ANNALES
POLONICI MATHEMATICI
94.3 (2008)

Warped compact foliations

by SZYMON M. WALCZAK (Łódź)

Abstract. The notion of the Hausdorffized leaf space $\widetilde{\mathcal{L}}$ of a foliation is introduced. A sufficient condition for warped compact foliations to converge to $\widetilde{\mathcal{L}}$ is given. Moreover, a necessary condition for warped compact Hausdorff foliations to converge to $\widetilde{\mathcal{L}}$ is shown. Finally, some examples are examined.

1. Introduction. The notion of *warped foliations* was developed by the author of this note as a generalization of the concept of Berger spheres. It was introduced in [14], where it was defined as follows:

Let $(M, \mathcal{F}, g)$ be a foliated Riemannian manifold, and $f : M \to (0, \infty)$ a smooth function on M constant along the leaves of the foliation $\mathcal{F}$. We modify the Riemannian structure g to g_f in the following way:

For any vectors v, w tangent to $\mathcal{F}$ we set $g_f(v, w) = f^2 g(v, w)$. If at least one of v, w is perpendicular to $\mathcal{F}$, then we set $g_f(v, w) = g(v, w)$. The foliated Riemannian manifold $(M, \mathcal{F}, g_f)$ is called the *warped foliation* and denoted by M_f. The function f is called the *warping function*.

In [14] a necessary and sufficient condition for a sequence of warped compact Riemannian submersions to converge to the base of a Riemannian submersion was given. The results obtained in [14] were published in [13]. Later on, this condition was generalized to Riemannian foliations with all leaves compact, and some results on compact Hausdorff foliations were obtained [12].

In this note, the concept of the Hausdorffized leaf space of an arbitrary foliation $\mathcal{F}$ on a compact Riemannian manifold (M, g) is developed (Section 2). It is defined as the quotient space of a certain equivalence relation in the space of leaves of $\mathcal{F}$.

In Sections 3–5 we recall some basic facts about compact foliations, Gromov–Hausdorff distance, and the Bishop volume estimation [1] for an arbitrary metric space with measure. Some facts about Bishop measures re-

2000 *Mathematics Subject Classification*: 57R30, 53C12, 54A20.
Key words and phrases: foliation, compact foliation, warped foliation, Gromov–Hausdorff topology, Hausdorffized leaf space.

lating to compact Riemannian foliations, compact Hausdorff foliations and Hausdorffized leaf spaces are given. In Section 6 we provide a sufficient condition for a sequence of warped compact foliations on a compact Riemannian manifold to converge in the Gromov–Hausdorff sense to the Hausdorffized leaf space (Theorem 6.1). Additionally, we prove a necessary condition for compact Hausdorff foliations to converge to the Hausdorffized leaf space (Theorem 6.5). These two theorems are the main results of this note. Finally, in Section 7, we examine an example of a compact 1-dimensional foliation of codimension 3 with nonempty bad set, and we study the topology of its Hausdorffized leaf space (Theorem 7.3 and Corollary 7.4).

2. Hausdorffized leaf space. Let $\mathcal{F}$ be an arbitrary foliation on a smooth compact Riemannian manifold (M, g). Let $\mathcal{L}$ denote the leaf space of $\mathcal{F}$. Set

$$\varrho(L, L') = \inf \Big\{ \sum_{i=1}^{n-1} \operatorname{dist}(L_i, L_{i+1}) \Big\}, \tag{1}$$

where the infimum is taken over all finite sequences of leaves beginning with $L_1 = L$ and ending with $L_n = L'$, and $\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$. It is easy to check that ϱ defines a pseudo-metric in $\mathcal{L}$. Next, let us introduce in $\mathcal{L}$ an equivalence relation $\sim$ as follows:

$$L \sim L' \ \Leftrightarrow\ \varrho(L, L') = 0, \qquad L, L' \in \mathcal{L}.$$

Set $\widetilde{\mathcal{L}} = \mathcal{L}/\sim$ and

$$\widetilde{\varrho}([L], [L']) = \varrho(L, L'), \qquad [L], [L'] \in \widetilde{\mathcal{L}}. \tag{2}$$

Then $(\widetilde{\mathcal{L}}, \widetilde{\varrho})$ is a metric space. We call it the *Hausdorffized leaf space* (briefly, the HLS) for the foliation $\mathcal{F}$.

Remark 2.1. One can ask if dist can be used above instead of ϱ. It is easy to check that this can lead to different metric spaces. For example, any two leaves of a compact foliation have a distance greater than zero. If the bad set of a compact foliation is nonempty, then there exist leaves $L \neq L'$ such that $\varrho(L, L') = 0$. Moreover, the distance of sets does not satisfy the triangle inequality, hence it is not a pseudo-metric.

Remark 2.2. Note that the topology of $\widetilde{\mathcal{L}}$ does not depend on the metric structure g. In fact, let $(M, \mathcal{F})$ be a foliated compact manifold, and let g and g' be any Riemannian structures on M. Since the Riemannian metrics d and d', induced by g and g' respectively, satisfy

$$\frac{1}{C}\, d'(x, y) \le d(x, y) \le C d'(x, y)$$

for some constant $C \ge 1$ and all $x \in L$ and $y \in L'$, we have $\operatorname{dist}(L, L') \le$

$C\,\mathrm{dist}'(L,L')$ and $\mathrm{dist}'(L,L') \le C\,\mathrm{dist}(L,L')$. Hence, $\varrho(L,L') \le \varrho'(L,L')$ and $\varrho'(L,L') \le \varrho(L,L')$. Finally, $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}'$ and $(1/C)\widetilde{\varrho}' \le \widetilde{\varrho} \le C\widetilde{\varrho}'$.

