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CENTRAL LIMIT THEOREM FOR SQUARE ERROR OF MULTIVARIATE NONPARAMETRIC BOX SPLINE DENSITY ESTIMATORS

Abstract. We prove the central limit theorem for the integrated square error of multivariate box-spline density estimators.

1. Introduction. Peter Hall [14] proved the central limit theorem for the integrated square error of multivariate density estimators. He considered the Rosenblatt–Parzen kernel estimator $f_{h,n}$ such that

$$Ef_{h,n} = \frac{1}{h^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h}\right) f(y)\, dy.$$

To state the assumptions imposed on the function K let us introduce the following standard notation: the convolution

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy,$$

the scaling operator

$$\sigma_h f(x) = f(x/h),$$

and the convolution operators

$$T_K f = K * f \quad \text{and} \quad T_{K,h} f = \sigma_h \circ T_K \circ \sigma_{1/h}$$

for $h > 0$. The assumptions imposed on $K \geq 0$:

$$\int_{\mathbb{R}^d} K(x)\, dx = 1,$$

$$\int_{\mathbb{R}^d} x_j K(x)\, dx = 0,$$

$$\int_{\mathbb{R}^d} x_j x_i K(x)\, dx = 2k\delta_{j,i} < \infty$$

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for all $i, j = 1, \ldots, d$ and $x = (x_1, \ldots, x_d)$, imply that for the operators $T_{K,h}$ we have the saturation theorem, i.e. for all sufficiently smooth functions f,

$$\frac{T_{K,h}f - f}{h^2} \to k\nabla^2 f, \tag{1}$$

where ∇^2 is the Laplacian, the convergence is (for example) in L^2 norm and k is a constant which depends on K. We set ourselves two aims:

(i) To prove P. Hall's theorem for box-spline estimators. For some box-spline operators we have the saturation theorem 1.13 of [11].

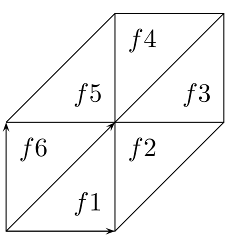
(ii) To weaken condition (1). Our earlier research [1], [2] on the weak saturation theorem will help us to achieve this aim. In this way the linear density estimator (see below) also satisfies the conditions of the main theorem.

The first approach to spline estimators was made by Ciesielski [6], and the research was continued by Krzykowski [15]. We develop and simplify their ideas. Why do we use box splines? The definition of a box spline is a generalization of a B-spline used in the univariate case. For details we refer to [5]. The simplest case of our estimators is a histogram. Unfortunately it does not satisfy the assumptions of the theorem, unlike the linear estimator which is a significant example of our estimators. Ciesielski's definition does not contain this important case. The linear estimator is also known in the literature [18]. We also know that spline estimators are better than Rosenblatt–Parzen kernel estimators from the numerical point of view since they localize the observation. The same concept is important in wavelet theory but the construction is different [13].

Let us present the linear estimator in $\mathbb{R}^2$ from our point of view. Let

$$B(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in [0,1)^2, \\ 0, & (x_1, x_2) \notin [0,1)^2, \end{cases}$$

and let us define a hat function B_1 by



$$B_1(x_1, x_2) = \begin{cases} x_2 & \text{on } f1, \\ x_2 - x_1 + 1 & \text{on } f2, \\ 2 - x_1 & \text{on } f3, \\ 2 - x_2 & \text{on } f4, \\ x_1 - x_2 + 1 & \text{on } f5, \\ x_1 & \text{on } f6. \end{cases}$$

Fig. 1. Support of the hat function

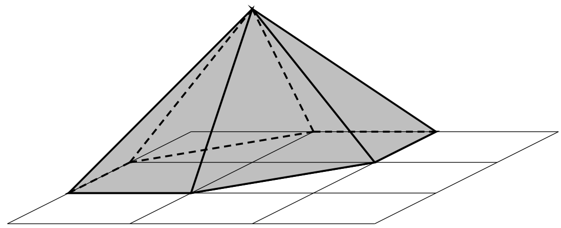


Fig. 2. Hat function

A spline estimator is defined as follows. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density f. Let

$$c = (1/2, 1/2), \quad B_2(x_1, x_2) = B((x_1, x_2) - c).$$

Then

$$f_{h,n}(x) = \frac{1}{h} \sum_{\alpha \in \mathbb{Z}^2} \left(\frac{1}{n} \sum_{j=1}^{n} B_2(X_j/h - \alpha) \right) B_1(x/h - \alpha).$$

Let us notice that B_1 gives the appropriate smoothness of the estimator.

EXAMPLE 1.1. As a random vector (X_1, X_2) let us consider the daily changes of the Warsaw Market Index (WIG) and the Dow Jones Industrial Index (DJI). 70 data are collected between January and April, 2000. An example of the data: 12.01.2000 WIG, 11.01.2000 DJI $(-1.93\%, -0.53\%)$. The estimator is visualized in Fig. 3.

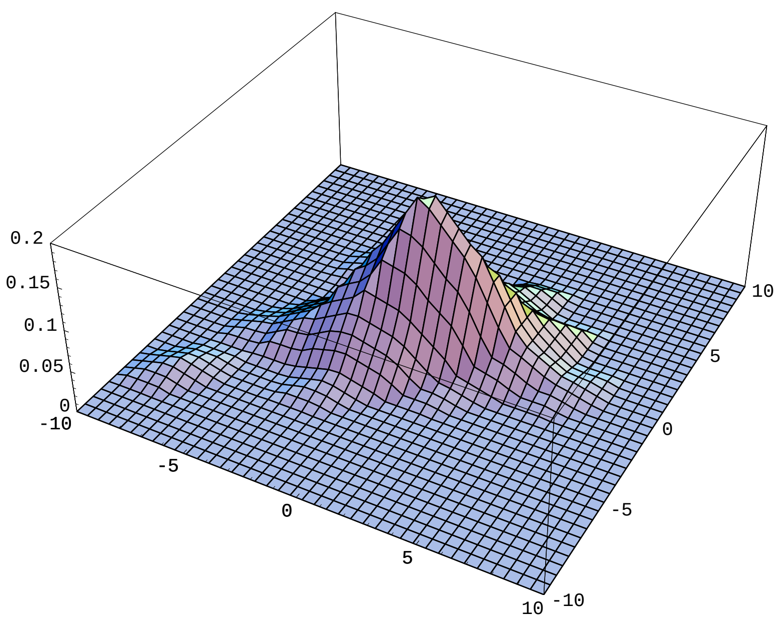


Fig. 3. Visualization of the estimator

The organization of the paper is as follows. In Section 2 we state the main result, the central limit theorem. In Section 3 we present without proof the asymptotic formula for the error of the Ciesielski operator. In Section 4 we give the proof of the main theorem.

