Hukuhara’s differentiable iteration semigroups of linear set-valued functions

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Abstract. Let $K$ be a closed convex cone with nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of $K$. A family $\{F_t: t \geq 0\}$ of continuous linear set-valued functions $F_t: K \rightarrow cc(K)$ is a differentiable iteration semigroup with $F^0(x) = \{x\}$ for $x \in K$ if and only if the set-valued function $\Phi(t, x) = F^t(x)$ is a solution of the problem

$$D_t \Phi(t, x) = \Phi(t, G(x)) := \bigcup \{\Phi(t, y): y \in G(x)\}, \quad \Phi(0, x) = \{x\},$$

for $x \in K$ and $t \geq 0$, where $D_t \Phi(t, x)$ denotes the Hukuhara derivative of $\Phi(t, x)$ with respect to $t$ and $G(x) := \lim_{s \to 0^+} (F^s(x) - x)/s$ for $x \in K$.

1. Let $X$ be a vector space. Throughout this paper all vector spaces are supposed to be real. We write

$$A + B = \{a + b: a \in A, b \in B\}, \quad \lambda A := \{\lambda a: a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$. A subset $K \subset X$ is called a cone if $tK \subset K$ for all positive $t$.

Let $X$ and $Y$ be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F: K \rightarrow n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of $Y$, is called linear if

$$F(x) + F(y) = F(x + y), \quad F(\lambda x) = \lambda F(x)$$

for all $x, y \in K$ and $\lambda > 0$.

Let $K$ be a convex cone in a normed vector space and let $b(K), c(K)$, and $cc(K)$ denote the sets of all bounded, compact, and convex compact members of $n(K)$, respectively. The difference $A - B$ of $A, B \in cc(K)$ is a set $C \in cc(K)$ such that $A = B + C$. If the difference exists, then it is unique. This is a consequence of a theorem of Rådström (see [7]).

Let $H: [0, \infty) \rightarrow cc(K)$ be a set-valued function such that the differences $H(t) - H(s)$ exist for $t, s \in [0, \infty)$ such that $t > s$. The Hukuhara derivative

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of $H$ at $t$ is defined by the formula (see [3])
\[ DH(t) = \lim_{{s \to t^+}} \frac{H(s) - H(t)}{s - t} = \lim_{{s \to t^-}} \frac{H(t) - H(s)}{t - s}, \]
whenever both limits exist with respect to the Hausdorff metric $d$ in $cc(K)$ derived from the norm in $X$. Moreover,
\[ DH(0) = \lim_{{s \to 0^+}} \frac{H(s) - H(0)}{s}. \]

Now, we will prove the following

**Lemma 1.** Let $X$ be a Banach space and $H : [0, \infty) \to cc(X)$ be a set-valued function. If $H$ is differentiable at $t \in [0, \infty)$, then $H$ is continuous at this point.

**Proof.** For $s > t \geq 0$ we have
\[ d(H(s), H(t)) = d(H(s) - H(t), \{0\}) = (s - t)d\left( \frac{H(s) - H(t)}{s - t}, \{0\} \right) \]
\[ \leq (s - t)\left[ d\left( \frac{H(s) - H(t)}{s - t}, DH(t) \right) + d(DH(t), \{0\}) \right]. \]
This implies that
\[ \lim_{{s \to t^+}} H(s) = H(t). \]
Similarly, for $t > 0$ and $0 < s < t$ we have
\[ d(H(s), H(t)) = d(\{0\}, H(t) - H(s)) = (t - s)d\left( \{0\}, \frac{H(t) - H(s)}{t - s} \right) \]
\[ \leq (t - s)\left[ d\left( \frac{H(t) - H(s)}{t - s}, DH(t) \right) + d(DH(t), \{0\}) \right], \]
whence
\[ \lim_{{s \to t^-}} H(s) = H(t). \]
Thus $H$ is continuous at $t$.

**2.** Let $K$ be a nonempty set. A family $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to n(K)$ is said to be an iteration semigroup if
\[ F^{s+t}(x) = F^t[F^s(x)]:= \bigcup \{F^t(y) : y \in F^s(x)\} \]
for all $x \in K$ and $t, s \geq 0$.