REMARK 2.3. It seems reasonable to ask if we can define the Hausdorffized leaf space $\widetilde{\mathcal{L}}$ in topological terms.

Let X be a compact Hausdorff space and let $\sim$ denote an equivalence relation in X. Let $\overline{X} = X/{\sim}$. Of course, $\overline{X}$ with the quotient topology is seldom Hausdorff. Let $\mathbf{x}, \mathbf{y} \in \overline{X}$. We define a relation $\approx$ in $\overline{X}$ putting $\mathbf{x} \approx \mathbf{y}$ iff $\mathbf{x}$ and $\mathbf{y}$ cannot be separated in $\overline{X}$, i.e. there do not exist open sets $U, V \subset \overline{X}$ such that $\mathbf{x} \in U$, $\mathbf{y} \in V$ and $U \cap V = \emptyset$.

The relation $\approx$ need not be an equivalence relation. Let $\simeq$ be the smallest equivalence relation containing $\approx$. Put $\widetilde{X} = \overline{X}/{\simeq}$. It is easy to check that $\widetilde{X}$ with the quotient topology is Hausdorff and compact.

Let $X = M$ be a compact Riemannian manifold equipped with a foliation $\mathcal{F}$, and let $\sim$ be defined by

$$x \sim y \;\Leftrightarrow\; L_x = L_y,$$

where L_x denotes the leaf through $x \in M$. In this case $\widetilde{X} = \widetilde{\mathcal{L}}$.

REMARK 2.4. The assumption of compactness of X in Remark 2.3 is essential. Consider the foliation $\mathcal{F}$ of the plane $\mathbb{R}^2$ with the standard metric given by the graphs of the functions $f_a : \mathbb{R} \setminus \{0\} \ni x \mapsto 1/x + a \in \mathbb{R}$, $a \in \mathbb{R}$, and the y-axis. Then any two leaves L, L' can be separated in the topology of $\mathbb{R}^2$, but $\varrho(L,L') = 0$.

Let $\pi : M \to \widetilde{\mathcal{L}}$ be the natural projection given by $\pi(x) = [L_x]$, where L_x again denotes the leaf through $x \in M$. Let $U \subset \widetilde{\mathcal{L}}$ be an open set and $[L] \in U$. Since U is open, there exists $\varepsilon > 0$ such that the open ball $B_{\widetilde{\mathcal{L}}}([L], \varepsilon)$ is contained in U. By (2),

$$B_M(x, \varepsilon) \subset \pi^{-1}(U)$$

for all $x \in L$. Hence the projection π is continuous, and the space $(\widetilde{\mathcal{L}}, \widetilde{\varrho})$ is compact because M is compact.

Due to the Reeb Stability Theorem [2], for any compact leaf L with finite holonomy the equivalence class $[L]$ of this leaf equals L.

3. Compact foliations. A foliation with all leaves compact is called a *compact foliation*. Let $\mathcal{F}$ be an arbitrary compact foliation on a manifold M. Let $\pi : M \to \mathcal{L}$ be a quotient map onto the space of leaves $\mathcal{L}$ which identifies each leaf to a point. The space of leaves is often non-Hausdorff. In fact, the following theorem [6, Theorem 4.1] describes the topology of such a foliation:

THEOREM 3.1. *The following conditions are equivalent.*

(1) π *is a closed map.*
(2) π *maps compact sets onto closed sets.*

(3) *Each leaf has arbitrarily small saturated neighborhoods.*
(4) $\mathcal{L}$ *with the quotient topology is Hausdorff.*
(5) *If* $K \subseteq M$ *is compact, then the saturation of* K *is also compact.*

Let M be a Riemannian manifold and N be a submanifold on M. One can consider the induced Riemannian structure on N and define the volume of N as its volume in the induced Riemannian structure.

The relation between the volume of leaves defined as above (briefly, the volume function), the holonomy group and the topology of the space of leaves of the foliation $\mathcal{F}$ on a Riemannian manifold (M, g) is also well known [6, Theorem 4.2].

THEOREM 3.2. *If* $(M, \mathcal{F}, g)$ *is a foliated Riemannian manifold and* L *is a compact leaf then the following conditions are equivalent.*

(1) *There exists a saturated neighborhood* N *of the leaf* L *such that the volume function is bounded on* N.
(2) *The holonomy group of* L *is finite.*

The above conditions imply that some neighborhood N of L consists of compact leaves, and in N the conditions of Theorem 3.1 are satisfied.

THEOREM 3.3. *Suppose that* M *is a smooth compact Riemannian manifold which is foliated by compact leaves of codimension one or two. Then there is an upper bound on the volumes of the leaves of* M.

The above result was obtained by R. Edwards, K. Millett, and D. Sullivan in [4, Theorem 2].

Let G be the set of all $x \in M$ near which the volume function is bounded, i.e. $x \in G$ if and only if there exists an open neighborhood U of x such that the volumes of all leaves passing through U are uniformly bounded. The set G is called the *good set* of the foliation $\mathcal{F}$ [5]. It is open, saturated, and dense in M. By Theorem 3.2, the holonomy groups of all leaves contained in G are finite.

