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Subextension of plurisubharmonic functions without increasing the total Monge–Ampère mass

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Abstract. We prove that subextension of certain plurisubharmonic functions is always possible without increasing the total Monge–Ampère mass.

1. Introduction. Bedford and Burns [5] (see also [9]) proved that any smooth bounded domain in $\mathbb{C}^n$ satisfying a certain non-degeneracy condition on the Levi form on the boundary is the domain of existence for plurisubharmonic functions, and El Mir [19] constructed an example of a plurisubharmonic function defined on the unit bidisc in $\mathbb{C}^2$ for which the restriction to any smaller bidisc admits no subextension to $\mathbb{C}^2$. Bedford and Taylor [6] improved an example by Fornæss and Sibony [20] by constructing a smooth negative plurisubharmonic function on an arbitrary bounded domain in $\mathbb{C}^n$ with C^2-boundary that does not subextend.

In this article we are interested in subextension without increasing the total Monge–Ampère mass. Before proceeding we need some background and notation. Let $\mathcal{PSH}(\Omega)$ denote the set of all plurisubharmonic functions defined on a domain $\Omega \subset \mathbb{C}^n$. Recall that a bounded domain $\Omega \subseteq \mathbb{C}^n$ is called *hyperconvex* if there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. Let $\mathcal{E}_0(\Omega)$ be the class of bounded plurisubharmonic functions u such that $\lim_{z\to\xi} u(z) = 0$ for all $\xi \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < \infty$. We say that a negative function $u \in \mathcal{PSH}(\Omega)$ is in the class $\mathcal{F}(\Omega)$ if there is a decreasing sequence $[u_j]$ of functions $u_j \in \mathcal{E}_0(\Omega)$ which converges pointwise to u on Ω and $\sup_j \int (dd^c u_j)^n < \infty$. The class $\mathcal{E}(\Omega)$ contains the functions in $\mathcal{PSH}(\Omega)$ that are locally in $\mathcal{F}(\Omega)$, and Theorem 4.2

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in [11] implies that $(dd^c)^n$ is well-defined on $\mathcal{E}(\Omega)$. For more details about these classes, see [11]. Note that, by [7, 8], a function $u \in \mathcal{E}(\Omega)$ is maximal if, and only if, $(dd^cu)^n = 0$. The set of all maximal plurisubharmonic functions defined on Ω will be denoted by $\mathcal{MPSH}(\Omega)$.

Similarly to [10, 12], if $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{N}\}$, where $\mathcal{N}$ will be defined later, we say that a plurisubharmonic function u defined on Ω belongs to the class $\mathcal{K}(\Omega, H)$ for some $H \in \mathcal{E}(\Omega)$ if there exists a function $\varphi \in \mathcal{K}(\Omega)$ such that

$$H \geq u \geq \varphi + H. \tag{1.1}$$

Note that $\mathcal{K}(\Omega, 0) = \mathcal{K}(\Omega)$ and that functions belonging to $\mathcal{K}(\Omega, H)$ not necessarily have finite total Monge–Ampère mass (see [3, 16]). Inequality (1.1) allows one to introduce, in some sense, generalized boundary values of certain plurisubharmonic functions ([14], see also [21]).

Our aim is to prove the following theorem:

THEOREM 1.1. *Let Ω_1 and Ω_2 be two bounded hyperconvex domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$, $n \geq 1$, and let $F \in \mathcal{E}(\Omega_1)$ and $G \in \mathcal{E}(\Omega_2) \cap \mathcal{MPSH}(\Omega_2)$ be such that*

$$F \geq G \quad \text{on } \Omega_1. \tag{1.2}$$

If $u \in \mathcal{F}(\Omega_1, F)$, then there exists $v \in \mathcal{F}(\Omega_2, G)$ such that $v \leq u$ on Ω_1 and

$$\int_{\Omega_2} (dd^cv)^n \leq \int_{\Omega_1} (dd^cu)^n.$$

Under the assumption that F and G are identically zero and Ω_1 is relatively compact in Ω_2, Theorem 1.1 was proved in [17], and when F and G are the Perron–Bremermann envelopes for certain continuous functions f and g that satisfiy (1.2), it was proved in [4]. Example 5.5 in [4] shows that condition (1.2) is necessary. In contrast to the corresponding results in [4, 17] we do not need the assumption that Ω_1 is relatively compact in Ω_2.

