APPLICATIONES MATHEMATICAE
40,1 (2013), pp. 99–108

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ON DISCRETE FOURIER SPECTRUM OF A HARMONIC WITH RANDOM FREQUENCY MODULATION

Abstract. Asymptotic properties of the Discrete Fourier Transform spectrum of a complex monochromatic oscillation with frequency randomly distorted at the observation times $t = 0, 1, \ldots, n-1$ by a series of independent and identically distributed fluctuations is investigated. It is proved that the second moments of the spectrum at the discrete Fourier frequencies converge uniformly to zero as $n \to \infty$ for certain frequency fluctuation distributions. The observed effect occurs even for frequency fluctuations with magnitude arbitrarily small in comparison to the original oscillation frequency.

1. Introduction. The well-known Discrete Fourier Transform (DFT) is widely used in signal analysis and processing, involving tasks like filtering oscillations with specified frequency-bands from an observed signal [14], [17]. Also the DFT-based periodogram is applied to detect hidden periodicities and estimate unknown oscillation parameters (amplitude and frequency) [5], [27].

If the time series of complex-valued signal observations at discrete equidistant times x_t, $t = 0, 1, \ldots, n-1$, is available, then its Discrete Fourier Transform is computed as follows [14]:

$$\tilde{x}_\nu = \frac{1}{n} \sum_{t=0}^{n-1} x_t \exp(-i2\pi\nu t/n) \tag{1}$$

for $\nu = -n/2+1, -n/2+2, \ldots, n/2$, and even integer $n > 0$. As mentioned

2010 *Mathematics Subject Classification*: 62M15, 62F12.
Key words and phrases: discrete Fourier transform, monochromatic harmonic, spectrum estimation, random frequency modulation.

DOI: 10.4064/am40-1-6

earlier, it can be used to calculate the values of the periodogram [16]

$$I_n(\lambda) = \left| \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t \exp(-i\lambda t) \right|^2, \qquad \lambda \in [-\pi, \pi],$$

at the discrete frequencies $\lambda_\nu = 2\pi\nu/n$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$. Algebraic as well as numerical and statistical properties of the DFT are described in time series analysis textbooks [5], [7], [8], [14], [17]. Frequently, the appropriate Fast Fourier Transform computer procedures are used to perform the relevant calculations [9], [21], [24].

A number of works concern finite sample properties [2], [11], [12], [15] and asymptotic properties [13], [16], [18], [25] of the DFT or periodogram ordinates at the discrete Fourier frequencies. Other approaches to spectrum estimation are also considered [1], [22]. Analysis of the influence of frequency modulation on time series spectra is the subject of only a few articles [3], [4], [20], [23], [26]. Electrocardiographic signals are an example of biological signals whose basic characteristics can be altered by environmental factors. Frequency modulation due to the influence of respiration and other factors may alter the signals and affect the characteristic power spectrum.

In this work the asymptotic properties of the DFT spectra of time series representing complex monochromatic oscillations with frequency randomly distorted at the observation times by a series of independent and identically distributed fluctuations are investigated.

A random frequency modulation modelling is described in Section 2. The results on the asymptotic properties of the DFT spectrum of randomly frequency modulated harmonics are presented in Section 3.

2. Random frequency modulation modelling. Let us consider a finite duration complex-valued time series $o_t = A\exp(i\omega t + i\phi)$, $t = 0, 1, \ldots, n-1$, that represents the values of a monochromatic oscillation with constant frequency $\omega \in \mathbb{R}$, amplitude $A > 0$, and phase $\phi \in [0, 2\pi)$, at discrete equidistant times.

Suppose the oscillation frequency is distorted at the observation times by independent and identically distributed random variables (fluctuations) δ_t, $t = 0, 1, \ldots, n-1$, according to the model

$$v_t = A\exp(i(\omega + \delta_t)t + i\phi) = r_t o_t, \tag{2}$$

where $r_t = \exp(it\delta_t)$. We shall investigate the asymptotic properties of the DFT spectra $\tilde{v}_\nu$ computed according to (1). Hence, we will consider discrete spectra of randomly frequency modulated harmonics.