The following gives a precise description of the good set [2, Theorem 2.4.3] ($\mathcal{H}(L)$ denotes the holonomy group of the leaf L):

THEOREM 3.4 (Reeb Stability Theorem). *For any compact leaf* L *with finite holonomy, there exists a tubular neighborhood* $p : V \to L$ *of* L *in* M *such that* $(V, \mathcal{F}|_V, p)$ *is a foliated bundle with all leaves compact. Moreover, each leaf* $L' \subset V$ *has finite holonomy group of rank at most the order of* $\mathcal{H}(L)$ *and the covering* $p : L' \to L$ *has* k *sheets, where* k *is less than or equal to rank of* $\mathcal{H}(L)$.

The set $B = M \setminus G$ is called the *bad set* of the foliation $\mathcal{F}$. The bad set of a compact foliation is closed and nowhere dense in M [6].

Let $(M,\mathcal{F})$ be a compact foliated manifold, and $\mathcal{F}$ a compact foliation on M. Let g and g' be arbitrary Riemannian structures on M such that the g-orthogonal bundle $\mathcal{F}_g^{\perp}$ and the g'-orthogonal bundle $\mathcal{F}_{g'}^{\perp}$ coincide, i.e. every vector v perpendicular to $\mathcal{F}$ in g is perpendicular in g' and vice versa. Let $g(v,w)=g'(v,w)$ for any $v,w\in\mathcal{F}_g^{\perp}$. Let ϱ and ϱ' be the pseudo-metrics defined by (1) using g and g' respectively.

LEMMA 3.5. *$\widetilde{\varrho}=\widetilde{\varrho'}$.*

Proof. Since any two metric structures on a compact Riemannian manifold are equivalent, we can assume that $g\le Cg'$, $C\ge 1$. Let ϱ and ϱ' be the pseudo-metrics given by (1). Since the geometry of a compact manifold is bounded, for every $A>0$ and $\varepsilon>0$ there exists $\delta>0$ such that for every curve $\gamma:[0,l(\gamma)]\to M$ satisfying

(1) $\dot\gamma(0)$ is perpendicular to $\mathcal{F}$,
(2) its g'-length satisfies $l'(\gamma)<\delta$,
(3) its g-geodesic curvature satisfies $|k_g(\gamma)|<A$,

the g-length of the component tangent to the foliation $\mathcal{F}$ satisfies

$$|\dot\gamma^{\top}|<\varepsilon,\quad\text{where}\quad |v|=\sqrt{g(v,v)}.$$

Let $\varepsilon>0$, $L,L'\in\mathcal{F}$, and let $\gamma:[0,d]\to M$ be a g'-geodesic curve realizing the distance $\operatorname{dist}'(L,L')=d<\delta$. Then

$$\begin{aligned}\operatorname{dist}(L,L')&\le\int_{[0,d]}|\dot\gamma|\le\int_{[0,d]}|\dot\gamma^{\top}|+\int_{[0,d]}|\dot\gamma^{\perp}|\le\varepsilon Cd+\int_{[0,d]}|\dot\gamma^{\perp}|'\\&\le\varepsilon Cd+l'(\gamma)=(1+C\varepsilon)\operatorname{dist}'(L,L'),\end{aligned}$$

where $|v|'=\sqrt{g'(v,v)}$. By (1), for every sequence of leaves satisfying

$$\sum_{i=1}^{n-1}\operatorname{dist}'(L_i,L_{i+1})\le\varrho'(L,L')+\varepsilon,$$

and such that $\operatorname{dist}'(L_i,L_{i+1})<\delta$ for every $1\le i\le n-1$, we obtain

$$\begin{aligned}\varrho(L,L')&\le\sum_{i=1}^{n-1}\operatorname{dist}(L_i,L_{i+1})\le(1+C\varepsilon)\sum_{i=1}^{n-1}\operatorname{dist}'(L_i,L_{i+1})\\&\le(1+C\varepsilon)(\varrho'(L,L')+\varepsilon).\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain $\varrho\le\varrho'$. Consequently, $\widetilde{\varrho}\le\widetilde{\varrho'}$. Similarly, we can show that $\varrho'\le\varrho$. ∎

4. Gromov–Hausdorff distance. Let us recall that the *Hausdorff distance* d_{H} of two compact subsets A,B of a metric space (X,d) is defined as

$$d_{\mathrm{H}}(A,B)=\inf\{\varepsilon>0: A\subset N(B,\varepsilon)\wedge B\subset N(A,\varepsilon)\},$$

where $N(Y,\varepsilon)=\{x\in X: d(x,Y)<\varepsilon\}$.

M. Gromov [8] generalized this notion to arbitrary compact metric spaces (X, d_X) and (Y, d_Y) introducing the distance of X and Y as

$$d_{\mathrm{GH}}(X, Y) := \inf d_{\mathrm{H}}(X, Y), \tag{3}$$

where d ranges over all admissible metrics on the disjoint sum $X \amalg Y$, i.e. d is an extension of d_X and d_Y, and d_{H} denotes the Hausdorff distance. The number $d_{\mathrm{GH}}(X, Y)$ is called the *Gromov–Hausdorff distance* of the metric spaces X and Y.

Note that d_{GH} defines a metric on the class $\mathcal{M}$ of all isometry classes of compact metric spaces [3].

