ANNALES
POLONICI MATHEMATICI
83.1 (2004)

On the index of contact

by M. Montserrat Alonso Ferrero (Bologna)

Abstract. We use the construction of the intersection product of two algebraic cones to prove that the multiplicity of contact of the cones at the vertex is equal to the product of their degrees. We give an example to show that in order to calculate the index of contact it is not sufficient to perform the analytic intersection algorithm with hyperplanes.

1. Introduction. The aim of this paper is to find a relation between two indices which characterize locally the intersection of analytic sets: the intersection multiplicity and the index of contact.

The index of contact has been introduced by E. Cygan [Cy] in connection with the study of Łojasiewicz inequalities and separation exponent for analytic subsets X, Y of a complex manifold N. If N is an open subset of $\mathbb{C}^n$, fix a norm on $\mathbb{C}^n$ and set $\operatorname{dist}(X,z) := \inf_{x\in X}\|z-x\|$. Consider a point $c \in X\cap Y$. Then

$$\operatorname{dist}(X,z)+\operatorname{dist}(Y,z) \geq \mathrm{const}\cdot \operatorname{dist}(X\cap Y,z)^p$$

in a neighbourhood of the point c for some $\mathrm{const}, p>0$ (see e.g. [Ł]). In this situation we say that X and Y are *p-separated* at c, and p is called a *separation exponent*.

In [T$_1$], [CyT] and [Cy] upper bounds for p have been given. In particular, the local degree $\nu(X\bullet Y,c)$ of the intersection cycle $X\bullet Y$ (see [T$_2$]) is an upper bound, which can be improved in the case of an isolated intersection (see [T$_1$]) or improper intersection (see [Cy]). In [Cy] this improvement is achieved by modifying the analytic intersection algorithm of Tworzewski for the construction of the intersection cycle. This leads to the following two numbers: the index of contact $p(X,Y)(c)$ in the case that Y is a submanifold, and the multiplicity of contact $r(X,Y)(c) := p(X\times Y,\triangle_N)(c,c)$, where $\triangle_N$

2000 *Mathematics Subject Classification*: Primary 32B10, 14C17.

Key words and phrases: multiplicity of proper and improper intersection, index and multiplicity of contact, Łojasiewicz exponent.

Investigation supported by the MIUR and the University of Bologna, Funds for Selected Research Topics.

is the diagonal in $N \times N$. Immediately from the definition one has

$$r(X, Y)(c) \leq \nu(X \bullet Y, c)$$

and in [Cy] it is proved that also $r(X, Y)(c)$ is a separation exponent. We note that the difference between $\nu(X \bullet Y, c)$ and $r(X, Y)(c)$ can be arbitrarily large (see our Remark 3.5), but in a Zariski open subset of each component of $X \cap Y$ the two numbers coincide (see [R]). Note also that the index of multiplicity is not the sharp upper bound for the separation exponent.

Whereas $\nu(X \bullet Y, c)$ can be expressed as a Samuel multiplicity of an associated graded ring (see [N_3], [AR]), at present there are no algebraic or geometric characterizations of the index or multiplicity of contact.

In this note we characterize the index of contact of an algebraic cone Z and a subspace S. As a consequence we determine the multiplicity of contact of two algebraic cones (see Section 3). In Section 2, we recall some basic notions of intersection theory. In the last section we give an example which shows that the "Linear Testing Theorem" of [N_3] and [AR] for the extended index of intersection fails to be true in the case of the index of contact.

2. Notation

2.1. *Analytic cycles.* Let $A = \sum_{j \in J} \alpha_j C_j$ be an analytic cycle on a complex manifold N of dimension n. As a natural extension of the local degree of analytic sets we can define the *degree* of the cycle A at the point c as

$$\nu(A, c) = \sum_{j \in J} \alpha_j \nu(C_j, c) \in \mathbb{N},$$

where $\nu(C_j, c)$ denotes the degree of the irreducible analytic set C_j at c.

Any cycle A has a unique decomposition $A = T_{(n)} + T_{(n-1)} + \ldots + T_{(0)}$, where $T_{(j)}$ is a j-cycle (i.e. a formal combination of irreducible analytic sets of dimension j), and the *extended degree* of the cycle A at c is

$$\widetilde{\nu}(A, c) = (\nu(T_{(n)}, c), \nu(T_{(n-1)}, c), \ldots, \nu(T_{(0)}, c)) \in \mathbb{N}^{n+1}.$$

The analytic set $\bigcup_{j \in J} C_j$ is called the *support* of A and is denoted by $|A|$.