2. Central theorem. Let $V = \{v_1, \ldots, v_s\}$ denote a set of not necessarily distinct vectors in $\mathbb{Z}^d \setminus \{0\}$ such that

$$\operatorname{span}(V) = \mathbb{R}^d.$$

We call such a set *admissible*. Let $\mathbf{V}$ denote the matrix whose columns are the vectors from V. Usually the *box spline* corresponding to V (denoted by $B(\cdot \mid V)$ or B_V) is defined by requiring that

$$\int_{\mathbb{R}^d} f(x) B(x \mid V)\, dx = \int_{[0,1]^s} f(\mathbf{V}u)\, du \tag{2}$$

for any continuous function f on $\mathbb{R}^d$. This definition gives a class of functions. In numerical applications we choose the smoothest box spline such that for all $x \in \mathbb{R}^d$,

$$\sum_{\alpha \in \mathbb{Z}^d} B_V(x - \alpha) = 1.$$

The best reference is [5].

Let us present a recurrent definition of a box spline in $\mathbb{R}^2$. If

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $B(\cdot \mid W_1)$ is the characteristic function of the cube $[0,1)^2$. If

$$W_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

then $B(\cdot \mid W_2)$ is the hat function (see Figs. 1 and 2). Moreover

$$B(x \mid W_2) = \int_0^1 B(x - t(1,1) \mid W_1)\, dt.$$

Thus we get a constructive definition of a box spline. Let V be an admissible family. The box spline corresponding to V is given by the recurrent formula

$$B(x \mid V) = \int_0^1 B(x - tv_j \mid V \setminus \{v_j\})\, dt \tag{3}$$

where the vector v_j is chosen in such a way that the family $V \setminus \{v_j\}$ is admissible.

Properties of box splines (see [5]). The function B_V is piecewise polynomial. The closed convex set

$$\langle V \rangle = \Big\{ \sum_{j=1}^{s} t_j v_j : 0 \le t_j \le 1,\ j = 1, \ldots, s \Big\} \tag{4}$$

is the support of the box spline $B(\cdot \mid V)$. For an admissible set V let

$$\varrho_V = \max\{r : \forall_{X \subset V,\ \#X = r} \operatorname{span}(V \setminus X) = \mathbb{R}^d\}. \tag{5}$$

If $\varrho_V \ge 1$, then

$$B_V \in C^{\varrho_V - 1}(\mathbb{R}^d) \setminus C^{\varrho_V}(\mathbb{R}^d). \tag{6}$$

If $\varrho_V = 0$ then B_V may not be continuous only on the border of its support $\langle V \rangle$.

For $1 \le p < \infty$ and $k \in \mathbb{N}$, W_p^k denotes the Sobolev space on $\mathbb{R}^d$, and W_∞^k stands for the closure of the smooth functions with compact support in the norm

$$\|f\|_{\infty,k} = \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \le k} |D^\alpha f(x)|,$$

where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}, \qquad \alpha = (\alpha_1, \ldots, \alpha_d), \qquad |\alpha| = \alpha_1 + \ldots + \alpha_d.$$

By $\|\cdot\|_p$ we denote the standard L^p norm on $\mathbb{R}^d$:

$$\|f\|_p = \Big(\int_{\mathbb{R}^d} |f(x)|^p \, dx \Big)^{1/p}.$$

Let D_v be the directional derivative

$$D_v f = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}.$$

Now we turn to the main theorem. Let us introduce a box spline estimator. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density f. Let V_1, V_2 be admissible sets. Then

$$f_{h,n}(x) = \frac{1}{n} \sum_{j=1}^{n} Q_h(X_j, x), \tag{7}$$

where Q_h is the kernel of the Ciesielski operator (see (23), (24) below), i.e.

$$Q_h(y, x) = h^{-d} \sum_{\alpha \in \mathbb{Z}^d} B(y/h - \alpha - c_Y \mid V_2) B(x/h - \alpha \mid V_1),$$

where

$$c_Y = \frac{1}{2} \sum_{v \in Y} v, \tag{8}$$

$$Y = V_1 \cup (-V_2), \tag{9}$$

and $-V_2$ is the admissible set such that $w \in -V_2 \Leftrightarrow -w \in V_2$.

We now state the main result. We follow the notation of [14]. Let

$$c(n) = \int_{\mathbb{R}^d} E\{f_{h,n}(x) - f(x)\}^2\,dx$$

and

$$d(n) = \begin{cases} n^{1/2}h^{-2} & \text{if } nh^{d+4} \to \infty, \\ nh^{d/2} & \text{if } nh^{d+4} \to 0, \\ n^{(d+8)/(2d+8)} & \text{if } nh^{d+4} \to \lambda, \end{cases}$$

where $0 < \lambda < \infty$. Let

$$Kf = \frac{1}{24}\sum_{v\in Y} D_v^2 f. \tag{10}$$

For $j = 1, 2$ define the admissible sets

$$U_j = V_j \cup (-V_j).$$

THEOREM 2.1. *Assume that* $f \in W_2^2(\mathbb{R}^d) \cap W_\infty^2(\mathbb{R}^d)$. *If* $\varrho_{V_1} \ge 1$ *and* $\varrho_{V_2} \ge 0$ *then assuming that* $h \to 0^+$ *and* $nh^d \to \infty$ *we have*

$$d(n)\Big[\int_{\mathbb{R}^d} \{f_{h,n}(x) - f(x)\}^2\,dx - c(n)\Big] \tag{11}$$
$$\to \begin{cases} 2\sigma_1 Z & \text{if } nh^{d+4} \to \infty, \\ \sqrt{2}\sigma_3 Z & \text{if } nh^{d+4} \to 0, \\ (4\sigma_1^2\lambda^{4/(d+4)} + 2\sigma_3^2\lambda^{-d/(d+4)})^{1/2} Z & \text{if } nh^{d+4} \to \lambda, \end{cases}$$

in distribution, *where* Z *is the standard normal distribution*,

$$\sigma_1^2 = \int (Kf(x))^2 f(x)\,dx - \Big(\int Kf(x)f(x)\,dx\Big)^2,$$
$$\sigma_3^2 = \sum_{\delta_1\in\mathbb{Z}^d}\sum_{\delta_2\in\mathbb{Z}^d}\sum_{\delta_3\in\mathbb{Z}^d} B(\delta_1 \mid U_1)B(\delta_2 \mid U_1) \times B(\delta_3 \mid U_2)B(\delta_2 - \delta_1 + \delta_3 \mid U_2)\int_{\mathbb{R}^d} f^2(x)\,dx.$$

REMARKS. 1. For the Rosenblatt–Parzen estimator this theorem was proved for the L^2 norm in [14], and for the L^p norm with $1 \le p < \infty$ in [7].

