Infinite systems of first order PFDEs with mixed conditions

by W. Czernous (Gdańsk)

Abstract. We consider mixed problems for infinite systems of first order partial functional differential equations. An infinite number of deviating functions is permitted, and the delay of an argument may also depend on the spatial variable. A theorem on the existence of a solution and its continuous dependence upon initial boundary data is proved. The method of successive approximations is used in the existence proof. Infinite differential systems with deviated arguments and differential integral systems can be derived from the general model by specializing the operators.

1. Introduction. For any normed linear spaces X and Y and for $M \subset X$, let $C(M, Y)$ denote the class of all continuous functions from M into Y. Inequalities between vectors will be understood componentwise.

Let $a > 0$, $h_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$, $h = (h_1, \ldots, h_n)$, and $b = (b_1, \ldots, b_n)$ $\in \mathbb{R}^n$ be given where $b_j > 0$ for $1 \leq j \leq n$. We define

$$E = [0, a] \times [-b, b], \quad D = [-h_0, 0] \times [-h, h].$$

Let $\bar{c} = (c_1, \ldots, c_n) = b + h$ and

$$E_0 = [-h_0, 0] \times [-\bar{c}, \bar{c}], \quad \partial_0 E = [0, a] \times ([-\bar{c}, \bar{c}] \setminus (-b, b)),$$
$$E^* = E_0 \cup E \cup \partial_0 E, \quad \Omega = E \times C(D, X) \times \mathbb{R}^n.$$

Moreover, for $\tau \in (0, a]$ and a set $U \subset \mathbb{R}^{1+n}$ we set

$$U_\tau = U \cap ((-\infty, \tau] \times \mathbb{R}^n).$$

Suppose that $z : E^* \to \mathbb{R}$ and $(t, x) \in E$ are fixed. We define the function $z_{(t,x)} : D \to \mathbb{R}$ by

$$z_{(t,x)}(\zeta, \xi) = z(t + \zeta, x + \xi), \quad (\zeta, \xi) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to $[t - h_0, t] \times [x - h, x + h]$ and this restriction is shifted to D.

Let $\mathcal{S}$ be an arbitrary set of indices and let

$$X = \{p = \{p_s\}_{s\in\mathcal{S}} : p_s \in \mathbb{R} \text{ for } s \in \mathcal{S} \text{ and } \|p\| = \sup\{|p_s| : s \in \mathcal{S}\} < \infty\}.$$

We will denote by $M_{m\times k}$ the class of all $m \times k$ real matrices. For $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $A = [a_{ij}]_{i=1,\ldots,m,\, j=1,\ldots,k} \in M_{m\times k}$ we put

$$\|y\| = \sum_{j=1}^{m} |y_j| \quad \text{and} \quad \|A\| = \max_{1\le j\le k} \sum_{i=1}^{m} |a_{ij}|.$$

The product of two matrices is denoted by "$*$", and the scalar product in $\mathbb{R}^m$ by "$\circ$". If $A \in M_{m\times k}$, then A^T is the transpose matrix. We next define

$$X^m = \{y = \{y_s\}_{s\in\mathcal{S}} : y_s \in \mathbb{R}^m,\ s \in \mathcal{S};\ \|y\| = \sup\{\|y_s\| : s \in \mathcal{S}\} < \infty\}$$

and

$$X^{m\times k} = \{A = \{A_s\}_{s\in\mathcal{S}} : A_s \in M_{m\times k},\ s \in \mathcal{S};$$
$$\|A\| = \sup\{\|A_s\| : s \in \mathcal{S}\} < \infty\}.$$

For $\eta \in X^m$ we will write $(\eta_{1,s}, \ldots, \eta_{m,s}) = \eta_s$. Let $\|\cdot\|_\xi$, $\|\cdot\|_{(\xi)}$, $\|\cdot\|_D$ denote the supremum norms in $C(E_\xi^*, X)$, $C(E_\xi, X^m)$, $C(D, X)$, respectively. For a domain $U \subset \mathbb{R}^{1+n}$ and for functions $z : U \to X$, $u : U \to \mathbb{R}^n$ of the variables (t, x) we will write

$$\partial_t z = \{\partial_t z_s\}_{s\in\mathcal{S}}, \quad \partial_x z = \{\partial_x z_s\}_{s\in\mathcal{S}} = \{(\partial_{x_1} z_s, \ldots, \partial_{x_n} z_s)\}_{s\in\mathcal{S}},$$
$$\partial z = (\partial_t z, \partial_x z), \quad \partial_t u = (\partial_t u_1, \ldots, \partial_t u_n)^T, \quad \partial_x u = [\partial_{x_j} u_i]_{i,j=1,\ldots,n},$$

provided that the derivatives exist.

Let

$$f : E^* \times C(D, X) \times \mathbb{R}^n \to X, \quad \varphi : E_0 \cup \partial_0 E \to X,$$
$$\alpha_0 = \{\alpha_{0,s}\}_{s\in\mathcal{S}} : E \to X, \quad \alpha' : E \to X^n$$

be given functions. We write $\alpha_s = (\alpha_{0,s}, \alpha'_s)$. For a function $z : E^* \to X$ and a point $(t, x) \in E$, we write

$$z_{\alpha(t,x)} = \{(z_s)_{\alpha_s(t,x)}\}_{s\in\mathcal{S}}.$$

We consider the system of functional differential equations

(1) $$\partial_t z_s(t, x) = f_s(t, x, z_{\alpha(t,x)}, \partial_x z_s(t, x)), \quad s \in \mathcal{S},$$

with the initial boundary condition

(2) $$z(t, x) = \varphi(t, x) \quad \text{on } E_0 \cup \partial_0 E.$$

For $\xi \in \mathbb{R}$, $0 < \xi \le a$, we define a classical solution $\widetilde{z} : E_\xi^* \to X$ of the system (1), (2) to be a continuous function satisfying the system (1) on E, the condition (2), and having the derivatives $\partial_t \widetilde{z}_s$, $\partial_x \widetilde{z}_s$, $s \in \mathcal{S}$, at every point of E. In fact, higher regularity of solutions is proved in this paper (see p. 213 for the definition of the relevant function space).

Note that our hereditary setting contains well known delay structures as particular cases.

EXAMPLE 1.1. Suppose that $\widetilde{f} : E \times X \times \mathbb{R}^n \to X$ is a given function. Set $f_s(t, x, w, q) = \widetilde{f}_s(t, x, w(0, 0), q)$, $s \in \mathcal{S}$, then

$$f_s(t, x, z_{\alpha(t,x)}, \partial_x z_s(t, x)) = \widetilde{f}_s(t, x, z(\alpha(t, x)), \partial_x z_s(t, x)), \quad s \in \mathcal{S},$$

and (1) becomes a system with deviated variables.

EXAMPLE 1.2. For a function $w \in C(D, X)$ and a measurable set $B \subset D$ write $\int_B w(\xi, y) \, dy \, d\xi = \{\int_B w_s(\xi, y) \, dy \, d\xi\}_{s \in \mathcal{S}}$. For the above $\widetilde{f}$ we put

$$f_s(t, x, w, q) = \widetilde{f}_s\Big(t, x, \int_B w(\xi, y) \, dy \, d\xi, q\Big), \quad s \in \mathcal{S}.$$

Then

$$(3) \quad f_s(t, x, z_{\alpha(t,x)}, \partial_x z_s(t, x)) = \widetilde{f}_s\Big(t, x, \int_B z_{\alpha(\xi,y)} \, dy \, d\xi, \partial_x z_s(t, x)\Big), \quad s \in \mathcal{S},$$

and (1) becomes a differential integral system.

We will discuss the existence of solutions of problem (1), (2).

In recent years, numerous papers have been published, concerning first order partial functional differential equations. The following subjects have been investigated: functional differential inequalities and their applications, uniqueness of solutions to initial or initial boundary value problems, existence of classical or generalized solutions, numerical methods for functional differential equations. It is not our aim to give a full review of papers concerning the above problems. We just mention some results on the existence of solutions and cite only a few items of the vast literature.

Nonlinear evolutionary equations with first order partial derivatives have the following property: any classical solutions to initial or initial boundary value problem exist locally with respect to the time variable. This leads in a natural way to weak or generalized solutions.

Continuous solutions satisfying integral systems obtained by integrating original equations along bicharacteristics are considered in [12]. The Schauder fixed point theorem is used to prove existence results. That paper initiated the theory of first order partial functional differential equations.

Existence results for initial value problems for equations with deviated variables can be found in [1]. The Baiada method is applied there and the unknown functions depend on two variables. The paper [6] brings an existence result for quasilinear differential integral equations. The proof is based on the method of bicharacteristics.

Carathéodory solutions of quasilinear differential functional systems with initial or initial boundary conditions are investigated in [5], [16]. The method of bicharacteristics and functional integral inequalities are used in existence

proofs. Weak solutions in the Cinquini Cibrario sense are studied in [3], [11]. Existence results for nonlinear equations are obtained by using the method of quasilinearization. It consists in constructing a quasilinear system for the unknown function and for its spatial derivatives. This system is then transformed into a system of integral functional equations of Volterra type. Continuous solutions of this integral problem lead to weak solutions of the original differential problem. Existence of Carathéodory solutions for nonlinear equations is considered in [4]; the constructive proof is based on the finite difference method. Classical solutions of nonlinear functional differential problems are considered in [2] and [8] (Chapter 2). These studies involve the method of successive approximations, introduced in [17] for systems without functional dependence. On each step of this method, differential problems without functional dependence arise. The existence of a sequence of successive approximations follows from a classical theorem. Partial differential inequalities are used in the proof of convergence of the successive approximations.

