H})$ be given by

$$\Lambda_\eta\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s)\big)\,ds \tag{4.16}$$

for $\boldsymbol{\sigma} \in L^2(0,T,\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(0,T,\mathcal{H})$ we use (4.16) and (3.14) to obtain, for all $t \in [0,T]$,

$$|\Lambda_\eta\boldsymbol{\sigma}_1(t) - \Lambda_\eta\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\,ds.$$

It follows that for p large enough, the power Λ_η^p is a contraction on the Banach space $L^2(0,T;\mathcal{H})$, and therefore there exists a unique $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ such that $\Lambda_\eta\boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem QV_η and, using (4.14), the regularity of $\mathbf{u}_\eta$, the regularity of $\mathbf{k}_\eta$ and the properties of the operators $\mathcal{E}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$.

Consider now $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0,T;V' \times Y)$ and, for $i = 1,2$, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\mathbf{k}_{\eta_i} = \mathbf{k}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\mathbf{u}_i(s)), \mathbf{k}_i(s)\big)\,ds \qquad \forall t \in [0,T],$$

and, using the properties (3.13) and (3.14) of $\mathcal{E}$ and $\mathcal{G}$, we find

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2 \le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\,ds + \int_0^t |\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2\,ds+\int_0^t |\mathbf{k}_1(s)-\mathbf{k}_2(s)|_Y^2\,ds\Big) \quad \forall t\in[0,T]. \tag{4.17}$$

Using now a Gronwall argument we deduce (4.15). ∎

Finally, as a consequence of these results and using the properties of $\mathcal{G}$, $\mathcal{E}$, ϕ and j, for $t\in[0,T]$, we consider the element

$$\Lambda\boldsymbol{\eta}(t)=(\Lambda^1\boldsymbol{\eta}(t),\Lambda^2\boldsymbol{\eta}(t))\in V'\times Y, \tag{4.18}$$

defined by

$$(\Lambda^1\boldsymbol{\eta}(t),\mathbf{v})_{V'\times V}=(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + \Big(\int_0^t \mathcal{G}(\boldsymbol{\sigma}_\eta(s),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)),\mathbf{k}_\eta(s))\,ds,\boldsymbol{\varepsilon}(\mathbf{v})\Big)_{\mathcal{H}}+j(\mathbf{u}_\eta(t),\mathbf{v}) \quad \forall \mathbf{v}\in V, \tag{4.19}$$

$$\Lambda^2\boldsymbol{\eta}(t)=\phi\big(\boldsymbol{\sigma}_\eta(t),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\mathbf{k}_\eta(t)\big). \tag{4.20}$$

Here, for every $\boldsymbol{\eta}\in L^2(0,T;V'\times Y)$, $\mathbf{u}_\eta$, $\boldsymbol{\sigma}_\eta$ represent the displacement field and the stress field obtained in Lemmas 4.2, 4.3 respectively, and $\mathbf{k}_\eta$ is the internal state variable given by (4.13). We have the following result.

Lemma 4.4. *The operator Λ has a unique fixed point $\boldsymbol{\eta}^*\in L^2(0,T;V'\times Y)$.*

Proof. Let $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in L^2(0,T;V'\times Y)$. Write

$$\mathbf{u}_{\eta_i}=\mathbf{u}_i,\qquad \dot{\mathbf{u}}_{\eta_i}=\mathbf{v}_{\eta_i}=\mathbf{v}_i,\qquad \boldsymbol{\sigma}_{\eta_i}=\boldsymbol{\sigma}_i,\qquad \mathbf{k}_{\eta_i}=\mathbf{k}_i$$

for $i=1,2$. Using (3.10), (3.13), (3.14) and (3.16) we have

$$|\Lambda^1\boldsymbol{\eta}_1(t)-\Lambda^1\boldsymbol{\eta}_2(t)|_{V'}^2 \le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+\int_0^t |\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2\,ds + \int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\,ds+\int_0^t |\mathbf{k}_1(s)-\mathbf{k}_2(s)|_Y^2\,ds\Big). \tag{4.21}$$

We use estimate (4.15) to obtain

$$
\begin{aligned}
(4.22)\qquad & |\Lambda^1\boldsymbol{\eta}_1(t)-\Lambda^1\boldsymbol{\eta}_2(t)|_{V'}^2\\
&\le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+\int_0^t|\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\,ds\\
&\qquad+\int_0^t|\mathbf{k}_1(s)-\mathbf{k}_2(s)|_Y^2\,ds\Big).
\end{aligned}
$$