Subextension without increasing the Monge–Ampère mass has proven to be a useful tool in applications, e.g. approximation of plurisubharmonic functions ([13]) and estimating the volume of plurisubharmonic sublevel sets ([2]). Without the control of the total Monge–Ampère mass, subextension in $\mathcal{F}(\Omega, H)$, $H \in \mathcal{E}(\Omega)$, would follow as the second part of the proof of Theorem 1.1 by using Theorem 2.2 in [17]. At present the authors do not know if the assumption that $G \in \mathcal{MPSH}(\Omega_2)$ is necessary but we observe that it is necessary that $\int_{\Omega_2} (dd^cG)^n \leq \int_{\Omega_1} (dd^cF)^n$. For further results concerning subextension of plurisubharmonic functions see e.g. [15, 22] and the references therein.

2. Proof of Theorem 1.1. In this paper, a *fundamental sequence* $[\Omega_j]$ in Ω is an increasing sequence of strictly pseudoconvex subsets of Ω such that $\Omega_j \Subset \Omega_{j+1}$ for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Here $\Subset$ denotes that Ω_j is relatively compact in Ω_{j+1}.

DEFINITION 2.1. Let $u \in \mathcal{PSH}(\Omega)$, $u \leq 0$, and let $[\Omega_j]$ be a fundamental sequence in Ω. Define

$$u^j = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq u \text{ on } \mathcal{C}\Omega_j\},$$

where $\mathcal{C}\Omega_j$ denotes the complement of Ω_j in Ω, and

$$\widetilde{u} = (\lim_{j\to\infty} u^j)^*,$$

where ω^* denotes the upper semicontinuous regularization of ω.

Definition 2.1 implies that $[u^j]$ is an increasing sequence and therefore $\widetilde{u}$ is plurisubharmonic on Ω. Moreover, if $u \in \mathcal{E}(\Omega)$, then by [11] we know that $\widetilde{u} \in \mathcal{E}(\Omega)$, since $u \leq \widetilde{u} \leq 0$, and from [7, 8] it follows that $\widetilde{u}$ is maximal on Ω. Let $\mathcal{N}(\Omega)$ be the class of all functions $u \in \mathcal{E}(\Omega)$ such that $\widetilde{u} = 0$. Note that $\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}(\Omega)$.

In the proof of the main theorem we will need some well known results. For the convenience of the reader we will formulate some of them in the proposition below.

For brevity, in the statements and proofs of Propositions 2.2 and 2.3, we write $\mathcal{E}$, $\mathcal{N}(H)$ etc. for $\mathcal{E}(\Omega)$, $\mathcal{N}(\Omega, H)$ etc.

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $H \in \mathcal{E}$.*

(a) *Let $u, v \in \mathcal{N}(H)$, $u \leq v$, and $\int_\Omega (dd^c u)^n < \infty$. Then $\int_\Omega (dd^c v)^n \leq \int_\Omega (dd^c u)^n$.*
(b) *If $[u_j] \subset u_j \in \mathcal{N}(H)$ is a decreasing sequence that converges pointwise to a function $u \in \mathcal{N}(H)$ as $j \to \infty$, then*

$$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n.$$

(c) *If $u \in \mathcal{N}(H)$ and $\int_\Omega (dd^c u)^n < \infty$, then $u \in \mathcal{F}(H)$.*