It is rather difficult to compute the exact value of the Gromov–Hausdorff distance between two given metric spaces. But it is possible to estimate it [3, 11]:

LEMMA 4.1. *Let (X, d_X) and (Y, d_Y) be arbitrary compact metric spaces, and let*

$$A = \{x_1, \ldots, x_k\} \subset X \quad \text{and} \quad B = \{y_1, \ldots, y_k\} \subset Y$$

be ε-nets satisfying for all $1 \le i, j \le k$ the condition

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \le \varepsilon.$$

Then $d_{\mathrm{GH}}(X, Y) \le 3\varepsilon$.

For the proof we refer to [11].

The following theorem is useful:

THEOREM 4.2. *Let $((X_i, d_{X_i}))_{i\in\mathbb{N}}$ and (Y, d_Y) be compact metric spaces. If $X_i \to Y$ as $i \to \infty$ in the Gromov–Hausdorff sense then for any $\eta > 0$ and for any η-net $A = \{y_1, \ldots, y_l\}$ on Y there exists a sequence $(A^i = \{x_1^i, \ldots, x_l^i\})_{i\in\mathbb{N}}$ of 2η-nets on X_i such that A is a quasi-isometric limit of A^i, i.e.*

$$|d_Y(y_j, y_k) - d_{X_i}(x_j^i, x_k^i)| \to 0 \quad \text{as } i \to \infty$$

for any $j, k \in \{1, \ldots, l\}$.

For the proof we refer to [3].

5. Bishop measures. We say that a measure μ on a metric space (X, d) *satisfies the Bishop inequalities* if there exist constants $\beta \ge 1$, $\eta_0 > 0$, and $p > 0$ such that for all $\eta < \eta_0$ and $x \in X$,

$$\frac{1}{\beta}\eta^p \le \mu(B_d(x, \eta)) \le \beta\eta^p, \tag{4}$$

where $B_d(x, \eta) = \{y \in X : d(x, y) < \eta\}$. We then call μ a *Bishop measure* on (X, d), and the number p the *dimension* of μ. The family of all p-dimensional Bishop measures on (X, d) satisfying the Bishop inequalities with constants

β, η_0 and p will be denoted by $\mathcal{B}(X, d, \beta, \eta_0, p)$. If the space X is known, we briefly write $\mu \in \mathcal{B}(d, \beta, \eta_0, p)$.

REMARK 5.1. Let (X, d) be a metric space and let $\mu \in \mathcal{B}(d, \beta, \eta_0, p)$. Let d' be another metric on X such that $(1/C)d \le d' \le Cd$ with some constant $C \ge 1$. Since $B_d(x, (1/C)\varepsilon) \subset B_{d'}(x, \varepsilon) \subset B_d(x, C\varepsilon)$ for all $\varepsilon > 0$, we have

$$\mu(B_{d'}(x, \eta)) \le \mu(B(x, C\eta)) \le \beta(C\eta)^p = \beta C^p \eta^p$$

for all $\eta < \eta_0/C$, and

$$\mu(B_{d'}(x, \eta)) \ge \mu\left(B\left(x, \frac{\eta}{C}\right)\right) \ge \frac{1}{C}\left(\frac{\eta}{C}\right)^p = \frac{1}{\beta C^p}\,\eta^p.$$

Hence $\mu \in \mathcal{B}(d', \beta C^p, \eta_0/C, p)$.

Let $(M, \mathcal{F}, g)$ be a compact Riemannian manifold equipped with a compact Riemannian foliation $\mathcal{F}$ of codimension q. The space $\mathcal{L}$ of leaves of $\mathcal{F}$ forms an orbifold [10], and, by the Reeb Stability Theorem, coincides with $\widetilde{\mathcal{L}}$.

Let us equip $\mathcal{L}$ with a metric defined by the Hausdorff distance d_{H} of leaves. Since all holonomy mappings are local isometries, $\widetilde{\varrho} = d_{\mathrm{H}}$. In [12, Corollary 1], it was shown that for every compact Riemannian foliation $\mathcal{F}$ of codimension q on a compact Riemannian manifold (M, g) there exists a Bishop measure $\lambda \in \mathcal{B}(\widetilde{\mathcal{L}}, \widetilde{\varrho}, \widetilde{\beta}, \widetilde{\eta}, q)$ with some constants $\widetilde{\beta} \ge 1$, $\widetilde{\eta} > 0$.

Let $(M, \mathcal{F}, g)$ be an arbitrary compact Riemannian manifold with a compact Hausdorff foliation of codimension q. Then M admits a Riemannian structure g_0 such that $(M, \mathcal{F}, g_0)$ becomes a Riemannian foliation [10]. Hence, there exists $\lambda \in \mathcal{B}(\widetilde{\mathcal{L}}, \widetilde{\varrho}_0, \widetilde{\beta}_0, \widetilde{\eta}_0, q)$ with some constants $\widetilde{\beta}_0 \ge 1$ and $\widetilde{\eta}_0 > 0$. Since M is compact, we have

$$\lambda \in \mathcal{B}(\widetilde{\mathcal{L}}, \widetilde{\varrho}, \widetilde{\beta}_0 C^q, \widetilde{\eta}_0/C, q)$$

for some constant $C \ge 1$, that is, there exists $\lambda \in \mathcal{B}(\widetilde{\mathcal{L}}, \widetilde{\varrho}, \widetilde{\beta}, \widetilde{\eta}, q)$ with $\widetilde{\beta} \ge 1$ and $\widetilde{\eta} > 0$.

