APPLICATIONES MATHEMATICAE
40,1 (2013), pp. 63–90

IOANNIS K. ARGYROS (Lawton, OK)
SAÏD HILOUT (Poitiers)

EXPANDING THE APPLICABILITY OF TWO-POINT NEWTON-LIKE METHODS UNDER GENERALIZED CONDITIONS

Abstract. We use a two-point Newton-like method to approximate a locally unique solution of a nonlinear equation containing a non-differentiable term in a Banach space setting. Using more precise majorizing sequences than in earlier studies, we present a tighter semi-local and local convergence analysis and weaker convergence criteria. This way we expand the applicability of these methods. Numerical examples are provided where the old convergence criteria do not hold but the new convergence criteria are satisfied.

1. Introduction. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Let $U(w,R)$ and $\overline{U}(w,R)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center w and radius $R > 0$. Denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

where $F : \overline{U}(w,R) \to \mathcal{Y}$ is Fréchet-differentiable around $x_0 \in U(w,R)$ and $G : \overline{U}(w,R) \to \mathcal{Y}$ is continuous.

Many problems from computational sciences can be brought into the form of equation (1.1) using mathematical modelling [8], [11], [19]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are usually iterative. In particular, the practice of numerical analysis for finding such solutions usually involves Newton-like methods [8], [11], [29], [30].

2010 *Mathematics Subject Classification*: 65J15, 65G99, 47H99, 49M15.
Key words and phrases: Newton-like methods, Banach space, majorizing sequence, Kantorovich hypothesis, Fréchet derivative, divided difference.

DOI: 10.4064/am40-1-4

In [4], [8], [11], [19] we used the single-step Newton-like method defined by

$$\begin{aligned} &x_0 \text{ is a starting point in } U(w,R), \\ &x_{n+1} = x_n - A(x_n)^{-1}(F(x_n)+G(x_n)) \quad \text{for each } n=0,1,\ldots, \end{aligned} \tag{1.2}$$

as well as the two-point Newton-like method given by

$$\begin{aligned} &y_{-1}, y_0 \text{ are starting points in } U(w,R), \\ &y_{n+1} = y_n - A(y_n,y_{n-1})^{-1}(F(y_n)+G(y_n)) \quad \text{for each } n=0,1,\ldots \end{aligned} \tag{1.3}$$

to generate respectively sequences $\{x_n\}$ and $\{y_n\}$ approximating $x^\star$, where $A(x)$, $A(v,w)$ belong in $\mathcal{L}(\mathcal{X},\mathcal{Y})$. If $A(x)=F'(x)$ for each $x\in U(w,R)$, we obtain the Krasnosel'skiĭ–Zinčenko iteration [11], [19], [50], [51]. Moreover, if $G(x)=0$ for each $x\in U(w,R)$, we obtain Newton's method [8], [11], [19], [29] given by

$$\begin{aligned} &x_0 \text{ is a starting point in } U(w,R), \\ &x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for each } n=0,1,\ldots. \end{aligned} \tag{1.4}$$

If $A(x,y)=F'(x)+[x,y;G]$ where $[x,y;G]$ is the divided difference operator of order one for G, we obtain a secant-type method studied in [8], [20]. Several other choices of operators A, F and G in (1.2)–(1.4) are given in [1]–[51].

The study of convergence of Newton methods usually centers around two types of convergence analysis: semi-local and local. Semi-local convergence analysis is based on information around the initial point, to give criteria ensuring the convergence of Newton methods; while local convergence analysis is based on information around a solution, to find estimates of the radii of convergence balls. There is a plethora of studies on weakening and/or extending of the hypotheses made on the underlying operators; see for example [8], [11], [19], [29], [30] and the references therein. Concerning semi-local convergence of Newton methods, one of the most important results is the celebrated Kantorovich theorem for solving nonlinear equations. It provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or with Lipschitz continuous first derivatives. The second type analysis for Newton methods is local convergence. Traub and Woźniakowski [47], Rheinboldt [44], [45], Rall [43], Argyros [8] and other authors gave estimates of the radii of local convergence balls when the Fréchet derivatives are Lipschitz continuous around a solution.

In the present paper, using more precise majorizing sequences we provide convergence criteria and a tighter semi-local and local convergence analysis for single and two-point Newton-like methods (1.2) and (1.3) than in [4], [50], [51]. This way we expand the applicability of these methods.

The paper is organized as follows. Section 2 concerns semi-local convergence for single-step methods. In Section 3 we deal with semi-local convergence for two-step Newton-like methods. Local convergence is also studied in this section. Special cases and applications are given in the concluding Section 4.

2. Semi-local convergence for single-step methods. Let $R > 0$ be a constant and $r \in [0, R]$. Suppose there exist $w \in \mathcal{X}$ such that $A(w)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and for any $x, y \in \overline{U}(w, r)$, $\theta \in [0, 1]$ the following conditions hold:

$$\|A(w)^{-1}(A(x) - A(w))\| \leq g_0(\|x - w\|) + \beta \tag{2.1}$$

and

$$\begin{aligned}\|A(w)^{-1}((F'(x + \theta(y - x)) - A(x))(y - x) + G(y) - G(x))\| \\ \leq (g_1(\|x - w\| + \theta\|y - x\|) - g_2(\|x - w\|) + g_3(r) + \gamma)\|y - x\|,\end{aligned} \tag{2.2}$$

where $g_0(r)$, $g_1(r + \overline{r}) - g_2(r)$ ($\overline{r} \geq 0$), $g_2(r)$, $g_3(r)$ are non-decreasing and continuous functions for r in $[0, R]$, $[0, R]^2$, $[0, R]$, $[0, R]$, respectively, $g_i(0) = 0$ for $i = 0, 1, 2, 3$ and β, γ are constants which satisfy $\beta \geq 0$, $\gamma \geq 0$.

Hypotheses (2.1) and (2.2) were used in [4] to provide a semi-local convergence analysis for single-step Newton-like methods. The majorizing sequence $\{t_n\}$ for $\{x_n\}$ was given by

$$\begin{aligned} &t_0 = r_0 \in [r, R], \qquad t_1 = r_0 + \eta \text{ for some } \eta \geq 0, \\ &t_{n+2} = t_{n+1} + \delta_n(t_{n+1} - t_n) \qquad \text{for each } n = 0, 1, \ldots, \end{aligned} \tag{2.3}$$

where

$$\delta_n = \frac{\int_0^1 (g_1(t_n + \theta(t_{n+1} - t_n)) - g_2(t_n) + \gamma)\, d\theta + g_3(t_{n+1})}{1 - \beta - g_0(t_{n+1})} \tag{2.4}$$

for each $n = 0, 1, \ldots.$ Under the sufficient convergence conditions of [4, Theorem 3], we showed convergence of $\{x_n\}$ to $x^\star$ with the following error bounds:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{2.5}$$

$$\|x_{n+1} - x^\star\| \leq t^\star - t_n, \tag{2.6}$$

where

$$t^\star = \lim_{n\to\infty} t_n.$$

Next, we show how to improve these results. Let $x_0 \in \mathcal{X}$. Assume $A(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and for $x_0, x_1 \in \overline{U}(w, r)$ with $x_1 = x_0 - A(x_0)^{-1}(F(x_0) + G(x_0))$,

$\theta \in [0,1]$, the following conditions hold:

$$\|A(x_0)^{-1}(A(x_1) - A(x_0))\| \le g_0^0(\|x_1 - x_0\|) + \beta_0 \tag{2.7}$$

and

$$\begin{aligned} \|A(x_0)^{-1}((F'(x_0+\theta(x_1-x_0)) - A(x_0))(x_1-x_0) + G(x_1) - G(x_0))\| \\ \le (g_1^0(\theta\|x_1 - x_0\|) - g_2^0(0) + g_3^0(\|x_1 - x_0\|) + \gamma_0)\|x_1 - x_0\|, \end{aligned} \tag{2.8}$$

where g_i^0 $(i = 0,1,2,3)$, β_0 and γ_0 are as g_i $(i = 0,1,2,3)$, β and γ, respectively. We shall denote by $(\mathcal{C}_1)$ the conjunction of conditions (2.1), (2.2), (2.7) and (2.8). Let us define a sequence $\{s_n\}$ by

$$\begin{aligned} &s_0 = r_0 \in [r, R], \quad s_1 = r_0 + \eta, \quad s_2 = s_1 + \delta\ (s_1 - s_0), \\ &s_{n+2} = s_{n+1} + \alpha_n(s_{n+1} - s_n) \quad \text{for each } n = 0,1,\dots, \end{aligned} \tag{2.9}$$

where

$$\delta = \frac{\int_0^1 (g_1^0(s_0 + \theta(s_1 - s_0)) - g_2^0(s_0) + \gamma_0)\, d\theta + g_3^0(s_1)}{1 - \beta_0 - g_0^0(s_1)}$$

and

$$\alpha_n = \frac{\int_0^1 (g_1(s_n + \theta(s_{n+1} - s_n)) - g_2(s_n) + \gamma)\, d\theta + g_3(s_{n+1})}{1 - \beta - g_0(s_{n+1})} \tag{2.10}$$

for each $n = 1,2,\dots$. Suppose that for each $r \in [0,R]$ and $r_1 \in [0, R - r]$, the following conditions hold:

$$g_1^0(r + \overline{r}) - g_2^0(r) \le g_1(r + \overline{r}) - g_2(r), \tag{2.11}$$

$$g_0^0(r) \le g_0(r), \quad g_1^0(r) \le g_1(r), \tag{2.12}$$

$$g_2^0(r) \le g_2(r), \quad g_3^0(r) \le g_3(r), \tag{2.13}$$

$$\beta_0 \le \beta, \tag{2.14}$$

$$\gamma_0 \le \gamma. \tag{2.15}$$

Then a simple inductive argument shows that for each $n = 0,1,\dots$,

$$s_n \le t_n, \tag{2.16}$$

$$s_{n+1} - s_n \le t_{n+1} - t_n, \tag{2.17}$$

$$s^\star = \lim_{n\to\infty} s_n \le t^\star. \tag{2.18}$$

Next, we first provide sufficient conditions for the convergence of $\{s_n\}$. Then we show $\{s_n\}$ is a majorizing sequence for $\{x_n\}$. We define functions p_n^N, q_n^N, p_∞^N on $[0,1)$ for each fixed $N = 1,2,\dots$ and each $n = N, N+1, \dots$

by

$$p_n^N(t) = \int_0^1 \left(g_1\left(\frac{1-t^{n-1}}{1-t}(s_{N+1}-s_N)+s_N+\theta t^{n-1}(s_{N+1}-s_N)\right) - g_2\left(\frac{1-t^{n-1}}{1-t}(s_{N+1}-s_N)+s_N\right)\right) d\theta + \gamma + g_3\left(\frac{1-t^n}{1-t}(s_{N+1}-s_N)+s_N\right) + tg_0\left(\frac{1-t^n}{1-t}(s_{N+1}-s_N)+s_N\right) - t(1-\beta), \tag{2.19}$$

$$q_n^N(t) = p_{n+1}^N(t) - p_n^N(t) \tag{2.20}$$

$$p_\infty^N(t) = g_1\left(\frac{s_{N+1}-s_N}{1-t}+s_N\right) - g_2\left(\frac{s_{N+1}-s_N}{1-t}+s_N\right) + \gamma + g_3\left(\frac{s_{N+1}-s_N}{1-t}+s_N\right) + tg_0\left(\frac{s_{N+1}-s_N}{1-t}+s_N\right) - t(1-\beta). \tag{2.21}$$