The work [14] initiated the theory of infinite systems of first order partial differential functional equations. Sufficient conditions are given in [14] for the existence of classical solutions to a generalized Cauchy problem

$$(4) \qquad \begin{cases} \partial_t z_i(t,x) = G_i(t,x,z,\partial_x z_i(t,x)), \\ z_i(a_i,x) = \chi_i(x) \quad \text{for } x \in \mathbb{R}^n, \end{cases} \quad i = 0,1,2,\dots.$$

The variable z represents the functional argument in the system. The proof of the existence of solution is based on the following idea. A set X_c is constructed, which is a closed subset of the Banach space consisting of sequences $z = \{z_i\}_{i=0}^{\infty}$ of the bounded continuous functions $z_i : [-c,c] \times \mathbb{R}^n \to \mathbb{R}$. For $u \in X_c$, consider the classical Cauchy problems

$$(5) \quad \partial_t z_i(t,x) = G_i(t,x,u,\partial_x z_i(t,x)), \qquad z_i(a_i,x) = \chi_i(x), \qquad x \in \mathbb{R}^n,$$

where $i = 0,1,\dots$. Let us denote by $Tu = \{T_i u\}_{i=0}^{\infty}$ the solution of (5). The set X_c has the following property: $Tu \in X_c$ and T has exactly one fixed point $\widetilde{u} \in X_c$. This $\widetilde{u}$ is a classical solution of (5).

It is clear that the result from [14] can be extended to initial boundary value problems. The above existence result can be characterized as follows: the theorem has simple assumptions and the proof is very natural. Unfortunately, only a small class of problems are covered by this theorem. It is not applicable to differential problems with deviated variables nor to differential integral systems with right-hand sides given by (3).

Note that we have different models of functional dependence in (1) and (3). There are existence results for nonlinear problems ([2], [13], [16]) where the right-hand sides of the equations are superpositions of functions defined on a finite-dimensional Euclidean space and operators of Volterra

type. The main assumptions in the existence theorems concern the operators of Volterra type, and are inequalities for norms in function spaces. Comparisons between different models of functional dependence are presented in [7].

Functional differential inequalities generated by infinite systems are investigated in [9], [15]. Uniqueness of solutions and continuous dependence on given functions are consequences of comparison theorems.

For further bibliography on partial differential functional equations and applications, see the monographs [8], [18].

This paper continues the author's study [3] of hyperbolic differential-functional equations, and generalizes some results of [10]. We now consider a more general form of functional dependence. Our approach admits equations with deviated argument, where the delay depends on all variables, not only on t, as was the case in former studies.

The paper is organized as follows. The notion of bicharacteristics for problem (1), (2) and their properties are presented in Section 2. Then the initial boundary value problem is transformed into a system of integral functional equations. This system is solved in Section 3 by the method of successive approximations. The main existence result and continuous dependence of solutions on the initial boundary functions are proved in Section 4.

2. Bicharacteristics. The following function spaces will be needed in our considerations.

Given $s = (s_1, s_2) \in \mathbb{R}^2_+$, we denote by $C^{1.L}[s]$ the set of all functions $\varphi \in C(E_0 \cup \partial_0 E, X)$ such that

$$|\partial_t \varphi(t, x)| + \|\partial_x \varphi(t, x)\| \leq s_1,$$

$$|\partial_t \varphi(t, x) - \partial_t \varphi(\bar{t}, \bar{x})| + \|\partial_x \varphi(t, x) - \partial_x \varphi(\bar{t}, \bar{x})\| \leq s_2[|t - \bar{t}| + \|x - \bar{x}\|]$$

on E_c^*.

For fixed $\varphi \in C^{1.L}[s]$, $0 < c \leq a$, and $d \in \mathbb{R}_+$ with $d \geq s_1$, we consider the space $C^L_{\varphi.c}[d]$ of all $z \in C(E_c^*, X)$ such that $z(t, x) = \varphi(t, x)$ on $E_0 \cup \partial_0 E$ and

$$\|z(t, x) - z(\bar{t}, \bar{x})\| \leq d[|t - \bar{t}| + \|x - \bar{x}\|] \quad \text{on } E_c^*.$$

Let $p = (p_0, p_1) \in \mathbb{R}^2_+$ with $p_0 \geq s_1$, $p_1 \geq s_2$. We denote by $C^L_{\varphi.c}[p; 1]$ the class of all functions $u_0 : E_c^* \to X$ such that $u_0 = \partial_t \varphi$ on $E_0 \cup \partial_0 E$ and

$$\|u_0(t, x)\| \leq p_0 \quad \text{and} \quad \|u_0(t, x) - u_0(\bar{t}, \bar{x})\| \leq p_1[|t - \bar{t}| + \|x - \bar{x}\|]$$

on E_c.

Similarly, for such p, we denote by $C^L_{\varphi.c}[p; n]$ the class of all functions $u : E_c^* \to X^n$ such that $u = \partial_x \varphi$ on $E_0 \cup \partial_0 E$ and

$\|u(t,x)\| \leq p_0$ and $\|u(t,x) - u(\bar{t}, \bar{x})\| \leq p_1[|t - \bar{t}| + \|x - \bar{x}\|]$ on E_c.

Write $C^L_{\varphi.c}[p; 1+n] = C^L_{\varphi.c}[p; 1] \times C^L_{\varphi.c}[p; n]$. We prove that, under suitable assumptions on f, α and φ and for sufficiently small c, there exists a solution $\bar{z}$ of problem (1), (2) such that $\bar{z} \in C^L_{\varphi.c}[d]$ and $\partial \bar{z} \in C^L_{\varphi.c}[p; 1 + n]$.

We begin with assumptions on f. For $1 \leq j \leq n$, define

$$\Delta^{(j)}_+ = \{(t,x) \in E : x_j = b_j\}, \quad \Delta^{(j)}_- = \{(t,x) \in E : x_j = -b_j\},$$

$$\Delta = \bigcup_{j=1}^{n}(\Delta^{(j)}_+ \cup \Delta^{(j)}_-).$$

ASSUMPTION H[$\partial_q f$]. The functions $f_s : \Omega \to \mathbb{R}$, $s \in \mathcal{S}$, of variables (t, x, w, q) are continuous and satisfy the following conditions:

1) the partial derivatives

$$(\partial_{q_1} f_s(P), \ldots, \partial_{q_n} f_s(P)) = \partial_q f_s(P), \quad P = (t, x, w, q), \quad s \in \mathcal{S},$$

exist for $(t, x, w, q) \in \Omega$;

2) the functions $\partial_q f_s : \Omega \to \mathbb{R}^n$, $s \in \mathcal{S}$, are continuous and there are B, $L \in \mathbb{R}_+$ such that

$$\|\partial_q f(t, x, w, q)\| \leq B \quad \text{on } \Omega$$

and

$$\|\partial_q f(t, x, w, q) - \partial_q f(t, \bar{x}, \bar{w}, \bar{q})\| \leq L[\|x - \bar{x}\| + \|w - \bar{w}\|_D + \|q - \bar{q}\|];$$

3) there is $\kappa > 0$ such that for $1 \leq j \leq n$ and $s \in \mathcal{S}$ we have

$$\partial_{q_j} f_s(t, x, w, q) \geq 2\kappa \quad \text{on } [0, a] \times \Delta^{(j)}_+ \times C(D, X) \times \mathbb{R}^n,$$

$$\partial_{q_j} f_s(t, x, w, q) \leq -2\kappa \quad \text{on } [0, a] \times \Delta^{(j)}_- \times C(D, X) \times \mathbb{R}^n.$$

ASSUMPTION H[α]. The functions $\alpha_s : E \to E$, $s \in \mathcal{S}$, satisfy the following conditions:

1) $0 \leq \alpha_{0,s}(t,x) \leq t$ for $s \in \mathcal{S}$ and there is $r_0 \in \mathbb{R}_+$ such that

$$\|\alpha(t,x) - \alpha(\bar{t}, \bar{x})\| \leq r_0[|t - \bar{t}| + \|x - \bar{x}\|];$$

2) the derivatives $\partial_x \alpha_{0,s}$, $\partial_x \alpha'_s$, $s \in \mathcal{S}$, exist on E and there is $r_1 \in \mathbb{R}_+$ such that

$$\|\partial_x \alpha(t,x) - \partial_x \alpha(t, \bar{x})\| \leq r_1 \|x - \bar{x}\|,$$

where $\partial_x \alpha = \{\partial_x \alpha_s\}_{s \in \mathcal{S}}$.

Suppose that $\varphi \in C^{1.L}[s]$, $z \in C^L_{\varphi.c}[d]$, $u \in C^L_{\varphi.c}[p; n]$. Fix $s \in \mathcal{S}$ and consider the Cauchy problem

(6) $\eta'(\tau) = -\partial_q f_s(\tau, \eta(\tau), z_{\alpha(\tau, \eta(\tau))}, u_s(\tau, \eta(\tau))), \quad \eta(t) = x,$

and denote by $g_s[z, u_s](\cdot, t, x)$ its solution in the classical sense. The function $g_s[z, u_s](\cdot, t, x)$ is the sth bicharacteristic of system (1) corresponding to

(z, u_s). Let $I_{(t,x);\, s}$ be its domain, and let $\delta_s[z, u_s](t, x)$ denote the left end of the maximal interval on which the bicharacteristic is defined. The set of all bicharacteristics of (1) corresponding to $(z, u) \in C^L_{\varphi.c}[d] \times C^L_{\varphi.c}[p; n]$ is denoted by $g[z, u] = \{g_s[z, u_s]\}_{s \in \mathcal{S}}$.