2. Using our methods one can prove the same theorem for estimators based on the orthogonal projection on scaled invariant spaces (see [9] and [10]). In this case $\sigma_1 = 0$ and $\sigma_3 = \int f^2$. Our methods are different from those used in the papers of Doukhan and León.

3. Nonparametric hypotheses connected with this theorem are considered in [16].

Note that we have (see Definition 3.1)

$$Ef_{h,n} = Q_h^{(V_1,V_2)} f. \tag{12}$$

THEOREM 2.2. *Under the assumptions of Theorem* 2.1,

$$\begin{aligned} c(n) &= \int_{\mathbb{R}^d} E\{f_{h,n}(x) - f(x)\}^2\,dx \\ &= E\int_{\mathbb{R}^d} (f_{h,n} - Ef_{h,n})^2 + \int_{\mathbb{R}^d} (Q_h^{V_1,V_2} f - f)^2 \\ &\sim \frac{1}{h^d n} \sum_{\alpha\in\mathbb{Z}^d} B(\alpha \mid U_1) B(\alpha \mid U_2) + h^4 C(V_1,V_2) \end{aligned}$$

where

$$C(V_1,V_2) = \|Kf\|_2^2$$

provided $\varrho_{V_1} \geq 2$, $\varrho_{V_2} \geq 0$ *and*

$$C(V_1,V_2) = \|Kf\|_2^2 + \frac{1}{16\pi^4} \sum_{W\in\Lambda} \|D_W f\|_2^2 \sum_{\alpha\perp(V_1\setminus W),\,\alpha\neq 0} \prod_{v\in W} \frac{1}{(\alpha\cdot v)^2}.$$

provided $\varrho_{V_1} = 1$, $\varrho_{V_2} \geq 0$ *where*

$$\Lambda = \{W \subset V_1 : |W| = 2,\ \mathrm{span}(V_1 \setminus W) \neq \mathbb{R}^d\}$$

and

$$D_W = \prod_{v\in W} D_v.$$

REMARK. To estimate $c(n)$ (denoted by MISE(f) in the literature) we need to calculate the asymptotic formula for $\int (Q_h^{V_1,V_2} f - f)^2$. This is quite simple for a convolution operator. For the orthogonal projection it was done in [8] and independently in [2] and for all L^p norms, $1 \leq p < \infty$, in [4].

3. Box-spline operators. The inner product in $L^2(\mathbb{R}^d)$ is denoted by

$$(f,g)_{\mathbb{R}^d} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx.$$

From the definition (2) of a box spline we deduce that

$$\int_{\mathbb{R}^d} B(x \mid V)\,dx = 1, \tag{13}$$

and

$$B(-x \mid V) = B(x \mid -V), \tag{14}$$

and

$$B_V * B_W(x) = B(x \mid V, W). \tag{15}$$

DEFINITION 3.1. The *Ciesielski–Dürrmeyer operator* is defined by

$$Q^{(V_1,V_2)} f = \sum_{\alpha\in\mathbb{Z}^d} (f, B(\cdot - \alpha - c_Y \mid V_2))_{\mathbb{R}^d} B(\cdot - \alpha \mid V_1), \tag{16}$$

where $c_Y = \frac{1}{2}\sum_{v\in Y} v$ (recall (8)) and

$$Q_h^{(V_1,V_2)} = \sigma_h \circ Q^{(V_1,V_2)} \circ \sigma_{1/h}, \tag{17}$$

where $h > 0$ and

$$\sigma_h f(\cdot) = f(\cdot/h). \tag{18}$$

Let us state a simple lemma without proof.

LEMMA 3.1. *For any integrable function f,*

$$\begin{gathered} Q_h^{(V_1,V_2)} f \ge 0 \quad \textit{for } f \ge 0, \\ \int_{\mathbb{R}^d} Q_h^{(V_1,V_2)} f = \int_{\mathbb{R}^d} f. \end{gathered} \tag{19}$$

The saturation theorem for the tensor product of B-spline operators is due to Ciesielski (see [6]). We formulate the theorem for general box splines. This theorem was announced in [11] (Theorem 1.13) without proof since it is similar to the proof of the saturation theorem for quasi-projections presented in [11]. We slightly improve one assumption by requiring that $\varrho_{V_2} \ge 0$. The proof is the same. As usual in approximation theory, $o(1) \to 0$ as $h \to 0^+$.

THEOREM 3.1. *Assume that V_1 and V_2 are admissible and*

$$\varrho_{V_1} \ge 2, \quad \varrho_{V_2} \ge 0.$$

Then for all $f \in W_p^2(\mathbb{R}^d)$ and $1 \le p \le \infty$,

$$\left\| \frac{Q_h^{(V_1,V_2)} f - f}{h^2} - \frac{1}{24}\sum_{v\in Y} D_v^2 f \right\|_p = o(1). \tag{20}$$

The situation is worse for simpler operators, if we assume that $\varrho_{V_1} = 1$.

The idea of the following theorem is taken from [1] and [2]. Define as before

$$\Lambda = \{W \subset V_1 : |W| = 2, \operatorname{span}\{V_1 \setminus W\} \ne \mathbb{R}^d\}$$

and $D_W = \prod_{v\in W} D_v$, and recall $Kf = \frac{1}{24}\sum_{v\in Y} D_v^2 f$ from (10).

THEOREM 3.2. *Assume that V_1 and V_2 are admissible and*

$$\varrho_{V_1} = 1, \quad \varrho_{V_2} \ge 0.$$

Then for all $f \in W_2^2(\mathbb{R}^d)$ and $h \to 0$,

$$\frac{Q_h^{(V_1,V_2)} f - f}{h^2} \to Kf \tag{21}$$

weakly in $L^2(\mathbb{R}^d)$. *Moreover for* $h \to 0$,

$$\left\| \frac{Q_h^{(V_1,V_2)} f - f}{h^2} \right\|_2^2 \to \|Kf\|^2 + \frac{1}{16\pi^4} \sum_{W \in \Lambda} \|D_W f\|_2^2 \sum_{\alpha \perp (V_1 \setminus W),\, \alpha \neq 0} \prod_{v \in W} \frac{1}{(\alpha \cdot v)^2}. \tag{22}$$

The proof will be given in [12] (see also [1], [2]).

