ANNALES
POLONICI MATHEMATICI
113.1 (2015)

Stein open subsets with analytic complements in compact complex spaces

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Abstract. Let Y be an open subset of a reduced compact complex space X such that $X - Y$ is the support of an effective divisor D. If X is a surface and D is an effective Weil divisor, we give sufficient conditions so that Y is Stein. If X is of pure dimension $d \geq 1$ and $X - Y$ is the support of an effective Cartier divisor D, we show that Y is Stein if Y contains no compact curves, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and for every point $x_0 \in X - Y$ there is an $n \in \mathbb{N}$ such that $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ is empty or has dimension 0, where $\Phi_{|nD|}$ is the map from X to the projective space defined by a basis of $H^0(X, \mathcal{O}_X(nD))$.

1. Introduction. In three very interesting papers [O1, O2, O3], Ohsawa proved the following theorems among other results:

(1) Let M be a compact Kähler manifold and let U be an open subset in M. Suppose that $B = M - U$ is a complex analytic subset of pure codimension 1 such that there exists an effective divisor A with support B for which the line bundle $[A]|_B$ is topologically trivial. Then U admits no C^∞ plurisubharmonic exhaustion function whose Levi form has at least three positive eigenvalues everywhere outside a compact subset of U. In particular, U is not Stein.

(2) Let M be a connected compact complex manifold of dimension 2, and let D be an effective divisor on M such that $[D]$ has a fiber metric whose curvature form is semipositive on $|D|$ and positive at some point of $|D|$. Then $M - |D|$ is holomorphically convex and properly bimeromorphic to a Stein space.

(3) Let M be a connected compact complex manifold of dimension 2 and let D be an effective divisor on M. If $D^2 > 0$, then there exists a connected component D^* such that $M - D^*$ is 1-convex.

A Stein manifold Y may have no nonconstant regular functions and the compact complex manifold X containing Y may have no nonconstant mero-

2010 *Mathematics Subject Classification*: Primary 32E10; Secondary 14E05.
Key words and phrases: bimeromorphic map, divisor, Stein space.

DOI: 10.4064/ap113-1-2

morphic functions with poles in $X-Y$. Serre constructed a smooth algebraic surface Y which is a Zariski open subset of a smooth projective surface X with Kodaira dimension $-\infty$ such that Y is biholomorphic to $\mathbb{C}^*\times\mathbb{C}^*$, but X has no nonconstant meromorphic functions with poles in $X-Y$ [H2, p. 232]. Based on Serre's surface, Neeman [N] constructed a Zariski open subset of an affine algebraic variety which is Stein but not an affine algebraic variety. Hwang and Varolin [HV] constructed a compact Kähler manifold X of dimension 4 and a smooth divisor D on X such that $X-D$ is biholomorphic to $(\mathbb{C}^*)^4$ and X has no nonconstant meromorphic functions. Recently, Forstnerič [Fo] investigated a long standing problem: If a complex manifold admits nonexhausting strongly plurisubharmonic function, does it admit a nonconstant holomorphic function? Is it Stein? He discovered a class of counterexamples and showed that there is a complex surface, a connected open set in $\mathbb{CP}^2$, that admits a strongly plurisubharmonic function but no nonconstant holomorphic functions. His examples show that it is worthwhile to give sufficient conditions for open subsets in a compact complex surface to be Stein.

In [Z1, Z2, Z3], we show that if Y is an irreducible algebraic Stein variety with dimension $d\geq 1$, then:

(1) $\kappa(D,X)\neq d-1$.
(2) If $d=2k$, then $\kappa(D,X)$ can be any even number $0,2,\dots,2k$.
(3) If $d=2k+1$, then $\kappa(D,X)$ can be any odd number $1,3,\dots,2k+1$.

This shows that a general algebraic Stein manifold is very complicated, by the classification of algebraic varieties. Here Y is an open subset of X, $X-Y$ is the support of the Cartier divisor D, and $\kappa(D,X)$ is the number of algebraically independent nonconstant meromorphic functions with poles in $X-Y$.

We are interested in the following question for complex spaces with singularities: Find sufficient conditions on X and Y such that Y is a Stein space, where Y is an open subset of a compact complex space X and $X-Y$ is the support of a Cartier divisor D.

A complex space $(X,\mathcal{O}_X)$ is *Stein* if X is *holomorphically separable* (i.e., for any two distinct points $x,y\in X$, there is a holomorphic function f in $H^0(X,\mathcal{O}_X)$ such that $f(x)\neq f(y)$) and *holomorphically convex* (i.e., for any discrete sequence $S=\{P_1,P_2,\dots,\}$ in X, there is a holomorphic function $f\in H^0(X,\mathcal{O}_X)$ such that f is not bounded on S) ([KaK, pp. 223, 228], [GR1, pp. 109–116]).

In this paper, we will focus on *Moishezon spaces*, the complex spaces with maximum algebraic dimensions [U, Chapter 1, Section 3]. We first consider *surfaces*, the complex spaces with dimension 2. A *Weil divisor* D on X is a finite sum $\sum_{i=1}^{N} n_iD_i$, where $n_i\in\mathbb{Z}$, D_i is an irreducible subvariety of codimension 1 in X, and D_i is not contained in the singular locus of X. The divisor D is *effective* if $n_i \geq 0$ for all i and $D \neq 0$, and *big* on X if X has $d = \dim X$ algebraically independent nonconstant meromorphic functions with poles in $X - Y$. Corresponding to Ohsawa's theorem on nonsingular surfaces, with a different approach, we have the following theorem for normal surfaces.

THEOREM 1.1. *Let Y be an open subset of an irreducible normal reduced compact complex surface X such that $X - Y$ is the support of a connected effective Weil divisor D. If D is a big divisor and Y contains no compact curves, then there is a Weil divisor P_1 with support $X - Y$ such that for every irreducible curve C in X, $P_1 \cdot C > 0$. If X is nonsingular, then P_1 is an ample divisor with support $X - Y$ and Y is a Stein surface.*

The proof of Theorem 1.1 is by using Zariski decomposition and the Nakai–Moishezon criterion of ampleness. Zariski decomposition does not hold in higher dimensions so the proof cannot be generalized to higher-dimensional complex spaces. In Section 2, we will give an example to show that Theorem 1.1 is not true for higher dimensional complex spaces.