Proof. (a) follows directly from Lemma 3.3 in [1], and (b) from Corollary 3.4 in [1]. To prove (c) note that since if $u \in \mathcal{N}(H)$ there exists $\varphi \in \mathcal{N}$ such that $H \geq u \geq \varphi + H$ and therefore $\widetilde{H} \geq \widetilde{u} \geq \widetilde{\varphi} + \widetilde{H}$, so $\widetilde{u} = \widetilde{H}$ from Theorem 2.1 in [12]. But since $\mathcal{N}(H) \subset \mathcal{E}$ we have $u \in \mathcal{F}(\widetilde{u})$, so $u \in \mathcal{F}(\widetilde{H})$. Hence, there exists $\psi \in \mathcal{F}$ such that $\widetilde{H} \geq u \geq \psi + \widetilde{H}$. Now, since $\widetilde{H} \geq H$ we know that

$$H \geq u \geq \psi + \widetilde{H} \geq \psi + H$$

and so $u \in \mathcal{F}(H)$. ∎

Using Proposition 2.2(b) we obtain the following characterization of $\mathcal{F}(H)$ that will be used in the proof of Theorem 1.1. Proposition 2.3 is a generalization of Theorem 3.7 in [4].

PROPOSITION 2.3. *Let $H \in \mathcal{E}$. If $u \in \mathcal{F}(H)$ is such that*

$$\int_{\Omega} (dd^c u)^n < \infty, \tag{2.1}$$

then there exists a decreasing sequence $[u_j] \subset \mathcal{E}_0(H)$ that converges pointwise to u as j tends to ∞, and

$$\sup_j \int_{\Omega} (dd^c u_j)^n < \infty. \tag{2.2}$$

Moreover, if $[u_j] \subset \mathcal{F}(H)$ is a decreasing sequence that converges pointwise to a function u as j tends to ∞, and (2.2) *is satisfied, then $u \in \mathcal{F}(H)$ and* (2.1) *holds.*

Proof. Assume that $u \in \mathcal{F}(H)$ satisfies (2.1). It follows from Proposition 2.5 in [1] that there exists a decreasing sequence $[u_j] \subset \mathcal{E}_0(H)$ that converges pointwise to u on Ω as $j \to \infty$. By Proposition 2.2(b) and assumption (2.1) we have

$$\sup_j \int_{\Omega} (dd^c u_j)^n < \infty.$$

Now assume first that $[u_j] \subset \mathcal{E}_0(H)$ is a decreasing sequence such that (2.2) holds and $[u_j]$ converges pointwise to a function u as $j \to \infty$. From (2.2) and Proposition 2.2(a) we find that $\int_{\Omega} (dd^c H)^n < \infty$, since $u_j, H \in \mathcal{F}(H)$ and $u_j \le H$. Theorem 2.1 in [12] implies that $H \in \mathcal{F}(\widetilde{H})$, where $\widetilde{H}$ is defined as in Definition 2.1. Hence, we can without loss of generality assume that $(dd^c H)^n = 0$. The measure $(dd^c u_j)^n$ has finite total mass and vanishes on pluripolar sets by Lemma 4.11 in [1]. Therefore Lemma 5.14 of [11] implies that there exists a unique function $\varphi_j \in \mathcal{F}$ such that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Furthermore,

$$(dd^c(\varphi_j + H))^n \ge (dd^c u_j)^n.$$

Thus, $u_j \ge \varphi_j + H$, by Corollary 3.2 in [1]. Set $\varphi_j' = (\sup_{k \ge j} \varphi_k)^*$. Then $[\varphi_j'] \subset \mathcal{F}$ is a decreasing sequence and

$$\sup_j \int_{\Omega} (dd^c \varphi_j')^n \le \sup_j \int_{\Omega} (dd^c \varphi_j)^n < \infty,$$

by (2.2) and the fact that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Thus, by Lemma 2.1 in [18], $\varphi = (\lim_{j\to\infty} \varphi_j') \in \mathcal{F}$. For every $k \in \mathbb{N}$ we have $u_j \ge u_{j+k} \ge \varphi_{j+k} + H$. Hence, $u_j \ge \varphi + H$ for all $j \in \mathbb{N}$. By letting $j \to \infty$ we see that $u \in \mathcal{F}(H)$. Now (2.2) and Proposition 2.2(b) imply that

$$\int_{\Omega} (dd^c u)^n = \lim_{j\to\infty} \int_{\Omega} (dd^c u_j)^n < \infty.$$