6. Collapsing compact foliations. We now ask if the HLS for a compact foliation $\mathcal{F}$ on a compact Riemannian manifold (M, g) can be a limit of a sequence of warped foliations $(M_{f_n})_{n\in\mathbb{N}}$ for some warping functions $(f_n : M \to (0, 1])_{n\in\mathbb{N}}$. In the following, we try to formulate the conditions on warping functions as weak as possible.

Let $(M, \mathcal{F}, g)$ be a foliated Riemannian manifold with $\mathcal{F}$ having all leaves compact. Let $\mathcal{A}$ be a family of leaves from $\mathcal{F}$. For every $L, L' \in \mathcal{A}$ choose a finite sequence $F^{\varepsilon}_{L,L'} = (F^{L,L'}_1, \dots, F^{L,L'}_k)$ of leaves such that $F^{L,L'}_1 = L$, $F^{L,L'}_k = L'$, and

$$\sum_{i=1}^{k-1} \operatorname{dist}(F_i, F_{i+1}) < \widetilde{\varrho}(L, L') + \varepsilon,$$

where k depends on ε, L, and L'. The family of leaves which appear in all $F^{\varepsilon}_{L,L'}$ will be denoted by $\Upsilon(\mathcal{A},\varepsilon)$. The number

$$\operatorname{diam}(\mathcal{A}) := \sup_{F\in\mathcal{A}} \operatorname{diam} F$$

is called the diameter of the family $\mathcal{A}$. Note that for $\mathcal{A}$ finite the family $\Upsilon(\mathcal{A},\varepsilon)$ is also finite.

Let $(M,\mathcal{F},g)$ be a compact Riemannian manifold equipped with a compact foliation, let G denote the good set of $\mathcal{F}$, and let $(f_n : M \to (0,1])_{n\in\mathbb{N}}$ be a sequence of warping functions on M.

THEOREM 6.1. *Suppose that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ there exists a family $F^n = \{F^n_1,\dots,F^n_{l_n}\}$ of leaves such that*

(1) $\bigcup F^n \subset G$ *is ε-dense in M,*
(2) $\bigcup \Upsilon(F^n,\varepsilon) \subset G$,
(3) $f_n|_{\bigcup \Upsilon(F^n,\varepsilon)} < \varepsilon/d$,

where $d = \max\{\sharp\Upsilon(F^n,\varepsilon)\cdot \operatorname{diam}(\Upsilon(F^n,\varepsilon)),1\}$. Then the sequence $(M_{f_n})_{n\in\mathbb{N}}$ of warped foliations converges, as $n\to\infty$, to the HLS $(\widetilde{\mathcal{L}},\widetilde{\varrho})$ for the foliation $\mathcal{F}$.

REMARK 6.2. On a compact foliated Riemannian manifold $(M,\mathcal{F},g)$ with compact foliation one can easily construct a sequence of nonconstant warping functions satisfying the conditions of the above theorem. Let B be the bad set of $\mathcal{F}$, and let $\pi : M \to \widetilde{\mathcal{L}}$ be the natural projection. Let $f_n : M \to (1/n,1]$ be a smooth function satisfying:

- $f_n(x) = 1$ for $x \in \pi^{-1}(N(\pi(B),1/n))$,
- $f_n(x) = 1/n$ for $x \in M \setminus \pi^{-1}(N(\pi(B),2/n))$,

where $N(A,\eta) = \{x \in \widetilde{\mathcal{L}} : \widetilde{\varrho}(A,x) < \eta\}$. Since $M\setminus \pi^{-1}(N(\pi(B),1/n))$ is saturated, and all the leaves in $M\setminus\pi^{-1}(N(\pi(B),1/n))$ have finite holonomy, we can assume that f_n is constant along the leaves of $\mathcal{F}$. Hence, f_n is a warping function. Moreover, $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonconstant functions and satisfies the conditions listed in Theorem 6.1.

REMARK 6.3. For any sequence of warping functions converging to zero the conditions of Theorem 6.1 are satisfied. The converse need not be true.

Let $(f_n : M \to (0,1])_{n\in\mathbb{N}}$ be a sequence of warping functions equal to 1 on a certain leaf F_0 contained in the good set G and such that $f_n|_{G\setminus F_0} \to 0$. One can construct families F^n satisfying the conditions of Theorem 6.1 which are contained in $G\setminus\{F_0\}$. Hence, the conditions of Theorem 6.1 are weaker than pointwise convergence to zero.

Proof of Theorem 6.1. Let $\varepsilon > 0$, $n > N$, and $\pi : M \to \widetilde{\mathcal{L}}$ denote the natural projection $\pi(x) = [L_x]$. Since $\bigcup F^n \subset G$ is ε-dense in M, and

$\widetilde{\varrho}(L,L') \le \operatorname{dist}(L,L')$ for all $L,L' \in \mathcal{F}$, it follows that $\pi(\bigcup F^n)$ is also ε-dense in $(\widetilde{\mathcal{L}},\widetilde{\varrho})$.