Then we can show the following results on majorizing sequences for single-step Newton-like methods.

Lemma 2.1. *Suppose*

$$g_0(s_1) < 1 - \beta_0 \tag{2.22}$$

and there exists $\alpha \in (0,1)$ *such that*

$$0 \le \alpha_1 \le \alpha, \tag{2.23}$$

$$q_n^1(\alpha) \ge 0, \tag{2.24}$$

$$p_\infty^1(\alpha) \le 0. \tag{2.25}$$

Then the sequence $\{s_n\}$ *given by* (2.9) *is well defined, non-decreasing, bounded from above by*

$$s_1^{\star\star} = s_1 + \frac{s_2 - s_1}{1-\alpha} \tag{2.26}$$

and converges to its unique least upper bound $s^\star \in [0, s_1^{\star\star}]$. *Moreover,*

$$0 \le s_{n+2} - s_{n+1} \le (s_2 - s_1)\alpha^n \quad \textit{for each } n = 1,2,\dots. \tag{2.27}$$

Proof. By (2.9) and (2.22), s_2 is well defined and $s_2 \ge s_1$. We use induction to prove that

$$0 < \alpha_k \le \alpha \quad \text{for each } k = 1,2,\dots. \tag{2.28}$$

Estimate (2.28) holds for $k=1$ by (2.23). Then by (2.9) and (2.23) we have

$$(2.29)\qquad \begin{aligned} 0 \le s_3 - s_2 \le \alpha(s_2 - s_1) &\Rightarrow s_3 \le s_2 + \alpha(s_2 - s_1)\\ &\Rightarrow s_3 \le s_2 + (1+\alpha)(s_2 - s_1) - (s_2 - s_1)\\ &\Rightarrow s_3 \le s_1 + \frac{1-\alpha^2}{1-\alpha}(s_2 - s_1) < s_1^{\star\star}. \end{aligned}$$

Assume that (2.28) holds for all $n \le k$. Then by (2.9) and (2.28) we get

$$(2.30)\qquad 0 \le s_{k+2} - s_{k+1} \le s^k (s_2 - s_1)$$

and

$$(2.31)\qquad s_{k+2} \le s_1 + \frac{1-\alpha^{k+1}}{1-\alpha}(s_2 - s_1) < s_1^{\star\star}.$$

Evidently estimate (2.28) is true with k replaced by $k+1$ provided that

$$(2.32)\qquad p_k^1(\alpha) \le 0 \quad \text{for each } k = 1, 2, \dots.$$

By (2.21) and (2.24) we have

$$(2.33)\qquad p_k^1(\alpha) \le p_{k+1}^1(\alpha) \quad \text{for each } k = 1, 2, \dots.$$

Define

$$(2.34)\qquad p_\infty^1(t) = \lim_{k\to\infty} p_k^1(t).$$

Using (2.20) and letting $n \to \infty$ we see that p_∞^1 is given by (2.21) for $N = 1$. Estimate (2.32) holds by (2.34) and (2.25). The induction is now complete. Hence, $\{s_n\}$ is increasing, bounded from above by $s_1^{\star\star}$ given by (2.26) and it converges to its unique least upper bound $s^\star \in [0, s_1^{\star\star}]$. ∎

Remark 2.2. The conclusions of Lemma 2.1 hold if (2.24) and (2.25) are replaced by

$$(2.35)\qquad q_n^1(\alpha) \le 0,$$

$$(2.36)\qquad p_1^1(\alpha) \le 0.$$

In this case, we have $p_{k+1}^1(\alpha) \le p_k^1(\alpha) \le p_1^1(\alpha) \le 0$.

We have the following useful and obvious extension of Lemma 2.1.

Lemma 2.3. *Suppose there exists a minimum natural integer $N > 1$ and $\alpha \in (0,1)$ such that* (2.24) *and* (2.25),

$$s_1 \le s_2 \le \dots \le s_{N+1}, \qquad 0 < \alpha_N \le \alpha, \qquad g_0(s_{N+1}) < 1 - \beta,$$
$$g_0\left(\frac{s_{N+1} - s_N}{1-t}(s_{N+1} - s_N) + s_N\right) < 1 - \beta.$$

Then the sequence $\{s_n\}$ given by (2.9) *is well defined, non-decreasing, bounded* *from above by*

$$s_N^{\star\star} = s_{N-1} + \frac{s_N - s_{N-1}}{1-\alpha}$$

and converges to its unique least upper bound $s_N^\star \in [0, s_N^{\star\star}]$. *Moreover,*

$$0 \le s_{N+n} - s_{N+n-1} \le (s_{N+1} - s_N)\alpha^n \quad \textit{for each } n = 1, 2, \ldots.$$

Next, we present upper bounds on the limit point $t^\star$ using $\{t_n\}$. Set

$$\nu_i(t) = \frac{g_i(t)}{1-\beta} \quad \text{for all } i = 0,1,2,3 \quad \text{and} \quad \gamma_1 = \frac{\gamma}{1-\beta}.$$

Then iteration (2.3) can be written in the form

$$\begin{aligned} &t_0 = r_0 \in [r, R], \qquad t_1 = r_0 + \eta, \\ &t_{n+2} = t_{n+1} + \delta_n^1 (t_{n+1} - t_n) \quad \text{for each } n = 0, 1, \ldots, \end{aligned} \tag{2.37}$$

where

$$\delta_n^1 = \frac{\int_0^1 (\nu_1(t_n + \theta(t_{n+1} - t_n)) - \nu_2(t_n) + \gamma_1)\, d\theta + \nu_3(t_{n+1})}{1 - \nu_0(t_{n+1})}$$

for each $n = 0, 1, \ldots$. Assume for the rest of this section that ν_4 (i.e. g_4) is strictly increasing on $[0, R]$. Define a function $\chi(s,t)$ on $\{(s,t) \in [0,R]^2 : s \le t\}$ by

$$\chi(s,t) = \Big(\int_0^1 (\nu_1(s + \theta(t-s)) - \nu_2(t-s) + \gamma_1)\, d\theta + \nu_3(t-s)\Big)(t-s).$$

The result on upper bounds on the limit point $t^\star$ using $\{t_n\}$ is as follows.

Lemma 2.4. *Let* $\lambda \in [0, \nu_4^{-1}(1)]$. *Let* f, ν_4 *be differentiable functions on* $[0, \nu_4^{-1}(1)]$. *Suppose* f *has a zero in* $[\lambda, \nu_4^{-1}(1)]$; *denote by* ϱ *the smallest such zero. Define functions* φ *and* g *on* $[0, \nu_4^{-1}(1))$ *by*

$$\varphi(t) = \frac{f(t)}{1 - \nu_4(t)} \quad \textit{and} \quad g(t) = t + \varphi(t).$$

Moreover, suppose

$$g'(t) > 0 \quad \textit{for each } t \in [\lambda, \varrho].$$

Then g *is strictly increasing and bounded above by* ϱ.

Proof. The function φ is well defined on $[\lambda, \varrho]$ with the possible exception when $\varrho = \nu_4^{-1}(1)$; but the L'Hospital theorem implies that f admits a continuous extension on the interval $[\lambda, \varrho]$. The function g is strictly increasing, since $g'(t) > 0$ on $[\lambda, \varrho]$. Therefore, for each $t \in [\lambda, \varrho]$ we have

$$g(t) = t + \varphi(t) \le \varrho + \varphi(\varrho) = \varrho. \ \blacksquare$$

Lemma 2.5. *Suppose that the hypotheses of Lemma 2.4 hold. Define functions ϕ and ψ on $\mathcal{I} := \{(s,t) \in [\lambda, \varrho]^2 : s \le t\}$ by*

$$\phi(s,t) = t + \frac{\chi(s,t)}{1-\nu_4(t)}, \qquad \psi(s,t) = \begin{cases} \chi(s,t) - f(t) & \text{if } t \neq \varrho, \\ 0 & \text{if } t = \varrho. \end{cases}$$

Moreover, suppose that $\psi(s,t) \le 0$ for each $(s,t) \in \mathcal{I}$. Then

$$\phi(s,t) \le g(t) \quad \text{for each } (s,t) \in \mathcal{I}.$$

Proof. The result follows immediately from the definitions of g, ϕ, ψ and the hypothesis of the lemma. ∎

Lemma 2.6. *Fix $N = 0,1,2,\ldots$. Under the hypotheses of Lemma 2.5 with $\lambda = t_N$, further suppose that*

$$t_1 \le \cdots \le t_{N+1} \le \varrho \quad \text{and} \quad f(t_{N+1}) \ge 0.$$

Then the sequence $\{t_n\}$ generated by (2.37) is non-decreasing, bounded by ϱ and converges to its unique least upper bound $t^\star$ which satisfies $t^\star \in [t_N, \varrho]$.