**Example 1.** Let $G : \mathbb{R}^n \to cc(\mathbb{R}^n)$ be a set-valued function. The attainable set $R(t, \xi)$ of the differential inclusion
\[ (*) \quad x'(s) \in G(x(s)) \quad \text{a.e. in } [0, t], \quad x(\cdot) \in AC[0, t], \quad x(0) = \xi, \]
at time $t$ from $\xi \in \mathbb{R}^n$ is defined by the formula
\[ R(t, \xi) = \{x(t) : x(\cdot) \text{ satisfies } (*) \text{ and } x(0) = \xi\}. \]
It is known that if the sets $R(t, \xi)$ are nonempty, then the set-valued functions $\xi \mapsto R(t, \xi)$ form an iteration semigroup (see e.g. [2]). Moreover, if $G$ is locally Lipschitz on $\mathbb{R}^n$ and the sets $R([0, t]; \xi)$ are compact, then $G$ is the infinitesimal generator of this semigroup (see [13]).

Let $K$ be a convex cone in a normed space. An iteration semigroup $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to cc(K)$ is said to be differentiable if all set-valued functions $t \mapsto F^t(x)$ $(x \in K)$ have Hukuhara’s derivative on $[0, \infty)$. 

**Example 2.** Let $K$ be closed convex cone with nonempty interior in a Banach space. Every concave iteration semigroup $\{F^t : t \geq 0\}$ of continuous linear set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$ for $x \in K$ is differentiable (see [10]).

**Example 3.** The family $\{F^t : t \geq 0\}$, where $F^t(x) = [e^t, e^{2t}]x$ for $t \in [0, \infty)$ and $x \in \mathbb{R}$, is a differentiable iteration semigroup of continuous linear set-valued functions. This semigroup is not concave.

Let $X$ and $Y$ be two vector spaces and let $K$ be a convex cone in $X$. A set-valued function $F : K \to n(Y)$ is called superadditive if

$$F(x) + F(y) \subset F(x + y)$$

for all $x, y \in K$. A set-valued function $F : K \to n(K)$ is said to be $\mathbb{Q}_+$-homogeneous if

$$F(\lambda x) = \lambda F(x)$$

for all $x \in K$ and all positive rational numbers $\lambda$.

We will use the following six lemmas.

**Lemma 2** (see Lemma 3 in [10]). Let $X$ and $Y$ be two topological vector spaces and let $K$ be a closed convex cone in $X$. Assume that $F : K \to cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If the difference $A - B$ exists, then $F(A) - F(B)$ exists and $F(A) - F(B) = F(A - B)$.

**Lemma 3** (Theorem 3 in [12], see also Lemma 4 in [9]). Let $X$ and $Y$ be two normed vector spaces and let $K$ be a convex cone in $X$. Suppose that $\{F_i : i \in I\}$ is a family of superadditive lower semicontinuous and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i : K \to n(Y)$. If $K$ is of the second category in $K$ and $\bigcup_{i \in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a positive constant $M$ such that

$$\|F_i(x)\| := \sup\{\|y\| : y \in F_i(x)\} \leq M\|x\|$$

for every $i \in I$ and $x \in K$.

**Corollary 1.** If $X$, $Y$ and $K$ are as in Lemma 3, then the functional

$$F \mapsto \|F\| := \sup\{\|F(x)\|/\|x\| : x \in K, x \neq 0\}$$
is finite for every $Q_+$-homogeneous superadditive lower semicontinuous set-valued function $F : K \rightarrow b(Y)$.

**Lemma 4** (Lemma 5 in [9]). Let $X$ and $Y$ be two normed spaces and let $d$ be the Hausdorff distance derived from the norm in $Y$. Suppose that $K$ is a convex cone in $X$ with nonempty interior. Then there exists a positive constant $M_0$ such that for every linear continuous set-valued function $F : K \rightarrow c(Y)$ the inequality
\[ d(F(x), F(y)) \leq M_0 \|F\| \|x - y\| \]
holds for every $x, y \in K$.

**Lemma 5** (Theorem 2 in [5]). Let $(X, \rho_X)$ and $(Y, \rho_Y)$ be two metric spaces and let $d_X$ and $d_Y$ be the corresponding Hausdorff metrics. If $F : X \rightarrow n(Y)$ is a set-valued function and $M$ is a positive constant such that
\[ d_Y(F(x), F(y)) \leq M \rho_X(x, y) \]
for any $x, y \in X$, then
\[ d_X(F(A), F(B)) \leq M d_X(A, B) \]
for any nonempty subsets $A, B$ of $X$.