We will denote by $Q_h(y,x)$ the kernel of the operator $Q_h^{(V_1,V_2)}$, i.e.

$$Q_h^{(V_1,V_2)} f(x) = \int_{\mathbb{R}^d} Q_h(y,x) f(y)\, dy, \tag{23}$$

$$Q_h(y,x) = \frac{1}{h^d} \sum_{\alpha \in \mathbb{Z}^d} B(y/h - \alpha - c_Y \mid V_2) B(x/h - \alpha \mid V_1). \tag{24}$$

The properties (21) and (22) help us understand the following lemma.

LEMMA 3.2. *Let* $1 \le p \le \infty$. *If*

$$\varrho_{V_1} \ge 1, \qquad \varrho_{V_2} \ge 0.$$

then there is a constant C_p *such that for all* $f \in W_p^2(\mathbb{R}^d)$,

$$\|Q_h^{(V_1,V_2)} f - f\|_p \le C_p h^2 \sum_{|\beta|=2}^{d} \|D^\beta f(x)\|_p.$$

Moreover assuming that $f \in W_\infty^2 \cap W_2^2$ *and* $h \to 0$ *we have*

$$\int_{\mathbb{R}^d} Q_h(u,x) \frac{Q_h^{(V_1,V_2)} f(x) - f(x)}{h^2}\, dx \to Kf(u)$$

uniformly for $u \in \mathbb{R}^d$.

The first part of the lemma follows from Proposition 3.4 of [5]. The "moreover" part will be proved in a forthcoming paper.

4. Proofs. To prove Theorems 2.1 and 2.2 we need three lemmas. We use almost the same notation as in [14]. The proof of Theorem 2.1 is a consequence of these lemmas. We follow the proof of Hall's theorem. To simplify notation let

$$Q_h(y,x) = h^{-d} \sum_{\alpha \in \mathbb{Z}^d} B_2(y/h - \alpha) B_1(x/h - \alpha),$$

where

$$B_1(x) = B(x \mid V_1), \quad B_2(x) = B(x - c_Y \mid V_2).$$

Moreover, set $Q_h^{1,2} = Q_h^{V_1,V_2}$, $\int = \int_{\mathbb{R}^d}$, $f_n = f_{h,n}$, and

$$
\begin{aligned}
I_{n,1} &= \frac{1}{n}\sum_{j=1}^{n} Z_{n,1,j},\\
Z_{n,1,j} &= Y_{n,1,j} - EY_{n,1,j}, \qquad (25)\\
Y_{n,1,j} &= \int Q_h(X_j,x)[Ef_n(x) - f(x)]\,dx\\
&= \int Q_h(X_j,x)[Q_h^{1,2}f(x) - f(x)]\,dx.
\end{aligned}
$$

We have the analogue of Lemma 1 of [14].

Lemma 4.1. *Under the assumptions of Theorem* 2.1,

$$EZ_{n,1,j} = 0,$$

$$EZ_{n,1,j}^2 \sim h^4\Big[\int (Kf(x))^2 f(x)\,dx - \Big(\int Kf(x)f(x)\,dx\Big)^2\Big], \tag{26}$$

$$EZ_{n,1,j}^4 \sim O(h^8). \tag{27}$$

Proof. Note that

$$
\begin{aligned}
EY_{n,1,j}^k &= \underbrace{\int\ldots\int}_{k} E\prod_{i=1}^{k} Q_h(X_j,u^i)[Q_h^{1,2}f(u^i) - f(u^i)]\,du^1\ldots du^k\\
&= \underbrace{\int\ldots\int}_{k}\int \prod_{i=1}^{k} Q_h(z,u^i)[Q_h^{1,2}f(u^i) - f(u^i)] \times f(z)\,dz\,du^1\ldots du^k.
\end{aligned}
$$

By Lemma 3.2 and since Kf is bounded,

$$
\begin{aligned}
EY_{n,1,j}^k &= \int \prod_{i=1}^{k}\int Q_h(z,u^i)[Q_h^{1,2}f(u^i) - f(u^i)]\,du^i\, f(z)\,dz\\
&\sim h^{2k}\int (Kf(z))^k f(z)\,dz.
\end{aligned}
$$

Thus

$$EY_{n,1,j} \sim h^2\int (Kf(z))f(z)\,dz, \tag{28}$$

$$EY_{n,1,j}^2 \sim h^4\int (Kf(z))^2 f(z)\,dz. \tag{29}$$

Now (28) and (29) imply (26). To prove (27) note that

$$EZ_{n,1,j}^4 = EY_{n,1,j}^4 - 4EY_{n,1,j}^3 EY_{n,1,j} + 6EY_{n,1,j}^2(EY_{n,1,j})^2 - 3(EY_{n,1,j})^4.$$

This finishes the proof. ∎

COROLLARY 4.1. *Under the assumptions of Theorem* 2.1,

$$EI_{n,1} = 0,$$

$$\operatorname{Var} I_{n,1} \sim h^4 \frac{\sigma_1^2}{n},$$

$$\frac{I_{n,1} - EI_{n,1}}{h^2\sigma_1/\sqrt{n}} \to N(0,1) \quad \textit{in distribution as } h \to 0.$$

The following lemma is a generalization of the Fejér–Mazur–Orlicz Theorem.

LEMMA 4.2. *Let g be a bounded $\mathbb{Z}^d$-periodic function, i.e. $g(x+\alpha) = g(x)$ for all $\alpha \in \mathbb{Z}^d$. Let f be integrable. Then*

$$\lim_{h\to 0^+} \int g(x/h) f(x)\,dx = \int f(x)\,dx \int_{[0,1]^d} g(u)\,du.$$

The proof is given in [2] and [3] or in [17]. Let us introduce

$$\begin{aligned} Z_{n,2,j} &= \int [Q_h(X_j,x) - EQ_h(X_j,x)]^2\,dx \\ &= \int [Q_h(X_j,x) - Q_h^{1,2} f(x)]^2\,dx, \\ I_{n,2} &= \frac{1}{n^2}\sum_{j=1}^{n} Z_{n,2,j}. \end{aligned} \tag{30}$$

We have the analogue of Lemma 2 of [14].