By similar arguments, from (4.20), (4.15) and (3.15) it follows that

$$
\begin{aligned}
(4.23)\qquad & |\Lambda^2\boldsymbol{\eta}_1(t)-\Lambda^2\boldsymbol{\eta}_2(t)|_{Y}^2\\
&\le C(|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+|\mathbf{k}_1(t)-\mathbf{k}_2(t)|_Y^2)\\
&\le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+\int_0^t|\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\,ds\\
&\qquad+|\mathbf{k}_1(t)-\mathbf{k}_2(t)|_Y^2+\int_0^t|\mathbf{k}_1(s)-\mathbf{k}_2(s)|_Y^2\,ds\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4.24)\qquad & |\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)|_{V'\times Y}^2\le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2\\
&+\int_0^t|\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\,ds+|\mathbf{k}_1(t)-\mathbf{k}_2(t)|_Y^2+\int_0^t|\mathbf{k}_1(s)-\mathbf{k}_2(s)|_Y^2\,ds\Big).
\end{aligned}
$$

Moreover, from (4.4) we obtain

$$
\begin{aligned}
&(\dot{\mathbf{v}}_1-\dot{\mathbf{v}}_2,\mathbf{v}_1-\mathbf{v}_2)_{V'\times V}+(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1)-\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2),\boldsymbol{\varepsilon}(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}\\
&\qquad+(\boldsymbol{\eta}_1^1-\boldsymbol{\eta}_2^1,\mathbf{v}_1-\mathbf{v}_2)_{V'\times V}=0,\qquad \text{a.e. } t\in(0,T).
\end{aligned}
$$

We integrate this equality with respect to time, and use the initial conditions $\mathbf{v}_1(0)=\mathbf{v}_2(0)=\mathbf{v}_0$, condition (3.10) and (3.12) to find

$$
m_{\mathcal{A}}\int_0^t|\mathbf{v}_1(s)-\mathbf{v}_2(s)|_V^2\,ds\le-\int_0^t(\boldsymbol{\eta}_1^1(s)-\boldsymbol{\eta}_2^1(s),\mathbf{v}_1(s)-\mathbf{v}_2(s))_{V'\times V}\,ds
$$

for all $t\in[0,T]$. Then, using the inequality $2ab\le a^2/\gamma+\gamma b^2$ we obtain

$$
(4.25)\qquad \int_0^t|\mathbf{v}_1(s)-\mathbf{v}_2(s)|_V^2\,ds\le C\int_0^t|\boldsymbol{\eta}_1^1(s)-\boldsymbol{\eta}_2^1(s)|_{V'}^2\,ds\qquad\forall t\in[0,T].
$$

On the other hand, from (4.13) we have

$$
(4.26)\qquad |\mathbf{k}_1(t)-\mathbf{k}_2(t))|_Y^2\le C\int_0^t|\boldsymbol{\eta}_1^2(s)-\boldsymbol{\eta}_2^2(s)|_Y^2\,ds.
$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$, we get

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \le C \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2 \, ds.$$

From this inequality and (4.24) we obtain

$$|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)|_{V'\times Y}^2 \le C\Big(\int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2 \, ds + |\mathbf{k}_1(t) - \mathbf{k}_2(t)|_Y^2 + \int_0^t |\mathbf{k}_1(s) - \mathbf{k}_2(s)|_Y^2 \, ds\Big).$$

It follows now from (4.25) and (4.26) that

$$|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)|_{V'\times Y}^2 \le C \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_{V'\times Y}^2 \, ds.$$

Reiterating this inequality m times leads to

$$|\Lambda^m\boldsymbol{\eta}_1 - \Lambda^m\boldsymbol{\eta}_2|_{L^2(0,T;V'\times Y)}^2 \le \frac{C^m T^m}{m!} |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|_{L^2(0,T;V'\times Y)}^2.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V' \times L^2(\Omega))$, and so Λ has a unique fixed point. ∎

Now, we have all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\boldsymbol{\eta}_* = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in L^2(0,T;V'\times Y)$ be the fixed point of Λ defined by (4.18)–(4.20) and denote

$$\mathbf{u} = \mathbf{u}_{\eta_*}, \qquad \mathbf{k} = \mathbf{k}_{\eta_*}, \tag{4.27}$$

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \boldsymbol{\sigma}_{\eta_*}. \tag{4.28}$$