Proof. We can write $t_{n+1} = \phi(t_{n-1}, t_n)$. Then

$$t_{N+2} = \phi(t_N, t_{N+1}) \le g(t_{N+1}) \le \varrho. \ ∎$$

Remark 2.7. (a) The hypotheses of Lemma 2.1 or Lemma 2.3 are satisfied in the interesting case of Newton's method [8], [29]. Set $\beta_0 = \gamma_0 = \beta = \gamma = 0$, $g_0(t) = L_0 t$, $g_1(t) = g_2(t) = Lt$ and $g_3(t) = 0$. Then by (2.19) and (2.20) we get

$$g_n^N(t) = \tfrac{1}{2}(2L_0t^2 + Lt - L)t^n(s_{N+1} - s_N),$$
$$p_\infty^N(t) = t\bigg(\frac{L_0(s_{N+1} - s_N)}{1-t} + s_N\bigg).$$

Set

$$\alpha = \frac{2L}{L + \sqrt{L^2 + 8LL_0}}.$$

Then the hypotheses of Lemma 2.1 are satisfied provided that

$$h_3 = L_3\eta \le 1/2, \tag{2.38}$$

where

$$L_3 = \tfrac{1}{8}((L_0L)^{1/2} + 4L_0 + (8L_0^2 + L_0L)^{1/2}).$$

Moreover, the hypotheses of Lemma 2.3 become

$$s_1 \le \cdots \le s_N < 1/L_0, \quad 0 < \alpha_N \le \alpha \le 1 - \frac{L_0(s_{N+1} - s_N)}{1 - L_0 s_N}.$$

Another interesting choice but not necessarily the best possible is given by the following scheme. Choose

$$f(t)=\frac{\frac{L_0}{2}t^2-t+\frac{L_3}{L_0}(t_N-t_{N-1})}{1-L_0t}\quad\text{and}\quad\lambda=t_N\text{ for }N=1,2,\dots.$$

Suppose that

$$h_3^N=L_3(t_N-t_{N-1})\le 1/2.$$

Set

$$\varrho_3^N=\frac{2(t_N-t_{N-1})L_3}{L_0(1+(1-2L_3(t_N-t_{N-1}))^{1/2})},\qquad r_3=\left(1+\frac{L_0\eta}{2(1-\alpha)(1-L_0\eta)}\right)\eta.$$

Moreover, suppose

$$\psi(t_N,t)\le 0\quad\text{for each } t=g(t_N)\in[t_N,\varrho_3^N].$$

Then ϱ_3^N are well defined. Elementary computations now show that all hypotheses of Lemma 2.6 are satisfied for $\varrho=\varrho_3^N$.

Moreover, the following estimates hold:

$$\frac{2(t_N-t_{N-1})}{1+(1-2L_3(t_N-t_{N-1}))^{1/2})}\le\varrho_3^N\le\frac{1}{L_0},\qquad\frac{L_3}{L_0}\ge 1.$$

Furthermore, if

$$h_\star=L\eta\le 1/2$$

and

$$(L_0^2+LL_3)(t_N-t_{N-1})\le L_0(1-(1-2h_3^N)^{1/2}(1-2h_\star)^{1/2}),$$

then

$$\varrho_3^N\le\varrho^\star=\frac{2(t_N-t_{N-1})}{1+(1-2h_\star)^{1/2}}.$$

We also see that for sufficiently small t_N-t_{N-1}, ϱ_3^N is smaller than r_3.

(b) The hypotheses of Lemma 2.6 are satisfied in the Kantorovich case [8], [11], [19], [29]. Indeed, set $\beta=\gamma=0$, $g_0(t)=g_1(t)=g_2(t)=Lt$ and $g_3(t)=0$. Then we must have

$$h_\star\le\frac{1}{2}\quad\text{and}\quad\varrho=\frac{1-\sqrt{1-2h_\star}}{L}\quad\text{for}\quad f(t)=\frac{L}{2}t^2-t+\eta.$$

(c) The set $\mathcal{I}$ can be replaced by the more practical $\mathcal{J}=[\lambda,\varrho]^2$ in Lemma 2.6.

We can now present our semi-local convergence results for single-step Newton-like methods. The proofs are omitted since they can be found in [4, Theorem 3] by simply replacing the hypotheses of [4, Lemma 2] by those of Lemma 2.1, 2.3 or 2.6 of the present paper.

Theorem 2.8. *Suppose that* (2.1), (2.2) *and the hypotheses of Lemma* 2.6 *hold. Moreover, suppose that there exists* $x_0 \in \overline{U}(w, r_0)$ *such that*

$$\|A(x_0)^{-1}(F(x_0) + G(x_0))\| \leq \eta. \tag{2.39}$$

Then the sequence $\{x_n\}$ *generated by the single-step Newton-like method is well defined, remains in* $\overline{U}(w, t^\star)$ *for each* $n = 0, 1, \ldots$ *and converges to a solution* $x^\star$ *of the equation* $F(x)+G(x) = 0$. *Moreover, for each* $n = 0, 1, \ldots,$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \quad \text{and} \quad \|x_n - x^\star\| \leq t^\star - t_n \leq \varrho - t_n.$$

Furthermore, the solution $x^\star$ *is unique in* $\overline{U}(x_0, t^\star)$ *if*

$$\int_0^1 (g_1(t^\star + \theta t^\star) - g_2(t^\star))\, d\theta + g_3(t^\star) + g_0(t^\star) + \beta + \gamma < 1,$$

and in $\overline{U}(x_0, R_0)$ *if* $t^\star < R_0 < R$ *and*

$$\int_0^1 (g_1(t^\star + \theta R_0) - g_2(t^\star))\, d\theta + g_3(t^\star + R_0) + g_0(t^\star) + \beta + \gamma < 1.$$

Theorem 2.9. *Suppose that conditions* $(\mathcal{C}_1)$, (2.39) *and the hypotheses of Lemma* 2.1 *or Lemma* 2.3 *hold. Then the conclusions of Theorem* 2.8 *hold with* $\{s_n\}$ *replacing* $\{t_n\}$.

The local convergence analysis of single-step Newton-like methods is given in the next section as a special case of the two-point Newton-like method.

3. Convergence for two-point Newton-like methods. We present our results for the semi-local and local convergence of two-point Newton-like methods. As in [4], suppose there exist $v, w \in \mathcal{X}$ such that $A(v, w)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and for each $x, y, z \in \overline{U}(w, r)$, $\theta \in [0, 1]$ the following conditions hold:

$$\|A(v, w)^{-1}(A(x, y) - A(v, w))\| \leq f_0(\|x - v\|, \|y - w\|) + \beta \tag{3.1}$$

and

$$\begin{aligned} &\|A(v, w)^{-1}((F'(y + \theta(z - y)) - A(x, y))(z - y) + G(x) - G(y))\| \\ &\quad \leq (f_1(\|y - w\| + \theta\|z - y\|) - f_2(\|y - w\|) + f_3(\|z - x\|) + \gamma)\|z - y\|, \end{aligned} \tag{3.2}$$

where $f_0(r, s)$ is continuous on $[0, R]^2$ and decreasing with respect to each argument, $f_1(r + \overline{r}) - f_2(r)$ $(\overline{r} \geq 0)$, $f_2(r)$, $f_3(r)$ are non-decreasing and continuous functions on $[0, R]$ with $f_0(0, 0) = f_1(0) = f_2(0) = f_3(0) = 0$, and the constants β, γ satisfy $\beta \geq 0$, $\gamma \geq 0$ and $\beta + \gamma < 1$. Using (3.1) and (3.2) a semi-local convergence analysis was given by us in [4, Theorem 2].

We now show how to improve these results under the same hypotheses (3.1) and (3.2). Let $r_0 \in [0, R]$, $y_{-1} \in \overline{U}(w, R)$ and $y_0 \in \overline{U}(w, r_0)$. It follows from (3.1) and (3.2) that

$$\|A(v,w)^{-1}(A(y_{-1},y_0)-A(v,w))\| \le f_0^0(\|v-y_{-1}\|,\|w-y_0\|)+\beta_0 \tag{3.3}$$

and

(3.4)
$$\|A(v,w)^{-1}((F'(y_0+\theta(y_1-y_0))-A(y_{-1},y_0))(y_1-y_0)+G(y_1)-G(y_0))\|$$
$$\le (f_1^0(\|y_0-w\|+\theta\|y_1-y_0\|)-f_2^0(\|y_0-w\|)+f_3^0(\|y_1-y_{-1}\|)+\gamma_0)\|y_1-y_0\|,$$

where f_i^0 $(i=0,1,2,3)$, β_0, γ_0 are as f_i $(i=0,1,2,3)$, β, γ, respectively. Clearly

$$f_1^0(r+\overline{r})-f_2^0(r) \le f_1(r+\overline{r})-f_2(r), \tag{3.5}$$

$$f_0^0(r,t) \le f_0(r,t), \tag{3.6}$$

$$f_1^0(r) \le f_1(r), \tag{3.7}$$

$$f_2^0(r) \le f_2(r), \tag{3.8}$$

$$f_3^0(r) \le f_3(r), \tag{3.9}$$

$$\beta_0 \le \beta \tag{3.10}$$

$$\gamma_0 \le \gamma. \tag{3.11}$$

In practice, the computation of the functions f_i and constants β, γ requires that of f_i^0, β_0, γ_0 for $i=0,1,2,3$. Given y_{-1}, y_0 in $\mathcal{X}$, define c_{-1}, c_0, c_1 by

$$\|y_{-1}-v\| \le c_{-1}, \quad \|y_{-1}-y_0\| \le c_0 \quad \text{and} \quad \|v-w\| \le c_1. \tag{3.12}$$

We shall refer to (3.1)–(3.4) and (3.12) as *conditions* $(\mathcal{C}_2)$. The majorizing sequence $\{t_n\}$ for $\{y_n\}$ was given in [4] by

$$\begin{aligned} &t_{-1}=r_0, \quad t_0=c_0+r_0, \quad t_1=c_0+r_0+\eta \quad \text{for some } \eta\ge 0,\\ &t_{n+2}=t_{n+1}+\delta_n(t_{n+1}-t_n) \quad \text{for each } n=0,1,\ldots, \end{aligned} \tag{3.13}$$

where

$$\delta_n = \frac{\int_0^1 (f_1(t_n-t_0+r_0+\theta(t_{n+1}-t_n))-f_2(t_n-t_0+r_0))\,d\theta+\gamma+f_3(t_{n+1}-t_{n-1})}{1-\beta-f_0(t_n-t_{n-1}+c_{-1},t_{n+1}-t_0+r_0)}$$

for each $n=0,1,\ldots$.