Obviously, the equalities

$$Er_t = E\exp(it\delta_t) = \chi_\delta(t) \tag{3}$$

hold for $t = 0, 1, \ldots, n-1$, where $\chi_\delta(s)$, $s \in \mathbb{R}$, denotes the characteristic function of the frequency distortions, and consequently

$$\text{(4)} \qquad \operatorname{Var}(r_t) = E|\exp(it\delta_t)|^2 - |\chi_\delta(t)|^2 = 1 - |\chi_\delta(t)|^2 \le 1.$$

For example, suppose the distribution of δ_t is uniform on the interval $(-\Delta, \Delta)$, i.e. $\delta_t \sim U(-\Delta, \Delta)$, where $0 < \Delta \le \pi$, which gives immediately

$$\text{(5)} \qquad \chi_{\delta\Delta}(t) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} \exp(itu)\,du = \frac{1}{2t\Delta}\int_{-t\Delta}^{t\Delta} \exp(iv)\,dv = \frac{\sin(t\Delta)}{t\Delta}.$$

Clearly, for $\Delta = \pi$ we have $\chi_{\delta\pi}(0) = 1$ and $\chi_{\delta\pi}(t) = 0$, $t = 1, 2, \ldots$.

In order to compute the DFT of the modulated signals of the form $v_t = r_t o_t$, $t = 0, 1, \ldots, n-1$, we apply the well-known circular convolution formula [14]

$$\text{(6)} \quad \tilde{v}_\mu = \sum_{j+k=\mu \bmod n} \tilde{r}_j \tilde{o}_k = \sum_{\substack{j+k=\mu \\ j+k=n+\mu}} \tilde{r}_j \tilde{o}_k = \sum_{j=0}^{\mu} \tilde{r}_j \tilde{o}_{\mu-j} + \sum_{j=\mu+1}^{n-1} \tilde{r}_j \tilde{o}_{n+\mu-j}$$

for $\mu = 0, 1, \ldots, n-1$. Of course, since the DFT is periodic (with period n), the values $\tilde{r}_\mu, \tilde{o}_\mu$, $\mu = 0, 1, \ldots, n-1$, can be obtained from $\tilde{r}_\nu, \tilde{o}_\nu$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$. Hence, if we want to analyze the DFT of the modulated signal $\tilde{v}_\nu$, it is necessary to characterize the statistical properties of the DFT $\tilde{r}_\nu$ of the modulating series and estimate the magnitude of $\tilde{o}_\nu$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$. For this purpose we need the following lemmas.

LEMMA 2.1. *If $r_t = \exp(it\delta_t)$, where the real-valued random variables δ_t, $t = 0, 1, \ldots, n-1$, are independent and identically distributed with characteristic function $\chi_\delta(s)$, $s \in \mathbb{R}$, then*

$$|E\tilde{r}_\nu| \le \sqrt{\frac{1}{n}\sum_{t=0}^{n-1} |\chi_\delta(t)|^2} \quad \textit{and} \quad |E(\tilde{r}_\nu - E\tilde{r}_\nu)(\overline{\tilde{r}}_\mu - E\overline{\tilde{r}}_\mu)| \le 1/n$$

for $\nu, \mu = -n/2+1, -n/2+2, \ldots, n/2$.

Proof. Definition (1) together with (3) and the Schwarz inequality yield immediately

$$|E\tilde{r}_\nu| = \left|\frac{1}{n}\sum_{t=0}^{n-1} \chi_\delta(t)\exp(-i2\pi\nu t/n)\right| \le \sqrt{\frac{1}{n}\sum_{t=0}^{n-1} |\chi_\delta(t)|^2}$$

for $\nu = -n/2+1, -n/2+2, \ldots, n/2$. The independence of the random variables δ_t, $t = 0, 1, \ldots, n-1$, together with (4), implies

$$\begin{aligned}
&|E(\tilde{r}_\nu - E\tilde{r}_\nu)(\overline{\tilde{r}}_\mu - E\overline{\tilde{r}}_\mu)| \\
&\quad = \frac{1}{n^2}\Big|E\sum_{t=0}^{n-1}(r_t - Er_t)\exp(-i2\pi\nu t/n)\sum_{s=0}^{n-1}(\overline{r}_s - E\overline{r}_s)\exp(i2\pi\mu s/n)\Big| \\
&\quad = \frac{1}{n^2}\Big|\sum_{t=0}^{n-1}\sum_{s=0}^{n-1}E(r_t - Er_t)(\overline{r}_s - E\overline{r}_s)\exp(-i2\pi(\nu t - \mu s)/n)\Big| \\
&\quad = \frac{1}{n^2}\Big|\sum_{t=0}^{n-1}\mathrm{Var}(r_t)\exp(-i2\pi(\nu-\mu)t/n)\Big| \le \frac{1}{n}
\end{aligned}$$

for $\nu, \mu = -n/2+1, -n/2+2, \ldots, n/2$. ■

Lemma 2.2. *For* $u \in [0, \pi/2]$ *the inequality* $\sin(u) \ge 2u/\pi$ *holds.*