Let S be a closed submanifold in N. For the analytic cycle $A = \sum_{j \in J} \alpha_j C_j$ the *part of A supported by S* is defined to be

$$A^S = \sum_{j \in J,\, C_j \subset S} \alpha_j C_j.$$

2.2. *Proper intersection of cycles.* Let $A^1, \ldots, A^k$ be analytic cycles on a complex manifold N of dimension n, with supports $|A^j|$ of respective pure dimensions d_j. We say that these cycles *intersect properly* if the dimension of $\bigcap_{j=1}^k |A^j|$ is $(\sum_{j=1}^k d_j) - n(k-1)$. In this case we have the *intersection* *product* $A^1 \cdot \ldots \cdot A^k$ of the analytic cycles $A^1, \ldots, A^k$ as defined in [D] (see also [Ch]).

2.3. *Algorithm.* Let N be an n-dimensional complex manifold, Z an analytic subset of pure dimension d of N, and S an s-dimensional closed submanifold of N. Let U be an open subset of N such that $U \cap S \neq \emptyset$. Then $(U \setminus S) \cap Z$ is an analytic subset of $U \setminus S$ of pure dimension d (or empty). In the open subset U of N we consider a system $\mathcal{H} = (H_1, \ldots, H_{n-s})$ that satisfies the following conditions:

1. H_i is a smooth hypersurface in U containing $U \cap S$ for all $i = 1, \ldots, n-s$.
2. $\bigcap_{i=1}^{n-s} \mathrm{T}_y H_i = \mathrm{T}_y S$ for each $y \in U \cap S$.
3. $((U \setminus S) \cap Z) \cap H_1 \cap H_2 \cap \ldots \cap H_i$ is an analytic subset of $U \setminus S$ of pure dimension $d - i$, or an empty set, for $i = 1, \ldots, d$.

We denote by $\mathcal{H}(U, Z)$ the set of all systems $\mathcal{H}$ which have these three properties. For an element $\mathcal{H} = (H_1, \ldots, H_{n-s}) \in \mathcal{H}(U, Z)$ we define an analytic cycle $Z \cdot \mathcal{H}$ in $S \cap U$ following the Tworzewski algorithm ([T_2, Algorithm 4.1]). Set $\ell = \ell(\mathcal{H}) = \max\{j \in \{1, 2, \ldots, n-s\} : |Z_{j-1} - Z_{j-1}^S| \cap H_j \neq \emptyset\}$, or $\ell = \ell(\mathcal{H}) = 0$ if $Z \cap U \subset S \cap U$.

Algorithm:

Step 0. Let $Z_0 = Z \cap U$. Then $Z_0 = (Z_0 - Z_0^S) + Z_0^S$, with $Z_0^S :=$ part of Z_0 supported by $S \cap U$.
Step 1. Let $Z_1 = (Z_0 - Z_0^S) \cdot H_1$. Then $Z_1 = (Z_1 - Z_1^S) + Z_1^S$, with $Z_1^S :=$ part of Z_1 supported by $S \cap U$.
Step 2. Let $Z_2 = (Z_1 - Z_1^S) \cdot H_2$. Then $Z_2 = (Z_2 - Z_2^S) + Z_2^S$, with $Z_2^S :=$ part of Z_2 supported by $S \cap U$.
. . . .
Step ℓ. Let $Z_\ell = (Z_{\ell-1} - Z_{\ell-1}^S) \cdot H_\ell$. Then $Z_\ell = (Z_\ell - Z_\ell^S) + Z_\ell^S$, with $Z_\ell^S :=$ part of Z_ℓ supported by $S \cap U$, and $|Z_\ell - Z_\ell^S| \cap S = \emptyset$.

The positive analytic cycle $Z \cdot \mathcal{H} = Z_0^S + \ldots + Z_\ell^S$ on $S \cap U$ is called the *result* of the above algorithm.