The following estimates hold for each $n=0,1,\ldots$:

$$\|y_{n+1}-y_n\| \le t_{n+1}-t_n, \tag{3.14}$$

$$\|y_n-x^\star\| \le t^\star-t_n, \tag{3.15}$$

where $t^\star=\lim_{n\to\infty} t_n$.

In the present paper, we use the more precise majorizing sequence $\{s_n\}$ given by

$$(3.16)\qquad \begin{aligned} &s_{-1}=t_{-1},\quad s_0=t_0,\quad s_1=t_1,\quad s_2=s_1+\alpha_0(s_1-s_0),\\ &s_{n+2}=s_{n+1}+\alpha_n(s_{n+1}-s_n)\quad \text{for each } n=1,2,\dots, \end{aligned}$$

where

$$\alpha_0=\frac{\int_0^1(f_1^0(s_0+\theta(s_1-s_0))-f_2^0(s_0))\,d\theta+\gamma_0+f_3^0(s_1-s_{-1})}{1-\beta_0-f_0^0(s_0-s_{-1}+c_{-1},s_1-s_0+r_0)}$$

and

$$\alpha_n=\frac{\int_0^1(f_1(s_n-s_0+r_0+\theta(s_{n+1}-s_n))-f_2(s_n-s_0+r_0))\,d\theta+\gamma+f_3(s_{n+1}-s_{n-1})}{1-\beta-f_0(s_n-s_{n-1}+c_{-1},s_{n+1}-s_0+r_0)}$$

for each $n=1,2,\dots$.

In view of (3.5)–(3.11), (3.13) and (3.16), a simple inductive argument shows for each $n=0,1,\dots$ that

$$(3.17)\qquad s_n\le t_n,$$

$$(3.18)\qquad s_{n+1}-s_n\le t_{n+1}-t_n,$$

$$(3.19)\qquad s^\star=\lim_{n\to\infty}s_n\le t^\star.$$

Hence, $\{s_n\}$ converges to $s^\star$ under the same convergence hypotheses for $\{t_n\}$ given in [4, Theorem 2]. However, the sequence $\{s_n\}$ is tighter than $\{t_n\}$. Next, we first provide weaker sufficient convergence conditions for $\{s_n\}$. Then we show that $\{s_n\}$ is indeed a majorizing sequence for $\{y_n\}$.

We define functions p_n^N, q_n^N, p_∞^N on $[0,1]$ for each fixed $N=1,2,\dots$ and each $n=N,N+1,\dots$ by

$$(3.20)\qquad \begin{aligned} p_n^N(t)=&\\ &\int_0^1\Big(f_1\Big(\frac{1-t^{n-1}}{1-t}(s_{N+1}-s_N)+s_N-s_0+r_0+\theta t^{n-1}(s_{N+1}-s_N)\Big)\\ &\qquad -f_2\Big(\frac{1-t^{n-1}}{1-t}(s_{N+1}-s_N)+s_N-s_0+r_0\Big)\Big)\,d\theta\\ &+\gamma+f_3((t^{n-1}+t^{n-2})(s_{N+1}-s_N))\\ &+tf_0\Big(\frac{1-t^{n-1}}{1-t}(s_{N+1}-s_N)+s_N-s_{-1}+c_{-1},\\ &\qquad \frac{1-t^{n}}{1-t}(s_{N+1}-s_N)+s_N-s_0+r_0\Big)\\ &-t(1-\beta), \end{aligned}$$

$$(3.21)\qquad q_n^N(t)=p_{n+1}^N(t)-p_n^N(t)$$

and

$$(3.22)\qquad p_\infty^N(t) = f_1\Big(\frac{s_{N+1}-s_N}{1-t} + s_N - s_0 + r_0\Big) - f_2\Big(\frac{s_{N+1}-s_N}{1-t} + s_N - s_0 + r_0\Big) + \gamma f_3(0) + t f_0\Big(\frac{s_{N+1}-s_N}{1-t} + s_N - s_{-1} + c_{-1}, \frac{s_{N+1}-s_N}{1-t} + s_N - s_0 + r_0\Big) - t(1-\beta).$$

Then we present the following results on majorizing sequences for two-point Newton-like methods. The proofs are similar to the proofs of Lemmas 2.1, 2.3 and Theorems 2.8, 2.9.

LEMMA 3.1. *Suppose*

$$f_0(s_0 - s_{-1} + c_{-1}, s_1 - s_0 + r_0) < 1 - \beta_0$$

and there exists $\alpha \in (0,1)$ *such that* $0 \le \alpha_1 \le \alpha$ *and*

$$(3.23)\qquad q_n^1(\alpha) \ge 0,$$

$$(3.24)\qquad p_\infty^1(\alpha) \le 0.$$

Then the sequence $\{s_n\}$ *given by* (3.16) *is well defined, non-decreasing, bounded from above by*

$$s_1^{\star\star} = s_1 + \frac{s_2 - s_1}{1-\alpha}$$

and converges to its unique least upper bound $s^\star \in [0, s_1^{\star\star}]$. *Moreover,*

$$0 \le s_{n+2} - s_{n+1} \le (s_2 - s_1)\alpha^n \quad \textit{for each } n = 1, 2, \ldots.$$

REMARK 3.2. The conclusions of Lemma 3.1 hold if (3.23) and (3.24) are replaced by

$$(3.25)\qquad q_n^1(\alpha) \le 0,$$

$$(3.26)\qquad p_1^1(\alpha) \le 0.$$

This time we have $p_{k+1}^1(\alpha) \le p_k^1(\alpha) \le p_1^1(\alpha) \le 0$.

We have the following useful and obvious extension of Lemma 3.1.

LEMMA 3.3. *Suppose there exists a minimum natural integer* $N > 1$ *and* $\alpha \in (0,1)$ *such that* (3.23) *and* (3.24) *hold and*

$$s_1 \le \cdots \le s_{N+1}, \qquad 0 < \alpha_N \le \alpha,$$

$$f_0(s_{N+1} - s_{-1} + c_{-1}, s_{N+1} - s_0 + r_0) < 1 - \beta.$$

$$f_0\Big(\frac{s_{N+1}-s_N}{1-t} s_N - s_{-1} + c_{-1}, \frac{s_{N+1}-s_N}{1-t}(s_N - s_N) + s_N - s_0 + r_0\Big) < 1 - \beta.$$

Then the sequence $\{s_n\}$ *given by* (3.16) *is well defined, non-decreasing, bounded from above by*

$$s_N^{\star\star} = s_{N-1} + \frac{s_N - s_{N-1}}{1-\alpha}$$

and converges to its unique least upper bound $s_N^\star$ *which satisfies* $s_N^\star \in [0, s_N^{\star\star}]$. *Moreover,*

$$0 \le s_{N+n} - s_{N+n-1} \le (s_{N+1} - s_N)\alpha^n \quad \text{for each } n = 1, 2, \ldots .$$

Remark 3.4. The upper bound ϱ on the limit point $s^\star$ can be computed as in Lemmas 2.4–2.6. Simply replace g_i $(i = 0, 1, 2, 3)$ by

$$f_0(t - t_{-1} + c_{-1}, t - t_0 + r_0), \quad f_1(t - t_0 + r_0), \quad f_2(t - t_0 + r_0), \quad f_3(t - s),$$

respectively.

We shall call Lemma 2.6* the result corresponding to Lemma 2.6 with the above changes.

Next, we present semi-local convergence results for two-step Newton-like methods. The proofs can be found in [4, Theorem 2].

Theorem 3.5. *Suppose that conditions* $(\mathcal{C}_2)$ *and the hypotheses of Lemma* 2.6* *hold. Moreover, suppose that there exist* $y_{-1} \in \overline{U}(w, R)$ *and* $y_0 \in \overline{U}(w, r_0)$ *such that*

$$\|A(y_{-1}, y_0)^{-1}(F(y_0) + G(y_0))\| \le \eta. \tag{3.27}$$

Then the sequence $\{y_n\}$ *generated by the two-step Newton-like method is well defined, remains in* $\overline{U}(w, t^\star)$ *for each* $n = 0, 1, \ldots$ *and converges to a solution* $x^\star$ *of the equation* $F(x) + G(x) = 0$. *Moreover, for each* $n = 0, 1, \ldots,$

$$\|y_{n+1} - y_n\| \le t_{n+1} - t_n \le \varrho - t_n \quad \text{and} \quad \|y_n - x^\star\| \le t^\star - t_n \le \varrho - t_n.$$

Furthermore, the solution $x^\star$ *is unique in* $\overline{U}(x_0, t^\star)$ *for* $x_{-1} = v$ *if*

$$\int_0^1 (f_1(t^\star + \theta t^\star) - f_2(t^\star))\, d\theta + f_3(t^\star) + f_0(t^\star + c_1, t^\star) + \beta + \gamma < 1,$$

and in $\overline{U}(x_0, R_0)$ *if* $t^\star \le R_0 < R$ *and*

$$\int_0^1 (f_1(t^\star + \theta R_0) - f_2(t^\star))\, d\theta + f_3(R_0) + f_0(t^\star + c_1, t^\star) + \beta + \gamma < 1.$$

Theorem 3.6. *Suppose that conditions* $(\mathcal{C}_2)$ *hold and there exist* $y_{-1} \in \overline{U}(w, R)$, $y_0 \in \overline{U}(w, r_0)$ *such that* (3.27) *holds. Moreover, suppose that the hypotheses of Lemma* 3.1 *or Lemma* 3.3 *hold. Then the conclusions of Theorem* 3.5 *hold with* $\{s_n\}$ *replacing* $\{t_n\}$.