Let

$$
\begin{aligned}
P_s[z, u_s]&(\tau, t, x) \\
&= (\tau, g_s[z, u_s](\tau, t, x), z_{\alpha(\tau, g_s[z, u_s](\tau, t, x))}, u_s(\tau, g_s[z, u_s](\tau, t, x))).
\end{aligned}
$$

We prove a lemma on bicharacteristics.

LEMMA 2.1. *Suppose that Assumptions* [H$\partial_q f$] *and* H[α] *are satisfied and* $\varphi, \bar{\varphi} \in C^{1.L}[s]$, $z \in C^L_{\varphi.c}[d]$, $\bar{z} \in C^L_{\bar{\varphi}.c}[d]$, $u \in C^L_{\varphi.c}[p; n]$, $\bar{u} \in C^L_{\bar{\varphi}.c}[p; n]$ *are given. Then for each* $s \in \mathcal{S}$ *the solutions* $g_s[z, u_s](\cdot, t, x)$ *and* $g_s[\bar{z}, \bar{u}_s](\cdot, t, x)$ *exist on the intervals* $I_{(t,x);\, s}$ *and* $\bar{I}_{(t,x);\, s}$, *respectively, such that, for* $\xi = \delta_s[z, u_s](t, x)$ *and* $\bar{\xi} = \delta_s[\bar{z}, \bar{u}_s](t, x)$, *we have* $g_s[z, u_s](\xi, t, x) \in \Delta$ *and* $g_s[\bar{z}, \bar{u}_s](\bar{\xi}, t, x) \in \Delta$. *The bicharacteristics are unique on* $I_{(t,x);\, s}$ *and* $\bar{I}_{(t,x);\, s}$. *Moreover,*

$$
\tag{7} \|g_s[z, u_s](\tau, t, x) - g_s[z, u_s](\tau, \bar{t}, \bar{x})\| \leq \overline{C}[|t - \bar{t}| + \|x - \bar{x}\|],
$$

$$
\tag{8} \|g_s[z, u_s](\tau, t, x) - g_s[\bar{z}, \bar{u}_s](\tau, t, x)\| \leq \overline{C}\Big| \int_t^\tau [\|z - \bar{z}\|_\xi + \|u - \bar{u}\|_{(\xi)}]\, d\xi \Big|,
$$

where $\overline{C} = \max\{1, B, L\} \exp(cL\bar{d})$ *and* $\bar{d} = 1 + r_0 d + p_1$. *Furthermore, the functions* $\delta_s[z, u_s](t, x)$ *and* $\delta_s[\bar{z}, \bar{u}_s](t, x)$ *are continuous on* E_c *and*

$$
\tag{9} |\delta_s[z, u_s](t, x) - \delta_s[z, u_s](\bar{t}, \bar{x})| \leq \widehat{C}[|t - \bar{t}| + \|x - \bar{x}\|],
$$

$$
\tag{10} |\delta_s[z, u_s](t, x) - \delta_s[\bar{z}, \bar{u}_s](t, x)| \leq \widehat{C} \int_0^t [\|z - \bar{z}\|_\xi + \|u - \bar{u}\|_{(\xi)}]\, d\xi,
$$

where $\widehat{C} = \overline{C}\kappa^{-1}$.

Proof. The existence and uniqueness of solutions of (6) follow from the classical theorems on the solutions of Cauchy problems. The function $g_s[z, u_s](\cdot, t, x)$ satisfies the integral equation

$$
\tag{11} g_s[z, u_s](\tau, t, x) = x - \int_t^\tau \partial_q f_s(P_s[z, u_s](\xi, t, x))\, d\xi.
$$

Since $z \in C^L_{\varphi.c}[d]$, condition 2) of Assumption H[α] shows that

$$
\|z_{\alpha(\tau, y)} - z_{\alpha(\tau, \bar{y})}\|_D \leq r_0 d\|y - \bar{y}\|
$$

for all $(\tau, y), (\tau, \bar{y}) \in E_c$. It follows from Assumption $H[\partial_q f]$ that

$$\|g_s[z, u_s](\tau, t, x) - g_s[z, u_s](\tau, \bar{t}, \bar{x})\| \leq \max\{1, B\}[|t - \bar{t}| + \|x - \bar{x}\|]$$
$$+ L\bar{d}\left|\int_t^\tau \|g_s[z, u_s](\xi, t, x) - g_s[z, u_s](\xi, \bar{t}, \bar{x})\| \, d\xi\right|$$

for $\tau \in I_{(t,x);\,s} \cap I_{(\bar{t},\bar{x});\,s}$. We now obtain (7) by the Gronwall inequality. For $(\tau, y), (\tau, \bar{y}) \in E_c$, we have

$$\|z_{\alpha(\tau,y)} - \bar{z}_{\alpha(\tau,\bar{y})}\|_D \leq \|z - \bar{z}\|_\tau + r_0 d\|y - \bar{y}\|.$$

It follows that

$$\|g_s[z, u_s](\tau, t, x) - g_s[\bar{z}, \bar{u}_s](\tau, t, x)\| \leq L\left|\int_t^\tau (\|z - \bar{z}\|_\xi + \|u - \bar{u}\|_{(\xi)}) \, d\xi\right|$$
$$+ L\bar{d}\left|\int_t^\tau \|g_s[z, u_s](\xi, t, x) - g_s[\bar{z}, \bar{u}_s](\xi, t, x)\| \, d\xi\right|,$$

for $\tau \in I_{(t,x);\,s} \cap \bar{I}_{(t,x);\,s}$, and we get (8) from the Gronwall inequality.

The continuity of $\delta_s[z, u_s]$ and $\delta_s[\bar{z}, \bar{u}_s]$ on E_c follows from the classical theorems on continuous dependence of Cauchy problems' solutions on initial conditions. We now prove (9). This estimate is obvious if $\delta_s[z, u_s](t, x) = \delta_s[z, u_s](\bar{t}, \bar{x}) = 0$ (i.e. the solutions of problem (6) are defined on $[0, t]$ and $[0, \bar{t}]$). Now assume that $0 \leq \delta_s[z, u_s](t, x) < \delta_s[z, u_s](\bar{t}, \bar{x})$. Then, for $\bar{\zeta} = \delta_s[z, u_s](\bar{t}, \bar{x})$, we have $g_s[z, u_s](\bar{\zeta}, \bar{t}, \bar{x}) \in \Delta$ and there exists i, $1 \leq i \leq n$, such that $|g_{i,s}(\bar{\zeta}, \bar{t}, \bar{x})| = b_i$. The following two cases are possible:

(i) $g_{i,s}(\bar{\zeta}, \bar{t}, \bar{x}) = b_i$;
(ii) $g_{i,s}(\bar{\zeta}, \bar{t}, \bar{x}) = -b_i$.

Consider case (i). Let $x = (x_1, \ldots, x_n)$, $\tilde{x} = (x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n)$. Then for $(t, x) \in E_c$,

$$(12) \quad |\partial_{q_i} f_s(t, x, z_{\alpha(t,x)}, u(t, x)) - \partial_{q_i} f_s(t, \tilde{x}, z_{\alpha(t,\tilde{x})}, u_s(t, \tilde{x}))| \leq \tilde{c}(b_i - x_i),$$

where $\tilde{c} = L\bar{d}$. Thus

$$\partial_{q_i} f_s(t, x, z_{\alpha(t,x)}, u_s(t, x)) \geq \kappa$$

for $(t, x) \in E_c$ such that $b_i - x_i \leq \kappa \tilde{c}^{-1}$. It follows from (7) that

$$b_i - g_{i,s}[z, u_s](\bar{\zeta}, t, x) = g_{i,s}[z, u_s](\bar{\zeta}, \bar{t}, \bar{x}) - g_{i,s}[z, u_s](\bar{\zeta}, t, x) \leq \kappa/\tilde{c}$$

for $(t, x), (\bar{t}, \bar{x}) \in E_c$ such that

$$(13) \qquad\qquad |t - \bar{t}| + \|x - \bar{x}\| \leq \kappa/\tilde{c}\overline{C}.$$