2.4. *Definitions.* Using the algorithm we can define, for $c \in S$:

$$\widetilde{g}(Z, S)(c) := \min\nolimits_{\mathrm{lex}}\{\widetilde{\nu}(Z \cdot \mathcal{H}, c) : \mathcal{H} \in \mathcal{H}(U, Z) \text{ and } U \ni c\} \in \mathbb{N}^{s+1},$$
$$g(Z, S)(c) := \text{sum of the coordinates of } \widetilde{g}(Z, S)(c),$$
$$p(Z, S)(c) := \min\{\nu(Z \cdot \mathcal{H}, c) : \mathcal{H} \in \mathcal{H}(U, Z) \text{ and } U \ni c\} \in \mathbb{N};$$

we call these the *extended index of intersection*, the *index of intersection* and the *index of contact* (this last has been introduced in [Cy]) of Z with the submanifold S at the point c, respectively.

2.5. *Improper intersection of analytic sets.* Let X and Y be irreducible analytic sets of an n-dimensional manifold N and let $c \in N$. By standard diagonal construction the *multiplicity of intersection* of X and Y at c is defined to be

$$d(X,Y)(c) = g(X \times Y, \triangle_N)(c,c).$$

The *intersection product* of X and Y is the unique analytic cycle $X \bullet Y$ in N such that $\nu(X \bullet Y) = d(X,Y)$ (see [T$_2$, Definition 6.1]). This definition can be naturally extended to the case of arbitrary analytic cycles by multilinearity.

Now, if X and Y are pure dimensional analytic subsets of a manifold N, for $c \in X \cap Y$, we introduce the following index (see [CyKT]):

$$r(c) = r(X,Y)(c) := p(X \times Y, \triangle_N)(c,c),$$

which we call the *multiplicity of contact* of X and Y at the point c.

3. Intersection of algebraic cones with subspaces. We will see that the index of contact and the index of intersection of a cone with a linear subspace at the vertex, and the degree of the cone at the vertex are all equal. Thus the index of contact and the index of intersection, in this case, do not depend on the linear subspace.

Let us start with the following lemmas.

LEMMA 3.1. *Let $A^1, \ldots, A^k$ be positive analytic cycles of pure dimensions in a domain $D \subset \mathbb{C}^n$, intersecting properly in D, and let $a \in \bigcap_{j=1}^k |A^j|$. Then*

$$\nu(A^1 \cdot \ldots \cdot A^k, a) \geq \nu(A^1, a) \cdot \ldots \cdot \nu(A^k, a).$$

Proof. See [Ch, 12.5].

LEMMA 3.2. *Let Z be a pure d-dimensional analytic subset of an n-dimensional complex manifold N, S an s-dimensional closed submanifold of N, and $c \in Z \cap S$ a point. Then*

$$p(Z,S)(c) \geq \nu(Z,c).$$

Proof. We choose a neighbourhood U of c and a system

$$\mathcal{H} = (H_1, \ldots, H_{n-s}) \in \mathcal{H}(U,Z)$$

such that $\nu(Z \cdot \mathcal{H}, c) = p(Z,S)(c)$. For this system of hypersurfaces the (positive analytic cycle) result of the Tworzewski Algorithm will be of the form

$$Z \cdot \mathcal{H} = Z_0^S + Z_1^S + \ldots + Z_\ell^S,$$

where $\ell = 0$ if $Z \cap U \subset S \cap U$, or $\ell \in \{1, \ldots, n-s\}$ is the maximum of the indices for which $|Z_{\ell-1} - Z_{\ell-1}^S| \cap H_\ell \neq \emptyset$.

Set $\ell' = \ell'(\mathcal{H}, c) = \max\{j \in \{1, \ldots, \ell\} : c \in |Z_{j-1} - Z_{j-1}^S|\}$, or $\ell' = \ell'(\mathcal{H}, c) = 0$ if $\ell = 0$. For simplicity let us still write $\ell' = \ell$. From the algorithm and by $\mathbb{Z}$-bilinearity of the degree we get the equalities

$$\begin{aligned}
\nu(Z_0,c) &= \nu(Z\cap U,c) = \nu(Z,c) = \nu(Z_0^S,c) + \nu(Z_0 - Z_0^S,c),\\
\nu(Z_1,c) &= \nu((Z_0 - Z_0^S)\cdot H_1,c) = \nu(Z_1^S,c) + \nu(Z_1 - Z_1^S,c),\\
\nu(Z_2,c) &= \nu((Z_1 - Z_1^S)\cdot H_2,c) = \nu(Z_2^S,c) + \nu(Z_2 - Z_2^S,c),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\nu(Z_\ell,c) &= \nu((Z_{\ell-1} - Z_{\ell-1}^S)\cdot H_\ell,c) = \nu(Z_\ell^S,c) + \nu(Z_\ell - Z_\ell^S,c).
\end{aligned}$$