Choose $x_i^n \in F_i^n$, $i=1,\dots,l_n$. Then $\{x_1^n,\dots,x_{l_n}^n\}$ is a 2ε-net on $M_n = (M,g_{f_n})$. Note that g and $g_n = g_{f_n}$ have identical orthogonal bundles and coincide on them, i.e. $g(v,w) = g_n(v,w)$ for all vectors v,w orthogonal to $\mathcal{F}$ either in g or g_n.

Let $1 \le i,j \le l_n$. By Lemma 3.5, we have

$$\widetilde{\varrho}(F_i^n,F_j^n) = \widetilde{\varrho}_n(F_i^n,F_j^n) \le d_n(x_i^n,x_j^n).$$

Next, let $(F_1,\dots,F_k) \in F^n$ be a sequence of leaves such that

$$\sum_{i=1}^{k-1} \operatorname{dist}(F_i,F_{i+1}) < \widetilde{\varrho}(F_i^n,F_j^n) + \varepsilon.$$

By the assumptions (1) and (2),

$$d_n(x_i^n,x_j^n) \le \sum_{\nu=1}^{k-1} \operatorname{dist}(F_\nu,F_{\nu+1}) + \sum_{\nu=1}^{k} \operatorname{diam}^n(F_\nu) \le \widetilde{\varrho}(F_i^n,F_j^n) + 2\varepsilon,$$

and Lemma 4.1 gives us the statement. ∎

Corollary 6.4. *For every compact foliation $\mathcal{F}$ on a compact Riemannian manifold (M,g) there exists a sequence $(f_n : M \to (0,1])_{n\in\mathbb{N}}$ of warping functions such that the Gromov–Hausdorff limit $\lim M_{f_n}$ of the sequence of warped foliations is equal to the HLS for the foliation $\mathcal{F}$.*

Suppose that $\mathcal{F}$ is Hausdorff, i.e. the space of leaves $\mathcal{L}$ with the quotient topology is Hausdorff.

Theorem 6.5. *Let $(f_n : M \to (0,1])_{n\in\mathbb{N}}$ be a sequence of warping functions on M. If $(M,g_{f_n}) \to (\widetilde{\mathcal{L}},\widetilde{\varrho})$ then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$ there exists a finite family $F^n = \{F_1^n,\dots,F_k^n\}$ of leaves such that*

(1) $\bigcup F^n \subset G$,
(2) $\bigcup F^n$ *is ε-dense in M,*
(3) $f_n|_{\bigcup F^n} < \varepsilon$.

Proof. Let $\pi : M \to \widetilde{\mathcal{L}}$ denote the natural projection given by $\pi(x) = [L_x]$, and suppose that there exists an $\varepsilon_0 > 0$ satisfying for all $N \in \mathbb{N}$ the following: there exists $n > N$ such that for any finite ε_0-dense family F^n of leaves from G one can find $L_n \in F^n$ satisfying

$$f_n|_{L_n} \ge \varepsilon_0.$$

It follows that one can choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$, a sequence $(r_k)_{k\in\mathbb{N}}$, leaves $(L_k)_{k\in\mathbb{N}}$, and a constant $r_0 > 0$ such that $r_k > r_0$ and

$$f_{n_k}|_{\pi^{-1}(B([L_k],r_k))} \ge \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

In fact, suppose that there is no such subsequence. Then for almost all n in $\mathbb{N}$, in every open saturated set $U \subset M$ one can find a point with f_n arbitrarily small. Since f_n is constant along the leaves it is possible to construct, for large enough $n \in \mathbb{N}$, an ε_0-dense finite family F_0^n of leaves from G such that $f_n|_{\bigcup F_0^n}$ is arbitrarily small. But this contradicts our assumption.

Since $\widetilde{\mathcal{L}}$ is compact, there exist $r_0 > 0$, $[L_0] \in \widetilde{\mathcal{L}}$, and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$,

$$f_{n_k}|_{\pi^{-1}(B([L_0],r_0))} \geq \varepsilon_0.$$

Suppose now that $M_{f_{n_k}} \to (\widetilde{\mathcal{L}}, \widetilde{\varrho})$ in the Gromov–Hausdorff topology. Let $\eta > 0$ be small enough and let $\mathcal{A} = \{[L_0], \ldots, [L_\nu]\} \subset \widetilde{\mathcal{L}}$ be a minimal η-net. Then, by Theorem 4.2, there exists a sequence of 2η-nets $A^{n_k} = \{x_0^{n_k}, \ldots, x_\nu^{n_k}\} \subset M_{f_{n_k}}$ such that A is a quasi-isometric limit of A^{n_k}. Moreover,

$$(\nu + 1) \min_{[L]\in\widetilde{\mathcal{L}}} \lambda(B([L], \eta/4)) \leq \lambda(\widetilde{\mathcal{L}}),$$

and

$$(\nu + 1) \max_{x\in M} \mathrm{vol}_{n_k}(B(x, 2\eta)) \geq \mathrm{vol}_{n_k} M_{f_{n_k}},$$

where $\lambda \in \mathcal{B}(\widetilde{\mathcal{L}}, \widetilde{\varrho}, \widetilde{\beta}, \widetilde{\eta}_0, q)$ is the measure mentioned in Section 5.