We prove that $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ satisfies (3.24)–(3.27) and (4.1)–(4.3). Indeed, we write (4.14) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and use (4.27)–(4.28) to deduce that (3.24) is satisfied. We use (4.4) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and the first equality in (4.27) to find

$$(\ddot{\mathbf{u}}(t), \mathbf{v})_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\eta}^1(t), \mathbf{v})_{V'\times V} = (\mathbf{f}(t), \mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T). \tag{4.29}$$

The equalities $\Lambda^1(\boldsymbol{\eta}_*) = \boldsymbol{\eta}^1$ and $\Lambda^2(\boldsymbol{\eta}_*) = \boldsymbol{\eta}^2$ combined with (4.19), (4.20), (4.27) and (4.28) show that

$$(\boldsymbol{\eta}^1(t), \mathbf{v})_{V'\times V} = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}(t), \mathbf{v}) + \Big(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)) \, ds, \boldsymbol{\varepsilon}(\mathbf{v})\Big)_{\mathcal{H}} \quad \forall \mathbf{v} \in V, \tag{4.30}$$

$$\boldsymbol{\eta}^2(t) = \phi\big(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\big). \tag{4.31}$$

From (4.31) and (4.27) we infer that (3.26) is satisfied. We now substitute

(4.30) in (4.29) and use (3.24) to see that $(\mathbf{u},\boldsymbol{\sigma},\mathbf{k})$ satisfies (3.25). Next, (3.27), the regularity (4.1) and (4.3) follow from Lemma 4.2 and (4.13). The regularity $\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H})$ follows from Lemmas 4.2, 4.3, assumption (3.12) and (4.28). Finally (3.25) implies that

$$\rho\ddot{\mathbf{u}}(t)=\operatorname{Div}\boldsymbol{\sigma}(t)+\mathbf{f}_0(t)\quad \text{in } V',\ \text{a.e. } t\in(0,T),$$

and so $\operatorname{Div}\boldsymbol{\sigma}\in L^2(0,T;V')$ by (3.17) and (3.19). We deduce that the regularity (4.2) holds, which proves the existence part of Theorem 4.1.

The uniqueness is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.18)–(4.20) and the unique solvability of Problems PV_η and QV_η. ■

5. Continuous dependence. In this section we study the dependence of the solution of Problem PV on perturbations of contact conditions.

We suppose that (3.12)–(3.20) hold and denote by $(\mathbf{u},\boldsymbol{\sigma},\mathbf{k})$ the solution of Problem PV obtained in Theorem 4.1. Also, for all $\alpha>0$ we denote by p^α a perturbation of p which satisfies (3.16) with L_p replaced by L_p^α.

We introduce the functional j^α defined by (3.23), replacing p with p^α and we consider the following variational problem.

Problem PV^α. *Find a displacement field $\mathbf{u}^\alpha:[0,T]\to V$, a stress field $\boldsymbol{\sigma}^\alpha:[0,T]\to\mathcal{H}$ and an internal state variable $\mathbf{k}^\alpha:[0,T]\to Y$ such that for a.e. $t\in(0,T)$,*

$$\boldsymbol{\sigma}^\alpha(t)=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t))+\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}^\alpha(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}^\alpha(s)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(s)),\boldsymbol{\varepsilon}(\mathbf{u}^\alpha(s)),\mathbf{k}^\alpha(s))\,ds,\tag{5.1}$$

$$(\ddot{\mathbf{u}}^\alpha(t),\mathbf{v})_{V'\times V}+(\boldsymbol{\sigma}^\alpha(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}+j^\alpha(\mathbf{u}^\alpha(t),\mathbf{v})=(\mathbf{f}(t),\mathbf{v})_{V'\times V}\quad\forall\mathbf{v}\in V,\tag{5.2}$$

$$\dot{\mathbf{k}}^\alpha(t)=\phi(\boldsymbol{\sigma}^\alpha(t)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t)),\boldsymbol{\varepsilon}(\mathbf{u}^\alpha(t)),\mathbf{k}^\alpha(t)),\tag{5.3}$$

$$\mathbf{u}^\alpha(0)=\mathbf{u}_0,\qquad\dot{\mathbf{u}}^\alpha(0)=\mathbf{v}_0,\qquad\mathbf{k}^\alpha(0)=\mathbf{k}_0.\tag{5.4}$$