Since the degree is $\mathbb{Z}$-linear, we can write

$$p(Z,S)(c) = \nu(Z_0^S,c) + \nu(Z_1^S,c) + \ldots + \nu(Z_\ell^S,c)$$

and by the above equalities this is equal to

$$\begin{aligned}
[\nu(Z,c) - \nu(Z_0 - Z_0^S,c)] + [\nu((Z_0 - Z_0^S)\cdot H_1,c) - \nu(Z_1 - Z_1^S,c)] + \ldots\\
+ [\nu((Z_{\ell-1} - Z_{\ell-1}^S)\cdot H_\ell,c) - \nu(Z_\ell - Z_\ell^S,c)].
\end{aligned}$$

We group in the following way:

$$\begin{aligned}
\nu(Z,c) &+ [-\nu(Z_0 - Z_0^S,c) + \nu((Z_0 - Z_0^S)\cdot H_1,c)]\\
&+ [-\nu(Z_1 - Z_1^S,c) + \nu((Z_1 - Z_1^S)\cdot H_2,c)] + \ldots\\
&+ [-\nu(Z_{\ell-1} - Z_{\ell-1}^S,c) + \nu((Z_{\ell-1} - Z_{\ell-1}^S)\cdot H_\ell,c)] - \nu(Z_\ell - Z_\ell^S,c).
\end{aligned}$$

Since $|Z_\ell - Z_\ell^S|\cap H_{\ell+1} = \emptyset$, or, in the case $\ell = n-s$, $|Z_{n-s} - Z_{n-s}^S|\cap S = \emptyset$, we have $\nu(Z_\ell - Z_\ell^S,c) = 0$; and, by Lemma 3.1, $\nu((Z_j - Z_j^S)\cdot H_{j+1},c) \geq \nu(Z_j - Z_j^S,c)$ for $j = 0,\ldots,\ell-1$. Thus

$$p(Z,S)(c) \geq \nu(Z,c). \ \blacksquare$$

THEOREM 3.3. *If $Z\subset\mathbb{C}^n$ is an algebraic cone of pure dimension d and $S\subset\mathbb{C}^n$ is a linear subspace of dimension s, then*

$$g(Z,S)(c) = \nu(Z,c) = p(Z,S)(c),$$

where $c = 0\in\mathbb{C}^n$ denotes the vertex of the cone.

Proof. By [N_2, Cor. 7] (see also [N_3, Chap. III, Sect. 3]), $g(Z,S)(c) = g(Z\times S,\triangle_N)(c,c)$, which is by definition $d(Z,S)(c)$, i.e. the multiplicity of intersection of the sets Z and S at the point c. Now by construction of the intersection product cycle we have

$$d(Z,S)(c) = \nu(Z\bullet S)(c)$$

and from Bézout's theorem for algebraic varieties in $\mathbb{P}^{n-1}$ (see [SV], also e.g. [FSV, Chap. 2]), that is, for algebraic cones in $\mathbb{C}^n$ (see e.g. [N_2, Cor. 7] and [N_4]) we get

$$\nu(Z\bullet S)(c) = \nu(Z,c)\cdot\nu(S,c) = \nu(Z,c)$$

(S is linear and therefore $\nu(S,c) = 1$). Thus we have proved $g(Z,S)(c) = \nu(Z,c)$.

We note that also in the general case, that is, Z an analytic subset of pure dimension d on an n-dimensional complex manifold N, S a closed submanifold of N of dimension s and $c \in Z \cap S$ a point, we have

$$p(Z,S)(c) \le g(Z,S)(c).$$

Thus, in our case, it is enough to prove $p(Z,S)(c) \ge g(Z,S)(c)$. But from Lemma 3.2, $p(Z,S)(c) \ge \nu(Z,c)$. So the theorem is proved. ∎

The next examples indicate that the index of contact may be less than the index of intersection if the analytic set is not a cone, and that the index of intersection may not be equal to the degree of the cone at the vertex if the submanifold is not linear.