A *Cartier divisor* D of a complex space X is a global section of the sheaf

$$\mathcal{M}_X^*/\mathcal{O}_X^*,$$

where $\mathcal{M}_X^*$ is the sheaf of germs of meromorphic functions which are not identically zero on X, and $\mathcal{O}_X^*$ is the sheaf of germs of nowhere vanishing holomorphic functions on X. A Cartier divisor D is *effective* if all local equations of D are holomorphic functions and at least one local equation has zeros. The *support* of a Cartier divisor D is a closed complex subspace of pure codimension 1 that is locally defined as the zeros of a single holomorphic function.

If D is a Cartier divisor on a compact complex space X, then $\mathcal{O}_X(D)$ is a coherent sheaf defined by

$$\mathcal{O}_X(D)_x = f_i^{-1}\mathcal{O}_{x,X} \subset \mathcal{M}_X, \quad \forall x \in X,$$

where f_i is the local equation of D on some $U_i \ni x$, $i \in I$ and $\{U_j\}_{j\in I}$ is an open cover of X. Letting $\mathbb{C}(X)$ be the meromorphic function field of X, we define the vector space [U, p. 39]

$$L(D) = \{f \in \mathbb{C}(X) : f = 0 \text{ or } (f) + D \geq 0\}.$$

Here (f) is the principal divisor defined by f. Then $L(D)$ is isomorphic to $H^0(X, \mathcal{O}_X(D))$ as a vector space over $\mathbb{C}$ [U, Lemma 4.14]. We do not distinguish between these two spaces and consider any element in $H^0(X, \mathcal{O}_X(D))$ as a meromorphic function on X which is holomorphic on Y.

We will construct a bimeromorphic map from X to a projective variety. If $h^0(X, \mathcal{O}_X(mD)) = 0$ for all $m \geq 0$, we define the *D-dimension* $\kappa(D, X)$

to be $-\infty$. If $h^0(X, \mathcal{O}_X(mD)) > 0$ for some $m \in \mathbb{Z}$ and X is a normal reduced compact complex space, choose a basis $\{f_0, f_1, \ldots, f_n\}$ of the vector space $H^0(X, \mathcal{O}_X(mD))$; it defines a meromorphic map $\Phi_{|mD|}$ from X to the projective space $\mathbb{P}^n$ sending a point x in X to $(f_0(x), f_1(x), \ldots, f_n(x))$ in $\mathbb{P}^n$. We define the *D-dimension* [U, Definition 5.1] by

$$\kappa(D, X) = \max_m \dim(\Phi_{|mD|}(X)).$$

If X is not normal, let $\pi : X' \to X$ be the normalization of X; then define $\kappa(D, X) = \kappa(\pi^* D, X')$, where $\pi^* D$ on X' is the pullback divisor of D [U, pp. 51, 53]

If D is an effective divisor, then $0 \le \kappa(D, X) \le d$, where d is the dimension of X; and D is a big divisor if $\kappa(D, X) = d$. Let K_X be the canonical divisor of X. The *Kodaira dimension* of X is $\kappa(X) = \kappa(K_X, X)$.

In Theorems 1.2 and 1.4, $\mathcal{O}_Y$ is the sheaf of holomorphic functions.

THEOREM 1.2. *Let Y be an open subset of a normal reduced compact complex space X of pure dimension $d \ge 1$ such that $X - Y$ is the support of an effective Cartier divisor D. If D is a big divisor on every irreducible component of X, if Y contains no compact curves, and if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then Y is holomorphically separable.*

THEOREM 1.3. *Let Y be a Zariski open subset of a normal projective variety X of pure dimension $d \ge 1$ defined over $\mathbb{C}$ such that $X - Y$ is the support of an effective Cartier divisor D. If D is a big divisor on every irreducible component of X, if Y contains no compact curves, and if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then Y is an algebraic affine variety, so it is Stein. Here $\mathcal{O}_Y$ is the sheaf of regular functions on Y.*

It is well-known that a domain Ω in $\mathbb{C}^n$ is Stein if and only if $H^i(\Omega, \mathcal{O}_\Omega) = 0$ for all $0 < i < n$ ([GR1, p. 159], [Gu, p. 156]). We have $H^n(\Omega, \mathcal{O}_\Omega) = 0$ for every domain Ω in $\mathbb{C}^n$ by a theorem of Siu [Siu2]. Laufer generalized it to Stein manifolds and proved that an open subset U of a Stein manifold M is Stein if and only if $H^i(U, \mathcal{O}_U) = 0$ for all $0 < i < n$ [GR1, p. 160]. This is no longer true for an open subset in a compact complex manifold (see Example 3.10). Naturally we would ask what conditions we should add so that an open subset in a complex space is Stein. It seems that many results and methods used to understand the Steinness of complex manifolds cannot be applied to open subsets in complex spaces with singularities. One reason is that the traditional approach heavily relies on the smoothness of the spaces: when the space is singular, there is no known method to construct strictly plurisubharmonic exhaustion functions from distance functions [AN, Siu3] unless the singularity is isolated [AN]. Fornæss and Narasimhan [FN] first considered the Levi problem for complex spaces with singularities: What open subsets in a Stein space are Stein? They gave several sufficient conditions with analytic methods. By using methods of birational geometry, we obtain the following theorem for normal spaces.

THEOREM 1.4. *Let Y be an open subset of a normal reduced compact complex space X of pure dimension $d \geq 1$ such that $X - Y$ is the support of an effective Cartier divisor D. Then Y is Stein if Y contains no compact curves, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and for every $x_0 \in X - Y$ there is an $n \in \mathbb{N}$ such that $\Phi_{|nD|}^{-1}(\Phi_{|nD|}(x_0)) \cap Y$ is empty or has dimension 0.*

We will consider surfaces in Section 2 and prove Theorems 1.2–1.4 in Section 3. The methods in Sections 2 and 3 are completely different. At the end of Section 3, we will give three examples to show that if one condition in Theorem 1.4 is not satisfied, then Y is not Stein in general.