LEMMA 4.3. *Under the assumptions of Theorem* 2.1,

$$EZ_{n,2,j} \sim h^{-d} \int_{[0,1]^d} \sum_{\alpha_1\in\mathbb{Z}^d} \sum_{\alpha_2\in\mathbb{Z}^d} B(\alpha_2 - \alpha_1 \mid U_1) \times B(u - \alpha_1 - c_Y \mid V_2) B(u - \alpha_2 - c_Y \mid V_2)\,du, \tag{31}$$

$$EZ_{n,2,j}^2 \sim O(1/h^{2d}). \tag{32}$$

Proof. Note that

$$\begin{aligned} EZ_{n,2,j} &= \int E[Q_h(X_j,x)]^2 - [EQ_h(X_j,x)]^2\,dx \\ &= \int E[Q_h(X_j,x)]^2 - [Q_h^{1,2} f(x)]^2\,dx. \end{aligned}$$

If $h \to 0^+$ then

$$\int (Q_h^{1,2} f)^2 \to \int f^2$$

in L^2 norm (see Lemma 3.2). Consequently, to prove (31) it is sufficient to show that

$$\int E[Q_h(X_j,x)]^2\,dx \sim h^{-d} \int_{[0,1]^d} \sum_{\alpha_1\in\mathbb{Z}^d} \sum_{\alpha_2\in\mathbb{Z}^d} B(\alpha_2 - \alpha_1 \mid U_1) \times B(u - \alpha_1 - c_Y \mid V_2) B(u - \alpha_2 - c_Y \mid V_2)\,du. \tag{33}$$

Now

$$\int E[Q_h(X_j,x)]^2\,dx = \iint h^{-d}\sum_{\alpha_1\in\mathbb{Z}^d} B_2(u/h-\alpha_1)B_1(x/h-\alpha_1) \times h^{-d}\sum_{\alpha_2\in\mathbb{Z}^d} B_2(u/h-\alpha_2)B_1(x/h-\alpha_2)f(u)\,du\,dx.$$

By (14) and (15) we have

$$\begin{aligned}(34)\qquad \int B_1(x-\alpha_1)B_1(x-\alpha_2)\,dx &= \int B(x\mid V_1)B(x+\alpha_1-\alpha_2\mid V_1)\,dx\\ &= \int B(x\mid V_1)B(\alpha_2-\alpha_1-x\mid -V_1)\,dx\\ &= B(\alpha_2-\alpha_1\mid U_1).\end{aligned}$$

Thus

$$\int E[Q_h(X_j,x)]^2\,dx = h^{-d}\int\sum_{\alpha_1\in\mathbb{Z}^d}\sum_{\alpha_2\in\mathbb{Z}^d} B(\alpha_2-\alpha_1\mid U_1) \times B_2(u/h-\alpha_1)B_2(u/h-\alpha_2)f(u)\,du\,dx.$$

The function

$$g(u)=\sum_{\alpha_1\in\mathbb{Z}^d}\sum_{\alpha_2\in\mathbb{Z}^d} B(\alpha_2-\alpha_1\mid U_1)B_2(u-\alpha_1)B_2(u-\alpha_2)$$

is bounded and $\mathbb{Z}^d$-periodic. Lemma 4.2 gives (31). To prove (32) note that

$$\begin{aligned}EZ_{n,2,j}^2 &= \iint E\{[Q_h(X_j,x)-EQ_h(X_j,x)]^2\\ &\quad\times[Q_h(X_j,y)-EQ_h(X_j,y)]^2\}\,dx\,dy\\ &\le 4\iint E\{[Q_h(X_j,x)]^2[Q_h(X_j,y)]^2\}\,dx\,dy\\ &\quad+4\iint E\{[Q_h(X_j,x)]^2\}[EQ_h(X_j,y)]^2\,dx\,dy\\ &\quad+4\iint [EQ_h(X_j,x)]^2E\{[Q_h(X_j,y)]^2\}\,dx\,dy\\ &\quad+4\iint [EQ_h(X_j,x)]^2[EQ_h(X_j,y)]^2\,dx\,dy.\end{aligned}$$

From (33) we have the estimate of the second and third integrals. The last is constant as $h\to 0^+$. Thus it is sufficient to show that

$$\iint E\{[Q_h(X_j,x)]^2[Q_h(X_j,y)]^2\}\,dx\,dy \sim O(1/h^{2d}).$$

Analogously to the proof of (33),

$$\begin{aligned}&\iint E\{[Q_h(X_j,x)]^2[Q_h(X_j,y)]^2\}\,dx\,dy\\ &\quad= h^{-4d}\int\sum_{\alpha_i\in\mathbb{Z}^d,\,i=1,2,3,4}\int B_1(x/h-\alpha_1)B_1(x/h-\alpha_2)\,dx\\ &\qquad\times\int B_1(x/h-\alpha_3)B_1(x/h-\alpha_4)\,dx\prod_{j=1}^4 B_2(u/h-\alpha_j)f(u)\,du.\end{aligned}$$

By (34),

$$\iint E\{[Q_h(X_j,x)]^2[Q_h(X_j,y)]^2\}\,dx\,dy$$
$$= h^{-2d}\int \sum_{\alpha_i\in\mathbb{Z}^d,\, i=1,2,3,4} B(\alpha_2-\alpha_1 \mid U_1)$$
$$\times B(\alpha_4-\alpha_3 \mid U_1)\prod_{j=1}^{4} B_2(u/h-\alpha_j)f(u)\,du.$$

By Lemma 4.2 we get

$$\iint E\{[Q_h(X_j,x)]^2[Q_h(X_j,y)]^2\}\,dx\,dy \sim O(1/h^{2d}). \ \blacksquare$$

COROLLARY 4.2. *Under the assumptions of Theorem* 2.1,

$$\begin{aligned}\int\{Ef_n^2(x)-(Ef_n(x))^2\}\,dx &= E\int\{f_n(x)-Ef_n(x)\}^2\,dx\\ &= E\frac{1}{n^2}\sum_{j=1}^{n}[Q_h(X_j,x)-EQ_h(X_j,x)]^2\\ &= EI_{n,2},\\ \operatorname{Var} I_{n,2} &= O\{(n^3h^{2d})^{-1}\}.\end{aligned}$$

We call f a *Lipschitz function* if there is K such that

$$|f(x)-f(y)|\le K\|x-y\| \quad \text{for all } x,y\in\mathbb{R}^d.$$

LEMMA 4.4. *Let the density f be a bounded Lipschitz function. Then for all* $z_1,z_2\in[-r,r]^d$,

$$\lim_{h\to 0^+} h^d \sum_{\alpha\in\mathbb{Z}^d} f(h\alpha+hz_1)f(h\alpha+hz_2)=\int f^2(x)\,dx.$$

Proof. Since f is a bounded Lipschitz function, f and f^2 are Riemann integrable. On the other hand

$$\lim_{h\to 0^+} h^d\Big|\sum_{\alpha\in\mathbb{Z}^d}[f(h\alpha+hz_1)f(h\alpha+hz_2)-(f(h\alpha+hz_1))^2]\Big|$$
$$\le 2Krh\lim_{h\to 0^+} h^d\sum_{\alpha\in\mathbb{Z}^d} f(h\alpha+hz_1)\to 0.$$