In order for us to cover the local convergence case for two-point Newton-like methods, let us suppose as in [4, Theorem 5] that $x^\star$ is a solution of (1.1),

$A(x^\star, x^\star)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and for any $x, y \in \overline{U}(x^\star, r)$, $\theta \in [0,1]$,

$$\|A(x^\star, x^\star)^{-1}(A(x,y) - A(x^\star, x^\star))\| \leq f_4(\|x - x^\star\|, \|y - x^\star\|) + \beta_4 \tag{3.28}$$

and

$$\begin{aligned}&\|A(x^\star, x^\star)^{-1}((F'(x^\star + \theta(y - x^\star)) - A(x,y))(y - x^\star) + G(y) - G(x^\star))\| \\ &\quad \leq (f_5((1+\theta)\|y - x^\star\|) - f_6(\|y - x^\star\|) + f_7(\|x - x^\star\|) + \gamma_4)\|y - x^\star\|,\end{aligned} \tag{3.29}$$

where f_i $(i = 4,5,6,7)$, β_4, γ_4 are as f_i $(i = 0,1,2,3)$, β_0, γ_0, respectively. These conditions were used in [4, Theorem 5] to provide a local convergence analysis for two-step Newton-like methods. We can improve the error bounds on the distances $\|y_n - x^\star\|$ for each $n = 0, 1, \ldots$. It follows from hypotheses (3.28) and (3.29) that

$$\|A(x^\star, x^\star)^{-1}(A(y_{-1}, y_0) - A(x^\star, x^\star))\| \leq f_4^0(\|y_{-1} - x^\star\|, \|y_{-1} - x^\star\|) + \beta_4^0, \tag{3.30}$$

$$\|A(x^\star, x^\star)^{-1}(A(y_0, y_1) - A(x^\star, x^\star))\| \leq f_4^1(\|y_0 - x^\star\|, \|y_1 - x^\star\|) + \beta_4^1, \tag{3.31}$$

$$\begin{aligned}&\|A(x^\star, x^\star)^{-1}((F'(x^\star + \theta(y_0 - x^\star)) - A(y_{-1}, y_0))(y_0 - x^\star) + G(y_0) - G(x^\star))\| \\ &\quad \leq (f_5^0((1+\theta)\|y_0 - x^\star\|) - f_6^0(\|y_0 - x^\star\|) + f_7^0(\|y_{-1} - x^\star\|) + \gamma_4^0)\|y_0 - x^\star\|\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\|A(x^\star, x^\star)^{-1}((F'(x^\star + \theta(y_1 - x^\star)) - A(y_0, y_1))(y_1 - x^\star) + G(y_1) - G(x^\star))\| \\ &\quad \leq (f_5^1((1+\theta)\|y_1 - x^\star\|) - f_6^1(\|y_1 - x^\star\|) + f_7^1(\|y_0 - x^\star\|) + \gamma_4^1)\|y_1 - x^\star\|,\end{aligned} \tag{3.33}$$

where f_i^j, β_4^j, γ_4^j $(i = 4,5,6,7$ and $j = 1,2)$ are tighter that f_i, β_4, γ_4 $(i = 4,5,6,7)$, respectively. Note that (3.30)–(3.33) require computations only involving the initial data. Using the approximation

$$\begin{aligned}&-(y_{n+1} - x^\star) \\ &= (A(y_{n-1}, y_n)^{-1} A(y_{-1}, y_0)) A(y_{-1}, y_0)^{-1} \Big(\int_0^1 (F'(x^\star + \theta(y_n - x^\star)) \\ &\qquad - A(y_{n-1}, y_n))\, d\theta (y_n - x^\star) + G(y_n) - G(x^\star) \Big)\end{aligned} \tag{3.34}$$

and (3.28)–(3.33) (where (3.30)–(3.33) are used for the first two distances $\|y_1 - x^\star\|$, $\|y_2 - x^\star\|$ and (3.28), (3.29) for $\|y_n - x^\star\|$ for each $n = 2, 3, \ldots$ as in [4, Theorem 5], where only (3.28) and (3.29) were used), we arrive at the following result.

THEOREM 3.7. *Suppose that there exists a solution of the equation* $f_8(t) = 0$ *in* $[0, R]$, *where*

$$f_8(t) = \int_0^1 (f_5((1+\theta)t) - f_6(t))\, d\theta + f_7(t) + f_4(t,t) + \beta_4 + \gamma_4 - 1.$$

Denote by R_1 *the smallest of the solutions in* $[0, R]$. *Then the sequence* $\{y_n\}$ *generated by the two-step Newton-like method is well defined, remains in* $\overline{U}(x^\star, R_1)$ *and converges to* $x^\star$ *provided that* $y_{-1}, y_0 \in \overline{U}(x^\star, R_1)$. *Moreover, the following error bounds hold for each* $n = 0, 1, \ldots$:

$$\|y_{n+1} - x^\star\| \le e_n \|y_n - x^\star\|,$$

where

$$e_n = \frac{\int_0^1 (f_5^n((1+\theta)\|y_n - x^\star\|) - f_6^n(\|y_n - x^\star\|))\, d\theta + f_7^n(\|y_{n-1} - x^\star\|) + \gamma_4^n}{1 - \beta_4^n - f_4^n(\|y_n - x^\star\|)}$$

for $n = 0, 1$ *and*

$$e_n = \frac{\int_0^1 (f_5((1+\theta)\|y_n - x^\star\|) - f_6(\|y_n - x^\star\|))\, d\theta + f_7(\|y_{n-1} - x^\star\|) + \gamma_4}{1 - \beta_4 - f_4(\|y_n - x^\star\|)}$$

for each $n = 2, 3, \ldots$.

REMARK 3.8. (a) We have $f_8(0) = \beta_4 + \gamma_4 - 1 < 0$. Therefore, if $f_8(R) \ge 0$, then it follows from the intermediate value theorem that $f_8 = 0$ has a solution in $[0, R]$. That is, $f_8(R) \ge 0$ can replace the hypothesis about the existence of the solution in Theorem 3.7.

(b) Let us define $\overline{e}_n$ as e_n but using f_i $(i = 4, 5, 6, 7)$, β_4, γ_4 for each $n = 0, 1, \ldots$. Then $e_n \le \overline{e}_n$, so our new estimates on the distances $\|y_n - x^\star\|$ are tighter than the ones in [4]. Note that these advantages are obtained under the same computational cost, since in practice the computation of f_i $(i = 4, 5, 6, 7)$, β_4, γ_4 requires that of f_i^j $(i = 4, 5, 6, 7)$, β_4^j, γ_4^j $(j = 0, 1)$.

(c) The local convergence results for single-step Newton-like methods are obtained immediately from the above if we replace $A(x^\star, x^\star)$, y_{-1}, y_0, y_1, $f_4(t,t)$, $f_4^j(t,t)$ $(j = 0, 1)$, y_n by $A(x^\star)$, x_0, x_0, x_1, $f_4(t)$, $f_4^j(t)$ $(j = 0, 1)$, x_n, respectively.

4. Special cases and applications. Let us consider single-step methods in the Lipschitz case for Newton's method. We set $A(x) = F'(x)$, $G(x) = 0$ for each $x \in \mathcal{D}$, $x_0 = w$, $\beta_0 = \beta = \gamma_0 = \gamma = r_0 = 0$, $g_0(t) = L_0 t$, $g_1(t) = g_2(t) = Lt$, $g_3(t) = 0$, $g_0^0(t) = L_{-2} t$, $g_1^0(t) = g_2^0(t) = L_{-1} t$. Then conditions $(\mathcal{C}_1)$ reduce to

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_0\|, \tag{4.1}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le L\|x - y\|, \tag{4.2}$$

$$\|F'(x_0)^{-1}(F'(x_0 + \theta(F'(x_0)^{-1}F(x_0))) - F'(x_0))\| \le L_{-1}\theta\|F'(x_0)^{-1}F(x_0)\|, \tag{4.3}$$

$$\|F'(x_0)^{-1}(F'(x_0 - F'(x_0)^{-1}F(x_0)) - F'(x_0))\| \le L_{-2}\|F'(x_0)^{-1}F(x_0)\|. \tag{4.4}$$

The iteration $\{s_n\}$ becomes

$$\begin{aligned} &s_0 = 0, \quad s_1 = \eta, \quad s_2 = \eta + \frac{L_{-2}\eta^2}{2(1 - L_{-1}\eta)}, \\ &s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)^2}{2(1 - L_0 s_{n+1})} \quad \text{for each } n = 1, 2, \ldots. \end{aligned} \tag{4.5}$$

Note that

$$L_{-2} \le L_{-1} \le L_0 \le L. \tag{4.6}$$

(a) The conditions of Lemma 2.3 will hold for $N = 2$ if

$$\eta + \frac{L_{-2}\eta^2}{2(1 - L_{-1}\eta)} < \frac{1}{L_0} \tag{4.7}$$

and

$$\alpha_1 = \frac{\dfrac{L}{2}\,\dfrac{L_{-2}\eta^2}{2(1 - L_{-1}\eta)}}{1 - L_0\left(\eta + \dfrac{L_{-2}\eta^2}{2(1 - L_{-1}\eta)}\right)} \le \alpha \le \alpha_2 = 1 - \frac{L_0\dfrac{L_{-2}\eta^2}{2(1 - L_{-1}\eta)}}{1 - L_0\eta}. \tag{4.8}$$

To solve the system of inequalites (4.7) and (4.8), it is convenient to define quadratic polynomials p_1 and p_2 by

$$p_1(t) = \beta_1 t^2 + \beta_2 t + \beta_3, \quad p_2(t) = \gamma_1 t^2 + \gamma_2 t + \gamma_3,$$

where

$$\beta_1 = LL_{-2} + 2\alpha L_0(2L_{-1} - L_{-2}), \quad \beta_2 = 4\alpha(L_{-1} - L_0), \quad \beta_3 = -4\alpha,$$

$$\gamma_1 = L_0(L_{-2} - 2(1-\alpha)L_{-1}), \quad \gamma_2 = 2(1-\alpha)(L_0 + L_{-1}), \quad \gamma_3 = -2(1-\alpha).$$