Hence

$$\partial_{q_i} f_s(\bar{\zeta}, g_s[z, u_s](\bar{\zeta}, t, x), z_{\alpha(\bar{\zeta}, g_s[z, u_s](\bar{\zeta}, t, x))}, u_s(\bar{\zeta}, g_s[z, u_s](\bar{\zeta}, t, x))) \geq \kappa > 0,$$

and consequently

$$\partial_t g_{i,s}[z, u_s](\delta_s[z, u_s](\bar{t}, \bar{x}), t, x) < 0$$

for $(t, x), (\bar{t}, \bar{x}) \in E_c$ satisfying (13). It is easy to see that $g_{i,s}(\cdot, t, x)$ is decreasing on the interval $(\delta_s[z, u_s](t, x), \delta_s[z, u_s](\bar{t}, \bar{x}))$. Therefore

$$b_i - g_{i,s}[z, u_s](\tau, t, x) \leq \kappa / \widetilde{c},$$

and

$$\partial_{q_i} f_s(P_s[z, u_s](\tau, t, x))$$
$$= \partial_{q_i} f_s(\tau, g_s[z, u_s](\tau, t, x), z_{\alpha(\tau, g_s[z, u_s](\tau, t, x))}, u_s(\tau, g_s[z, u_s](\tau, t, x))) \geq \kappa$$

for $\tau \in (\delta_s[z, u_s](t, x), \delta_s[z, u_s](\bar{t}, \bar{x}))$ and $(t, x), (\bar{t}, \bar{x}) \in E_c$ such that (13) is satisfied. Hence

$$-\kappa[\delta_s[z, u_s](\bar{t}, \bar{x}) - \delta_s[z, u_s](t, x)]$$
$$\geq - \int_{\delta_s[z,u_s](t,x)}^{\delta_s[z,u_s](\bar{t},\bar{x})} \partial_{q_i} f_s(P_s[z, u_s](\tau, t, x)) \, d\tau$$
$$= g_{i,s}[z, u_s](\delta_s[z, u_s](\bar{t}, \bar{x}), t, x) - g_{i,s}[z, u_s](\delta_s[z, u_s](t, x), t, x)$$
$$\geq g_{i,s}[z, u_s](\delta_s[z, u_s](\bar{t}, \bar{x}), t, x) - g_{i,s}[z, u_s](\delta_s[z, u_s](\bar{t}, \bar{x}), \bar{t}, \bar{x})$$
$$\geq -\overline{C}[|t - \bar{t}| + \|x - \bar{x}\|].$$

Thus, the proof of (9) for $(t, x), (\bar{t}, \bar{x}) \in E_c$ such that (13) holds is complete in case (i). In case (ii) we proceed in a similar way. Note that we have proved a local Lipschitz property of $\delta_s[z, u_s]$ with respect to (t, x) with a Lipschitz constant independent of (t, x). Hence $\delta_s[z, u_s]$ satisfies a global Lipschitz condition with respect to (t, x) with the same constant. That proves (9).

The proof of (10) is similar. ∎

3. Integral functional equations. We denote by $CL(D, X)$ the set of all linear and continuous real functions defined on $C(D, X)$ and we denote by $\| \cdot \|$ the norm in $CL(D, X)$ generated by the supremum norm $\| \cdot \|_D$ in the space $C(D, X)$.

ASSUMPTION H[f]. The Assumption H[$\partial_q f$] is satisfied and:

1) there is $B_0 \in \mathbb{R}_+$ such that $\|f(t, x, w, q)\| \leq B_0$ on Ω,
2) for each $s \in \mathcal{S}$ the partial derivatives $(\partial_{x_1} f_s(P), \ldots, \partial_{x_n} f_s(P)) = \partial_x f_s(P)$, $P = (t, x, w, q)$, and the Fréchet derivative $\partial_w f_s(P) \in CL(D, X)$ exist on Ω,
3) the estimates $\|\partial_x f(t, x, w, q)\| \leq B$ and

$$\|\partial_w f(t, x, w, q)\| = \sup\{\|\partial_w f_s(t, x, w, q)\| : s \in \mathcal{S}\} \leq B$$

are satisfied on Ω, where $\partial_x f = \{\partial_x f_s\}_{s \in \mathcal{S}}$, $\partial_w f = \{\partial_w f_s\}_{s \in \mathcal{S}}$,

4) there is $B_1 \in \mathbb{R}_+$ such that for $t, \bar{t} \in [0, a]$ and $(x, w, q) \in \mathbb{R}^n \times C(D, X) \times \mathbb{R}^n$ we have

$$\|f(t, x, w, q) - f(\bar{t}, x, w, q)\| \leq B_1 |t - \bar{t}|,$$

5) for $(t, x, w, q) \in \Omega$ and $\bar{w} \in C(D, X)$ we have

$$\|\partial_x f(t, x, w, q) - \partial_x f(t, \bar{x}, \bar{w}, \bar{q})\| \leq L[\|x - \bar{x}\| + \|w - \bar{w}\|_D + \|q - \bar{q}\|],$$
$$\|\partial_w f(t, x, w, q) - \partial_w f(t, \bar{x}, \bar{w}, \bar{q})\| \leq L[\|x - \bar{x}\| + \|w - \bar{w}\|_D + \|q - \bar{q}\|].$$

REMARK 3.1. We will give a theorem on the existence of solutions of problem (1), (2). For simplicity of formulation, we have assumed the same estimates for the derivatives $\partial_x f$, $\partial_w f$ and $\partial_q f$. We have also assumed the Lipschitz condition for these derivatives with the same coefficient.

We now exhibit a system of integral equations generated by (1), (2). We write

$$Q_s[z, u_s](t, x) = (\delta_s[z, u_s](t, x), g_s[z, u_s](\delta_s[z, u_s](t, x), t, x)),$$
$$\Phi_s[z, u_s](t, x) = \varphi_s(Q[z, u_s](t, x)),$$
$$\Psi_s[z, u_s](t, x) = \partial_x \varphi_s(Q[z, u_s](t, x)),$$
$$\overline{W}_s[u_{0,s}, u_s](t, x) = (u_{0,s}, u_s)_{\alpha_s(t,x)} * \partial_x \alpha_s(t, x),$$
$$\overline{W}[u_0, u](t, x) = \{\overline{W}_s[u_{0,s}, u_s](t, x)\}_{s \in \mathcal{S}},$$
$$W_{\{s\}}[z, u_0, u](\tau, t, x) = \overline{W}[u_0, u](\tau, g_s[z, u_s](\tau, t, x)) \in C(D, X).$$

For $u \in C(D, X^n)$, $P \in \Omega$ and $s \in \mathcal{S}$, we set

$$\partial_w f_s(P) u = (\partial_w f_s(P) u_1, \ldots, \partial_w f_s(P) u_n).$$

Given $\varphi \in C^{1.L}[s]$, $z \in C^L_{\varphi.c}[d]$ and $u_0 \in C^L_{\varphi.c}[p; 1]$, $u \in C^L_{\varphi.c}[p; n]$, where $0 < c \leq a$, we define

$$(14) \qquad F_s[z, u_s](t, x) = \Phi_s[z, u_s](t, x)$$

$$+ \int_{\delta_s[z,u_s](t,x)}^{t} \Big[f_s(P_s[z, u_s](\tau, t, x))$$

$$- u(\tau, g_s[z, u_s](\tau, t, x)) \circ \partial_q f_s(P_s[z, u_s](\tau, t, x)) \Big] d\tau,$$

$$(15) \quad G_s[z, u_0, u](t, x) = \Psi_s[z, u_s](t, x)$$

$$+ \int_{\delta_s[z,u_s](t,x)}^{t} \Big[\partial_x f_s(P_s[z, u_s](\tau, t, x))$$

$$+ \partial_w f_s(P_s[z, u_s](\tau, t, x)) W_{\{s\}}[z, u_0, u](\tau, t, x) \Big] d\tau,$$

for $s \in \mathcal{S}$. Write

$$F[z, u] = \{F_s[z, u_s]\}_{s \in \mathcal{S}} \quad \text{and} \quad G[z, u_0, u] = \{G_s[z, u_0, u]\}_{s \in \mathcal{S}}.$$

Consider the system of integral functional equations

(16) $$z(t,x) = F[z,u](t,x), \quad u(t,x) = G[z,u_0,u](t,x),$$

where

(17) $$u_{0,s}(t,x) = f_s(t,x,z_{\alpha(t,x)},u_s(t,x)), \quad s \in \mathcal{S},$$

with the initial boundary condition

(18) $$z = \varphi, \quad (u_0,u) = \partial\varphi \quad \text{on } E_0 \cup \partial_0 E.$$

REMARK 3.2. The integral functional system (16)–(18) is obtained in the following way. We first introduce an additional unknown function u, where $u = \partial_x z$. Then we consider the linearization of (1) with respect to u, i.e.

(19) $$\partial_t z_s(t,x) = f_s(U_s) + \partial_q f_s(U_s) \circ (\partial_x z_s(t,x) - u_s(t,x)), \quad s \in \mathcal{S},$$

where $U_s = (t,x,z_{\alpha(t,x)},u_s(t,x))$. By virtue of equation (1) we get a differential system for the unknown function u:

(20) $$[\partial_t u_s(t,x)]^T = \partial_x f_s(U_s) + \partial_q f_s(U_s) * [\partial_x u_s(t,x)]^T$$
$$+ \partial_w f(U_s)\overline{W}[\partial_t z,\partial_x z](t,x), \quad s \in \mathcal{S},$$

where $\partial_t z = \{\partial_t z_s\}_{s\in\mathcal{S}}$, $\partial_x z = \{\partial_x z_s\}_{s\in\mathcal{S}}$. Finally, we put $\partial_t z = u_0$ and $\partial_x z = u$ in (20). System (19), (20) has the following property: the differential equations of bicharacteristics for (19) and for (20) are the same and they have the form (6).

