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Two remarks on the Suita conjecture

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Abstract. It is shown that the weak multidimensional Suita conjecture fails for any bounded non-pseudoconvex domain with $C^{1+\varepsilon}$-smooth boundary. On the other hand, it is proved that the weak converse to the Suita conjecture holds for any finitely connected planar domain.

Let D be a domain in $\mathbb{C}^n$. Denote by K_D and A_D the Bergman kernel on the diagonal and the Azukawa metric of D (cf. [JP]). Let

$$I_D^A(z) = \{X \in \mathbb{C}^n : A_D(z;X) < 1\}$$

be the indicatrix of A_D at $z \in D$.

Z. Błocki and W. Zwonek have recently proved the following (see [BZ, Theorem 2] and [B2, Theorem 7.5]).

Theorem 1. *If D is a pseudoconvex domain in $\mathbb{C}^n$, then*

$$K_D(z) \geq \frac{1}{\lambda(I_D^A(z))}, \qquad z \in D \ (^1).$$

Theorem 1 for $n = 1$ is known as the Suita conjecture (see [S]). The first proof of this conjecture is given in [B1].

Theorem 1 can hold for some bounded non-pseudoconvex domains. To see this, note that if M is a closed pluripolar subset of a domain D in $\mathbb{C}^n$, then $K_{D\setminus M} = K_D$ and $A_{D\setminus M} = A_D$.

On the other hand, our first remark says that even a weaker version of Theorem 1 fails for every bounded non-pseudoconvex domain with $C^{1+\varepsilon}$-smooth boundary.

Proposition 2. *Let D be a bounded non-pseudoconvex domain in $\mathbb{C}^n$ with $C^{1+\varepsilon}$-smooth boundary ($\varepsilon > 0$). Then there exists a sequence $(z_j)_j \subset D$*

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(1) If $K_D(z) = 0$, then $\lambda(I_D^A(z)) = \infty$.

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such that

$$\lim_{j\to\infty} K_D(z_j)\lambda(I_D^A(z_j))=0.$$

Proof. Since D is non-pseudoconvex, we can find a sequence $(z_j)_j\subset D$ approaching $a\in\partial D$ such that

$$\lim_{j\to\infty} K_D(z_j)<\infty \tag{1}$$

(otherwise, $\log K_D$ would be a plurisubharmonic exhaustion function for D). On the other hand, if $\varepsilon\le 1$, then, by [DNT, Proposition 2(i)], there exists a constant $c_1>0$ such that

$$c_1A_D(z;X)\ge\frac{|X_N|}{(d_D(z))^{\varepsilon/(1+\varepsilon)}}+\|X\|,\qquad z\text{ near }a,$$

where $d_D(z)=\operatorname{dist}(z,\partial D)$ and X_N is the projection of X on the complex normal to ∂D at a point a' such that $\|z-a'\|=d_D(z)$. Thus, one can find a constant $c_2>0$ for which

$$\lambda(I_D^A(z))\le c_2(d_D(z))^{2\varepsilon/(1+\varepsilon)},\qquad z\text{ near }a.$$

This inequality and (1) imply the wanted result. ∎

Proposition 3 ([BZ, Proposition 4]). *Let* $0<r<1$ *and* $P_r=\{z\in\mathbb{C}: r<|z|<1\}$. *Then*

$$K_{P_r}(\sqrt{r})\ge-\frac{2\log r}{\pi^2}\cdot\frac{1}{\lambda(I_{P_r}^A(\sqrt{r}))}.$$

So, the converse to the Suita conjecture is not true with any universal constant instead of 1. However, our second remark says that any finitely connected planar domain has its own constant.

Proposition 4. *For any finitely connected planar domain D there exists a constant $c>0$ such that*

$$K_D(z)\le\frac{c}{\lambda(I_D^A(z))},\qquad z\in D.$$

Proof. It follows from the removable singularity theorem and the uniformization theorem that it is enough to consider the case when $D=\mathbb{D}\setminus E$, where E is a union of disjoint closed discs contained in the open unit disc $\mathbb{D}$. Since now ∂D is C^1-smooth, [JN, Proposition 2] (see also [JP, Lemma 20.3.1]) and the remark at the end of [JN] show that

$$\lim_{z\to\partial D}\frac{d_D^2(z)}{\lambda(I_D^A(z))}=\frac{1}{4\pi}=\lim_{z\to\partial D}K_D(z)d_D^2(z).$$

Hence

$$\lim_{z\to\partial D}K_D(z)\lambda(I_D^A(z))=1,$$

which leads to the desired inequality. ∎

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