Since M is compact, we have $\mathrm{vol} \in \mathcal{B}(M, d, \beta_M, \eta_M, \dim M)$. Hence,

$$\begin{aligned}0 &< \mathrm{vol}(\pi^{-1}(B([L_0], r_0))) \cdot \varepsilon_0^p \\ &\leq \mathrm{vol}_{n_k} M_{f_n} \leq \lambda(\mathcal{L}) \frac{\max_{x\in M} \mathrm{vol}_{f_{n_k}} B(x, 2\eta)}{\min_{[L]\in\widetilde{\mathcal{L}}} \lambda(B([L], \eta/4))} \\ &\leq \lambda(\widetilde{\mathcal{L}}) \frac{\beta_M \widetilde{\beta}(2\eta)^{\dim M}}{(\eta/4)^q}.\end{aligned}$$

Letting $\eta \to 0$ yields a contradiction. ∎

7. Example. We will now study an example of a compact foliation of dimension 1 and codimension 3 on a 4-dimensional manifold, which was described by D. B. A. Epstein and E. Vogt [7] in 1978. The codimension of that foliation is the smallest one with possible bad set nonempty. The bad set of the foliation consists of four spheres of dimension 3 with Hopf fibration on each of them, and four tori of dimension 2 with circular foliations. It is connected and compact. We only recall the precise analytic description of the manifold and the foliation given in [7, Sections 3, 4, and 6].

Let us consider the octagon

$$D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 2,\ |y| \leq 2,\ |x - y| \leq 3,\ |x + y| \leq 3\},$$

and let

$$\psi(x,y) = (2-x)(2+x)(2-y)(2+y)(3+x+y)(3-x-y)(3+x-y)(3-x+y).$$

Notice that $\psi > 0$ on $\operatorname{int} D$ and $\psi|_{\partial D} = 0$. Let

$$A = D \cap \{(x,y) \in \mathbb{R} : \psi(x,y) \le 1\},$$

and let $\xi = (x, y, u_1, u_2, w_1, w_2, z) \in \mathbb{R}^7$. We define $F : \mathbb{R}^7 \to \mathbb{R}^3$ by

$$\begin{aligned} F_1(\xi) &= u_1^2 + u_2^2 - 4 + x^2, \\ F_2(\xi) &= w_1^2 + w_2^2 - 4 + y^2, \\ F_3(\xi) &= z^2 - \varrho(x,y), \end{aligned}$$

where

$$\varrho(x,y) = (1 - \psi(x,y))(3 - x - y)(3 + x + y)(3 + x - y)(3 - x + y).$$

We see that $F(\xi) \in \mathbb{R}^3$. We define $M = F^{-1}(0)$. By [7, Lemma 4.1], the projection of $\mathbb{R}^7$ onto the first two coordinates maps M onto A, and M is a 4-dimensional compact manifold ([7, Lemmas 4.2 and 4.3]).

Define a vector field X on $\mathbb{R}^7$ by

$$\begin{aligned} X_\xi = \psi \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \psi \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} + (Ku_1 - pu_2)\frac{\partial}{\partial u_1} + (pu_1 + Ku_2)\frac{\partial}{\partial u_2} \\ + (Lw_1 - qw_2)\frac{\partial}{\partial w_1} + (qw_1 + Lw_2)\frac{\partial}{\partial u_2} + z\sigma\frac{\partial}{\partial z}, \end{aligned}$$

where

$$\begin{aligned} K(x,y) &= -x\frac{\partial\psi}{\partial y}(4 - y^2)(9 - (x+y)^2)(9 - (x-y)^2), \\ L(x,y) &= y\frac{\partial\psi}{\partial y}(4 - x^2)(9 - (x+y)^2)(9 - (x-y)^2), \\ p(x,y) &= (9 + x^2 - y^2)y, \\ q(x,y) &= (9 - x^2 + y^2)x, \\ \sigma(x,y) &= \left(\frac{\partial\psi(x,y)}{\partial x} - \frac{\partial\psi(x,y)}{\partial y}\right)\frac{(x+y)\psi}{9 - (x+y)^2} \\ &\quad + \left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi(x,y)}{\partial y}\right)\frac{(y-x)\psi(x,y)}{9 - (x-y)^2}. \end{aligned}$$

By [7, Lemma 6.1], if $\xi \in M$, then $X(\xi) \in T_\xi M$. Hence, X is a nowhere vanishing vector field on M. Moreover, by [7, Lemma 6.5], the orbit of X through ξ is diffeomorphic to a circle if only $\psi(x,y) > 0$, and the length of that orbit tends to infinity if $\psi(x,y)$ tends to zero. By [7, Lemmas 6.6 and 6.7], all other orbits are also diffeomorphic to circles.

We will now examine the topology of the HLS for the foliation $\mathcal{F}$ defined by the orbits of X on M. Let $\pi : M \to \widetilde{\mathcal{L}}$ be the natural projection.

LEMMA 7.1. *The bad set B of the foliation $\mathcal{F}$ projects to a singleton.*

Proof. Let $\pi_1 : M \ni \xi = (x, y, u_1, u_2, w_1, w_2, z) \mapsto (x, y) \in A$. Since X is invariant under rigid rotation about the origin in the u-plane or the w-plane [7, Lemma 6.3], the vector field X is constant along the set $\pi_1^{-1}(x, y)$ for any $(x, y) \in A$. Moreover, for the level surface $\psi^{-1}(a)$ with $0 < a \leq 1$, there exists $(x_a, y_a) \in A$ such that the component of X tangent to $\pi_1^{-1}(x_a, y_a)$ is tangent to a circle in the u-plane.