We deduce from Theorem 4.1 that for every $\alpha>0$, Problem PV^α has a unique solution $(\mathbf{u}^\alpha,\boldsymbol{\sigma}^\alpha,\mathbf{k}^\alpha)$ which satisfies (4.1)–(4.3). Assume that the contact function satisfies the following assumptions:

$$\begin{cases}\text{There exist } \beta\in\mathbb{R}_+ \text{ and } \theta:\,]0,\infty[\,\to[0,\infty[\text{ such that}\\ \text{(a) } |p^\alpha(\mathbf{x},r)-p(\mathbf{x},r)|\le\theta(\alpha)(|r|+\beta) \text{ for all } \alpha>0,\ r\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_3.\\ \text{(b) } \lim_{\alpha\to0}\theta(\alpha)=0.\\ \text{(c) There exists } L_0>0 \text{ such that } L_p^\alpha\le L_0 \text{ for all } \alpha>0.\end{cases}\tag{5.5}$$

Under these assumptions, we have the following convergence result.

THEOREM 5.1. *The solution* $(\mathbf{u}^\alpha, \boldsymbol{\sigma}^\alpha, \mathbf{k}^\alpha)$ *of Problem* PV^α *converges to the solution* $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ *of Problem* PV, *i.e.*

$$\mathbf{u}^\alpha \to \mathbf{u} \quad \text{in } W^{1,2}(0,T;V) \quad \text{as } \alpha \to 0, \tag{5.6}$$

$$\boldsymbol{\sigma}^\alpha \to \boldsymbol{\sigma} \quad \text{in } L^2(0,T;\mathcal{H}) \quad \text{as } \alpha \to 0, \tag{5.7}$$

$$\mathbf{k}^\alpha \to \mathbf{k} \quad \text{in } W^{1,2}(0,T;Y) \quad \text{as } \alpha \to 0. \tag{5.8}$$

In addition to the interest of this convergence result from the asymptotic analysis point of view, it is important from the mechanical point of view since it shows that small perturbations of contact conditions lead to small perturbations of the weak solution of the dynamic contact Problem P.

Proof. Let $\alpha > 0$. Below, C always denotes a positive constant which may depend on the data and on the solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ but does not depend on α, nor on the time variable, and whose value may change from place to place. Using (3.25) and (5.2) we obtain

$$\begin{aligned}(\ddot{\mathbf{u}}^\alpha(t) - \ddot{\mathbf{u}}(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t))_{V'\times V} + (\boldsymbol{\sigma}^\alpha(t) - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \\ + j^\alpha(\mathbf{u}^\alpha(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)) - j(\mathbf{u}(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)) = 0, \quad \text{a.e. } t \in (0,T).\end{aligned} \tag{5.9}$$

We define

$$\boldsymbol{\sigma}^{\alpha R}(t) = \boldsymbol{\sigma}^\alpha(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t)), \boldsymbol{\sigma}^R(t) = \boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) \tag{5.10}$$

and note that (3.24) and (5.1) yield

$$\boldsymbol{\sigma}^{\alpha R}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}^\alpha(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}^{\alpha R}(s), \boldsymbol{\varepsilon}(\mathbf{u}^\alpha(s)), \mathbf{k}^\alpha(s))\, ds, \tag{5.11}$$

$$\boldsymbol{\sigma}^{R}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}^{R}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds. \tag{5.12}$$

We use (3.26), (5.3) and since $\mathbf{k}^\alpha(0) = \mathbf{k}(0) = \mathbf{k}_0$ we have

$$\begin{aligned}&\mathbf{k}^\alpha(t) - \mathbf{k}(t) \\ &\quad = \int_0^t (\phi(\boldsymbol{\sigma}^{\alpha R}(s), \boldsymbol{\varepsilon}(\mathbf{u}^\alpha(s)), \mathbf{k}^\alpha(s)) - \phi(\boldsymbol{\sigma}^{R}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)))\, ds.\end{aligned} \tag{5.13}$$

We combine (5.9) and (5.10) to obtain

$$\begin{aligned}(\ddot{\mathbf{u}}^\alpha(t) - \ddot{\mathbf{u}}(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t))_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \\ = -(\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^R(t), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \\ + j(\mathbf{u}(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)) - j^\alpha(\mathbf{u}^\alpha(t), \dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)), \text{ a.e. } t \in (0,T).\end{aligned} \tag{5.14}$$

It follows from (3.12) that for a.e. $t \in (0,T)$,

$$\big(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t))\big)_{\mathcal{H}} \ge m_{\mathcal{A}} |\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)|_V^2. \tag{5.15}$$