If we consider (19), (20) along the bicharacteristics $g_s[z,u_s](\cdot,t,x)$, we obtain, for $s \in \mathcal{S}$,

$$\frac{d}{d\tau}z_s(\tau,g_s[z,u_s](\tau,t,x))$$
$$= f_s(P_s[z,u_s](\tau,t,x)) - \partial_q f_s(P_s[z,u_s](\tau,t,x)) \circ u_s(\tau,g_s[z,u_s](\tau,t,x))$$

and

$$\frac{d}{d\tau}u_s(\tau,g_s[z,u_s](\tau,t,x))$$
$$= \partial_x f_s(P_s[z,u_s](\tau,t,x)) + \partial_w f_s(P_s[z,u_s](\tau,t,x))W_{\{s\}}[z,u_0,u](\tau,t,x).$$

By integrating the above equations on $[\delta_s[z,u_s](t,x),t]$ with respect to τ, we get (16). Again by virtue of equation (1), we get the formula (17) for u_0.

The existence results for (16)–(18) are based on the following method of successive approximations. Assume that $\varphi \in C^{1.L}[s]$ and Assumption H[f] is satisfied. We define a sequence $\{z^{(m)},u_0^{(m)},u^{(m)}\}$ in the following way: Write $(z,u_0,u)^{(m)} = (z^{(m)},u_0^{(m)},u^{(m)})$ and $\mathcal{C} = C_{\varphi.c}^L[d] \times C_{\varphi.c}^L[p;1] \times C_{\varphi.c}^L[p;n]$. Let $(z,u_0,u)^{(0)} \in \mathcal{C}$ be arbitrary differentiable functions such that $\partial z^{(0)} = (u_0^{(0)},u^{(0)})$ on E_c. Next, if $(z,u_0,u)^{(m)} \in \mathcal{C}$ are known functions, then $u^{(m+1)}$

is a solution of the equation

$$u = G^{(m)}[u] \tag{21}$$

and

$$z^{(m+1)}(t,x) = F[z^{(m)}, u^{(m+1)}](t,x) \\
u_{0,s}^{(m+1)}(t,x) = f_s(t,x, z_{\alpha(t,x)}^{(m)}, u_s^{(m+1)}(t,x)), \quad s \in \mathcal{S} \Big\} \quad \text{on } E_c, \tag{22}$$

$$z^{(m+1)}(t,x) = \varphi(t,x) \\
u_0^{(m+1)}(t,x) = \partial_t \varphi(t,x) \Big\} \quad \text{on } E_0 \cup \partial_0 E_c, \tag{23}$$

where $G^{(m)} = \{G_s^{(m)}\}_{s \in \mathcal{S}}$ is defined by

$$G_s^{(m)}[u](t,x) = \Psi_s[z^{(m)}, u_s](t,x) \tag{24}$$

$$+ \int_{\delta_s[z^{(m)}, u_s](t,x)}^{t} \Big[\partial_x f_s(P_s[z^{(m)}, u_s](\tau, t, x))$$

$$+ \partial_w f_s(P_s[z^{(m)}, u_s](\tau, t, x)) W_{\{s\}}^{(m)}[u_s](\tau, t, x) \Big] d\tau$$

for $(t,x) \in E_c$, $s \in \mathcal{S}$, and by

$$G^{(m)}[u](t,x) = \partial_x \varphi(t,x) \tag{25}$$

for $(t,x) \in E_0 \cup \partial_0 E$, and where

$$W_{\{s\}}^{(m)}[u_s](\tau, t, x) = \overline{W}[u_0^{(m)}, u^{(m)}](\tau, g_s[z^{(m)}, u_s](\tau, t, x)) \in C(D, X).$$

We wish to emphasize that the main difficulty in carrying out this construction is to prove the existence of the sequence $\{z^{(m)}, u_0^{(m)}, u^{(m)}\}$.

We set $\bar{p} = 1 + np_0 r_0$, $\bar{\omega} = p_1 r_0^2 + p_0 r_1$, $\bar{Q} = \overline{C} + \widehat{C}(1 + B)$ and $A_0 = B\bar{p}\widehat{C} + c\overline{C}[L\bar{d}\bar{p} + n\bar{\omega}B]$.

ASSUMPTION H[c, d, p]. The constants c, d and $p = (p_0, p_1)$ satisfy the conditions

$$d = p_0 \geq \max\{s_1 + cB\bar{p}, B_0\},$$
$$p_1 \geq \max\{B\bar{p} + A_0 + s_2 \bar{Q}, B_1 + B\bar{d}\}.$$

It is easy to see that there are d, p, and $c \in (0, a]$ sufficiently small that satisfy Assumption H[c, d, p].

LEMMA 3.3. *If $\varphi \in C^{1.L}[s]$ and Assumptions* H[α], H[f] *and* H[c, d, p] *are satisfied, then*

$$G^{(m)} : C_{\varphi.c}^{L}[p; n] \to C_{\varphi.c}^{L}[p; n].$$

Proof. Suppose that $\varphi \in C^{1.L}[s]$, $(z, u_0, u)^{(m)} \in \mathcal{C}$ and $u \in C_{\varphi.c}^{L}[p; n]$. Then

$$\|G^{(m)}[u](t,x)\| \leq s_1 + cB\bar{p} \quad \text{on } E_c. \tag{26}$$

We now prove that the function $G^{(m)}[u]$ satisfies the Lipschitz condition with constant p_1. If $(t, x), (\bar{t}, \bar{x}) \in E_c$, then for each $s \in \mathcal{S}$,

$$\|G_s^{(m)}[u](t, x) - G_s^{(m)}[u](\bar{t}, \bar{x})\| \le A_\varphi + A_f,$$

where

$$A_\varphi = \|\partial_x \varphi_s(Q_s[z^{(m)}, u_s](t, x)) - \partial_x \varphi_s(Q_s[z^{(m)}, u_s](\bar{t}, \bar{x}))\|,$$

$$A_f = \Big\| \int\limits_{\delta_s[z^{(m)}, u_s](t, x)}^{t} \big[\partial_x f_s(P_s[z^{(m)}, u_s](\tau, t, x))$$

$$+ \partial_w f_s(P_s[z^{(m)}, u_s](\tau, t, x)) W_s^{(m)}[u_s](\tau, t, x) \big] d\tau$$

$$- \int\limits_{\delta_s[z^{(m)}, u_s](\bar{t}, \bar{x})}^{\bar{t}} \big[\partial_x f_s(P_s[z^{(m)}, u_s](\tau, \bar{t}, \bar{x}))$$

$$+ \partial_w f_s(P_s[z^{(m)}, u_s](\tau, \bar{t}, \bar{x})) W_s^{(m)}[u_s](\tau, \bar{t}, \bar{x}) \big] d\tau \Big\|.$$

It follows from Lemma 2.1 that

$$A_\varphi \le s_2 \bar{Q}[|t - \bar{t}| + \|x - \bar{x}\|], \qquad A_f \le B\bar{p}|t - \bar{t}| + A_0[|t - \bar{t}| + \|x - \bar{x}\|].$$

Hence, using Assumption H$[c, d, p]$, we get

$$\|G^{(m)}[u](t, x) - G^{(m)}[u](\bar{t}, \bar{x})\| \le p_1[|t - \bar{t}| + \|x - \bar{x}\|]$$

on E_c. This inequality, together with (26) and (25), yields Lemma 3.3. ∎

LEMMA 3.4. *Suppose that $\varphi \in C^{1.L}[s]$ and Assumptions H$[\alpha]$, H$[f]$, H$[c, d, p]$ are satisfied. Then there exists exactly one function $\bar{u} \in C_{\varphi.c}^{L}[p; n]$ satisfying the equation $u = G^{(m)}[u]$.*

Proof. Lemma 3.3 shows that $G^{(m)} : C_{\varphi.c}^{L}[p; n] \to C_{\varphi.c}^{L}[p; n]$. It follows that there is $A > 0$ such that, for $u, \bar{u} \in C_{\varphi.c}^{L}[p; n]$, we have

$$(27) \qquad \|G^{(m)}[u](t, x) - G^{(m)}[\bar{u}](t, x)\| \le A \int\limits_0^t \|u - \bar{u}\|_{(\tau)} \, d\tau$$

on E_c. We now define the norm in the space $C(E_c^*, X^n)$ as follows:

$$\|u\|_\lambda = \max\{\|u(t, x)\| e^{-\lambda t} : (t, x) \in E_c^*\},$$

where $\lambda > A$. It is easy to see that $(C(E_c^*, X^n), \|\cdot\|_\lambda)$ is a Banach space and $C_{\varphi.c}^{L}[p; n]$ is its closed subset. Let us prove that there exists $q \in (0, 1)$ such that

$$(28) \qquad \|G^{(m)}[u] - G^{(m)}[\bar{u}]\|_\lambda \le q\|u - \bar{u}\|_\lambda \quad \text{for } u, \bar{u} \in C_{\varphi.c}^{L}[p; n].$$

Since, from (25), $\|G^{(m)}[u](t,x) - G^{(m)}[\bar{u}](t,x)\|e^{-\lambda t}$ vanishes on $E_0 \cup \partial_0 E_c$, it is sufficient to estimate it on E_c. According to (27), we have

$$\|G^{(m)}[u](t,x) - G^{(m)}[\bar{u}](t,x)\| \leq A \int_0^t \|u - \bar{u}\|_{(\xi)}\, d\xi$$

$$= A \int_0^t \|u - \bar{u}\|_{(\xi)} e^{-\lambda\xi} e^{\lambda\xi}\, d\xi \leq A\|u - \bar{u}\|_\lambda \int_0^t e^{\lambda\xi}\, d\xi$$

$$= \frac{A}{\lambda}\|u - \bar{u}\|_\lambda(e^{\lambda t} - 1) \leq \frac{A}{\lambda}\|u - \bar{u}\|_\lambda e^{\lambda t}$$

for $(t,x) \in E_c$. Hence

$$\|G^{(m)}[u](t,x) - G^{(m)}[\bar{u}](t,x)\|e^{-\lambda t} \leq \frac{A}{\lambda}\|u - \bar{u}\|_\lambda, \quad (t,x) \in E_c.$$

It follows that estimate (28) holds with $q = A\lambda^{-1}$. By the Banach fixed-point theorem, there exists exactly one $\bar{u} \in C^L_{\varphi.c}[p;n]$ satisfying the equation $u = G^{(m)}[u]$. ∎

We now prove the main lemma in this section.