In view of (4.6)–(4.8), the polynomial p_1 has a positive root

$$\lambda_1 = \frac{-\beta_2 + \sqrt{\beta_2^2 - 4\beta_1\beta_3}}{2\beta_1}$$

and a negative root. Inequalities (4.7) and (4.8) are satisfied if

$$p_1(\eta) \le 0, \tag{4.9}$$

$$p_2(\eta) \le 0. \tag{4.10}$$

Therefore, (4.9) is satisfied if

$$\eta \le \lambda_1. \tag{4.11}$$

If $\gamma_1 < 0$ and $\varDelta_2 := \gamma_2^2 - 4\gamma_1\gamma_3 \le 0$, (4.10) always holds. If $\gamma_1 < 0$ and $\varDelta_2 > 0$, then p_2 has two positive roots. The smaller is denoted by λ_2 and is given by

$$\lambda_2 = \frac{-\gamma_2 - \sqrt{\varDelta_2}}{2\gamma_1}. \tag{4.12}$$

If $\gamma_1 > 0$, the polynomial p_2 has a positive root λ_3 given by

$$\lambda_3 = \frac{-\gamma_2 + \sqrt{\varDelta_2}}{2\gamma_1} \tag{4.13}$$

and a negative root. Let us define

$$\frac{L_4^{-1}}{2} = \begin{cases} \lambda_1 & \text{if } \gamma_1 < 0 \text{ and } \varDelta_2 \le 0, \\ \min\{\lambda_1, \lambda_2\} & \text{if } \gamma_1 < 0 \text{ and } \varDelta_2 > 0, \\ \min\{\lambda_1, \lambda_3\} & \text{if } \gamma_1 > 0\ . \end{cases} \tag{4.14}$$

Summarizing we conclude that (4.7) and (4.8) are satisfied if

$$h_4 = L_4\eta \le 1/2. \tag{4.15}$$

In the special case when $L_{-2} = L_{-1} = L_0$, elementary computations show $L_3 = L_4$. That is, (4.15) reduces to (2.38), which is the weaker than the "h" conditions given in [17]. The rest of the "h" conditions are

$$h_\star = L\eta \le 1/2, \qquad h_1 = L_1\eta \le 1/2, \qquad h_2 = L_2\eta \le 1/2,$$

where

$$L_1 = \tfrac{1}{2}(L_0 + L) \quad \text{and} \quad L_2 = \tfrac{1}{8}(L + 4L_0 + (L^2 + 8L_0L)^{1/2}).$$

Note that

$$h_\star \le 1/2 \Rightarrow h_1 \le 1/2 \Rightarrow h_2 \le 1/2 \Rightarrow h_3 \le 1/2 \Rightarrow h_4 \le 1/2$$

but not necessarily vice versa unless $L_{-2} = L_{-1} = L_0 = L$.

In view of the definition of β_1, the condition on h_4 can be improved if L can be replaced by $L_\star$ such that $0 < L_\star < L$. That requires the verification of the condition

$$\|F'(x_0)^{-1}(F'(x_1+\theta(x_2-x_1))-F'(x_1))\| \le L_\star\theta\|x_2-x_1\| \quad \text{for each } \theta \in [0,1],$$

where x_1 and x_2 are given by Newton's iterations. The computation of $L_\star$ is possible and requires computations with the initial data (see Example 4.1). Note also that s_3 will be given by

$$s_3 = s_2 + \frac{L_\star(s_2 - s_1)^2}{2(1 - L_0 s_2)}$$

instead of

$$s_3 = s_2 + \frac{L(s_2 - s_1)^2}{2(1 - L_0 s_2)}.$$

Moreover, the condition on h_4 is then replaced by the at least as weak

$$h_5 = L_5 \eta \le 1/2,$$

where L_5 is defined as L_4 with $L_\star$ replacing L in the definition of β_1.

(b) Another extension of our results is as follows. Let $N = 1, 2, \ldots$ and $R \in (0, 1/L_0)$. Assume $x_1, \ldots, x_N$ can be computed by Newton's iterations, and $F'(x_N)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ with $\|F'(x_N)^{-1}F(x_N)\| \le R - \|x_N - x_0\|$. Set $\mathcal{D} = \overline{U}(x_0, R)$ and $\mathcal{D}_N = \overline{U}(x_N, R - \|x_N - x_0\|)$. Then, for all $x \in \mathcal{D}_N$,

$$\begin{aligned}\|F'(x_N)^{-1}(F'(x) - F'(y))\| &\le \|F'(x_N)^{-1}F'(x_0)\|\|F'(x_0)^{-1}(F'(x) - F'(y))\| \\ &\le \frac{L}{1 - L_0\|x_N - x_0\|}\|x - y\|\end{aligned}$$

and

$$\|F'(x_N)^{-1}F(x_N)\| \le \frac{L}{2}\frac{L}{1 - L_0\|x_N - x_0\|}\|x_N - x_{N-1}\|^2.$$

Set

$$L^N = L_0^N = \frac{L}{1 - L_0\|x_N - x_0\|}, \qquad \eta^N = \frac{LL_N}{2}\|x_N - x_{N-1}\|^2.$$

Then the Kantorovich hypothesis becomes

$$h_\star^N = L^N \eta^N \le 1/2.$$

Clearly, the most interesting case is when $N = 1$. In this case the Kantorovich condition becomes

$$h_\star^1 = L_6 \eta \le 1/2,$$

where

$$L_6 = \tfrac{1}{2}(L_0 + L\sqrt{L}).$$

Note also that L_6 can be smaller than L.

Let us compare the "h" conditions on concrete examples.

Example 4.1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $\mathcal{D} = [\xi, 2 - \xi]$, $\xi \in [0, .5)$. Define a function F on $\mathcal{D}$ by

$$F(x) = x^3 - \xi. \tag{4.16}$$

(a) Using the Lipschitz conditions (4.1) and (4.2), we obtain

$$\eta = \tfrac{1}{3}(1 - \xi), \quad L_0 = 3 - \xi, \quad L = 2(2 - \xi).$$

We have

$$h_\star = \tfrac{2}{3}(1 - \xi)(2 - \xi) > .5 \quad \text{for all } \xi \in (0, .5).$$

Hence, there is no guarantee that Newton's method starting at $x_0 = 1$ converges to $x^\star$. However, one can easily see that if for example $\xi = .49$, Newton's method converges to $x^\star = \sqrt[3]{.49}$.

(b) Consider our "h" conditions given in this section. We obtain

$$h_1 = \frac{1}{6}(7-3\xi)(1-\xi) \le .5 \qquad \text{for all } \xi \in [.4648162415, .5),$$
$$h_2 = \frac{1}{12}(8-3\xi+(5\xi^2-24\xi+28)^{1/2})(1-\xi) \le .5 \quad \text{for all } \xi \in [.450339002, .5),$$
$$h_3 = \frac{1}{24}(1-\xi)(12-4\xi+(84-58\xi+10\xi^2)^{1/2}(12-10\xi+2\xi^2)^{1/2}) \le .5$$
$$\text{for all } \xi \in [.4271907643, .5).$$

Next, we pick some values of ξ such that all hypotheses are satisfied, so we can compare the "h" conditions using Maple 13 (see Table 1).

Table 1

ξ	$x^\star$	$h_\star$	h_1	h_2	h_3
.486967	.6978302086	.5174905727	.4736234295	.4584042632	.4368027442
.5245685	.7242710128	.4676444075	.4299718890	.4169293786	.3983631448
.452658	.6727986326	.5646168433	.5146862992	.4973315343	.4727617854
.435247	.6597325216	.5891326340	.5359749755	.5174817371	.4913332192
.425947	.6526461522	.6023932312	.5474704233	.5283539600	.5013421729
.7548589	.8688261621	.2034901726	.1934744795	.1900595014	.1850943832

In Fig. 1, we compare the "h" conditions for $\xi \in (0, .9)$.

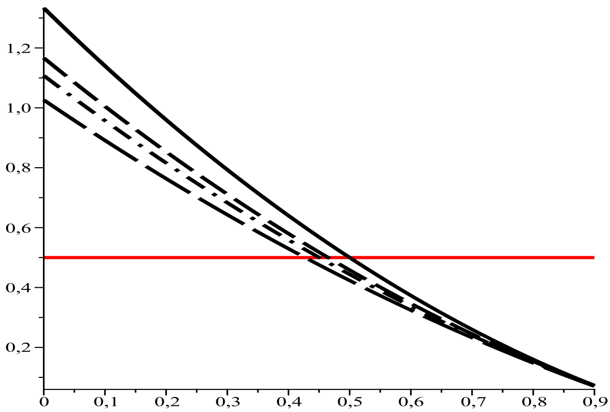

Fig. 1. Functions $h_\star$, h_1, h_2 and h_3 (from top to bottom) with respect to ξ in interval $(0, .9)$, respectively. The horizontal line has equation $y = .5$.

(c) Consider the case $\xi = .7548589$ where our "h" are satisfied. Let

$$L_{-1} = 2 \quad \text{and} \quad L_{-2} = \frac{\xi+5}{3} = 1.918286300.$$

Hence,

$$\alpha = .5173648648, \quad \alpha_1 = .01192474572 \quad \text{and} \quad \alpha_2 = .7542728992.$$

Thus, conditions (4.6)–(4.8) hold. We also have

$$\beta_1 = 9.613132975, \qquad \beta_2 = -.5073095684, \qquad \beta_3 = -2.069459459,$$
$$\gamma_1 = -.02751250012, \qquad \gamma_2 = 4.097708498, \qquad \gamma_3 = -.965270270,$$
$$\lambda_1 = .4911124649, \qquad \Delta_2 = 16.68498694, \qquad \lambda_2 = 148.7039440.$$

Since $\gamma_1 < 0$, $\Delta_2 > 0$ and using (4.14), we get in turn

$$\frac{1}{2L_4} = \min\{\lambda_1, \lambda_2\} = .4911124649.$$

We deduce that condition (4.15) holds provided that

$$L_4 = 1.018096741 \quad \text{and} \quad h_4 = .08319245167 < .5.$$

If we consider $L_\star = 1.836572600$, we get

$$h_5 = .07234026489 < h_4.$$

Finally, we pick the same values of ξ as in Table 1, so we can compare the h_4 and h_5 conditions (see Table 2).