This finishes the proof. ∎

Let

$$(35)\quad \begin{aligned}&H_n(X_1,X_2)\\ &\quad=\int[Q_h(X_1,u)-EQ_h(X_1,u)][Q_h(X_2,u)-EQ_h(X_2,u)]\,du\end{aligned}$$

and

$$G_n(X_1,X_2)=E\{H_n(X_0,X_1)H_n(X_0,X_2)\}.$$

LEMMA 4.5. *Under the assumptions of Theorem* 2.1,

$$EH_n^2(X_1,X_2)\sim h^{-d}\sum_{\delta_i\in\mathbb{Z}^d\, i=1,2,3} B(\delta_1\mid U_1)B(\delta_2\mid U_1) \times B(\delta_3\mid U_2)B(\delta_2-\delta_1+\delta_3\mid U_2)\int f^2(x)\,dx, \tag{36}$$

$$EH_n^4(X_1,X_2)\sim O(1/h^{3d}), \tag{37}$$

$$EG_n^2(X_1,X_2)\sim O(1/h^{d}). \tag{38}$$

Proof. Note that for $k\in\mathbb{N}$,

$$EH_n^k(X_1,X_2)=\underbrace{\int\ldots\int}_{k}\Big(E\Big\{\prod_{j=1}^{k}[Q_h(X_1,u^j)-EQ_h(X_1,u^j)]\Big\}\Big)^2\,du^1\ldots du^k.$$

Since (Lemma 3.2)

$$EQ_h(X_1,u^j)=Q_h^{1,2}f(u^j)\to f(u^j)$$

in L^p norm ($p\ge 2$) as $h\to 0^+$ it follows that to calculate the asymptotic estimate of $EH_n^k(X_1,X_2)$ it is sufficient to consider

$$\begin{aligned}
I &:= \underbrace{\int\ldots\int}_{k}\Big(E\Big\{\prod_{j=1}^{k}[Q_h(X_1,u^j)]\Big\}\Big)^2\,du^1\ldots du^k\\
&= h^{-2dk}\underbrace{\int\ldots\int}_{k}\sum_{\alpha_1\in\mathbb{Z}^d}\cdots\sum_{\alpha_k\in\mathbb{Z}^d}\int\prod_{j=1}^{k}B_2(w/h-\alpha_j)f(w)\,dw\\
&\quad\times\prod_{j=1}^{k}B_1(u^j/h-\alpha_j)\sum_{\beta_1\in\mathbb{Z}^d}\cdots\sum_{\beta_k\in\mathbb{Z}^d}\int\prod_{j=1}^{k}B_2(w/h-\beta_j)\\
&\quad\times f(w)\,dw\prod_{j=1}^{k}B_1(u^j/h-\beta_j)\,du^1\ldots du^k\\
&= h^{-dk}\sum_{\alpha_1\in\mathbb{Z}^d}\cdots\sum_{\alpha_k\in\mathbb{Z}^d}\sum_{\beta_1\in\mathbb{Z}^d}\cdots\sum_{\beta_k\in\mathbb{Z}^d}\int\prod_{j=1}^{k}B_2(w/h-\alpha_j)f(w)\,dw\\
&\quad\times\int\prod_{j=1}^{k}B_2(w/h-\beta_j)f(w)\,dw\prod_{j=1}^{k}\int B_1(u^j)B_1(u^j+\beta_j-\alpha_j)\,du^j.
\end{aligned}$$

Let $\delta_j=\alpha_j-\beta_j$ for $j=1,\ldots,k$. Then by (34),

$$I=(h^{2d}/h^{kd})\sum_{\delta_1\in\mathbb{Z}^d}\cdots\sum_{\delta_k\in\mathbb{Z}^d}\prod_{j=1}^{k}B(\delta_j\mid U_1)\sum_{\beta_1\in\mathbb{Z}^d}\cdots\sum_{\beta_k\in\mathbb{Z}^d}$$
$$\iint\prod_{j=1}^{k}B_2(w^1-\delta_j-\beta_j)\prod_{j=1}^{k}B_2(w^2-\beta_j)f(hw^1)f(hw^2)\,dw^1\,dw^2.$$

Let $w^1 - \delta_1 - \beta_1 = v^1$ and $w^2 - \beta_1 = v^2$. Then

$$I = (h^{2d}/h^{kd}) \sum_{\delta_1 \in \mathbb{Z}^d} \cdots \sum_{\delta_k \in \mathbb{Z}^d} \prod_{j=1}^{k} B(\delta_j \mid U_1) \sum_{\beta_1 \in \mathbb{Z}^d} \cdots \sum_{\beta_k \in \mathbb{Z}^d}$$
$$\iint B_2(v^1) B_2(v^2) \prod_{j=2}^{k} B_2(v^1 + \delta_1 - \delta_j + \beta_1 - \beta_j)$$
$$\times \prod_{j=2}^{k} B_2(v^2 + \beta_1 - \beta_j) f(h(v^1 + \delta_1 + \beta_1)) f(h(v^2 + \beta_1))\, dv^1\, dv^2.$$

Next let $\gamma_1 = \beta_1 - \beta_2$. This yields

$$I = (h^{2d}/h^{kd}) \sum_{\delta_1 \in \mathbb{Z}^d} \cdots \sum_{\delta_k \in \mathbb{Z}^d} \prod_{j=1}^{k} B(\delta_j \mid U_1) \sum_{\gamma_1 \in \mathbb{Z}^d} \sum_{\beta_2 \in \mathbb{Z}^d} \cdots \sum_{\beta_k \in \mathbb{Z}^d}$$
$$\iint B_2(v^1) B_2(v^2) B_2(v^1 + \gamma_1 + \delta_1 - \delta_2) B_2(v^2 + \gamma_1)$$
$$\times \prod_{j=3}^{k} B_2(v^1 + \delta_1 - \delta_j + \gamma_1 + \beta_2 - \beta_j) \prod_{j=3}^{k} B_2(v^2 + \gamma_1 + \beta_2 - \beta_j)$$
$$\times f(h(v^1 + \delta_1 + \gamma_1 + \beta_2)) f(h(v^2 + \gamma_1 + \beta_2))\, dv^1\, dv^2.$$

If $k = 2$ then

$$\iint \Big(E\Big\{ \prod_{j=1}^{2} [Q_h(X_1, u^j)] \Big\} \Big)^2 du^1\, du^2$$
$$= \sum_{\delta_1 \in \mathbb{Z}^d} \sum_{\delta_2 \in \mathbb{Z}^d} \prod_{j=1}^{2} B(\delta_j \mid U_1)$$
$$\times \sum_{\gamma_1 \in \mathbb{Z}^d} \iint B_2(v^1) B_2(v^2) B_2(v^1 + \gamma_1 + \delta_1 - \delta_2) B_2(v^2 + \gamma_1)$$
$$\times \sum_{\beta_2 \in \mathbb{Z}^d} f(h(v^1 + \delta_1 + \gamma_1 + \beta_2)) f(h(v^2 + \gamma_1 + \beta_2))\, dv^1\, dv^2.$$

By Lemma 4.4 (all parameters $v^1, v^2, \delta_1, \gamma_1$ are uniformly bounded),

$$h^d \sum_{\beta_2 \in \mathbb{Z}^d} f(h(v^1 + \delta_1 + \gamma_1 + \beta_2)) f(h(v^2 + \gamma_1 + \beta_2))\, dv^1\, dv^2 \to \int f^2(x)\, dx.$$

Replacing $\gamma_1 = \delta_3$ and applying (34) for the function B_2 we get (36).