Using (5.11) and (5.12) we deduce that

$$\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^{R}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}^{\alpha}(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}^{\alpha R}(s), \boldsymbol{\varepsilon}(\mathbf{u}^{\alpha}(s)), \mathbf{k}^{\alpha}(s)\big)\, ds - \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}^{R}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)\big)\, ds \quad \forall t \in [0, T]. \tag{5.16}$$

Combining (3.13), (3.14) and (3.10) we obtain

$$|\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^{R}(t)|_{\mathcal{H}} \le C\Big(|\mathbf{u}^{\alpha}(t) - \mathbf{u}(t)|_V + \int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds + \int_0^t |\boldsymbol{\sigma}^{\alpha R}(s) - \boldsymbol{\sigma}^{R}(s)|_{\mathcal{H}}\, ds + \int_0^t |\mathbf{k}^{\alpha}(s) - \mathbf{k}(s)|_Y\, ds\Big) \quad \forall t \in [0, T].$$

From (5.13), (3.15) and (3.10) we get

$$|\mathbf{k}^{\alpha}(t) - \mathbf{k}(t)|_Y \le C\Big(\int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds + \int_0^t |\boldsymbol{\sigma}^{\alpha R}(s) - \boldsymbol{\sigma}^{R}(s)|_{\mathcal{H}}\, ds + \int_0^t |\mathbf{k}^{\alpha}(s) - \mathbf{k}(s)|_Y\, ds\Big) \quad \forall t \in [0, T].$$

Adding the last two inequalities we find

$$|\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^{R}(t)|_{\mathcal{H}} + |\mathbf{k}^{\alpha}(t) - \mathbf{k}(t)|_Y \le C\Big(|\mathbf{u}^{\alpha}(t) - \mathbf{u}(t)|_V + \int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds + \int_0^t |\boldsymbol{\sigma}^{\alpha R}(s) - \boldsymbol{\sigma}^{R}(s)|_{\mathcal{H}}\, ds + \int_0^t |\mathbf{k}^{\alpha}(s) - \mathbf{k}(s)|_Y\, ds\Big).$$

Using Gronwall arguments we see that

$$|\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^{R}(t)|_{\mathcal{H}} + |\mathbf{k}^{\alpha}(t) - \mathbf{k}(t)|_Y \le C\Big(|\mathbf{u}^{\alpha}(t) - \mathbf{u}(t)|_V + \int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds\Big)$$

and

$$|\boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^{R}(t)|_{\mathcal{H}} \le C\Big(|\mathbf{u}^{\alpha}(t) - \mathbf{u}(t)|_V + \int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds\Big), \tag{5.17}$$

$$|\mathbf{k}^{\alpha}(t) - \mathbf{k}(t)|_Y \le C\Big(|\mathbf{u}^{\alpha}(t) - \mathbf{u}(t)|_V + \int_0^t |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)|_V\, ds\Big). \tag{5.18}$$

The inequality (5.17) shows that

$$(5.19)\qquad -(\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^{R}(t),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)))_{\mathcal{H}}$$
$$\le C\Big(|\mathbf{u}^{\alpha}(t)-\mathbf{u}(t)|_V+\int_0^t|\mathbf{u}^{\alpha}(s)-\mathbf{u}(s)|_V\,ds\Big)|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V,\quad \text{a.e. } t\in(0,T).$$

From the definition of the functionals j and j^{α} it follows that

$$j(\mathbf{u}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))-j^{\alpha}(\mathbf{u}^{\alpha}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))$$
$$=\int_{\Gamma_3}(p(u_\nu(t)-g)-p^{\alpha}(u_\nu(t)-g))(\dot{u}^{\alpha}_\nu(t))-\dot{u}_\nu(t))\,da$$
$$+\int_{\Gamma_3}(p^{\alpha}(u_\nu(t)-g)-p^{\alpha}(u^{\alpha}_\nu(t)-g))(\dot{u}^{\alpha}_\nu(t))-\dot{u}_\nu(t))\,da,\quad \text{a.e. } t\in(0,T).$$

Using (5.5), (3.11) and the fact that p^{α} satisfies (3.16) with L_p^{α} in place of L_p we deduce that

$$(5.20)\qquad j(\mathbf{u}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))-j^{\alpha}(\mathbf{u}^{\alpha}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))$$
$$\le C\big(\theta(\alpha)+|\mathbf{u}^{\alpha}(t)-\mathbf{u}(t)|_V\big)|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V,\quad \text{a.e. } t\in(0,T).$$