Example 1. (See [N_3, Chap. III, Sect. 3, Ex. 3].) In $\mathbb{C}^3$ take the intersection of $Z = \{(x,y,z) \in \mathbb{C}^3 : yx^2 = z^2\}$ with the linear subspace $S = \{y = z = 0\}$. Using the computer program CALI, [G], together with the script *Segre*, [AA], for commutative algebra, we calculate $\widetilde{g}(Z,S)(0,0,0) = (0,1,2)$ and $g(Z,S)(0,0,0) = 3$. But if we do the algorithm with the system $\mathcal{H} = (H_1,H_2)$, where $H_1 = \{y = 0\}$, the total result of the intersection algorithm is the cycle $2 \cdot S$, so $p(Z,S)(0,0,0) \le 2$. But $\widetilde{g}(Z,S)(0,0,0) := \min_{\text{lex}}\{\widetilde{\nu}(Z \cdot \mathcal{H},(0,0,0)) : \mathcal{H} \in \mathcal{H}(Z,U),\ U \text{ a neighbourhood of } (0,0,0)\} = (0,1,2)$ and the germ of Z at $(0,0,0)$ is irreducible. Hence $p(Z,S)(0,0,0) = 2$ (if $p(Z,S)(0,0,0) = 1$ then it would come from a cycle $Z \cdot \overline{\mathcal{H}}$ with $\widetilde{\nu}(Z \cdot \overline{\mathcal{H}},(0,0,0)) = (1,0,0)$, which is not possible).

Example 2. Let $N = \mathbb{C}^3$, $Z = \{x^2 + y^2 = xz\}$ and $S = \{y^2 + x = z^2\}$. By computer (see [G] and [AA]), we calculate $\widetilde{g}(Z,S)(0,0,0) = (0,4,0)$. Hence $g(Z,S)(0,0,0) = 4$, and obviously in this case $p(Z,S)(0,0,0) = 4$. But we have $\nu(Z,(0,0,0)) = 2$.

As a consequence of the previous theorem we obtain our main result:

Theorem 3.4. *Let X, Y be pure dimensional algebraic cones of $\mathbb{C}^n$. Then*

$$r(X,Y)(0) = \nu(X,0) \cdot \nu(Y,0).$$

Proof. The definition of the multiplicity of contact and Theorem 3.3 give

$$r(X,Y)(0) = p(X \times Y, \triangle_{\mathbb{C}^n})(0,0) = \nu(X \times Y,(0,0)).$$

Now, the Bézout Theorem for algebraic cones (see e.g. [N_3]) shows that this is equal to $\nu(X,0) \cdot \nu(Y,0)$, and the proof is complete. ∎

As the following example shows, we cannot extend Theorem 3.4 to general analytic sets.

Example 3. In $\mathbb{C}^6$ take coordinates (x,y,z,u,v,w) and calculate $r(X,Y)(c)$ for $X = \{(x,y,z) \in \mathbb{C}^3 : yx^2 = z^2\}$, $Y = \{y = z = 0\}$ and $c = (0,0,0)$. By definition $r(0) = p(X \times Y, \triangle_{\mathbb{C}^3})(0,0)$, where $X \times Y =$

$\{(x,y,z,u,v,w) \in \mathbb{C}^6 : yx^2 = z^2,\ v = w = 0\}$ and $\triangle_{\mathbb{C}^3} = \{(x,y,z,u,v,w) \in \mathbb{C}^6 : x = u,\ y = v,\ z = w\}$. Let $\mathcal{H} = (H_1, H_2, H_3)$ be defined by $H_1 = \{y = v\}$, $H_2 = \{x = u\}$ and $H_3 = \{z = w\}$. The total result of the intersection algorithm for this system is the cycle $2 \cdot T$, where $T = \{(x,y,z,u,v,w) \in \mathbb{C}^6 : x = u,\ y = z = v = w = 0\}$. Thus $p(X \times Y, \triangle_{\mathbb{C}^3})(0,0) \leq 2$. So $r(X,Y)(0) \neq \nu(X,0) \cdot \nu(Y,0) = 3$.