2. Surfaces. Let $\mathrm{Div}(X)$ be the group of Weil divisors on an irreducible normal reduced compact surface X. Let $\mathrm{Div}(X, \mathbb{Q}) = \mathrm{Div}(X) \otimes \mathbb{Q}$ be the group of $\mathbb{Q}$-divisors. The intersection pairing

$$\mathrm{Div}(X, \mathbb{Q}) \times \mathrm{Div}(X, \mathbb{Q}) \to \mathbb{Q}$$

is defined in the following way. Let $\pi : X' \to X$ be a resolution of singularities and let $A = \bigcup E_i$ denote the set of exceptional divisors of π, i.e., $\pi(E_i)$ is a point of X for all i. For a $\mathbb{Q}$-divisor D on X we define the inverse image $\pi^* D$ by

$$\pi^* D = \bar{D} + \sum a_i E_i$$

where $\bar{D}$ is the strict transform of D by π and the rational numbers a_i are uniquely determined by the equations $\bar{D} \cdot E_j + \sum a_i E_i \cdot E_j = 0$ for all j. For two divisors D and D' on X, their *intersection number* is defined to be

$$D \cdot D' = \pi^* D \cdot \pi^* D'.$$

A $\mathbb{Q}$-divisor $D = \sum_i r_i D_i$ on X is *effective* if $r_i \geq 0$ for all i; *nef* if $D \cdot C \geq 0$ for all irreducible curves C on X; and *pseudoeffective* if $D \cdot P \geq 0$ for all nef divisors P on X. The Zariski decomposition is due to Sakai [Sa2].

LEMMA 2.1 (Zariski Decomposition). *Let D be a pseudoeffective $\mathbb{Q}$-divisor on an irreducible normal reduced compact surface X. Then there exists a unique decomposition*

$$D = P + N$$

satisfying the following conditions:

(1) *N is an effective $\mathbb{Q}$-divisor and either $N = 0$ or the intersection matrix of the irreducible components of N is negative definite;*
(2) *P is a nef $\mathbb{Q}$-divisor and the intersection of P with each irreducible component of N is zero.*

Lemma 2.2 was proved by Sakai [Sa1] for nonsingular projective surfaces. The idea of the proof works for normal compact complex surfaces since the Riemann–Roch formula [Fu] and Zariski decomposition hold for normal compact surfaces which are not projective.

Let $S(X)$ be the set of singular points of X. Then $X' = X - S(X)$ is a complex manifold, $D' = D|_{X'}$ is a Cartier divisor on X', and $\mathcal{O}_{X'}(D')$ is an invertible sheaf on X'. Let $e : X' \hookrightarrow X$ be the inclusion map. If D is a Weil divisor, then $\mathcal{O}_X(D)$ is defined to be $e_*\mathcal{O}_{X'}(D')$ [Sa2].

Lemma 2.2. *Let D be an effective Weil divisor on an irreducible normal reduced compact surface X, and $D = P + N$ be the Zariski decomposition. Then D is a big divisor if and only if $P^2 > 0$.*

Proof. Since $h^0(X, \mathcal{O}_X(nD)) = h^0(X, \mathcal{O}_X(nP))$, D is a big divisor if and only if P is a big divisor [Sa1]. We will show that if $P^2 > 0$, then P is a big divisor. Let $\pi : X' \to X$ be the resolution of singularity. Note that P is big if and only if $P' = \pi^*P$ is big [U, Lemma 5.3].

By Serre duality for connected compact complex manifolds,

$$H^2(X', \mathcal{O}_{X'}(nP')) = H^0(X', \mathcal{O}_{X'}(K_{X'} - nP')),$$

where $K_{X'}$ is the canonical divisor of X'. Hence the dimensions of these vector spaces are equal:

$$h^2(X', \mathcal{O}_{X'}(nP')) = h^0(X', \mathcal{O}_{X'}(K_{X'} - nP')).$$

As P' is an effective divisor, it is pseudoeffective: for every nef divisor N on X', $N \cdot P' \geq 0$. If there is a sequence $n_k \to \infty$ such that

$$h^0(X', \mathcal{O}_{X'}(K_{X'} - n_kP')) > 0,$$

then $(1/n_k)K_{X'} - P'$ is linearly equivalent to a pseudoeffective divisor and the limit is $-P'$. This is a contradiction. So for sufficiently large n, we have

$$h^2(X', \mathcal{O}_{X'}(nP')) = h^0(X', \mathcal{O}_{X'}(K_{X'} - nP')) = 0.$$

By Riemann–Roch, for sufficiently large n,

$$h^0(X', \mathcal{O}_{X'}(nP')) = h^1(X', \mathcal{O}_{X'}(nP')) + \tfrac{1}{2}n^2P'^2 - \tfrac{1}{2}nP' \cdot K_{X'} + \chi(X', \mathcal{O}_{X'}).$$

Now $P'^2 = P^2 > 0$ [H1, p. 387] implies that $h^0(X', \mathcal{O}_{X'}(nP')) \geq cn^2$ for some constant $c > 0$. So P' is a big divisor on X' [I, Theorem 10.2] and hence D is big.

Conversely, if $\kappa(D, X) = 2$, then $\kappa(P, X) = 2$. If $P^2 = 0$, let mP be a Weil divisor and $|mP| = |M| + F$, where M is the movable part and F the fixed part. Then

$$0 = (mP)^2 = mP \cdot (M + F) = (M + F) \cdot M + mP \cdot F = M^2 + M \cdot F + mP \cdot F.$$

Since $M^2 \geq 0$ and $M \cdot F \geq 0$ and $P \cdot F \geq 0$, we have $M^2 = 0$. Then the image $\Phi_{|mP|}(X) = \Phi_{|M|}(X)$ is not a surface but a curve. This is not possible. So $P^2 > 0$. ∎

THEOREM 2.3. *Let Y be an open subset of an irreducible normal reduced compact complex surface X such that $X - Y$ is the support of a connected effective Weil divisor D. If D is a big divisor and Y contains no compact curves, then $X - Y$ is the support of a divisor P_1 such that for all irreducible curves C in X, $P_1 \cdot C > 0$.*