Table 2

ξ	h_4	h_5
.486967	.1767312629	.1377052696
.5245685	.1634740591	.1290677624
.452658	.1888584478	.1454742725
.435247	.1950234005	.1493795529
.425947	.1983192281	.1514558980
.7548589	.08319245167	.07234026489

EXAMPLE 4.2. We also consider Example 4.1 in one more case. Let $\xi = .74137931$. Then $L = 2.51724138$, $L_0 = 2.25862069$ and $\eta = .0862068966$. Using the hypotheses of Lemma 2.6 and Remark 2.7(a), we have

$$\varrho = .1218276252, \qquad \alpha = .5181703378, \qquad 1/L_0 = .4427480915,$$
$$r_3 = .1078366107.$$

Using (2.3), we get

$$t_2 = .09782207463, \quad t_3 = .09804003491, \quad t_4 = .09804011171,$$

and for all $n \geq 5$,

$$t_n = t_4 = .09804011171 = t^\star < \varrho^\star = .09839144902 < .4427480915 = 1/L_0.$$

Moreover, if we use the choices in Remark 2.7(a) with $N = 1$, we get

$$L_3 = 2.279717419 \quad \text{and} \quad \varrho_3^1 = .09781769768.$$

For $\varrho := \varrho_3^1$, we obtain

$$\psi(\eta, t) = .1293103450\, t^2 + .7829964326\, t - .08891007985 \quad \text{for } \eta \le t < \varrho_3^1.$$

Hence, the hypotheses of Lemma 2.6 are satisfied for $\lambda = \eta$ and $\varrho := \varrho_3^1$. Finally, we have $r_3 \ge \varrho_3^1$.

EXAMPLE 4.3. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, the space of continuous functions on $[0,1]$, equipped with the max-norm. Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 \mathcal{G}(s,t) u(t)\, dt + y(s) - \theta. \tag{4.17}$$

Nonlinear integral equations of the form (4.17) are called Chandrasekhar-type equations [8], [11], [19] and they arise in the theories of radiative transfer, neutron transport, and in the kinetic theory of gases. Here, the kernel $\mathcal{G}(s,t)$ is a continuous function of two variables $(s,t) \in [0,1] \times [0,1]$ satisfying

(i) $0 < \mathcal{G}(s,t) < 1$,
(ii) $\mathcal{G}(s,t) + \mathcal{G}(t,s) = 1$.

The parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. For simplicity, we choose

$$u_0(s) = y(s) = 1,$$
$$\mathcal{G}(s,t) = \frac{s}{s+t} \quad \text{for all } (s,t) \in [0,1] \times [0,1] \ (s + t \ne 0).$$

Let $\mathcal{D} = U(u_0, 1-\theta)$ and define an operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 \mathcal{G}(s,t) x(t)\, dt + y(s) - \theta \quad \text{for all } s \in [0,1]. \tag{4.18}$$

Then every zero of F satisfies (4.17). Using (4.18) we obtain (cf. [8], [11])

$$[F'(x)v](s) = \lambda x(s) \int_0^1 \mathcal{G}(s,t) v(t)\, dt + \lambda v(s) \int_0^1 \mathcal{G}(s,t) x(t)\, dt + 3x^2(s) v(s) - I(v(s)).$$

Therefore, the operator F' satisfies the conditions of Theorem 2.8 with

$$\eta = \frac{|\lambda| \ln 2 + 1 - \theta}{2(1 + |\lambda| \ln 2)}, \quad L = \frac{|\lambda| \ln 2 + 3(2-\theta)}{1 + |\lambda| \ln 2}, \quad L_0 = \frac{2|\lambda| \ln 2 + 3(3-\theta)}{2(1 + |\lambda| \ln 2)}.$$

It follows from our main results that if one of our "h" conditions holds, then problem (4.17) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis. Note also that $L_0 < L$ for all $\theta \in [0,1]$ (see also Fig. 2).

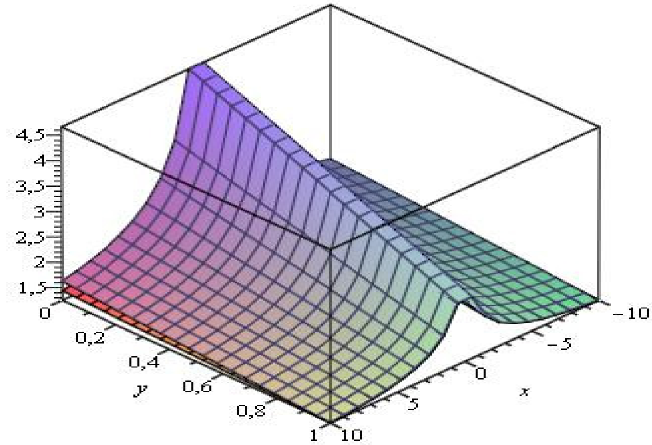

Fig. 2. Functions L_0 and L in 3d with respect to (λ,θ) in $(-10,10)\times(0,1)$; L is above L_0.

Next, we pick some values of λ and θ such that all hypotheses are satisfied, so we can compare the "h" conditions (see Table 3).

Table 3

λ	θ	$h_\star$	h_1	h_2	h_3
.97548	.954585	.4895734208	.4851994045	.4837355633	.4815518345
.8457858	.999987	.4177974405	.4177963046	.4177959260	.4177953579
.3245894	.815456854	.5156159025	.4967293568	.4903278739	.4809439506
.3569994	.8198589998	.5204140737	.5018519741	.4955632842	.4863389899
.3789994	.8198589998	.5281518448	.5093892893	.5030331107	.4937089648
.458785	.5489756	1.033941504	.9590659445	.9332478337	.8962891928

Example 4.4. Let $\mathcal{X}$ and $\mathcal{Y}$ as in Example 4.3. Consider the nonlinear boundary value problem [8]

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)(u^3(t) + \gamma u^2(t))\,dt \tag{4.19}$$

where $\mathcal{Q}$ is the Green function given by

$$\mathcal{Q}(s,t) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s < t. \end{cases}$$

Then problem (4.19) is in the form (1.1), where $F : \mathcal{D} \to \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)(x^3(t) + \gamma x^2(t))\,dt.$$

Set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R_0)$. The Fréchet derivative of F is given by (cf. [8])

$$[F'(x)v](s) = v(s) - \int_0^1 \mathcal{Q}(s,t)(3x^2(t) + 2\zeta x(t))v(t)\,dt.$$

It is easy to verify that $U(u_0, R_0) \subset U(0, R_0 + 1)$ since $\|u_0\| = 1$. If $2\gamma < 5$, the operator F' satisfies the Lipschitz conditions, with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \quad L = \frac{\gamma + 6R_0 + 3}{4(5-2\gamma)} \quad \text{and} \quad L_0 = \frac{2\gamma + 3R_0 + 6}{8(5-2\gamma)}.$$

Note that $L_0 < L$ (see also Fig. 3).

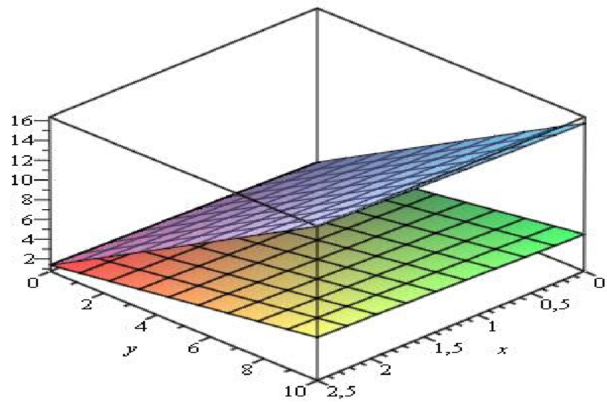

Fig. 3. Functions L_0 and L in 3d with respect to (γ, R_0) in $(0, 2.5) \times (0, 10)$; L is also above L_0.

Next, we pick some values of γ and R_0 such that all hypotheses are satisfied, so we can compare the "h" conditions (see Table 4).

Table 4

γ	R_0	$h_\star$	h_1	h_2	h_3
.00025	1	.4501700201	.3376306412	.2946446274	.2413108547
.25	.986587	.6367723612	.4826181423	.4240511567	.3508368298
.358979	.986587	.7361726023	.5600481163	.4932612622	.4095478068
.358979	1.5698564	1.013838328	.7335891949	.6245310288	.4927174588
.341378	1.7698764	1.084400750	.7750792917	.6539239239	.5088183074

Finally, we provide two examples in the local case, where $\ell^\star < \ell$ ($\ell^\star$, ℓ are the Lipschitz constants as in (4.1) and (4.2) where x_0 is replaced by $x^\star$).

EXAMPLE 4.5. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$. Define a function F on $\mathcal{D} = [-1, 1]$ by

$$F(x) = e^x - 1. \tag{4.20}$$

Then, using (4.20) for $x^\star = 0$, we find that $F(x^\star) = 0$ and $F'(x^\star) = e^0 = 1$. Moreover, the hypotheses of Theorem 3.7 hold for $\ell = e$ and $\ell^\star = e - 1$. Note that $\ell^\star < \ell$.

EXAMPLE 4.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Example 4.3. Define a function F on $\mathcal{D}$ by

$$F(h)(x) = h(x) - 5\int_0^1 x\theta h(\theta)^3 \, d\theta. \tag{4.21}$$

Then

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\theta h(\theta)^2 u(\theta) \, d\theta \quad \text{for all } u \in \mathcal{D}.$$

Using (4.21) we see that the hypotheses of Theorem 3.7 hold for $x^\star(x) = 0$, where $x \in [0, 1]$, $\ell = 15$ and $\ell^\star = 7.5$.

We conclude this section with an example where $G \neq 0$ in (1.1).