Remark 3.5. If in the previous example we take $X = \{(x,y,z) \in \mathbb{C}^3 : yx^n = z^2\}$, $n \in \mathbb{N}$, we obtain $r(X,Y)(0) \leq 2$ and $g(X,Y)(0) = \nu(X \bullet Y, 0) = n+1$. So the difference between $\nu(X \bullet Y, c)$ and $r(X,Y)(c)$ can be arbitrarily large.

Remark 3.6. In the previous example, using the Reduction Theorem for improper intersections (see [N_3], [AR]), we can write $g(X,Y)(0) = g(X \times Y, \triangle_{\mathbb{C}^3})(0,0) = 3$, i.e., $g(X,Y)(0) = d(X,Y)(0)$. By the same reasons of Example 1, we must have $p(X \times Y, \triangle_{\mathbb{C}^3})(0,0) = 2$, so in this case $p(X,Y)(0) = p(X \times Y, \triangle_{\mathbb{C}^3})(0,0)$. We have not been able to settle this question in general: given a pure dimensional analytic subset Z, a closed submanifold S and $c \in Z \cap S$ of a complex manifold N, is it true that $p(Z,S)(c) = r(Z,S)(c)$?

We have seen that in some cases we have the equality

$$p(Z,S)(c) = g(Z,S)(c),$$

if for example Z is an algebraic cone and S a subspace (Theorem 3.3), if $Z = \{x^2 + y^2 = xz\}$, $S = \{y^2 + x = z^2\}$ and $c = 0$ (Example 2), or for all c in a Zariski open set of each component of $Z \cap S$ (see [R]).

We have

$$g(Z,S)(c) = g(Z \times S, \triangle_N)(c,c),$$

and, as noted in the proof of Theorem 3.3,

$$g(Z \times S, \triangle_N)(c,c) \geq p(Z \times S, \triangle_N)(c,c).$$

But $p(Z \times S, \triangle_N)(c,c)$ is by definition $r(Z,S)(c)$. Thus, in the cases mentioned above, we have

$$p(Z,S)(c) \geq r(Z,S)(c).$$

So we can say that in the general case, that is, Z an analytic subset of pure dimension d on an n-dimensional complex manifold N, S a closed submanifold of N of dimension s, and $c \in Z \cap S$ a point, the multiplicity of contact of Z and S at c cannot be strictly greater than the index of contact of Z and S at c.

4. Index of contact and Linear Testing Theorem. Let N be an n-dimensional complex manifold, Z an analytic subset of pure dimension d of N, and S an s-dimensional closed submanifold of N. As the analytic intersection algorithm (of [T$_2$]) is local in the vicinity of the fixed point $c \in S$, we may assume that N is the germ of $\mathbb{C}^n$ at $c = 0$:

$$N = \mathbb{C}^n = \mathbb{C}^s_u \times \mathbb{C}^{n-s}_v,$$

where $u = (u_1, \dots, u_s)$ and $v = (v_1, \dots, v_{n-s})$ are the coordinates in $\mathbb{C}^s$ and $\mathbb{C}^{n-s}$, respectively, and

$$S = \{(u, v) : v = 0\} \subset \mathbb{C}^s_u \times \mathbb{C}^{n-s}_v.$$

As Nowak has shown in his Linear Testing Theorem ([N$_2$], [N$_3$]), or Achilles and Rams have shown in [AR], to obtain the extended index of intersection of Z with S at the point $c \in S$, $\widetilde{g}(Z, S)(c)$, we can reduce the analytic intersection algorithm for systems $\mathcal{H} = (H_1, \dots, H_{n-s})$ of smooth hypersurfaces near c to the case of systems of linear hyperplanes. More precisely we have the following theorem.

Theorem 4.1 (Linear Testing Theorem). *Let N be an affine space, S a vector subspace of N, and Z a pure dimensional analytic subset of N. Then the extended index of intersection of Z with S at the point $c = 0$ is realized by the intersection algorithm for systems $\mathcal{H}$ of linear hyperplanes in N.*

Proof. See [N$_2$, p. 138, Corollary 6] or [AR, p. 396, Corollary 3].

Remark 4.2. As an immediate consequence of the previous theorem, also the index of intersection of Z with S at the point $c \in S$, $g(Z, S)(c)$, is realized by the intersection algorithm for suitable systems of linear hyperplanes in N.

As we will see in the next example, the index of contact of an analytic set Z with a subspace S at a point $c \in S$, $p(Z, S)(c)$, cannot be realized using only linear hyperplanes.