**Lemma 6** (Lemma 4 in [8]). Let $D$ be a nonempty set and $Y$ be a normed space. If $F_0, F_n : D \rightarrow c(Y)$ are set-valued functions and the sequence $(F_n)$ uniformly converges to $F_0$ on $D$, then
\[ \lim_{n \to \infty} F_n(D) = F_0(D). \]

**Lemma 7.** Let $X$ be a Banach space, $Y$ a normed space, and $K$ a closed convex cone in $X$ with nonempty interior. Suppose that $F_0, F_n : K \rightarrow c(Y)$ are continuous linear set-valued functions. If
\[ F_0(y) = \lim_{n \to \infty} F_n(y) \]
for $y \in K$, then the sequence $(F_n)$ uniformly converges to $F_0$ on every $D \in c(K)$.

**Proof.** By (1) the set $\bigcup\{F_n(y) : n = 0, 1, 2, \ldots \}$ is bounded for every $y \in K$ and by Lemma 3 there exists a positive constant $M$ such that
\[ \|F_n\| \leq M \]
for $n = 0, 1, 2, \ldots$ According to Lemma 4 there exists a positive constant $M_0$ such that
\[ d(F_n(x), F_n(y)) \leq M_0 \|F_n\| \|x - y\| \leq M_0 M \|x - y\| \]
for $x, y \in K$ and $n = 0, 1, 2, \ldots$ This implies that
\[
\begin{align*}
|d(F_n(x), F_0(x)) - d(F_n(y), F_0(y))| &\leq d(F_n(x), F_n(y)) + d(F_0(x), F_0(y)) \\
&\leq 2M M_0 \|x - y\|.
\end{align*}
\]
Consequently, the family \( \{d((F_n(\cdot), F_0(\cdot)) : n = 0, 1, 2, \ldots \} \) is equicontinuous in \( K \) and by (1), \( F_0 \) is the uniform limit of \( (F_n) \) on every compact subset \( D \) of \( K \) (see Theorem 3.2.4 in [4]).

3. Now, we can prove our main results.

**Theorem 1.** Let \( X \) be a Banach space and let \( K \) be a closed convex cone in \( X \) with nonempty interior. Suppose that \( \{F^t : t \geq 0\} \) is a differentiable iteration semigroup of linear continuous set-valued functions \( F^t : K \to c\ell(K) \) with \( F^0(x) = \{x\} \). Then the set-valued function \( (t, x) \mapsto F^t(x) \) is continuous and

\[
D_tF^t(x) = F^t(G(x))
\]

for \( x \in K, t \geq 0 \), where \( D_t \) denotes the Hukuhara derivative of \( F^t(x) \) with respect to \( t \) and

\[
G(x) := \lim_{s \to 0^+} \frac{F^s(x) - x}{s}
\]

for \( x \in K \).

**Proof.** It is obvious that the differences \( F^s(x) - x \) exist for \( s > 0 \) and \( x \in K \), and hence, according to Lemma 2, so do the differences

\[
F^{t+s}(x) - F^t(x) = F^t[F^s(x)] - F^t(x) = F^t(F^s(x) - x)
\]

and

\[
F^t(x) - F^{t-s}(x) = F^{t-s}[F^s(x)] - F^{t-s}(x) = F^{t-s}(F^s(x) - x)
\]

whenever \( t > 0, s \in (0, t) \) and \( x \in K \).

By Lemmas 4 and 5,

\[
d\left(\frac{F^{t+s}(x) - F^t(x)}{s}, F^t(G(x))\right) = d\left(F^t\left(\frac{F^s(x) - x}{s}\right), F^t(G(x))\right)
\]

\[
\leq M_0\|F^t\|d\left(\frac{F^s(x) - x}{s}, G(x)\right)
\]

for \( x \in K, t > 0, s \in (0, t) \), so (2) implies

\[
\lim_{s \to 0^+} \frac{F^{t+s}(x) - F^t(x)}{s} = F^t(G(x))
\]

for \( t > 0 \) and \( x \in K \).