To prove (37) for $k = 4$ we continue setting $\gamma_2 = \beta_2 - \beta_3$, $\gamma_3 = \beta_3 - \beta_4$. We get

$$
\underbrace{\int\dots\int}_{4}\Big(E\Big\{\prod_{j=1}^{4}[Q_h(X_1,u^j)]\Big\}\Big)^2\,du^1\dots du^4
$$

$$
\begin{aligned}
&= h^{-2d}\sum_{\delta_1\in\mathbb{Z}^d}\dots\sum_{\delta_4\in\mathbb{Z}^d}\prod_{j=1}^{4}B(\delta_4\mid U_1)\sum_{\gamma_1\in\mathbb{Z}^d}\sum_{\gamma_2\in\mathbb{Z}^d}\sum_{\gamma_3\in\mathbb{Z}^d}\iint B_2(v^1)\\
&\quad\times B_2(v^1+\gamma_1+\delta_1-\delta_2)B_2(v^1+\gamma_1+\gamma_2+\delta_1-\delta_3)\\
&\quad\times B_2(v^1+\gamma_1+\gamma_2+\gamma_3+\delta_1-\delta_4)B_2(v^2)B_2(v^2+\gamma_1)\\
&\quad\times B_2(v^2+\gamma_1+\gamma_2)B_2(v^2+\gamma_1+\gamma_2+\gamma_3)\\
&\quad\times\sum_{\beta_4\in\mathbb{Z}^d}f(h(v^1+\delta_1+\gamma_1+\gamma_2+\gamma_3+\beta_4))\\
&\quad\times f(h(v^2+\gamma_1+\gamma_2+\gamma_3+\beta_4))\,dv^1\,dv^2.
\end{aligned}
$$

By Lemma 4.4,

$$
h^d\sum_{\beta_4\in\mathbb{Z}^d}f(h(v^1+\delta_1+\gamma_1+\gamma_2+\gamma_3+\beta_4))f(h(v^2+\gamma_1+\gamma_2+\gamma_3+\beta_4))\,dv^1\,dv^2 \to \int f^2(x)\,dx.
$$

This finishes the proof of (37).

As in the proof of Lemma 3 of [14] (see (5.4)) we have

$$
\begin{aligned}
EG_n^2(X_1,X_2) = \iiiint &F_n(u^1,u^2)F_n(v^1,v^2)\\
&\times F_n(u^1,v^1)F_n(u^2,v^2)\,du^1\,du^2\,dv^1\,dv^2,
\end{aligned}
$$

where

$$
F_n(u^1,u^2)=E\{[Q_h(X_1,u^1)-EQ_h(X_1,u^1)][Q_h(X_1,u^2)-EQ_h(X_1,u^2)]\}.
$$

Since

$$
EQ_h(X_1,u^j)=Q_h^{1,2}f(u^j)\to f(u^j)
$$

in L^2 norm as $h\to 0^+$ (Lemma 3.2) it follows that to calculate the asymptotic estimate of $EG_n^2(X_1,X_2)$ it is sufficient to consider

$$
\begin{aligned}
J :=& \iiiint E\{Q_h(X_1,u^1)Q_h(X_1,u^2)\}\\
&\times E\{Q_h(X_1,v^1)Q_h(X_1,v^2)\}E\{Q_h(X_1,u^1)Q_h(X_1,v^1)\}\\
&\times E\{Q_h(X_1,u^2)Q_h(X_1,v^2)\}\,du^1\,du^2\,dv^1\,dv^2\\
=& \iiiint (h^{-2d})^4\sum_{\alpha_1\in\mathbb{Z}^d}\sum_{\beta_1\in\mathbb{Z}^d}\int B_2(w/h-\alpha_1)B_2(w/h-\beta_1)f(w)dw\\
&\times B_1(u^1/h-\alpha_1)B_1(u^2/h-\beta_1)
\end{aligned}
$$

$$\times \sum_{\alpha_2\in\mathbb{Z}^d}\sum_{\beta_2\in\mathbb{Z}^d}\int B_2(w/h-\alpha_2)B_2(w/h-\beta_2)f(w)dw$$
$$\times B_1(v^1/h-\alpha_2)B_1(v^2/h-\beta_2)\sum_{\alpha_3\in\mathbb{Z}^d}\sum_{\beta_3\in\mathbb{Z}^d}\int B_2(w/h-\alpha_3)$$
$$\times B_2(w/h-\beta_3)f(w)\,dwB_1(u^1/h-\alpha_3)B_1(v^1/h-\beta_3)$$
$$\times \sum_{\alpha_4\in\mathbb{Z}^d}\sum_{\beta_4\in\mathbb{Z}^d}\int B_2(w/h-\alpha_4)B_2(w/h-\beta_4)f(w)\,dw$$
$$\times B_1(u^2/h-\alpha_4)B_1(v^2/h-\beta_4)\,du^1\,du^2\,dv^1\,dv^2.$$

By (34),

$$\begin{aligned} J = &\sum_{\alpha_i,\beta_i\in\mathbb{Z}^d, i=1,\ldots,4} B(\alpha_1-\alpha_3\mid U_1)B(\alpha_4-\beta_1\mid U_1)\\ &\times B(\beta_3-\alpha_2\mid U_1)B(\beta_2-\beta_4\mid U_1)\\ &\times \int B_2(z)B_2(z+\alpha_1-\beta_1)f(hz+h\alpha_1)\,dz\\ &\times \int B_2(z)B_2(z+\alpha_2-\beta_2)f(hz+h\alpha_2)\,dz\\ &\times \int B_2(z)B_2(z+\alpha_3-\beta_3)f(hz+h\alpha_3)\,dz\\ &\times \int B_2(z)B_2(z+\alpha_4-\beta_4)f(hz+h\alpha_4)\,dz. \end{aligned}$$