LEMMA 3.5. *If $\varphi \in C^{1.L}[s]$ and Assumptions* H[α], H[f] *and* H[c,d,p] *are satisfied, then, for any $m \geq 0$, we have*

$$(29) \qquad \partial z^{(m)}(t,x) = (u_0^{(m)}, u^{(m)})(t,x) \quad on\ E_c$$

and $z^{(m)} \in C^L_{\varphi.c}[d]$, $u_0^{(m)} \in C^L_{\varphi.c}[p;1]$.

Proof. We prove (29) by induction. It follows from the definition of the sequence $\{(z,u_0,u)^{(m)}\}$ that (29) is satisfied for $m = 0$. Assuming that it holds for a given $m \geq 0$, we prove that $z^{(m+1)}$, $u_0^{(m)}$ given by (22) satisfy (29). We fix $s \in \mathcal{S}$ and set

$$(30) \quad \Delta^{(m)}(t,\bar{t},x,\bar{x}) = z_s^{(m+1)}(\bar{t},\bar{x}) - z_s^{(m+1)}(t,x)$$
$$- u_{0,s}^{(m+1)}(t,x)(\bar{t} - t) - u_s^{(m+1)}(t,x) \circ (\bar{x} - x),$$

where $(t,x), (\bar{t},\bar{x}) \in E_c$. We prove that there exists $\widetilde{C} \in \mathbb{R}_+$ such that

$$(31) \qquad |\Delta^{(m)}(t,\bar{t},x,\bar{x})| \leq \widetilde{C}\|\bar{x} - x\|^2 \quad on\ E_c.$$

Fix $(t,x), (\bar{t},\bar{x}) \in E_c$ and set, for brevity,

$$g(\tau) = g_s[z^{(m)}, u_s^{(m+1)}](\tau,t,x), \qquad \bar{g}(\tau) = g_s[z^{(m)}, u_s^{(m+1)}](\tau,\bar{t},\bar{x}),$$
$$P(\tau) = P_s[z^{(m)}, u_s^{(m+1)}](\tau,t,x), \qquad \overline{P}(\tau) = P_s[z^{(m)}, u_s^{(m+1)}](\tau,\bar{t},\bar{x}),$$
$$\delta = \delta_s[z^{(m)}, u_s^{(m+1)}](t,x), \qquad \bar{\delta} = \delta_s[z^{(m)}, u_s^{(m+1)}](\bar{t},\bar{x}),$$

and

$$Q(\theta,\tau) = \theta\overline{P}(\tau) + (1-\theta)P(\tau), \quad W^{(m)}(\tau) = W_s^{(m)}[u_s^{(m+1)}](\tau,t,x).$$

It follows from (21) and (22) that

$$(32) \quad \Delta^{(m)}(t,\bar{t},x,\bar{x}) = F_s[z^{(m)}, u_s^{(m+1)}](\bar{t},\bar{x}) - F_s[z^{(m)}, u_s^{(m+1)}](t,x)$$
$$- u_{0,s}^{(m+1)}(t,x)(\bar{t} - t) - G_s^{(m)}[u^{(m+1)}](t,x) \circ (\bar{x} - x).$$

To simplify the formulation of properties of the function $\Delta^{(m)}$ we define

$$A^{(m)}(t,\bar{t},x,\bar{x}) = \varphi_s(\bar{\delta}, \bar{g}(\bar{\delta})) - \varphi_s(\delta, g(\delta))$$
$$- \partial_t \varphi_s(\delta, g(\delta))(\bar{\delta} - \delta) - \partial_x \varphi_s(\delta, g(\delta)) \circ [\bar{g}(\bar{\delta}) - g(\delta)],$$
$$B^{(m)}(t,\bar{t},x,\bar{x}) = \partial_t \varphi_s(\delta, g(\delta))(\bar{\delta} - \delta)$$
$$+ \partial_x \varphi_s(\delta, g(\delta)) \circ [\bar{g}(\bar{\delta}) - g(\delta) - \bar{x} + x],$$
$$C^{(m)}(\tau,t,\bar{t},x,\bar{x}) = \int_\tau^t [\partial_q f_s(\overline{P}(\xi)) - \partial_q f_s(P(\xi))]\, d\xi + \int_t^{\bar{t}} \partial_q f_s(\overline{P}(\xi))\, d\xi,$$
$$\Lambda^{(m)}(t,\bar{t},x,\bar{x}) = \Big(\int_{\bar{\delta}}^{\delta} + \int_t^{\bar{t}} \Big) [f_s(\overline{P}(\tau)) - u_s^{(m+1)}(\tau, \bar{g}(\tau)) \circ \partial_q f_s(\overline{P}(\tau))]\, d\tau.$$

It follows from (32) that

$$\Delta^{(m)}(t,\bar{t},x,\bar{x}) = \varphi_s(\bar{\delta}, \bar{g}(\delta)) - \varphi_s(\delta, g(\delta)) + \int_\delta^t [f_s(\overline{P}(\tau)) - f_s(P(\tau))]\, d\tau$$

$$+ \int_\delta^t [-u_s^{(m+1)}(\tau, \bar{g}(\tau)) \circ \partial_q f_s(\overline{P}(\tau)) + u_s^{(m+1)}(\tau, g(\tau)) \circ \partial_q f_s(P(\tau))]\, d\tau$$

$$+ \Lambda^{(m)}(t,\bar{t},x,\bar{x}) - u_{0,s}^{(m+1)}(t,x)(\bar{t} - t) - G_s^{(m)}[u^{(m+1)}](t,x) \circ (\bar{x} - x).$$

Having disposed of this preliminary step, we apply the Hadamard mean value theorem to the difference

$$f_s(P(\tau)) - f_s(\overline{P}(\tau)).$$

We thus get

$$\Delta^{(m)}(t,\bar{t},x,\bar{x}) = \overline{\Delta}^{(m)}(t,\bar{t},x,\bar{x}) + \widetilde{\Delta}^{(m)}(t,\bar{t},x,\bar{x})$$

where

$$\overline{\Delta}^{(m)}(t,\bar{t},x,\bar{x}) = A^{(m)}(t,\bar{t},x,\bar{x})$$

$$+ \int_\delta^t \int_0^1 [\partial_x f_s(Q(\theta,\tau)) - \partial_x f_s(P(\tau))]\, d\theta \circ [\bar{g}(\tau) - g(\tau)]\, d\tau$$

$$+ \int_\delta^t \int_0^1 [\partial_w f_s(Q(\theta,\tau)) - \partial_w f_s(P(\tau))]\, d\theta [z_{\alpha(\tau,\bar{g}(\tau))}^{(m)} - z_{\alpha(\tau,g(\tau))}^{(m)}]\, d\tau$$

$$+ \int\limits_{\delta}^{t} [u_s^{(m+1)}(\tau, \overline{g}(\tau)) - u_s^{(m+1)}(\tau, g(\tau))]$$

$$\circ \int\limits_{0}^{1} [\partial_q f_s(Q(\theta, \tau)) - \partial_q f_s(\overline{P}(\tau))]\, d\theta\, d\tau$$

$$+ \int\limits_{\delta}^{t} \partial_w f_s(P(\tau))[z_{\alpha(\tau, \overline{g}(\tau))}^{(m)} - z_{\alpha(\tau, g(\tau))}^{(m)} - W^{(m)}(\tau) \circ (\overline{g}(\tau) - g(\tau))]\, d\tau$$

and

$$\widetilde{\Delta}^{(m)}(t, \overline{t}, x, \overline{x}) = B^{(m)}(t, \overline{t}, x, \overline{x})$$

$$+ \int\limits_{\delta}^{t} [\partial_x f_s(P(\tau)) + \partial_w f_s(P(\tau))W^{(m)}(\tau)] \circ [\overline{g}(\tau) - g(\tau) - \overline{x} + x]\, d\tau$$

$$- \int\limits_{\delta}^{t} u_s^{(m+1)}(\tau, g(\tau)) \circ [\partial_q f_s(\overline{P}(\tau)) - \partial_q f_s(P(\tau))]\, d\tau$$

$$+ \Lambda^{(m)}(t, \overline{t}, x, \overline{x}) - u_{0,s}^{(m+1)}(t, x)(\overline{t} - t).$$

Let us estimate $|\widetilde{\Delta}^{(m)}(t, \overline{t}, x, \overline{x})|$. It follows from (11) that

$$\overline{g}(\tau) - g(\tau) - \overline{x} + x = C^{(m)}(\tau, t, \overline{t}, x, \overline{x})$$

and

$$\overline{g}(\overline{\delta}) - g(\delta) - \overline{x} + x = \Big(\int\limits_{\overline{\delta}}^{\delta} + \int\limits_{t}^{\overline{t}}\Big) \partial_q f_s(\overline{P}(\tau))\, d\tau + \int\limits_{\delta}^{t} [\partial_q f_s(\overline{P}(\tau)) - \partial_q f_s(P(\tau))]\, d\tau.$$