We substitute (5.15), (5.19) and (5.20) in (5.14) to obtain, for a.e. $t\in(0,T)$,

$$(5.21)\qquad (\ddot{\mathbf{u}}^{\alpha}(t)-\ddot{\mathbf{u}}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))_{V'\times V}+m_{\mathcal{A}}|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V^2$$
$$\le C\Big(\theta(\alpha)+|\mathbf{u}^{\alpha}(t)-\mathbf{u}(t)|_V+\int_0^t|\mathbf{u}^{\alpha}(s)-\mathbf{u}(s)|_V\,ds\Big)|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V,$$

and, using the inequality

$$ab\le\frac{1}{2m_{\mathcal{A}}}a^2+\frac{m_{\mathcal{A}}}{2}b^2,$$

after some algebra we find

$$(\ddot{\mathbf{u}}^{\alpha}(t)-\ddot{\mathbf{u}}(t),\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t))_{V'\times V}+\frac{m_{\mathcal{A}}}{2}|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V^2$$
$$\le C\Big(\theta^2(\alpha)+|\mathbf{u}^{\alpha}(t)-\mathbf{u}(t)|_V^2+\int_0^t|\mathbf{u}^{\alpha}(s)-\mathbf{u}(s)|_V^2\,ds\Big),\quad \text{a.e. } t\in(0,T).$$

We integrate this inequality on $[0,s]$ and use the initial condition $\dot{\mathbf{u}}^{\alpha}(0)=\dot{\mathbf{u}}(0)=\mathbf{v}_0$ to find

$$(5.22)\qquad \frac{m_{\mathcal{A}}}{2}\int_0^s\|\dot{\mathbf{u}}^{\alpha}(t)-\dot{\mathbf{u}}(t)|_V^2\,dt\le C\Big(\theta^2(\alpha)+\int_0^s|\mathbf{u}^{\alpha}(t)-\mathbf{u}(t)|_V^2\,dt\Big)\qquad\forall s\in[0,T].$$

Since $\dot{\mathbf{u}}^\alpha(0) = \dot{\mathbf{u}}(0) = \mathbf{v}_0$ we see that

$$(5.23)\qquad |\mathbf{u}^\alpha(s) - \mathbf{u}(s)|_V^2 \le C\int_0^s |\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)|_V^2\,dt \quad \forall s \in [0,T].$$

We substitute now (5.22) into (5.23) and use again the Gronwall inequality to find that

$$(5.24)\qquad |\mathbf{u}^\alpha(s) - \mathbf{u}(s)|_V^2 \le C\theta^2(\alpha) \quad \forall s \in [0,T].$$

From (5.22) and (5.23) we infer that

$$(5.25)\qquad \int_0^s |\dot{\mathbf{u}}^\alpha(t) - \dot{\mathbf{u}}(t)|_V^2\,dt \le C\theta^2(\alpha) \quad \forall s \in [0,T].$$

We combine now (5.24), (5.25) and assumption (5.5)(b) to see that (5.6) is satisfied. It follows from (5.10) that

$$\boldsymbol{\sigma}^\alpha(t) - \boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^{\alpha R}(t) - \boldsymbol{\sigma}^R(t) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \quad \text{a.e. } t \in (0,T).$$

Using this inequality, (5.17), the properties (3.12) of the operator $\mathcal{A}$ and (5.6) we see that (5.7) is satisfied. From (3.26), (5.3), the assumption (3.15) on ϕ, (5.17) and (5.18) we deduce that

$$(5.26)\qquad |\dot{\mathbf{k}}^\alpha(t) - \dot{\mathbf{k}}(t)|_Y^2 \le C\Big(\int_0^t |\dot{\mathbf{u}}^\alpha(s) - \dot{\mathbf{u}}(s)|_V^2\,ds\Big).$$

Since $\mathbf{k}^\alpha(0) = \mathbf{k}(0) = \mathbf{k}_0$ we find

$$(5.27)\qquad |\mathbf{k}^\alpha(t) - \mathbf{k}(t)|_Y^2 \le C\Big(\int_0^t |\dot{\mathbf{k}}^\alpha(s) - \dot{\mathbf{k}}(s)|_Y^2\,ds\Big).$$

We conclude now from (5.25)–(5.27) and (5.5)(b) that (5.8) is also satisfied. ■

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