Similarly, for \( t > 0, s \in (0, t) \) and \( x \in K \) we have

\[
d\left(\frac{F^t(x) - F^{t-s}(x)}{s}, F^t(G(x))\right)
\]

\[
= d\left(F^{t-s}\left(\frac{F^s(x) - x}{s}\right), F^{t-s}(F^s(G(x)))\right)
\]

\[
\leq M_0\|F^{t-s}\|d\left(\frac{F^s(x) - x}{s}, F^s(G(x))\right).
\]
By Lemma 1 the function $s \mapsto F^s(y)$ is continuous for every $y \in K$. Therefore the image $\bigcup_{0 \leq s \leq t} F^s(y)$ of the interval $[0, t]$ under this set-valued function is compact (see [1, p. 110]), whence it is bounded. According to Lemma 3 there exists a positive constant $M$ such that 

$$\|F^{t-s}(y)\| \leq M\|y\|$$

for $0 \leq s \leq t$ and $y \in K$. Consequently, 

$$\|F^{t-s}\| \leq M$$

for $s \in [0, t]$. This inequality and (3) imply that 

$$d\left(\frac{F^t(x) - F^{t-s}(x)}{s}, F^t(G(x))\right) \leq M_0Md\left(\frac{F^s(x) - x}{s}, G(x)\right) + M_0Md(G(x), F^s(G(x)))$$

for $t > 0$, $s \in (0, t)$ and $x \in K$. By Lemmas 7 and 6 we have 

$$\lim_{s \to 0^+} F^s(G(x)) = G(x),$$

and by (2) and (4), 

$$D_tF^t(x) = \lim_{s \to 0^+} \frac{F^t(x) - F^{t-s}(x)}{s} = F^t(G(x)).$$

It remains to prove that the multifunction $(t, x) \mapsto F^t(x)$ is continuous. Fix $t \geq 0$, $x \in K$ and $y \in G(x)$. By Lemmas 3–5 there are two positive constants $M_0$ and $M$ such that 

$$d(F^{t+s}(z), F^t(y)) \leq d(F^{t+s}(z), F^{t+s}(y)) + d(F^{t+s}(y), F^t(y)) \leq M_0\|F^t\|(M\|z - y\| + d(F^s(y), \{y\}))$$

for every $s \in (0, 1)$ and $z \in G(x)$. Therefore 

$$\limsup_{(s, z) \to (0^+, y)} d(F^{t+s}(z), F^t(y)) = 0.$$ 

Similarly, fix $t > 0$, $x \in K$ and $y \in G(x)$. There exist two positive constants $M_0$ and $M_1$ for which 

$$d(F^{t-s}(z), F^t(y)) \leq d(F^{t-s}(z), F^{t-s}(y)) + d(F^{t-s}(y), F^t(y)) \leq M_0M_1(\|z - y\| + d(F^s(y), \{y\})),$$

for every $s \in (0, t)$ and $z \in G(x)$. By (5) and (6) the set-valued function $(t, y) \mapsto F^t(y)$ is continuous.

**Definition 1.** Let $K$ be a convex cone in a Banach space $X$ and let $G, \Psi : K \to cc(K)$ be two continuous linear maps. A map $\Phi : [0, \infty) \times K \to cc(K)$ is said to be a solution of the problem 

$$D_t\Phi(t, x) = \Phi(t, G(x)) := \bigcup\{\Phi(t, y) : y \in G(x)\},$$
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\( \Phi(0, x) = \Psi(x) \),

if \( \Phi \) is continuous in \([0, \infty) \times K\) and differentiable with respect to \( t \), and satisfies (7) and (8) everywhere in \([0, \infty) \times K\) and \( K \), respectively.

**Lemma 8** (Theorem 2 in [11]). Let \( K \) be a closed convex cone with nonempty interior in a Banach space and let \( G, \Psi : K \to ccc(K) \) be two continuous linear maps. Then there exists exactly one solution of problem (7)–(8). This solution is linear with respect to the second variable.

**Theorem 2.** Let \( X \) be a Banach space, let \( K \) be a closed convex cone in \( X \) with nonempty interior, and let \( G : K \to ccc(K) \) be a continuous linear set-valued function. Suppose that \( \Phi : [0, \infty) \times K \to ccc(K) \) is a solution of problem (7)–(8) with \( \Psi(x) = \{x\} \) for \( x \in K \), such that every set-valued function \( x \mapsto \Phi(t, x) \), \( t \geq 0 \), is linear. Then the family \( \{F^t : t \geq 0\} \), where \( F^t(x) := \Phi(t, x) \) for \( (t, x) \in [0, \infty) \times K \), is a differentiable iteration semigroup.