Proof. Let $D = P + N$ be the Zariski decomposition, where N is effective and negative definite, P is effective and nef, and no prime component of N intersects P. Since both P and N are $\mathbb{Q}$-divisors (D is a Weil divisor but P and N may have rational coefficients), there is an integer $l > 0$ such that $lD = lP + lN$ and lP and lN are effective Weil divisors. We may assume that both P and N are integral. Let $\operatorname{Supp} D = \{D_1, \ldots, D_n\} = X - Y$. Since $\kappa(D, X) = 2$, we have $P^2 > 0$. First we claim that $\operatorname{Supp} P = \operatorname{Supp} D = X - Y$. If $\operatorname{Supp} P \neq X - Y$, then there is a prime component, say D_1, in $X - Y$ such that $P \cdot D_1 > 0$ and D_1 is not a component of P since $X - Y$ is connected. Let

$$Q = mP + D_1,$$

where m is a large positive integer. Then Q is an effective divisor and $\operatorname{Supp} Q = \operatorname{Supp} P \cup D_1$. Since $P^2 > 0$, we may choose m such that

$$Q^2 = m^2 P^2 + 2mP \cdot D_1 + D_1^2 > 0.$$

For every prime component E in P, since P is nef and D_1 is not contained in $\operatorname{Supp} P$, for sufficiently large m we have

$$Q \cdot E = mP \cdot E + D_1 \cdot E \geq 0, \qquad D_1 \cdot Q = mD_1 \cdot P + D_1^2 > 0.$$

Since Y contains no compact curves, any irreducible compact curve outside $X - Y$ intersects $X - Y$. Thus we get a new effective divisor Q such that Q is nef and $Q^2 > 0$. We may replace P by Q and still call it P. By finitely many such replacements, we can find an effective nef divisor P such that $P^2 > 0$ and $\operatorname{Supp} P = \operatorname{Supp} D = X - Y$.

Next we will show that $X - Y$ is the support of a divisor P_1 such that for every irreducible curve C in X, $P_1 \cdot C > 0$. This is obvious if C is not a component of $X - Y$ since Y has no compact curves. We only need to show that for every irreducible component C of D, $C \cdot P_1 > 0$.

If the above nef divisor P does not satisfy the positive intersection condition, then there is an irreducible compact curve C in X such that $P.C = 0$. Since Y has no compact curves, C must be one of the D_i's. We may change the order and assume $D_i.P = 0$ for $i = 1, \ldots, r$ and $D_j.P > 0$

for $j = r+1, \ldots, n$. Write

$$P = \sum_{i=1}^{r} a_i D_i + \sum_{j=r+1}^{n} b_j D_j = A + B,$$

where $A = \sum_{i=1}^{r} a_i D_i$, $B = \sum_{j=r+1}^{n} b_j D_j$. Then for $i = 1, \ldots, r$,

$$0 = P.D_i = A.D_i + B.D_i.$$

Since D_i is not a component of B, for all $i = 1, \ldots, r$ we have $B.D_i \geq 0$. So $A.D_i \leq 0$ for all $i = 1, \ldots, r$. If A is connected, this implies that the intersection matrix $[D_s.D_t]_{1\leq s,t\leq r}$ is negative semidefinite [Ar]. Since $A \cup B = X - Y$ is connected, there is at least one component D_{i_0} of A such that $D_{i_0}.B > 0$. Hence $D_{i_0}.A < 0$ and $A^2 < 0$. This implies that the intersection matrix $[D_s.D_t]_{1\leq s,t\leq r}$ is negative definite [Ar]. So there is an effective divisor $E = \sum_{i=1}^{r} \alpha_i D_i$ such that $E.D_i < 0$ for all $i = 1, \ldots, r$ [Ar]. If A is not connected, by induction, we may assume that $A = \sum_{i=1}^{r_1} a_i D_i + \sum_{i=r_1+1}^{r} a_i D_i = A_1 + A_2$, where $A_1 = \sum_{i=1}^{r_1} a_i D_i$ and $A_2 = \sum_{i=r_1+1}^{r} a_i D_i$ are disjoint and each of them is connected. Then there are $E_1 = \sum_{i=1}^{r_1} c_i D_i$ and $E_2 = \sum_{k=r_1+1}^{r} d_k D_k$ such that $D_i \cdot E_1 < 0$ and $D_k \cdot E_2 < 0$, for $i = 1, \ldots, r_1$ and $k = r_1+1, \ldots, r$ [Ar]. Let $E = E_1 + E_2$; then $E.D_i < 0$ for all $i = 1, \ldots, r$ since E_1 and E_2 are disjoint.

We have shown that there are positive numbers α_i, $i = 1, \ldots, r$, such that $E.D_i < 0$, $i = 1, \ldots, r$. Let $P_1 = mP - E$. If m is sufficiently large, then $P_1^2 > 0$, P_1 is nef and if $1 \leq i \leq r$,

$$P_1.D_i = -E.D_i > 0.$$

If $r + 1 \leq j \leq n$, then for sufficiently large m,

$$P_1.D_j = mP.D_j - E.D_j > 0. \ \blacksquare$$

D is an ample divisor if there is an $n \in \mathbb{N}$ such that nD is very ample, i.e., $\Phi_{|nD|}$ is a biholomorphic map.

Corollary 2.4. *In Theorem* 2.3, *if X is nonsingular, then P_1 is an ample divisor and Y is a Stein surface.*

Proof. In the proof of Theorem 2.3, by the Nakai–Moishezon criterion, P_1 is an ample divisor with support $X - Y$. So Y is Stein. $\blacksquare$

In general, Zariski decomposition does not hold for normal reduced compact spaces of dimension greater than 2. The proof for surfaces cannot be applied to higher dimensional spaces. In fact, Theorem 2.3 and Corollary 2.4 are not true if the dimension of X is greater than 2 even if X is a complex manifold.

Example 2.5. Let H be a hyperplane in $\mathbb{P}^3$ through a point O. Let L be a line through O and not contained in H. Blow up $\mathbb{P}^3$ along L and let

$\pi : X \to \mathbb{P}^3$ be the blowup. Define $D = \pi^* H + E$, where E is the exceptional divisor. Then $Y = X - D$ has no compact curves and D is a connected big divisor. But Y is not Stein since it is isomorphic to $\mathbb{C}^3 - L$ and L is of codimension 2 in $\mathbb{C}^3$ [GR1, p. 128] and D is not ample.