Example 4. Let

$$f : \mathbb{C}^3 \ni (x, y, z) \mapsto (x, y + xz, z) \in \mathbb{C}^3.$$

It is easily seen that f is a biholomorphism.

As in Example 1, let $Z = \{(x, y, z) \in \mathbb{C}^3 : yx^2 = z^2\}$, $S = \{y = z = 0\}$, and $c = 0$ in $\mathbb{C}^3$. Then $f(Z)$, $f(S)$, and $f(c)$ are $Z' = \{(x, y, z) \in \mathbb{C}^3 : (y - xz)x^2 = z^2\}$, $S' = \{(x, y, z) \in \mathbb{C}^3 : y - xz = z = 0\} = \{y = z = 0\}$ and $c' = 0$, respectively.

Since the extended index of intersection is invariant under biholomorphism, it follows that

$$\widetilde{g}(Z, S)(0) = (0, 1, 2) = \widetilde{g}(Z', S')(0);$$

also the index of contact is invariant under biholomorphism (this can be seen step by step in the algorithm), and we have

$$p(Z, S)(0) = 2 = p(Z', S')(0).$$

As we have seen in Example 1, the index of contact of Z and S at $c = 0$ is realized by the intersection algorithm for a suitable system $\mathcal{H} = (H_1, H_2)$

such that the hypersurfaces H_1, H_2 are linear, and

$$p(Z,S)(0) = 2 = \nu(Z \cdot \mathcal{H}, 0),$$

where $Z \cdot \mathcal{H}$ is the result of the intersection algorithm using the system $\mathcal{H} = (H_1, H_2)$ with $H_1 = \{y = 0\}$. But we will see that the index of contact of Z' and S' at $c' = 0$, although these are the biholomorphic images of Z, S and $c = 0$, respectively, cannot be realized by the intersection algorithm for an appropriate system of hyperplanes in $\mathbb{C}^3$, i.e.,

$$p(Z', S')(0) < \nu(Z' \cdot \mathcal{H}', 0)$$

for every suitable system $\mathcal{H}' = (H_1', H_2')$, where H_1' and H_2' are linear. Hence there is no Linear Testing Theorem for the index of contact.

Let U' be some neighbourhood of $c' = 0$. Take a system $\widetilde{\mathcal{H}} = (\widetilde{H}_1, \widetilde{H}_2) \in \mathcal{H}(U', Z')$, where $\widetilde{H}_1$ is a hyperplane, i.e., $\widetilde{H}_1 = \{(x,y,z) \in \mathbb{C}^3 : ax + by + cz = 0\}$ for some a, b, c in $\mathbb{C}$. Note that as the hyperplane $\widetilde{H}_1$ contains $S' = \{y = z = 0\}$, it must be of the form $\widetilde{H}_1 = \{(x,y,z) \in \mathbb{C}^3 : by + cz = 0\}$. Now we do the intersection algorithm with a system $\widetilde{\mathcal{H}} = (\widetilde{H}_1, \widetilde{H}_2)$, where $\widetilde{H}_1 = \{(x,y,z) \in \mathbb{C}^3 : by + cz = 0\}$.

Step 0. Let $Z_0' = Z' \cap U' = (Z_0' - {Z_0'}^{S'}) + {Z_0'}^{S'}$. It is clear that S' does not contain components of the cycle Z_0'.

Step 1. Let $Z_1' = (Z_0' - {Z_0'}^{S'}) \cdot \widetilde{H}_1 = (Z_1' - {Z_1'}^{S'}) + {Z_1'}^{S'}$. Straightforward calculation shows that the cycle Z_1' has always two components, one of them S' with multiplicity 1 and the other an irreducible analytic set with multiplicity 1 which is in the support of $(Z_1' - {Z_1'}^{S'})$.

Step 2. Let $Z_2' = (Z_1' - {Z_1'}^{S'}) \cdot \widetilde{H}_2 = (Z_2' - {Z_2'}^{S'}) + {Z_2'}^{S'}$. We do not need to calculate it.