Proof. The function $g(u) = \sin(u) - 2u/\pi$ is differentiable and satisfies $g(0) = g(\pi/2) = 0$, $g'(u) = \cos(u) - 2/\pi$. Since $0 < 2/\pi < 1$, the function $g(u)$ increases for $u \in [0, \arccos(2/\pi)]$ and decreases for $u \in (\arccos(2/\pi), \pi/2]$, which yields $g(u) \ge g(0) = g(\pi/2) = 0$ for $u \in [0, \pi/2]$. ■

Lemma 2.3. *If* $o_t = \exp(i\omega t)$, $t = 0, 1, \ldots, n-1$, *where* $\omega \in \mathbb{R}$, *and* n *is even, then*

$$\sum_{\nu=-n/2+1}^{n/2} |\tilde{o}_\nu| \le \ln(n/2) + 4.$$

Proof. For $\omega = 2\pi\mu/n$, $\mu = -n/2+1, -n/2+2, \ldots, n/2$, we have $\tilde{o}_\nu = \delta_{\nu\mu}$ (Kronecker delta), so the assertion holds. It remains to prove the lemma for $0 < \omega < 2\pi/n$, since for $\omega' = \omega + k2\pi/n$, $k = \pm 1, \pm 2, \ldots$, we have $\tilde{o}'_\nu = \tilde{o}_{\nu-k}$ and periodicity of the DFT ensures that

$$\sum_{\nu=-n/2+1}^{n/2} |\tilde{o}'_\nu| = \sum_{\mu=-n/2+1-k}^{n/2-k} |\tilde{o}_\mu| = \sum_{\nu=-n/2+1}^{n/2} |\tilde{o}_\nu|.$$

For $0 < \omega < 2\pi/n$ we have

$$\begin{aligned}
\tilde{o}_\nu &= \frac{1}{n}\sum_{t=0}^{n-1}\exp(i(\omega - 2\pi\nu/n)t) = \frac{1}{n}\,\frac{\exp(i(n\omega - 2\pi\nu)) - 1}{\exp(i(\omega - 2\pi\nu/n)) - 1} \\
&= \frac{1}{n}\,\frac{\exp(-i(\omega/2 - \pi\nu/n))(\exp(in\omega) - 1)}{\exp(i(\omega/2 - \pi\nu/n)) - \exp(-i(\omega/2 - \pi\nu/n))} \\
&= \frac{1}{n}\,\frac{\exp(-i(\omega/2 - \pi\nu/n))(\exp(in\omega) - 1)}{2i\sin(\omega/2 - \pi\nu/n)},
\end{aligned}$$

and consequently

$$|\tilde{o}_\nu| \le \frac{1}{n}\frac{1}{|\sin(\omega/2 - \pi\nu/n)|} \tag{7}$$

for $\nu = -n/2+1, -n/2+2, \ldots, n/2$. Now, for $0 < \omega < 2\pi/n$ and $\nu = 2, 3, \ldots, n/2$, we have $\pi\nu/n - \omega/2 \in (\pi/n, \pi/2)$, so by (7) and Lemma 2.2 we obtain, for even $n > 4$,

$$\begin{aligned} \sum_{\nu=3}^{n/2} |\tilde{o}_\nu| &\le \frac{1}{n}\sum_{\nu=3}^{n/2} \frac{1}{|\sin(\omega/2 - \pi\nu/n)|} \le \frac{2\pi}{2n}\sum_{\nu=3}^{n/2}\frac{1}{2\pi\nu/n - \omega} \\ &\le \frac{1}{2}\int_{4\pi/n}^{\pi} \frac{1}{u-\omega}\,du = \frac{1}{2}\ln\biggl(\frac{\pi-\omega}{4\pi/n - \omega}\biggr) \le \frac{1}{2}\ln(n/2). \end{aligned} \tag{8}$$