3. Higher-dimensional complex spaces. In this section, Y is a proper open subset of a reduced compact complex space X such that $X - Y$ is the support of an effective Cartier divisor D. Then D is a big divisor if and only if there are constants $b > a > 0$ such that [I, Theorem 10.2]

$$an^d \le h^0(X, \mathcal{O}_X(nD)) \le bn^d.$$

There is a one-to-one correspondence between the complete linear system $|D|$ and $H^0(X, \mathcal{O}_X(D))/\sim$, where $\phi_1, \phi_2 \in H^0(X, \mathcal{O}_X(D))$ and $\phi_1 \sim \phi_2$ if there is a constant c such that $\phi_1 = c\phi_2$ [U, Lemma 4.16].

A Cartier divisor on a normal space defines a Weil divisor [U, p. 36]. Let F be an effective Weil divisor on a normal complex space X. We say that F is a *fixed component* of a linear system L of effective Cartier (or Weil) divisors if $E > F$ (i.e., $E - F$ is an effective nonzero Weil divisor) for all $E \in L$. And F is the *fixed part* of a linear system if every irreducible component of F is a fixed component of the system and F is maximal with respect to the order $\ge$. If F is the fixed part of L, then every element E in the system can be written in the form $E = E' + F$. We say that E' is the *variable* (or *movable*) *part* of E. A point $x \in X$ is a *base point* of the linear system if x is contained in the supports of the variable parts of all divisors in the system. The set of all base points of L, called the *base locus* of L, is an analytic subset of X [U, p. 42]. Here the definition of base point is different from the definition in Hartshorne's book [H1, p. 158].

If D is an effective Cartier divisor on X, then the base locus of the complete linear system $|nD|$ for any positive integer n is contained in the boundary $X - Y$. Let $\{f_0, f_1, \dots, f_m\}$ be a basis of $H^0(X, \mathcal{O}_X(nD))$. It defines a meromorphic map $\Phi_{|nD|}$ from X to $\mathbb{P}^m$ for sufficiently large n. We will consider the algebraic variety $\Phi_{|nD|}(Y)$ in order to understand the complex space Y. If D is a big divisor on every irreducible component of X, we may choose n such that $\Phi_{|nD|}$ is bimeromorphic. By the correspondence between $|nD|$ and $H^0(X, \mathcal{O}_X(nD))$, for any $y \in Y$ there is at least one f_i so that $f_i(y) \ne 0$. So $\Phi_{|nD|}$ is a holomorphic map on Y. By Hironaka's elimination of indeterminacy, we may blow up the subsets on the boundary $X - Y$ finitely many times so that $\pi : X' \to X$ is the blowup and the new map from X' to $\mathbb{P}^m$ is holomorphic and its restriction to Y is equal to $\Phi_{|nD|}$. In the process, Y and the restriction map $\Phi_{|nD|}|_Y$ do not change. Since we are only interested in the open subset Y, without loss of generality we may assume that $\Phi_{|nD|}$ is holomorphic on X. Therefore for sufficiently large n, we may assume that $\Phi_{|nD|}$ is a proper bimeromorphic holomorphic map if D is a big divisor.

A holomorphic map $f : X_1 \to X_2$ of complex spaces is a *modification* if f is proper surjective and bimeromorphic.

LEMMA 3.1. *If Y is a complex space with $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then every hypersurface $Z = \{y \in Y : f(y) = 0,\ f \in H^0(Y, \mathcal{O}_Y)\}$ satisfies $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$, where f is the zero divisor of $\mathcal{O}_P$ for no $P \in Y$.*

Proof. We may assume that X is connected. We use the definition of complex subspace from the book of Grauert and Remmert [GR1, p. 16]. The underlying topological space Z is the zero set of the holomorphic function f and the structure sheaf $\mathcal{O}_Z$ is $\mathcal{O}_Y/(f)$, where (f) is the ideal sheaf generated by f. The global holomorphic function f on Y times an element in $\mathcal{O}_Y$ is still an element in $\mathcal{O}_Y$. Let $y \in Y$ and U an open neighborhood of y in Y. Let (U, g) be a germ at y and $fg = 0$ on U. Then $(U, \mathcal{O}_U)$ is an open subspace of X. Since f is not a zero divisor of $\mathcal{O}_{y,U}$ and $f \neq 0$ on U, $g = 0$ on U. This implies that f gives an injective map by multiplication from $\mathcal{O}_Y$ to itself. We have a short exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{f} \mathcal{O}_Y \xrightarrow{r} \mathcal{O}_Z \to 0,$$

where the first map is defined by f. The corresponding long exact sequence is

$$\cdots \to H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y) \to H^i(Z, \mathcal{O}_Z) \to H^{i+1}(Y, \mathcal{O}_Y) \to \cdots.$$

Since $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, we have $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. ∎

If Y is a reduced complex space, then the set $S(Y)$ of singular points of Y is a nowhere dense analytic subset of Y [GR2, p. 117]. Let f be a holomorphic function on Y which is not a constant and has zeros on every irreducible component of Y. In the above proof, if $fg = 0$ on the open subset U, then $fg = 0$ on the manifold $U - S(Y)$. This implies $g = 0$ and the short exact sequence still holds.

COROLLARY 3.2. *If Y is a reduced complex space with $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then every hypersurface $Z = \{y \in Y : f(y) = 0,\ f \in H^0(Y, \mathcal{O}_Y)\}$ satisfies $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$.*

LEMMA 3.3. *Let Y be an open subset of a normal reduced compact complex space X of pure dimension $d \geq 1$. Assume that $X - Y$ is the support of an effective Cartier divisor D. If D is a big divisor on every irreducible component of X, then D is a big divisor on every irreducible component $\bar{Z}_i$ of the hypersurface $\bar{Z}$ defined by $f \in H^0(X, \mathcal{O}_X(nD))$ for sufficiently large n, where f is not a constant and has zeros on every irreducible component of X, and $Z_i = \bar{Z}_i \cap Y$ is an open subset of $\bar{Z}_i$.*

Proof. We may assume that X is irreducible. The lemma is a direct consequence of the one-to-one correspondence between the hyperplane sections in the projective space and movable parts of effective divisors in $|nD|$ [U, p. 45]. Here we will give a different proof.