EXAMPLE 4.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Example 4.3. Consider the integral equation on $\mathcal{D} = \overline{U}(x_0, r/2)$ ($r \in [0, R]$) given by

$$x(t) = \int_0^1 k(t, s, x(s)) \, ds, \tag{4.22}$$

where the kernel $k(t, s, x(s))$ with $(t, s) \in [0, 1] \times [0, 1]$ is a non-differentiable operator on $\mathcal{D}$. Define operators F, G on $\mathcal{D}$ by

$$F(x)(t) = Ix(t) \quad (I \text{ the identity operator on } \mathcal{X}), \tag{4.23}$$

$$G(x)(t) = -\int_0^1 k(t, s, x(s)) \, ds. \tag{4.24}$$

Choose $x_0 = 0$ and assume there exists a constant $k_0 \in [0, 1)$, a real function $k_1(t, s)$ such that

$$\|k(t, s, x) - k(t, s, y)\| \le k_1(t, s)\|x - y\| \tag{4.25}$$

and

$$\sup_{t \in [0,1]} \int_0^1 k_1(t, s) \, ds \le k_0 \tag{4.26}$$

for each $t, s \in [0, 1]$ and $x, y \in \mathcal{D}$. Moreover choose $r_0 = 0$, $y_0 = y_{-1}$, $A(x) = I(x)$, $v_0(r) = r$, $\beta = \gamma = 0$, $v = 0$ and $v_1(r) = k_0$ for each $x, y \in \mathcal{D}$

and $r, s \in [0, 1]$. It can be easily seen that the conditions of Theorem 2.8 hold if

$$t^\star = \frac{\eta}{1 - k_0} \le \frac{r}{2}.$$

Conclusion. New convergence criteria for semi-local/local convergence of single-point and two-point Newton-like methods using majorant functions are presented. We use Lipschitz and center-Lipschitz conditions on the first Fréchet derivative. Our results expand the Kantorovich analysis used in earlier studies [2], [4], [8], [50], [51]. Numerical examples, special cases and applications are also provided.

References

[1] S. Amat, S. Busquier and M. Negra, *Adaptive approximation of nonlinear operators*, Numer. Funct. Anal. Optim. 25 (2004), 397–405.

[2] J. Appell, E. De Pascale, J. V. Lysenko and P. P. Zabrejko, *New results on Newton–Kantorovich approximations with applications to nonlinear integral equations*, Numer. Funct. Anal. Optim. 18 (1997), 1–17.

[3] J. Appell, E. De Pascale and P. P. Zabrejko, *On the application of the Newton–Kantorovich method to nonlinear integral equations of Urysohn type*, Numer. Funct. Anal. Optim. 12 (1991), 271–283.

[4] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. 298 (2004), 374–397.

[5] I. K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. 169 (2004), 315–332.

[6] I. K. Argyros, *Concerning the "terra incognita" between convergence regions of two Newton methods*, Nonlinear Anal. 62 (2005), 179–194.

[7] I. K. Argyros, *Approximating solutions of equations using Newton's method with a modified Newton's method iterate as a starting point*, Rev. Anal. Numér. Théor. Approx. 36 (2007), 123–138.

[8] I. K. Argyros, *Computational Theory of Iterative Methods*, Stud. Comput. Math. 15, Elsevier, New York, 2007.

[9] I. K. Argyros, *On a class of Newton-like methods for solving nonlinear equations*, J. Comput. Appl. Math. 228 (2009), 115–122.

[10] I. K. Argyros, *A semilocal convergence analysis for directional Newton methods*, Math. Comput. 80 (2011), 327–343.

[11] I. K. Argyros and S. Hilout, *Efficient Methods for Solving Equations and Variational Inequalities*, Polimetrica, Milano, 2009.

[12] I. K. Argyros and S. Hilout, *Enclosing roots of polynomial equations and their applications to iterative processes*, Surveys Math. Appl. 4 (2009), 119–132.

[13] I. K. Argyros and S. Hilout, *Extending the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. 234 (2010), 2993–3006.

[14] I. K. Argyros and S. Hilout, *Improved generalized differentiability conditions for Newton–like methods*, J. Complexity 26 (2010), 316–333.

[15] I. K. Argyros and S. Hilout, *Convergence domains under Zabrejko–Zinčenko conditions using recurrent functions*, Appl. Math. (Warsaw) 38 (2011), 193–209.

[16] I. K. Argyros and S. Hilout, *On the convergence of inexact two-step Newton-like algorithms using recurrent functions*, J. Appl. Math. Comput. 38 (2012), 41–61.

[17] I. K. Argyros and S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity 28 (2012), 364–387.

[18] I. K. Argyros and S. Hilout, *The majorant method in the theory of Newton–Kantorovich approximations and generalized Lipschitz conditions*, submitted.

[19] I. K. Argyros, S. Hilout and M. A. Tabatabai, *Mathematical Modelling with Applications in Biosciences and Engineering*, Nova Publ., New York, 2011.

[20] E. Cătinaş, *The inexact, inexact perturbed, and quasi-Newton methods are equivalent models*, Math. Comp. 74 (2005), 291–301.

[21] X. Chen and T. Yamamoto, *Convergence domains of certain iterative methods for solving nonlinear equations*, Numer. Funct. Anal. Optim. 10 (1989), 37–48.

[22] P. Deuflhard, *Newton Methods for Nonlinear Problems. Affine Invariance and Adaptive Algorithms*, Springer Ser. Comput. Math. 35, Springer, Berlin, 2004.

[23] J. A. Ezquerro, J. M. Gutiérrez, M. A. Hernández, N. Romero and M. J. Rubio, *The Newton method: from Newton to Kantorovich*, Gac. R. Soc. Mat. Esp. 13 (2010), 53–76 (in Spanish).

[24] J. A. Ezquerro and M. A. Hernández, *On the R-order of convergence of Newton's method under mild differentiability conditions*, J. Comput. Appl. Math. 197 (2006), 53–61.

[25] J. A. Ezquerro and M. A. Hernández, *An improvement of the region of accessibility of Chebyshev's method from Newton's method*, Math. Comp. 78 (2009), 1613–1627.

[26] J. A. Ezquerro, M. A. Hernández and N. Romero, *Newton-type methods of high order and domains of semilocal and global convergence*, Appl. Math. Comput. 214 (2009), 142–154.

[27] W. B. Gragg and R. A. Tapia, *Optimal error bounds for the Newton–Kantorovich theorem*, SIAM J. Numer. Anal. 11 (1974), 10–13.

[28] M. A. Hernández, *A modification of the classical Kantorovich conditions for Newton's method*, J. Comput. Appl. Math. 137 (2001), 201–205.

[29] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[30] L. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[31] A. M. Ostrowski, *Sur la convergence et l'estimation des erreurs dans quelques procédés de résolution des équations numériques*, in: Memorial Volume dedicated to D. A. Grave, publisher unknown, Moscow, 1940, 213–234.

[32] A. M. Ostrowski, *La méthode de Newton dans les espaces de Banach*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), 1251–1253.

[33] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*, Academic Press, New York, 1973.

[34] I. Păvăloiu, *Introduction in the Theory of Approximation of Equations Solutions*, Dacia Ed., Cluj-Napoca, 1976.

[35] F. A. Potra, *The rate of convergence of a modified Newton's process*, Apl. Mat. 26 (1981), 13–17.

[36] F. A. Potra, *An error analysis for the secant method*, Numer. Math. 38 (1981/82), 427–445.

[37] F. A. Potra, *On the convergence of a class of Newton-like methods*, in: Iterative Solution of Nonlinear Systems of Equations (Oberwolfach, 1982), Lecture Notes in Math. 953, Springer, Berlin, 1982, 125–137.

[38] F. A. Potra, *Sharp error bounds for a class of Newton-like methods*, Libertas Math. 5 (1985), 71–84.
[39] F. A. Potra and V. Pták, *Sharp error bounds for Newton's process*, Numer. Math. 34 (1980), 63–72.
[40] F. A. Potra and V. Pták, *Nondiscrete Induction and Iterative Processes*, Res. Notes in Math. 103, Pitman, Boston, MA, 1984.
[41] P. D. Proinov, *General local convergence theory for a class of iterative processes and its applications to Newton's method*, J. Complexity 25 (2009), 38–62.
[42] P. D. Proinov, *New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems*, J. Complexity 26 (2010), 3–42.
[43] L. B. Rall, *Computational Solution of Nonlinear Operator Equations*, Wiley, New York, 1969.
[44] W. C. Rheinboldt, *A unified convergence theory for a class of iterative processes*, SIAM J. Numer. Anal. 5 (1968), 42–63.
[45] W. C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, in: Banach Center Publ. 3, PWN–Polish Sci. Publ., 1977, 129–142.
[46] R. A. Tapia, *Classroom Notes: The Kantorovich theorem for Newton's method*, Amer. Math. Monthly 78 (1971), 389–392.
[47] J. F. Traub and H. Woźniakowski, *Convergence and complexity of Newton iteration for operator equations*, J. Assoc. Comput. Mach. 26 (1979), 250–258.
[48] Q. Wu and H. Ren, *A note on some new iterative methods with third-order convergence*, Appl. Math. Comput. 188 (2007), 1790–1793.
[49] T. Yamamoto, *A convergence theorem for Newton-like methods in Banach spaces*, Numer. Math. 51 (1987), 545–557.
[50] P. P. Zabrejko and D. F. Nguen, *The majorant method in the theory of Newton–Kantorovich approximations and the Pták error estimates*, Numer. Funct. Anal. Optim. 9 (1987), 671–684.
[51] A. I. Zinčenko, *Some approximate methods of solving equations with non-differentiable operators*, Dopovidi Akad. Nauk Ukraïn. RSR 1963, 156–161 (in Ukrainian).

Ioannis K. Argyros
Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, U.S.A.
E-mail: iargyros@cameron.edu

Saïd Hilout
Poitiers University
Laboratoire de Mathématiques et Applications
Bd. Pierre et Marie Curie
Téléport 2, B.P. 30179
86962 Futuroscope Chasseneuil Cedex, France
E-mail: said.hilout@math.univ-poitiers.fr

Received on 30.4.2012;
revised version on 10.7.2012 (2130)