Since for $0 < \omega < 2\pi/n$ and $\nu = -n/2+1, -n/2+2, \ldots, -1$, we also have $\omega/2 - \pi\nu/n \in (\pi/n, \pi/2)$, we obtain analogously, for even $n > 4$,

$$\begin{aligned} \sum_{\nu=-n/2+1}^{-2} |\tilde{o}_\nu| &\le \frac{1}{n}\sum_{\nu=-n/2+1}^{-2} \frac{1}{|\sin(\omega/2 - \pi\nu/n)|} \\ &\le \frac{2\pi}{2n}\sum_{\nu=-n/2+1}^{-2}\frac{1}{\omega - 2\pi\nu/n} \le \frac{1}{2}\int_{-\pi+2\pi/n}^{-2\pi/n}\frac{1}{\omega-u}\,du \\ &= \frac{1}{2}\ln\biggl(\frac{\omega+\pi-2\pi/n}{\omega+2\pi/n}\biggr) \le \frac{1}{2}\ln(n/2). \end{aligned} \tag{9}$$

In view of definition (1), $|\tilde{o}_\nu| \le 1$ for any $\omega \in \mathbb{R}$, so taking into account (8) and (9) finally gives

$$\sum_{\nu=-n/2+1}^{n/2} |\tilde{o}_\nu| = |\tilde{o}_{-1}| + |\tilde{o}_0| + |\tilde{o}_1| + |\tilde{o}_2| + \sum_{\nu=-n/2+1}^{-2}|\tilde{o}_\nu| + \sum_{\nu=3}^{n/2}|\tilde{o}_\nu| \le \ln(n/2) + 4,$$

which completes the proof (including the case of $n = 2, 4$). ■

3. Asymptotic properties of randomly frequency modulated harmonic spectra. Formula (6) together with Lemmas 2.1 and 2.3 allows us to characterize the DFT spectra corresponding to the frequency modulation model considered. Namely, in the following lemma we derive upper bounds for the mean values and variances of the random variables $\tilde{v}_\nu$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$, representing a randomly frequency modulated harmonic spectrum. The corollary that follows concerns the asymptotic behaviour of such spectra.

LEMMA 3.1. *If $o_t = \exp(i\omega t)$, where $\omega \in \mathbb{R}$, and $r_t = \exp(it\delta_t)$, where the real-valued random variables δ_t, $t = 0, 1, \ldots, n-1$, are independent and* *identically distributed with characteristic function* $\chi_\delta(s)$, $s \in \mathbb{R}$, *then the DFT of the series* $v_t = r_t o_t$, $t = 0, 1, \ldots, n-1$, *satisfies*

$$|E\tilde{v}_\nu| \le \frac{\ln(n/2)+4}{n^{1/2}} \sqrt{\sum_{t=0}^{n-1} |\chi_\delta(t)|^2} \quad \text{and} \quad E|\tilde{v}_\nu - E\tilde{v}_\nu|^2 \le \frac{(\ln(n/2)+4)^2}{n}$$

for $\nu = -n/2+1, -n/2+2, \ldots, n/2$.

Proof. Formula (6) yields immediately, for $\mu = 0, 1, \ldots, n-1$,

$$|E\tilde{v}_\mu| \le \sum_{j=0}^{\mu} |E\tilde{r}_j|\,|\tilde{o}_{\mu-j}| + \sum_{j=\mu+1}^{n-1} |E\tilde{r}_j|\,|\tilde{o}_{n+\mu-j}|,$$

which together with Lemma 2.1 and periodicity of the DFT shows that

$$|E\tilde{v}_\mu| \le \frac{1}{n^{1/2}} \sqrt{\sum_{t=0}^{n-1} |\chi_\delta(t)|^2} \sum_{j=0}^{n-1} |\tilde{o}_j|,$$

so Lemma 2.3 and periodicity of the DFT imply the first inequality of the assertion. Using (6) again we obtain

$$\tilde{v}_\mu - E\tilde{v}_\mu = \sum_{j=0}^{\mu} (\tilde{r}_j - E\tilde{r}_j)\tilde{o}_{\mu-j} + \sum_{j=\mu+1}^{n-1} (\tilde{r}_j - E\tilde{r}_j)\tilde{o}_{n+\mu-j} = \sum_{j=0}^{n-1} (\tilde{r}_j - E\tilde{r}_j)\tilde{o}_{m(j)},$$