Since D is an effective big divisor on X, X is a Moishezon space and there is an irreducible smooth projective variety X' of dimension d and a modification $\pi : X' \to X$ such that every fiber of π is connected [U, p. 26]. Let $D^* = \pi^* D$. Then D^* is an effective big divisor on X' [U, p. 51]. Since X' is a projective variety and D^* is big, there are an effective ample divisor A and an effective divisor E on X' such that $n_0 D^*$ is linearly equivalent to $A + E$ for some $n_0 \in \mathbb{N}$ [KM, pp. 67–68]. Let $M + F \in |nn_0 D|$, where M is the movable part and F is the fixed component. Then the support of F is contained in $X - Y$ since nD is effective. Let M_i be an irreducible component of M, and M_i' be the strict transform of M_i. Then A is also ample on M_i' [H2, p. 23]. Any ample divisor is a big divisor so A is big on M_i'. For sufficiently large n, there is a positive constant $c > 0$ such that [I, Theorem 10.2]

$$h^0(M_i', \mathcal{O}_{M_i'}(nA)) \geq cn^{d-1}.$$

So

$$h^0(M_i', \mathcal{O}_{M_i'}(nn_0 D^*)) = h^0(M_i', \mathcal{O}_{M_i'}(nA + nE)) \geq h^0(M_i', \mathcal{O}_{M_i'}(nA)) \geq cn^{d-1},$$

which implies that D^* is a big divisor on M_i', i.e., there are $d-1$ algebraically independent nonconstant rational functions on M_i' which are regular on $M_i' \cap Y'$, where $Y' = \pi^{-1}(Y)$. Then on every irreducible component M_i, $D|_{M_i}$ is a big divisor:

$$\kappa(D|_{M_i}, M_i) = \kappa(D^*|_{M_i'}, M_i') = d - 1.$$

The lemma is proved. ∎

THEOREM 3.4. *Let Y be an open subset of a normal reduced compact complex space X of pure dimension $d \geq 1$ such that $X - Y$ is the support of an effective Cartier divisor D. If D is a big divisor on every irreducible component of X, if Y contains no compact curves, and if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then Y is holomorphically separable.*

Proof. If Y is a curve, then Y has no compact components. This implies that for every coherent analytic sheaf $\mathcal{F}$ on Y, $H^i(Y, \mathcal{F}) = 0$ [Siu2]. By Cartan's Theorem B, Y is Stein and hence holomorphically separable. Let $P_1 \neq P_2$ be two points on Y. Let $f \in H^0(X, \mathcal{O}_X(nD))$ be such that $P_1 \in Z = \{y \in Y : f(y) = 0\}$. Here f is not a constant and has zeros on every component of X. If $P_2 \notin Z$, then $f(P_2) \neq 0$. Assume $P_2 \in Z$. We may choose n sufficiently large such that $\Phi_{|nD|}$ is a proper holomorphic bimeromorphic map. By Corollary 3.2, $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. Moreover, Z contains no compact curves and D is a big divisor on every irreducible component of $\bar{Z}$, where $\bar{Z}$ the closure of Z in X and is the movable part of the effective divisor defined by f. By induction, we may assume that Z is holomorphically separable. There is a holomorphic function ϕ on Z such that $\phi(P_1) \neq \phi(P_2)$. By the short exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{f} \mathcal{O}_Y \xrightarrow{r} \mathcal{O}_Z \to 0,$$

and $H^1(Z, \mathcal{O}_Z) = 0$, we have

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y) \to H^0(Z, \mathcal{O}_Z) \to 0.$$

So ϕ can be lifted to a holomorphic function on Y, and thus Y is holomorphically separable. ∎

In Theorem 3.4, if X is a projective variety, then Y is a Stein variety.

Theorem 3.5. *Let Y be a Zariski open subset of a normal projective variety X of pure dimension $d \geq 1$ defined over $\mathbb{C}$ such that $X - Y$ is the support of an effective Cartier divisor D. If D is a big divisor on every irreducible component of X, if Y contains no compact curves, and if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then Y is an algebraic affine variety, so it is Stein. Here $\mathcal{O}_Y$ is the sheaf of regular functions on Y.*

Proof. If X is a projective curve, then Y is a proper Zariski open subset of X, and Y contains no compact curves. So Y is an affine curve [H2, p. 68].

Since D is an effective big divisor, we may assume that $\Phi_{|nD|}$ is a proper birational regular map. Let $\bar{Z}$ be the movable part of a hypersurface defined by a rational function $f \in H^0(X, \mathcal{O}_X(nD))$ for sufficiently large n, where f is not a constant and has zeros on every component of X. Let $Z = Y \cap \bar{Z}$. By Corollary 3.2 and Lemma 3.3, $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$ and $D|_{\bar{Z}}$ is a big divisor on every irreducible component of $\bar{Z}$. By induction, we may assume that Z is an affine variety. Similar to the proof of Theorem 3.4, by induction and the exact sequence of global regular functions, Y is regularly separable: for any two distinct points P_1, P_2 on Y, there is a regular function g on Y such that $g(P_1) \neq g(P_2)$. This implies that Y is a Zariski open subset of an affine variety (a quasi-affine variety) by Goodman and Hartshorne's theorem [GH]. A quasi-affine variety Y is an affine variety if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, by Neeman's theorem [N]. So Y is an affine variety. Every affine variety is Stein [H2, p. 232]. The theorem is proved. ∎

Let $S = \{P_1, P_2, \ldots\}$ be a sequence in Y without accumulation point in Y. In the following lemmas and theorems, we assume that the following three conditions hold:

(1) Y contains no compact curves.
(2) $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

(3) For every point $x_0 \in X - Y$, there is a positive integer n such that

$$\Phi_{|nD|}^{-1}(\Phi_{|nD|}(x_0)) \cap Y$$

is empty or has dimension 0.

Let $\bar{C}$ be an irreducible curve on X such that $x_0 \in \bar{C}$ and $C = Y \cap \bar{C}$ is a curve on Y. Then C is not compact and $\Phi_{|nD|}(\bar{C})$ is not a point. So D is a big divisor on every irreducible component of X. We may assume that $\Phi_{|nD|}$ is a proper holomorphic bimeromorphic map as discussed at the beginning of this section.