**Proof.** Fix \( t \geq 0 \) and define

\[ \alpha(s, x) := \Phi(s + t, x), \quad \beta(s, x) := \Phi(t, \Phi(s, x)) \]

for \( x \in K, s \geq 0 \). We see that

\[ \alpha(0, x) = \Phi(t, x), \quad \beta(0, x) = \Phi(t, \Phi(0, x)) = \Phi(t, x) \]

for \( x \in K \). Now, we have

\[
\begin{align*}
\lim_{u \to s^+} \frac{\alpha(u, x) - \alpha(s, x)}{u - s} &= \lim_{u \to s^+} \frac{\Phi(u + t, x) - \Phi(s + t, x)}{u - s} = \Phi(s + t, G(x)) \\
&= \bigcup \{\Phi(s + t, y) : y \in G(x)\} = \bigcup \{\alpha(s, y) : y \in G(x)\} = \alpha(s, G(x))
\end{align*}
\]

for \( x \in K, s \geq 0 \), and

\[
\begin{align*}
\lim_{u \to s^-} \frac{\alpha(s, x) - \alpha(u, x)}{s - u} &= \lim_{u \to s^-} \frac{\Phi(s + t, x) - \Phi(u + t, x)}{s - u} \\
&= \Phi(s + t, G(x)) = \alpha(s, G(x))
\end{align*}
\]

for \( x \in K, s > 0 \). Thus

\[ D_s \alpha(s, x) = \alpha(s, G(x)) \]

for \( x \in K, s \geq 0 \). Further, by Lemma 2 we have

\[
\begin{align*}
\frac{\beta(u, x) - \beta(s, x)}{u - s} &= \frac{\Phi(t, \Phi(u, x)) - \Phi(t, \Phi(s, x))}{u - s} \\
&= \Phi \left( t, \frac{\Phi(u, x) - \Phi(s, x)}{u - s} \right)
\end{align*}
\]
for \( u > s \geq 0, \, x \in K \), and according to Lemma 4,
\[
\left( \frac{\beta(u, x) - \beta(s, x)}{u - s}, \Phi(t, D_s \Phi(s, x)) \right) \\
= \left( \frac{\Phi(t, \frac{\Phi(u, x) - \Phi(s, x)}{u - s})}{u - s}, \Phi(t, D_s \Phi(s, x)) \right) \\
\leq M_0 \| \Phi(t, \cdot) \| d\left( \frac{\Phi(u, x) - \Phi(s, x)}{u - s}, D_s \Phi(s, x) \right).
\]

Thus
\[
\lim_{u \to s^+} \frac{\beta(u, x) - \beta(s, x)}{u - s} = \Phi(t, D_s \Phi(s, x)) = \Phi(t, \Phi(s, G(x))) \\
= \Phi(t, \bigcup \{ \Phi(s, y) : y \in G(x) \}) = \bigcup \{ \Phi(t, \Phi(s, y)) : y \in G(x) \} \\
= \bigcup \{ \beta(s, y) : y \in G(x) \} = \beta(s, G(x)).
\]

Similarly for \( s > u \geq 0, \, x \in K \), we have
\[
\frac{\beta(s, x) - \beta(u, x)}{s - u} = \Phi(t, \Phi(s, x)) - \Phi(t, \Phi(u, x)) \\
= \Phi(t, \frac{\Phi(s, x) - \Phi(u, x)}{s - u}) \\
\text{and}
\]
\[
\left( \frac{\beta(s, x) - \beta(u, x)}{s - u}, \Phi(t, D_s \Phi(s, x)) \right) \\
= \left( \frac{\Phi(t, \frac{\Phi(s, x) - \Phi(u, x)}{s - u})}{s - u}, \Phi(t, D_s \Phi(s, x)) \right) \\
\leq M_0 \| \Phi(t, \cdot) \| d\left( \frac{\Phi(s, x) - \Phi(u, x)}{s - u}, D_s \Phi(s, x) \right).
\]

Thus
\[
\lim_{u \to s^-} \frac{\beta(s, x) - \beta(u, x)}{s - u} = \Phi(t, D_s \Phi(s, x)) = \Phi(t, \Phi(s, G(x))) \\
= \Phi(t, \bigcup \{ \Phi(s, y) : y \in G(x) \}) = \bigcup \{ \Phi(t, \Phi(s, y)) : y \in G(x) \} \\
= \bigcup \{ \beta(s, y) : y \in G(x) \} = \beta(s, G(x)).
\]