Substituting the above relations into $\widetilde{\Delta}^{(m)}(t, \overline{t}, x, \overline{x})$ and changing the order of integrals where necessary we obtain

$$\widetilde{\Delta}^{(m)}(t, \overline{t}, x, \overline{x}) = D^{(m)}(t, \overline{t}, x, \overline{x})$$

$$+ \int\limits_{\delta}^{t} E^{(m)}(\tau, t, x) \circ [\partial_q f_s(\overline{P}(\tau)) - \partial_q f_s(P(\tau))]\, d\tau,$$

where

$$D^{(m)}(t, \overline{t}, x, \overline{x}) = \int\limits_{t}^{\overline{t}} \big[(u_s^{(m+1)}(t, x) - u_s^{(m+1)}(\tau, \overline{g}(\tau))) \circ \partial_q f_s(\overline{P}(\tau))$$

$$+ f_s(\overline{P}(\tau)) - u_{0,s}^{(m+1)}(t, x)\big]\, d\tau$$

$$+ \int\limits_{\overline{\delta}}^{\delta} \big[(\partial_x \varphi_s(\delta, g(\delta)) - u_s^{(m+1)}(\tau, \overline{g}(\tau))) \circ \partial_q f_s(\overline{P}(\tau))$$

$$+ f_s(\overline{P}(\tau)) - \partial_t \varphi_s(\delta, g(\delta))\big]\, d\tau$$

and

$$E^{(m)}(\tau, t, x) = -u_s^{(m+1)}(\tau, g(\tau)) + \partial_x \varphi_s(\delta, g(\delta))$$
$$+ \int_\delta^\tau [\partial_x f_s(P(\xi)) + \partial_w f_s(P(\xi)) W^{(m)}(\xi)] \, d\xi.$$

The next claim is that $E(\tau, t, x) = 0$ for $(t, x), (\bar{t}, \bar{x}) \in E_c$ and $\tau \in [0, c]$. To see this, write (21) at the point $(\tau, g(\tau))$ and note that

$$g_s[z^{(m)}, u_s^{(m+1)}](\xi, \tau, g(\tau)) = g_s[z^{(m)}, u_s^{(m+1)}](\xi, t, x) = g(\xi).$$

Thus $\widetilde{\Delta}^{(m)}(t, \bar{t}, x, \bar{x}) = D^{(m)}(t, \bar{t}, x, \bar{x})$. We conclude from (22) and Lemma 2.1 that there is $C_1 \in \mathbb{R}_+$ such that

$$(33) \qquad |\widetilde{\Delta}^{(m)}(t, \bar{t}, x, \bar{x})| \le C_1[|\bar{t} - t| + \|\bar{x} - x\|]^2.$$

We will now estimate $\overline{\Delta}^{(m)}(t, \bar{t}, x, \bar{x})$. It follows from Assumption H[f] that

$$\|\partial_x f_s(Q(s, \tau)) - \partial_x f_s(P(\tau))\| \le L\bar{d}\|\overline{g}(\tau) - g(\tau)\|$$

and similar estimates hold for $\partial_w f_s$, $\partial_q f_s$. An easy computation shows that

$$|A^{(m)}(t, \bar{t}, x, \bar{x})| \le s_2(|\bar{\delta} - \delta| + \|\overline{g}(\bar{\delta}) - g(\delta)\|)^2,$$

$$\|z_{\alpha(\tau, \overline{g}(\tau))}^{(m)} - z_{\alpha(\tau, g(\tau))}^{(m)}\|_D \le dr_0\|\overline{g}(\tau) - g(\tau)\|,$$

$$\|u_s^{(m+1)}(\tau, \overline{g}(\tau)) - u_s^{(m+1)}(\tau, g(\tau))\| \le p_1\|\overline{g}(\tau) - g(\tau)\|.$$

Since $\partial z^{(m)} = (u_0^{(m)}, u^{(m)})$ on E_c^*, we have

$$\|z_{\alpha(\tau, \overline{g}(\tau))}^{(m)} - z_{\alpha(\tau, g(\tau))}^{(m)} - W^{(m)}(\tau) \circ [\overline{g}(\tau) - g(\tau)]\|_D \le n\bar{\omega}\|\overline{g}(\tau) - g(\tau)\|^2.$$

The above estimates and the definition of $\overline{\Delta}^{(m)}(t, \bar{t}, x, \bar{x})$ yield

$$(34) \quad \overline{\Delta}^{(m)}(t, \bar{t}, x, \bar{x}) \le s_2(|\bar{\delta} - \delta| + \|\overline{g}(\bar{\delta}) - g(\delta)\|)^2 + A^* \int_\delta^t \|\overline{g}(\tau) - g(\tau)\|^2 \, d\tau,$$

where $A^* = Bn\bar{\omega} + L\bar{d}^2$. It follows from Lemma 2.1 that there is $C_2 \in \mathbb{R}_+$ such that

$$(35) \qquad \overline{\Delta}^{(m)}(t, \bar{t}, x, \bar{x}) \le C_2[|\bar{t} - t| + \|\bar{x} - x\|]^2.$$

Adding inequalities (33) and (35), we get (31), and from arbitrariness of $(t, x), (\bar{t}, \bar{x}) \in E_c$, and $s \in \mathcal{S}$,

$$\partial z^{(m+1)} = (u_0^{(m+1)}, u^{(m+1)}) \quad \text{on } E_c.$$

This completes the proof of (29).

We now prove that $z^{(m+1)} \in C_{\varphi.c}^L[d]$. It is clear that $z^{(m+1)}$ is continuous on E_c and $z^{(m+1)} = \varphi$ on $E_0 \cup \partial_0 E$. Moreover, it follows from Assumption H[c, d, p] and (29) that $\|z^{(m+1)}(t, x) - z^{(m+1)}(\bar{t}, \bar{x})\| \le d[|\bar{t} - t| + \|\bar{x} - x\|]$.

It remains to prove $u_0^{(m+1)} \in C_{\varphi.c}^L[p; 1]$. Clearly, $u_0^{(m+1)}$ is continuous on E_c and $u_0^{(m+1)} = \partial_t \varphi$ on $E_0 \cup \partial_0 E$. Furthermore, Assumption H$[c, d, p]$ implies $\|u_0^{(m+1)}(t, x)\| \leq p_0$ and $\|u_0^{(m+1)}(t, x) - u_0^{(m+1)}(\bar{t}, \bar{x})\| \leq p_1[|\bar{t} - t| + \|\bar{x} - x\|]$. ∎

Now we prove that the sequence $\{(z, u_0, u)^{(m)}\}$ is uniformly convergent on E_c.

LEMMA 3.6. *If Assumptions* H$[\alpha]$, H$[f]$ *and* H$[c]$ *are satisfied, then the sequence* $\{(z, u_0, u)^{(m)}\}$ *is uniformly convergent on* E_c.

Proof. For $t \in [0, c]$ and $m \geq 1$, we set
$$Z^{(m)}(t) = \|z^{(m)} - z^{(m-1)}\|_t, \quad U^{(m)}(t) = \|u^{(m)} - u^{(m-1)}\|_{(t)}.$$
It follows from Lemma 2.1 that
$$\|g[z^{(m)}, u^{(m+1)}](\tau, t, x) - g[z^{(m-1)}, u^{(m)}](\tau, t, x)\|$$
$$\leq \overline{C} \left| \int_t^\tau [Z^{(m)}(\xi) + U^{(m+1)}(\xi)] \, d\xi \right|$$
and
$$|\delta_s[z^{(m)}, u_s^{(m+1)}](t, x) - \delta_s[z^{(m-1)}, u_s^{(m)}](t, x)|$$
$$\leq \widehat{C} \int_0^t [Z^{(m)}(\tau) + U^{(m+1)}(\tau)] \, d\tau, \quad s \in \mathcal{S}.$$
Hence we obtain the integral inequality
$$U^{(m+1)}(t) \leq \widetilde{A} \int_0^t U^{(m+1)}(\tau) \, d\tau$$
$$+ \int_0^t [\widetilde{B} Z^{(m-1)}(\tau) + \widetilde{A} Z^{(m)}(\tau) + \widetilde{B} U^{(m)}(\tau)] \, d\tau$$
for some constants $\widetilde{A}, \widetilde{B} \in \mathbb{R}_+$, independent of m. The above estimate and the Gronwall inequality yield
$$U^{(m+1)}(t) \leq e^{\widetilde{A}c} \int_0^t [\widetilde{B} Z^{(m-1)}(\tau) + \widetilde{A} Z^{(m)}(\tau) + \widetilde{B} U^{(m)}(\tau)] \, d\tau, \quad m \geq 1.$$
An easy computation shows that there is a constant $\overline{B} \in \mathbb{R}_+$ independent of m, such that
$$Z^{(m+1)}(t) \leq \overline{B} \int_0^t [Z^{(m)}(\tau) + U^{(m+1)}(\tau)] \, d\tau.$$