Since M is compact and X is a nonzero vector field, the length $|X|$ is bounded below by a positive constant. Recall that the component of X tangent to A is constant along the level surfaces of ψ and tends to zero as $\psi \to 0$. Hence there exists $\eta_0 > 0$ such that for any $0 < \eta < \eta_0$ one can find a point $(x_\eta, y_\eta) \in \psi^{-1}(\eta)$ with every leaf η-dense in

$$\pi_1^{-1}(B((x_\eta, y_\eta), \eta) \cap \psi^{-1}(\eta)).$$

It follows that for small enough $\eta > 0$ the distance $\operatorname{dist}(L_\xi, L_{\xi'})$ is smaller than η for any $\xi, \xi' \in \pi_1^{-1}(\psi(\eta))$, where L_ξ denotes the leaf through $\xi \in M$.

Finally, for small enough $\eta > 0$ and arbitrary $\xi, \xi' \in B$ we have

$$\widetilde{\varrho}(L_\xi, L_{\xi'}) < \eta.$$

This ends the proof. ■

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $T^2 = S^1 \times S^1$, and $\pi_1 : M \to A$ be as in Lemma 7.1.

LEMMA 7.2. *The projection $\pi(G)$ of the good set of $\mathcal{F}$ is homeomorphic to $(S^1 \setminus \{1\}) \times T^2$.*

Proof. Let L_ξ, $\xi = (x, y, u_1, u_2, w_1, w_2, z)$, be a leaf contained in the good set G of $\mathcal{F}$, and let z be fixed. Then L_ξ is contained in $\pi_1^{-1}(\psi(\eta))$ for a certain $0 < \eta \leq 1$, for ψ is constant on every orbit. As mentioned before, the vector field X is invariant under rigid rotation about the origin in the u-plane or the w-plane and it is constant along $\pi_1^{-1}(x, y)$ for any $(x, y) \in A$. Hence L_ξ intersects each torus in $\pi_1^{-1}(x, y)$ the same number of times, and so does any other leaf in $\pi^{-1}(\psi(\eta))$.

By the Reeb Stability Theorem and Theorem 3.2, $[L] = \{L\}$ for any leaf L from G. Hence, the space of leaves of $\pi_1^{-1}(\psi(\eta))$ is homeomorphic to the torus T^2, and $\pi(G)$ is homeomorphic to $S^1 \setminus \{1\} \times T^2$. ■

Let X be a topological space. Let $\sim$ be the equivalence relation in $S^1 \times X$ defined by

$$(z, x) \sim (w, y) \iff (z = w = 1 \text{ or } (z, x) = (w, y)).$$

Let $\Sigma(X)$ denote $(S^1 \times X)/{\sim}$ with the quotient topology.

THEOREM 7.3. *$\widetilde{\mathcal{L}}$ is homeomorphic to $\Sigma(T^2)$.*

Proof. This is a simple consequence of Lemmas 7.1 and 7.2. ∎

Let $h : \Sigma(T^2) \to \widetilde{\mathcal{L}}$ be the homeomorphism of Theorem 7.3. For any $x, y \in \Sigma(T^2)$ set $d(x,y) = \widetilde{\varrho}(h(x), h(y))$. Let $G' \subset G$ be a dense saturated subset of the good set of the foliation $\mathcal{F}$, and let $f_n : M \to (0,1]$ be a sequence of basic functions (i.e. functions constant along the leaves) converging to zero on G'.

As an easy corollary of Theorems 7.3 and 6.1 we have the following:

Corollary 7.4. *The sequence of warped foliations* $(M, \mathcal{F}, g_{f_n})$ *converges in the Gromov–Hausdorff topology to* $(\Sigma(T^2), d)$.

References

[1] R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York, 1964.

[2] A. Candel and L. Conlon, *Foliations I*, Amer. Math. Soc., Providence, 2000.

[3] D.-P. Chi and G. Yun, *Gromov–Hausdorff Topology and its Applications to Riemannian Manifolds*, Seoul National Univ., Seoul, 1998.

[4] R. Edwards, K. Millett and D. Sullivan, *Foliations with all leaves compact*, Topology 16 (1977), 13–32.

[5] D. B. A. Epstein, *Periodic flows on 3-manifolds*, Ann. of Math. 95 (1972), 66–82.

[6] —, *Foliations with all leaves compact*, Ann. Inst. Fourier (Grenoble) 26 (1976), 265–282.

[7] D. B. A. Epstein and E. Vogt, *A counterexample to the Periodic Orbit Conjecture in codimension 3*, Ann. of Math. 108 (1978), 539–552.

[8] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston, 1999.

[9] K. Kuratowski, *Topologie*, Polskie Towarzystwo Matematyczne, Warszawa, 1952.

[10] P. Molino, *Riemannian Foliations*, Birkhäuser, Boston, 1988.

[11] P. Petersen, *Riemannian Geometry*, Grad. Texts in Math. 171, Springer, New York, 1997.

[12] Sz. M. Walczak, *Collapse of warped foliations*, Differential Geom. Appl. 25 (2007), 649–654.

[13] —, *Collapse of warped submersions*, Ann. Polon. Math. 89 (2006), 139–146.

[14] —, *On deformations of foliations*, thesis, University of Łódź, 2005 (in Polish).

Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: sajmonw@math.uni.lodz.pl

Received 8.12.2007
and in final form 16.6.2008 (1839)