Let

$$\alpha_1=\delta_1+\alpha_3, \qquad \alpha_4=\delta_2+\beta_1,$$
$$\beta_3=\delta_3+\alpha_2, \qquad \beta_2=\delta_4+\beta_4.$$

Then

$$\begin{aligned} J = &\sum_{\delta_i\in\mathbb{Z}^d,\, i=1,2,3,4} B(\delta_1\mid U_1)B(\delta_2\mid U_1)B(\delta_3\mid U_1)B(\delta_4\mid U_1)\\ &\times \sum_{\beta_1\in\mathbb{Z}^d}\sum_{\alpha_2\in\mathbb{Z}^d}\sum_{\alpha_3\in\mathbb{Z}^d}\sum_{\beta_4\in\mathbb{Z}^d}\iiiint\\ &\quad B_2(z^1)B_2(z^1+\delta_1+\alpha_3-\beta_1)B_2(z^2)B_2(z^2-\delta_4+\alpha_2-\beta_4)\\ &\times B_2(z^3)B_2(z^3-\delta_3+\alpha_3-\alpha_2)B_2(z^4)B_2(z^4+\delta_2+\beta_1-\beta_4)\\ &\times f(hz^1+h\delta_1+h\alpha_3)f(hz^2+h\alpha_2)\\ &\times f(hz^3+h\alpha_3)f(hz^4+h\delta_2+h\beta_1)\,dz^1\,dz^2\,dz^3\,dz^4. \end{aligned}$$

Now we set $\beta_1=\alpha_3-\gamma_1$, next $\alpha_2=\alpha_3-\gamma_2$, and finally $\beta_4=\alpha_3-\gamma_3$ to obtain

$$J = \sum_{\delta_i \in \mathbb{Z}^d,\, i=1,2,3,4} B(\delta_1 \mid U_1)B(\delta_2 \mid U_1)B(\delta_3 \mid U_1)B(\delta_4 \mid U_1)$$
$$\times \sum_{\gamma_1 \in \mathbb{Z}^d} \sum_{\gamma_2 \in \mathbb{Z}^d} \sum_{\gamma_3 \in \mathbb{Z}^d} \iiiint B_2(z^1)B_2(z^1+\delta_1+\gamma_1)$$
$$\times B_2(z^2)B_2(z^2-\delta_4-\gamma_2+\gamma_3)B_2(z^3)$$
$$\times B_2(z^3-\delta_3+\gamma_2)B_2(z^4)B_2(z^4+\delta_2-\gamma_1+\gamma_3)$$
$$\times \sum_{\alpha_3 \in \mathbb{Z}^d} f(hz^1+h\delta_1+h\alpha_3)f(hz^2+h\alpha_3-h\gamma_2)f(hz^3+h\alpha_3)$$
$$\times f(hz^4+h\delta_2+h\alpha_3-h\gamma_1)\,dz^1\,dz^2\,dz^3\,dz^4.$$

We have z^j and δ_j, γ_j uniformly bounded. By an analogous argument to Lemma 4.4,

$$h^d \sum_{\alpha_3 \in \mathbb{Z}^d} f(hz^1+h\delta_1+h\alpha_3)f(hz^2+h\alpha_3-h\gamma_2)f(hz^3+h\alpha_3)$$
$$\times f(hz^4+h\delta_2+h\alpha_3-h\gamma_1) \to \int f^4(x)\,dx.$$

This finishes the proof of (38). ∎

Proof of Theorem 2.1. Set

$$I_{h,n} = \int \{f_{h,n}(x)-f(x)\}^2\,dx \tag{39}$$
$$= \int \{f_{h,n}(x)-Ef_{h,n}(x)\}^2\,dx + \int \{Ef_{h,n}(x)-f(x)\}^2\,dx$$
$$+ 2\int \{f_{h,n}(x)-Ef_{h,n}(x)\}\{Ef_{h,n}(x)-f(x)\}\,dx.$$

We have

$$EI_{h,n} = E\int (f_{h,n}(x)-Ef_{h,n}(x))^2\,dx + \int (Ef_{h,n}(x)-f(x))^2\,dx. \tag{40}$$

Thus (39) and (40) give

$$I_{h,n}-EI_{h,n} \tag{41}$$
$$= \int \{f_{h,n}(x)-Ef_{h,n}(x)\}^2\,dx - E\int \{f_{h,n}(x)-Ef_{h,n}(x)\}^2\,dx$$
$$+ 2\int \{f_{h,n}(x)-Ef_{h,n}(x)\}\{Ef_{h,n}(x)-f(x)\}\,dx.$$

To prove the asymptotic error of $I_{h,n}-EI_{h,n}$ it is enough to consider the asymptotic error of $I_{n,1}$ (recall (25)) and $I_{n,2}$ (recall (30)) and

$$I_{n,3} = 2(1/n)^2 \sum\sum_{1\le i<j\le n} H_n(X_i,X_j) \tag{42}$$

where $H_n(X_1,X_2)$ is defined in (35).

Note that

$$\int \{f_{h,n}(x) - Ef_{h,n}(x)\}^2\,dx - E\int \{f_{h,n}(x) - Ef_{h,n}(x)\}^2\,dx = I_{n,2} + I_{n,3}.$$

Thus

$$I_{h,n} - EI_{h,n} = 2I_{n,1} + I_{n,2} + I_{n,3}.$$

The asymptotic errors for $I_{n,1}$ and $I_{n,2}$ are given in Corollaries 4.1 and 4.2. It follows from Theorem 1 of [14] (central limit theorem for degenerate U-statistics) and Lemma 4.5 that $I_{n,3}$ is asymptotically normal with zero mean and variance equal to

$$\sum_{\delta_i\in\mathbb{Z}^d,\, i=1,2,3} B(\delta_1 \mid U_1)B(\delta_2 \mid U_1)B(\delta_3 \mid U_2)B(\delta_2 - \delta_1 + \delta_3 \mid U_2)\int f^2(x)\,dx.$$

Now the same considerations as in [14] give the assertion. ∎

Theorem 2.2 is just a consequence of Corollary 4.2, Theorems 3.1 and 3.2 and Lemma 4.2. Moreover

$$\int\limits_{[0,1]^d} \sum_{\alpha_1\in\mathbb{Z}^d} \sum_{\alpha_2\in\mathbb{Z}^d} B(\alpha_2 - \alpha_1 \mid U_1)B(u - \alpha_1 - c_Y \mid V_2)B(u - \alpha_2 - c_Y \mid V_2)\,du$$
$$= \sum_{\alpha\in\mathbb{Z}^d} B(\alpha \mid U_1)B(\alpha \mid U_2).$$

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