As we have

$$\widetilde{g}(Z', S')(0) = \min\nolimits_{\mathrm{lex}}\{\widetilde{\nu}(Z' \cdot \mathcal{H}', 0) : \mathcal{H}' \in \mathcal{H}(U', Z') \text{ and } U' \ni 0\} = (0,1,2),$$

the extended degree of the cycle result of the algorithm with $\widetilde{\mathcal{H}} = (\widetilde{H}_1, \widetilde{H}_2)$ for $\widetilde{H}_1 = \{(x,y,z) \in \mathbb{C}^3 : by + cz = 0\}$ and any $\widetilde{H}_2$ will be of the form

$$\widetilde{\nu}(Z' \cdot \widetilde{\mathcal{H}}, 0) = (\nu({Z_0'}^{S'}, 0), \nu({Z_1'}^{S'}, 0), \nu({Z_2'}^{S'}, 0)) = (0, 1, m)$$

with $m \in \mathbb{N}$ and $m \geq 2$.

Therefore we cannot obtain the index of contact

$$p(Z', S')(0) = 2 = \min\{\nu(Z' \cdot \mathcal{H}', 0) : \mathcal{H}' \in \mathcal{H}(U', Z') \text{ and } U' \ni 0\}$$

if we test only with hyperplanes.

Acknowledgements. I am grateful to Piotr Tworzewski for his essential help, useful suggestions and comments. I also thank the Jagiellonian University, in particular, the Institute of Mathematics, where this paper was prepared, for the hospitality during my stay.

References

[AA] R. Achilles and D. Aliffi, *Segre: a script for the REDUCE package CALI.* Bologna, 1999–2001. Available at http://www.dm.unibo.it/~achilles/segre/.

[AR] R. Achilles and S. Rams, *Intersection numbers, Segre numbers and generalized Samuel multiplicities*, Arch. Math. (Basel) 77 (2001), 391–398.

[ATW] R. Achilles, P. Tworzewski and T. Winiarski, *On improper isolated intersection in complex analytic geometry*, Ann. Polon. Math. 51 (1990), 21–36.

[Ch] E. M. Chirka, *Complex Analytic Sets*, Kluwer, 1989.

[Cy] E. Cygan, *Intersection theory and separation exponent in complex analytic geometry*, Ann. Polon. Math. 69 (1998), 287–299.

[CyKT] E. Cygan, T. Krasiński and P. Tworzewski, *Separation of algebraic sets and the Łojasiewicz exponent of polynomial mappings*, Invent. Math. 136 (1999), 75–87.

[CyT] E. Cygan and P. Tworzewski, *Proper intersection multiplicity and regular separation of analytic sets*, Ann. Polon. Math. 59 (1994), 293–298.

[D] R. Draper, *Intersection theory in analytic geometry*, Math. Ann. 180 (1969), 175–204.

[FSV] H. Flenner, L. O'Carroll and W. Vogel, *Joins and Intersections*, Springer Monogr. Math., Springer, Berlin, 1999.

[G] H. G. Gräbe, *CALI-A REDUCE package for commutative algebra*, Version 2.2.1 (1995). Available through the REDUCE library redlib@rand.org.

[Ł] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, 1991.

[N_1] K. J. Nowak, *Analytic improper intersections I: deformation to the normal cone*, Bull. Polish Acad. Sci. Math. 48 (2000), 121–130.

[N_2] —, *Analytic improper intersections II: deformation to an algebraic bicone and applications*, ibid., 131–140.

[N_3] —, *Improper intersections in complex analytic geometry*, Dissertationes Math. 391 (2001).

[N_4] —, *Supplement to the paper: "Improper intersections in complex analytic geometry"*, Ann. Polon. Math. 76 (2001), 303.

[R] S. Rams, *On the index of contact and multiplicities for bigraded rings*, Manuscripta Math. 106 (2001), 339–347.

[SV] J. Stückrad and W. Vogel, *An algebraic approach to the intersection theory*, in: The Curves Seminar at Queens, Vol. II, Queen's Papers in Pure and Appl. Math. 61, Kingston, ON, 1982, 1–32.

[T_1] P. Tworzewski, *Isolated multiplicity and regular separation of analytic sets*, Ann. Polon. Math. 58 (1993), 213–219.

[T_2] —, *Intersection theory in complex analytic geometry*, ibid. 62 (1995), 177–191.

Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
I-40126 Bologna, Italy
E-mail: alonso@dm.unibo.it

Reçu par la Rédaction le 14.8.2003
Révisé le 8.10.2003 (1461)