Let $f \in H^0(X, \mathcal{O}_X(nD))$ be nonconstant and have zeros on every irreducible component of X. Let $\bar{Z}$ be the movable part of the divisor defined by f, and $Z = Y \cap \bar{Z} = \{y \in Y : f(y) = 0\}$. By Lemma 3.1,

$$H^i(Z, \mathcal{O}_Z) = 0$$

for all $i > 0$ and Z and $\bar{Z}$ satisfy the above three conditions so we may assume that Z is Stein.

LEMMA 3.6. *If the image $\Phi_{|nD|}(S)$ is a finite set, then there is a global holomorphic function on Y that is not bounded on S.*

Proof. If $\Phi_{|nD|}(S)$ is a finite set, then there is a point $q \in \Phi_{|nD|}(S)$ such that $S' = \Phi_{|nD|}^{-1}(q) \cap S$ is not a finite subset of S, that is, $\Phi_{|nD|}$ maps infinitely many points in S to q.

Let H be a hyperplane section passing through q in $\mathbb{P}^m$. Pulling H back to Y by $\Phi_{|nD|}$, we obtain a hypersurface Z containing S' [U, Lemma 4.20.3]. By induction, we may assume that Z is Stein. Since S has no accumulation point on Y, its subset S' has no accumulation point on Z. Since Z is Stein, there is a holomorphic function ψ on Z that is not bounded on S'. Using the surjective map from $H^0(Y, \mathcal{O}_Y)$ to $H^0(Z, \mathcal{O}_Z)$ from the proof of Lemma 3.1, we can lift ψ to Y. In this way we obtain a global holomorphic function on Y that is not bounded on the subset S' of S. So it is unbounded on S. ■

LEMMA 3.7. *If $\Phi_{|nD|}(S)$ has an accumulation point w_0 in $W - \Phi_{|nD|}(Y)$, then there is a global holomorphic function on Y that is not bounded on S, where W is the affine variety containing $\Phi_{|nD|}(Y)$.*

Proof. $\Phi_{|nD|}(Y)$ is contained in an algebraic affine variety W in the projective space ([I, p. 302], [U, Lemma 4.20.3]). An algebraic affine variety is Stein [H2, p. 232]. We have

$$\Phi_{|nD|}(S) \subset \Phi_{|nD|}(Y) \subset W \subset \mathbb{P}^m,$$

where $m = h^0(X, \mathcal{O}_X(nD)) - 1$.

If $\Phi_{|nD|}(S)$ has an accumulation point w_0 in $W - \Phi_{|nD|}(Y)$, let H be a hyperplane in $\mathbb{P}^m$ passing through w_0 with $H \cap W \neq \emptyset$. Let h be a regular (hence holomorphic) function on W defining $H \cap W$. Since $H \cap W$ is a hypersurface of the Stein variety W, it is Stein. Pull H back to X; then the preimage of H in X is a hypersurface $\bar{Z}$. Let $Z = \bar{Z} \cap Y$. Then Z is defined by the holomorphic function determined by the pullback of h. By inductive assumption, Z is Stein.

Since both W and $H \cap W$ are Stein, there are holomorphic functions $h_1, \ldots, h_a$ on W such that $h_1, \ldots, h_a$ have no common zeros on $H \cap \Phi_{|nD|}(Y)$ and vanish at w_0 [Gu, p. 143, Theorem 5]. Pulling them back to Y by $\Phi_{|nD|}$, we have $a+1$ holomorphic functions $f = h \circ \Phi_{nD}, f_1 = h_1 \circ \Phi_{nD}, \ldots, f_a = h_a \circ \Phi_{nD}$ on Y that have no common zeros on Z and approach zero when $y \in Y$ approaches $x_0 \in X - Y$, where $\Phi_{|nD|}(x_0) = w_0$.

The hypersurface Z is defined by f. By their choice, $f, f_1, \ldots, f_a$ have no common zeros on Z. Let $\alpha_1 = f_1|_Z, \ldots, \alpha_a = f_a|_Z$ be the restrictions of holomorphic functions on Z. Since Z is Stein, there are holomorphic functions $\beta_1, \ldots, \beta_a$ in $H^0(Z, \mathcal{O}_Z)$ such that on Z we have [GR1, p. 161]

$$\alpha_1\beta_1 + \cdots + \alpha_a\beta_a = 1.$$

By the short exact sequence in the proof of Lemma 3.1, we have an exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \xrightarrow{f} H^0(Y, \mathcal{O}_Y) \xrightarrow{r} H^0(Z, \mathcal{O}_Z) \to 0.$$

Let g_i be the lifting of β_i on Y. Then $1 - (f_1g_1 + \cdots + f_ag_a)$ is holomorphic on Y and vanishes on Z, hence is contained in the kernel of the second map r. By the above exact sequence, $1 - (f_1g_1 + \cdots + f_ag_a)$ is in the image of the first map, which is defined by the holomorphic function f on Y by multiplication. Thus there is a $g \in H^0(Y, \mathcal{O}_Y)$ such that

$$1 - (f_1g_1 + \cdots + f_ag_a) = fg,$$

or on Y,

$$fg + (f_1g_1 + \cdots + f_ag_a) = 1.$$

Let $\{P_{i_j}\}$ be a subsequence in S such that $\Phi_{|nD|}(P_{i_j})$ approaches w_0. Then $f(P_{i_j}) = h \circ \Phi_{|nD|}(P_{i_j}), f_1(P_{i_j}) = h_1 \circ \Phi_{|nD|}(P_{i_j}), \ldots, f_a(P_{i_j}) = h_a \circ \Phi_{|nD|}(P_{i_j})$ approach zero since the $a+1$ functions $h, h_1, \ldots, h_a$ vanish at w_0. So there is at least one function among $g, g_1, \ldots, g_a$ that is not bounded on the subsequence $\{P_{i_j}\}$. Therefore it is not bounded on S. ∎

Theorem 3.8. *Let Y be an open subset of a normal reduced compact complex space X of pure dimension $d \geq 1$ such that $X - Y$ is the support of an effective Cartier divisor D. Then Y is holomorphically convex if Y contains no compact curves, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and for every $x_0 \in X - Y$ there is an $n \in \mathbb{N}$ such that $\Phi_{|nD|}^{-1}(\Phi_{|nD|}(x_0)) \cap Y$ is empty or has dimension 0.*