where $m(j) = \mu - j$ for $0 \le j \le \mu$ and $m(j) = n+\mu-j$ for $\mu+1 \le j \le n-1$, and consequently by Lemmas 2.1 and 2.3,

$$\begin{aligned}
E|\tilde{v}_\mu - E\tilde{v}_\mu|^2 &= E\sum_{j=0}^{n-1} (\tilde{r}_j - E\tilde{r}_j)\tilde{o}_{m(j)} \sum_{k=0}^{n-1} (\overline{\tilde{r}}_k - E\overline{\tilde{r}}_k)\overline{\tilde{o}}_{m(k)} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} E(\tilde{r}_j - E\tilde{r}_j)(\overline{\tilde{r}}_k - E\overline{\tilde{r}}_k)\tilde{o}_{m(j)}\overline{\tilde{o}}_{m(k)} \le \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} |\tilde{o}_{m(j)}|\,|\tilde{o}_{m(k)}| \\
&= \frac{1}{n}\sum_{j=0}^{n-1} |\tilde{o}_{m(j)}| \sum_{k=0}^{n-1} |\tilde{o}_{m(k)}| = \frac{1}{n}\sum_{j=0}^{n-1} |\tilde{o}_j| \sum_{k=0}^{n-1} |\tilde{o}_k| \le \frac{(\ln(n/2)+4)^2}{n}
\end{aligned}$$

for $\mu = 0, 1, \ldots, n-1$, and the proof is complete in view of the DFT periodicity. ∎

COROLLARY 3.1. *Under the assumptions of Lemma* 3.1 *the DFT of the series* $v_t = r_t o_t$, $t = 0, 1, \ldots, n-1$, *satisfies*

$$E|\tilde{v}_\nu|^2 \le \frac{(\ln(n/2)+4)^2}{n}\Big[1 + \sum_{t=0}^{n-1} |\chi_\delta(t)|^2\Big]$$

for $\nu = -n/2+1, -n/2+2, \ldots, n/2$. *If moreover* $\sum_{t=0}^{\infty} |\chi_\delta(t)|^2 < \infty$, *then*

$$\lim_{n\to\infty} E|\tilde{v}_\nu|^2 = 0$$

uniformly for $\nu = -n/2+1, -n/2+2, \ldots, n/2$.

The condition $\sum_{t=0}^{\infty} |\chi_\delta(t)|^2 < \infty$ is satisfied for the class of fluctuation distributions considered in the introduction. Namely, if $\delta_t \sim U(-\Delta, \Delta)$, where $0 < \Delta \le \pi$, then according to (5) we have

$$\sum_{t=0}^{\infty} |\chi_{\delta\Delta}(t)|^2 = 1 + \sum_{t=1}^{\infty} \frac{|\sin(t\Delta)|^2}{t^2\Delta^2} \le 1 + \frac{1}{\Delta^2}\sum_{t=1}^{\infty}\frac{1}{t^2} = 1 + \frac{\pi^2}{6\Delta^2}.$$

Another example of frequency fluctuation distribution which satisfies an even stronger condition $\chi(t) = 0$ for $|t| \ge 1$ is the distribution on $\mathbb{R}$ with density $p(t) = (1-\cos(t))/\pi t^2$ and characteristic function $\chi(t) = (1-|t|)_+$, $t \in \mathbb{R}$ (the formulae are easily derived from the characteristic function and density of the well-known triangular distribution).

Now, since $v_t = r_t \exp(i\omega t)$, where $r_t = \exp(it\delta_t)$, $t = 0, 1, \ldots, n-1$, the well-known equality [14]

$$\sum_{\nu=0}^{n-1} |\tilde{v}_\nu|^2 = \frac{1}{n}\sum_{t=0}^{n-1} |v_t|^2$$

yields immediately

$$E\sum_{\nu=0}^{n-1} |\tilde{v}_\nu|^2 = \frac{1}{n}\sum_{t=0}^{n-1} E|r_t|^2 |\exp(i\omega t)|^2 = 1.$$

Consequently, the average mean square value of the random variables $\tilde{v}_\nu$, $\nu = 0, 1, \ldots, n-1$, representing the spectrum of the frequency modulated harmonic $o_t = \exp(i\omega t)$, $t = 0, 1, \ldots, n-1$, equals

$$\frac{1}{n}E\sum_{\nu=0}^{n-1} |\tilde{v}_\nu|^2 = \frac{1}{n}$$

and converges to zero (with convergence rate n^{-1}) as $n \to \infty$ for any distribution of the frequency fluctuations.