Therefore
\[
(12) \quad D_s \beta(s, x) = \beta(s, G(x)).
\]

Equalities (9)–(12) mean that \( \alpha \) and \( \beta \) are solutions of problem (7)–(8) with \( \Psi(x) = \Phi(t, x) \). By Lemma 8 a solution of (7)–(8) is unique. Consequently, \( \beta = \alpha \), which means that
\[
F^t(F^s(x)) = \Phi(t, \Phi(s, x)) = \Phi(t + s, x) = F^{t+s}(x).
\]

This completes the proof.
4. Now, we give some applications.

**Corollary 2.** Let $K$ be a closed convex cone with nonempty interior in a Banach space and let $\{F^t : t \geq 0\}$ be a concave iteration semigroup of continuous linear set-valued functions $F^t : K \to {\text{cc}}(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then the set-valued function $\Phi : [0, \infty) \times K \to {\text{cc}}(K)$, $\Phi(t, x) = F^t(x)$, is a solution of problem (7)–(8) with $\Psi(x) = \{x\}$ for $x \in K$, where $G$ is given by (2), and the set-valued functions $t \mapsto \Phi(t, G(x))$, $x \in K$, are increasing.

**Proof.** By the Theorem in [10] the iteration semigroup $\{F^t : t \geq 0\}$ is differentiable and the set-valued function $\Phi(t, x) := F^t(x)$ is a solution of problem (7)–(8) with $\Psi(x) = \{x\}$, where $G$ is given by (2). Since the set-valued functions $t \mapsto \Phi(t, x)$, $x \in K$, are concave, Theorem 3.1 in [6] implies that the set-valued functions $t \mapsto D_t \Phi(t, x)$ are increasing. Thus the set-valued functions $t \mapsto F^t(G(x))$, $x \in K$, are also increasing.

**Corollary 3.** Let $K$ be a closed convex cone with nonempty interior in a Banach space and let $G : K \to {\text{cc}}(K)$ be a continuous linear set-valued function. If $\Phi : [0, \infty) \times K \to {\text{cc}}(K)$ is a solution of problem (7)–(8) with $\Psi(x) = \{x\}$ for $x \in K$, such that the set-valued functions $x \mapsto \Phi(t, x)$, $t \geq 0$, are linear and the functions $t \mapsto \Phi(t, G(x))$, $x \in K$, are increasing, then the family $\{F^t : t \geq 0\}$, where $F^t(x) := \Phi(t, x)$, is a concave iteration semigroup of continuous linear set-valued functions.

**Proof.** By Theorem 2 the family $\{F^t : t \geq 0\}$ is a differentiable iteration semigroup. According to the Proposition in [11],

$$F^t(x) = \Phi(t, x) = x + \int_0^t \Phi(s, G(x)) \, ds$$

for $x \in K$, $t \geq 0$, where $\int_0^t$ denotes a Riemann-type integral. Since the set-valued functions $s \mapsto \Phi(s, G(x))$ are increasing, Corollary 4.4 in [6] implies that the set-valued functions $t \mapsto F^t(x)$, $x \in K$, are concave. That means that the iteration semigroup $\{F^t : t \geq 0\}$ is concave.

**Corollary 4.** Let $K$ be a closed convex cone with nonempty interior in a Banach space. Suppose that $\{F^t : t \geq 0\}$ is a differentiable iteration semigroup of continuous linear set-valued functions $F^t : K \to {\text{cc}}(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then this semigroup is increasing if and only if $0 \in G(x)$ for $x \in K$.

**Proof.** Suppose that $\{F^t : t \geq 0\}$ is increasing. Then $x \in F^t(x)$ for $x \in K$ and

$$0 \in G(x) = \lim_{t \to 0^+} \frac{F^t(x) - x}{t}$$

for $x \in K$, $t \geq 0$. 


To prove the converse, note that by Theorem 1 the set-valued function 
\( \Phi(t,x) := F^t(x) \) is a solution of problem (7)–(8) with \( \Psi(x) = \{x\} \) and by the Proposition in [11],

\[
F^t(x) = x + \int_0^t F^s(G(x)) \, ds.
\]

If (13) holds then by (14), \( x \in F^t(x) \) for \( x \in K, t \geq 0 \), so this semigroup is increasing.

References