Proof. If $\Phi_{|nD|}(S)$ has no accumulation point in the affine variety W, then there is a regular function f on W such that f is not bounded on $\Phi_{|nD|}(S)$. Pulling f back to Y by $\Phi_{|nD|}$, we obtain a holomorphic function which is not bounded on S. Let $x_0 \in X - Y$ be an accumulation point of S in X. Then $\Phi_{|nD|}(x_0)$ is an accumulation point of $\Phi_{|nD|}(S)$ if $\Phi_{|nD|}(S)$ is not a finite set. If $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ is empty, then $\Phi_{|nD|}(x_0) \notin \Phi_{|nD|}(Y)$. By Lemmas 3.5 and 3.6, there is a holomorphic function which is not bounded on S.

Since $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0))$ is a compact subset of X, if $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ has dimension zero, then it is a finite set. Let $X \overset{\alpha}{\to} X' \overset{\beta}{\to} \Phi_{|nD|}(X)$ be the Stein factorization, where β is a finite surjective holomorphic map and α is a surjective holomorphic map such that every fiber of α is connected [U, p. 9]. Let $x_0' = \alpha(x_0)$. Then $\alpha^{-1}(x_0') \cap Y = \emptyset$. Otherwise, $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ has dimension at least 1 by the connectedness of the fibers of α. Since $\beta^{-1}(W)$ is a Stein space and $\alpha(x_0) \notin \alpha(Y) \subset W' = \beta^{-1}(W)$, similar to the proof Lemma 3.6, there is a holomorphic function f on W' whose pullback to X by α is not bounded on S. ■

THEOREM 3.9. *Let Y be an open subset of a normal reduced compact complex space X with pure dimension $d \geq 1$ such that $X - Y$ is the support of an effective Cartier divisor D. Then Y is Stein if Y contains no compact curves, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and for every $x_0 \in X - Y$ there is an $n \in \mathbb{N}$ such that $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ is empty or has dimension* 0.

Proof. By the assumption, D is a big divisor. By Theorems 3.4 and 3.8, Y is Stein. ■

EXAMPLE 3.10. In Theorem 3.9, the condition that Y contains no compact curves is necessary: There is a surface satisfying the other two conditions but not Stein.

Let $\pi : X \to \mathbb{P}^2$ be the blowup at a point $O \in \mathbb{P}^2$ and E the exceptional curve on X. Then E is isomorphic to $\mathbb{P}^1$. Let L be a line in $\mathbb{P}^2$ such that $O \notin L$. Let $D = \pi^*L$ and $Y = X - D$. Then D is an irreducible big divisor on X. For sufficiently large n, $\Phi_{|nD|}(Y) \cap \Phi_{|nD|}(X - Y) = \emptyset$. Since $\pi|_Y : Y \to \mathbb{P}^2 - L \cong \mathbb{C}^2$ is a proper holomorphic map, for every point $x \in \mathbb{C}^2$, the stalk of the ith direct image for $i = 1, 2$ vanishes [BaS, p. 93]:

$$(R^i\pi_*\mathcal{O}_Y)_x = H^i(\pi^{-1}(x), \mathcal{O}_Y) = 0.$$

So $R^i\pi_*\mathcal{O}_Y = 0$. Since $\pi_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{C}^2}$ [U, p. 11], we have [Gu, p. 73]

$$H^1(Y, \mathcal{O}_Y) = H^1(\mathbb{C}^2, \pi_*\mathcal{O}_Y) = H^1(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = 0,$$

$$H^2(Y, \mathcal{O}_Y) = H^2(\mathbb{C}^2, \pi_*\mathcal{O}_Y) = H^2(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = 0.$$

But Y is not holomorphically separable since it contains a compact curve $E \cong \mathbb{P}^1$.

Example 3.11. In Theorem 3.9, the condition $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ is necessary.

Let $\pi : X \to \mathbb{P}^2$ be the blowup at a point O and E the exceptional curve on X. Let L be a line such that $O \notin L \subset \mathbb{P}^2$. Define an effective divisor $D = \pi^*L + E$. Let $Y = X - E - \pi^*L$. Then Y contains no compact curves, $\pi^*L \cdot E = 0$ and $E^2 = -1$. For all integers $j \geq 0$ and $n > 0$, we have

$$0 \to \mathcal{O}_X(n\pi^*L + jE) \to \mathcal{O}_X(n\pi^*L + (j+1)E) \to \mathcal{O}_E(n\pi^*L + (j+1)E) \to 0.$$

Since

$$H^0(E, \mathcal{O}_E(n\pi^*L + (j+1)E)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-j-1)) = 0,$$

$H^0(X, \mathcal{O}_X(nD))$ is equal to

$$H^0(X, \mathcal{O}_X(n\pi^*L + nE)) = H^0(X, \mathcal{O}_X(n\pi^*L)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(nL)).$$

This implies that $\Phi_{|nD|}(Y) \cap \Phi_{|nD|}(X - Y) = \emptyset$ for sufficiently large n such that nL is very ample. Since Y is isomorphic to $\mathbb{C}^2 - \{O\}$, $H^1(Y, \mathcal{O}_Y)$ is not a finite-dimensional vector space [GR1, p. 131], and Y is not holomorphically convex [GR1, p. 159].

Example 3.12. Let Y be an open subset of a nonsingular projective threefold X defined over $\mathbb{C}$ such that $X - Y$ is the support of an effective divisor D, $\kappa(D, X) = 1$ and $H^i(Y, \Omega^j_Y) = 0$ for all $i > 0$ and $j \geq 0$, where Ω^j_Y is the sheaf of regular j-forms [Z1, Z2]. Then Y contains no compact curves and for a general point $x_0 \in X - Y$, $\Phi^{-1}_{|nD|}(\Phi_{|nD|}(x_0)) \cap Y$ is a surface on Y. We do not know if Y is Stein. This question was raised by Serre for complex manifolds and is still open except for curves [Se, Z1, Z2, Z3].

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Received 15.9.2013
and in final form 13.1.2014 (3222)