Let us now assume that the measurements of the frequency modulated monochromatic oscillation from model (2) are corrupted by random observation errors, i.e. we observe the time series

$$y_t = v_t + \eta_t, \qquad t = 0, 1, \ldots, n-1,$$

where η_t are uncorrelated complex-valued random variables having zero mean $E_\eta \eta_t = 0$ and finite second moment $\sigma_\eta^2 = E_\eta|\eta_t|^2 < \infty$. We further assume that the observation errors are independent of the frequency fluctuations. Then, inspection of the proof of Lemma 2.1 shows that $E_\eta \tilde{\eta}_\nu = 0$ and $E_\eta|\tilde{\eta}_\nu|^2 = \sigma_\eta^2/n$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$. Consequently, in view of linearity of the DFT, $E_\delta E_\eta \tilde{y}_\nu = E_\delta \tilde{v}_\nu + E_\eta \tilde{\eta}_\nu = E_\delta \tilde{v}_\nu$ and $E_\delta E_\eta |\tilde{y}_\nu|^2 =$

$E_\delta E_\eta|\tilde{v}_\nu + \tilde{\eta}_\nu|^2 = E_\delta|\tilde{v}_\nu|^2 + \sigma_\eta^2/n$, $\nu = -n/2+1, -n/2+2, \ldots, n/2$. So it is easy to see that the relevant bound for $E_\delta E_\eta|\tilde{y}_\nu|^2$ differs from the one in Corollary 3.1 by the additive term σ_η^2/n, which occurs because the observation errors have non-zero second moment. Hence, the uniform convergence rate of $E_\delta E_\eta|\tilde{y}_\nu|^2$, $\nu = 0, 1, \ldots, n-1$, as well as of their average value remains the same.

4. Conclusions. The asymptotic properties of the DFT spectrum of a complex monochromatic oscillation, investigated in this work, take into account the effects of random frequency modulation at the observation times by a series of independent and identically distributed fluctuations. It is proved that for certain fluctuation distributions the second moments of the DFT spectrum at the discrete Fourier frequencies converge uniformly to zero as the number of observations grows. So for such distributions the randomly frequency modulated harmonic spectrum asymptotically does not distinguish any frequency, whereas the DFT amplitude spectrum of the original monochromatic oscillation resembles the Dirac delta function $\delta(s-\omega)$, $s \in [-\pi, \pi]$, where $-\pi \le \omega \le \pi$ denotes the oscillation frequency [8]. The observed effect can occur both for frequency fluctuations with large and arbitrarily small magnitudes compared with the original oscillation frequency ω. Thus, it is shown that occurrence of random frequency modulation of monochromatic oscillations can completely change the character of the corresponding DFT spectra. However, it should be noted that certain linear combinations of frequency modulated monochromatic signals may produce a signal with amplitude modulation only, as can be seen from the simple example

$$\frac{1}{4}\exp(i(\omega+\delta_t)t) + \frac{1}{4}\exp(i(\omega-\delta_t)t) + \frac{5}{2}\exp(i\omega t) = [2+\cos^2(t\delta_t/2)]\exp(i\omega t),$$

where δ_t, $t = 0, 1, \ldots, n-1$, denote modulating random variables characterized in the introduction.

Similar conclusions are deduced also in the case of a randomly frequency modulated monochromatic oscillation which is corrupted at the observation times by uncorrelated random errors with zero mean and finite second moment.

Since the DFT is linear, the results obtained also help to understand the influence a particular signal component frequency modulation can have on the DFT spectrum of a signal which is a sum of several monochromatic oscillations.

The same model of random distortions as the one considered here for frequency modulation is applied in the Fourier Bootstrap Method for random phase modulation [6] and in surrogate data sets generation with phase randomization [19].

Acknowledgments. This research work was supported by the Polish National Science Centre through the grant No. N 526 157040 carried out at the Space Research Centre of Polish Academy of Sciences.

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Received on 9.7.2012;
revised version on 1.8.2012 (2142)