The above two inequalities yield

$$
(36) \qquad Z^{(m+1)}(t) + U^{(m+1)}(t) \le D_1 \int_0^t [Z^{(m)}(\tau) + U^{(m)}(\tau)] \, d\tau
$$

$$
+ D_2 \int_0^t Z^{(m-1)}(\tau) \, d\tau
$$

for all $m \ge 1$ and some $D_1, D_2 \in \mathbb{R}_+$. We define the norm in the space $C([0, c], \mathbb{R})$ as follows:

$$
\|\zeta\|_\lambda = \max\{|\zeta(t)|e^{-\lambda t} : t \in [0, c]\},
$$

where $\lambda > D_1 + D_2$. According to (36), we have

$$
Z^{(m+1)}(t) + U^{(m+1)}(t)
$$

$$
\le D_1 \int_0^t [Z^{(m)}(\tau) + U^{(m)}(\tau)] \, d\tau + D_2 \int_0^t Z^{(m-1)}(\tau) \, d\tau
$$

$$
= D_1 \int_0^t [Z^{(m)}(\tau) + U^{(m)}(\tau)]e^{-\lambda\tau} e^{\lambda\tau} \, d\tau + D_2 \int_0^t Z^{(m-1)}(\tau)e^{-\lambda\tau} e^{\lambda\tau} \, d\tau
$$

$$
\le D_1 \|Z^{(m)} + U^{(m)}\|_\lambda \int_0^t e^{\lambda\tau} \, d\tau + D_2 \|Z^{(m-1)}\|_\lambda \int_0^t e^{\lambda\tau} \, d\tau
$$

$$
= \frac{D_1}{\lambda} \|Z^{(m)} + U^{(m)}\|_\lambda (e^{\lambda t} - 1) + \frac{D_2}{\lambda} \|Z^{(m-1)}\|_\lambda (e^{\lambda t} - 1)
$$

$$
\le \left(\frac{D_1}{\lambda} \|Z^{(m)} + U^{(m)}\|_\lambda + \frac{D_2}{\lambda} \|Z^{(m-1)}\|_\lambda \right) e^{\lambda t}
$$

for $t \in [0, c]$. It follows that

$$
|Z^{(m+1)}(t) + U^{(m+1)}(t)|e^{-\lambda t} \le \frac{D_1}{\lambda} \|Z^{(m)} + U^{(m)}\|_\lambda + \frac{D_2}{\lambda} \|Z^{(m-1)}\|_\lambda
$$

for $t \in [0, c]$. Writing $y^{(m)} = \|Z^{(m)} + U^{(m)}\|_\lambda$ yields

$$
0 \le y^{(m+1)} \le \gamma_1 y^{(m)} + \gamma_2 y^{(m-1)}.
$$

Moreover, from Lemmata 3.3 and 3.5 we get $y^{(1)}, y^{(2)} \le 2(d\bar{b} + \widetilde{p})$, where $\widetilde{p} = \min\{p_0, p_1\bar{b}\}$ and $\bar{b} = \min\{b, c\}$. From the above and from the stability theory for difference equations (using the fact that $\gamma_1 + \gamma_2 < 1$), there exist $C \in \mathbb{R}_+$ and $q \in (0, 1)$ such that

$$
y^{(m)} \le Cq^m \quad \text{for } m \in \mathbb{N}.
$$

By the above inequality, $\{z^{(m)}\}$, $\{u^{(m)}\}$ are Cauchy sequences in $C^L_{\varphi.c}[d]$, $C^L_{\varphi.c}[p;n]$, hence are uniformly convergent. Now, define

$$U_0^{(m)}(t) = \|u_0^{(m)} - u_0^{(m-1)}\|_t.$$

Note that from (22) we get

$$U_0^{(m+1)}(t) \leq B[Z^{(m)}(t) + U^{(m+1)}(t)] \leq BC(1+q)q^m.$$

Since Lemma 3.3 implies $U_0^{(1)}(t) \leq 2\widetilde{p}$, $\{u_0^{(m)}\}$ is a Cauchy sequence (in $C^L_{\varphi.c}[p;1]$) as well, and hence is uniformly convergent. ∎

4. Existence of solutions of nonlinear mixed problems. We can now state the main result on the existence of classical solutions to problem (1), (2). For a function $\varphi \in C^{1.L}[s]$ and $\tau \in [0,a]$ we write

$$\|\varphi\|_\tau = \max\{\|\varphi(t,x)\| : (t,x) \in E_0 \cup \partial_0 E_\tau\},$$
$$\|\partial_t\varphi\|_\tau = \max\{\|\partial_t\varphi(t,x)\| : (t,x) \in E_0 \cup \partial_0 E_\tau\},$$
$$\|\partial_x\varphi\|_\tau = \max\{\|\partial_x\varphi(t,x)\| : (t,x) \in E_0 \cup \partial_0 E_\tau\}.$$

THEOREM 4.1. *If Assumptions* H[α], H[f] *and* H[c] *are satisfied, then, for every* $\varphi \in C^{1.L}[s]$, *there exists a solution* $v : E_c^* \to X$ *of problem* (1), (2). *Moreover,* $v \in C^L_{\varphi.c}[d]$ *and* $\partial v \in C^L_{\varphi.c}[p;1+n]$.

If $\bar{\varphi} \in C^{1.L}[s]$ *and* $\bar{v} \in C^L_{\bar{\varphi}.c}[d]$ *is a solution of system* (1) *with initial boundary condition* $z = \bar{\varphi}$ *on* $E_0 \cup \partial_0 E$, *then there is* $\Lambda_c \in \mathbb{R}_+$ *such that for* $0 \leq t \leq c$,

$$(37) \quad \|\bar{v} - v\|_t + \|\partial_t v - \partial_t \bar{v}\|_{(t)} + \|\partial_x v - \partial_x \bar{v}\|_{(t)}$$
$$\leq \Lambda_c[\|\bar{\varphi} - \varphi\|_t + \|\partial_t\varphi - \partial_t\bar{\varphi}\|_t + \|\partial_x\varphi - \partial_x\bar{\varphi}\|_t].$$

Proof. It follows from Lemmata 3.5 and 3.6 that there is $v \in C^L_{\varphi.c}[d]$ such that

$$v(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \quad \partial z(t,x) = \lim_{m\to\infty} u^{(m)}(t,x)$$

uniformly on E_c. Thus, we get

$$v(t,x) = F[v, \partial_x v](t,x), \quad \partial_x v(t,x) = G[v, \partial_t v, \partial_x v](t,x),$$

and

$$g_s[v, \partial_x v_s](\tau, t, x) = x + \int_\tau^t \partial_q f_s(P_s[v, \partial_x v_s](\xi, t, x))\, d\xi.$$

Moreover, the initial boundary conditions

$$v = \varphi, \quad \partial v = \partial\varphi \quad \text{on } E_0 \cup \partial_0 E_c$$

are satisfied. It follows that v is a classical solution of problem (1), (2) on E_c^*. The proof is similar to the proof of the corresponding properties

for initial-value or initial boundary value problems considered in [3] and [8] (Chapter 4). The details are omitted.

We now prove (37). To this end, we first prove

$$(38) \quad \|\bar{v} - v\|_t + \|\partial_x v - \partial_x \bar{v}\|_{(t)} \leq \overline{\Lambda}_c [\|\bar{\varphi} - \varphi\|_t + \|\partial_t \varphi - \partial_t \bar{\varphi}\|_t + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_t]$$

for $0 \leq t \leq c$. The functions $(\bar{v}, \partial_x \bar{v})$ satisfy the integral functional system (16), (17) and the initial boundary condition (18) with $\bar{\varphi}$ instead of φ. It easily follows that there are $\Lambda_0, \Lambda_1 \in \mathbb{R}_+$ such that

$$\|\bar{v} - v\|_t + \|\partial_x v - \partial_x \bar{v}\|_{(t)} \leq \Lambda_0 [\|\bar{\varphi} - \varphi\|_t + \|\partial_t \varphi - \partial_t \bar{\varphi}\|_t + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_t]$$
$$+ \Lambda_1 \int_0^t [\|\bar{v} - v\|_\tau + \|\partial_x v - \partial_x \bar{v}\|_{(\tau)}] \, d\tau, \quad 0 \leq t \leq c.$$

Using the Gronwall inequality, we get (38) with $\overline{\Lambda}_c = \Lambda_0 \exp(\Lambda_1 c)$. Moreover, from (17) we have

$$\|\partial_t v - \partial_t \bar{v}\|_{(t)} \leq \|\partial_t \varphi - \partial_t \bar{\varphi}\|_t + B[\|\bar{v} - v\|_t + \|\partial_x v - \partial_x \bar{v}\|_{(t)}], \quad 0 \leq t \leq c,$$

which, together with (38), yields (37) with $\Lambda_c = 1 + (1 + B)\overline{\Lambda}_c$. ∎

REMARK 4.2. Note that a corresponding existence result for systems of equations with deviated argument, or integral-differential equations, of the form given in Examples 1.1, 1.2, may be easily obtained from the preceding theorem.

REMARK 4.3. The theory presented in this paper may be easily extended to systems of equations with several function variables:

$$\partial_t z_s(t, x) = f_s(t, x, z_{\alpha_1(t,x)}, z_{\alpha_2(t,x)}, \dots, z_{\alpha_n(t,x)}, \partial_x z_s(t, x)), \quad s \in \mathcal{S},$$

with the initial-boundary condition (2). Such a model would allow one to consider mixed problems with both deviated argument and a classical argument:

$$\partial_t z_s(t, x) = f_s(t, x, z_{\alpha(t,x)}, z(t, x), \partial_x z_s(t, x)), \quad s \in \mathcal{S},$$

and to obtain existence results for them.

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Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: Wojciech.Czernous@math.univ.gda.pl