If $u_j \in \mathcal{F}(H)$ only, we can take $\psi \in \mathcal{E}_0(\Omega)$, $\psi \neq 0$ and define

$$u'_j = \max\{u_j, j\psi + H\}.$$

Since $j\psi + H \in \mathcal{E}_0(H)$ for every fixed j, we know that $u'_j \in \mathcal{E}_0(H)$. By construction, $u'_j \searrow u$ as $j \to \infty$, and then Proposition 2.2(a) and (2.2) imply that $\int_\Omega (dd^c u'_j)^n \leq \int_\Omega (dd^c u_j)^n$. It follows from (2.2) that

$$\sup_j \int_\Omega (dd^c u'_j)^n < \infty,$$

and the result follows. ■

Proof of Theorem 1.1. Let $u \in \mathcal{F}(\Omega_1, F)$. First assume that

$$\int_{\Omega_1} (dd^c u)^n < \infty. \tag{2.3}$$

This assumption and Proposition 2.2(a) imply that $\int_{\Omega_1} (dd^c F)^n < \infty$, since $u, F \in \mathcal{F}(\Omega_1, F)$ and $u \leq F$. Proposition 2.2(c) implies that $F \in \mathcal{F}(\Omega_1, \widetilde{F})$, where $\widetilde{F}$ is defined as in Definition 2.1. Hence, we can assume that $(dd^c F)^n = 0$. Proposition 2.3 implies that there exists a decreasing sequence $[u_j] \subset \mathcal{E}_0(\Omega_1, F)$ which converges pointwise to u on Ω_1 as $j \to \infty$, and

$$\sup_j \int_{\Omega_1} (dd^c u_j)^n < \infty. \tag{2.4}$$

Consider the measure $\mu_j = \chi_{\Omega_1}(dd^c u_j)^n$ defined on Ω_2, where χ_{Ω_1} is the characteristic function for Ω_1 defined in Ω_2. The measure μ_j is a Borel measure in Ω_2 and it vanishes on pluripolar sets by Lemma 4.11 in [1]. Moreover, from (2.4) it follows that $\mu_j(\Omega_2) < \infty$. Theorem 3.7 in [1] together with Proposition 2.2(c) shows that there exists a unique $\psi_j \in \mathcal{F}(\Omega_2, G)$ such that $(dd^c\psi_j)^n = \mu_j$ on Ω_2. Theorem 5.11 in [11] implies that there exist $w_j \in \mathcal{E}_0(\Omega_2, 0)$ and $\varphi_j \in L^1(\Omega_2, (dd^c w_j)^n)$, $\varphi_j \geq 0$, such that $\mu_j = \varphi_j(dd^c w_j)^n$ on Ω_2. For $k \in \mathbb{N}$ define the measure μ_{jk} on Ω_2 by

$$\mu_{jk} = \min(\varphi_j, k)(dd^c w_j)^n.$$

It follows from Theorem 3.7 in [1] and Proposition 2.2(c) that there exist decreasing sequences $[\psi_{jk}]_{k=1}^{\infty} \subset \mathcal{F}(\Omega_2, G)$ and $[\varphi_{jk}]_{k=1}^{\infty} \subset \mathcal{F}(\Omega_1, F)$ such that

$$(dd^c\psi_{jk})^n = \mu_{jk} \quad \text{on} \quad \Omega_2 \quad \text{and} \quad (dd^c\varphi_{jk})^n = \mu_{jk} \quad \text{on } \Omega_1.$$

Furthermore, $[\psi_{jk}]_{k=1}^{\infty}$ converges pointwise to ψ_j on Ω_2 and $[\varphi_{jk}]_{k=1}^{\infty}$ converges pointwise to u_j on Ω_1 as $k \to \infty$. Corollary 3.2 in [1] and (1.2) imply that

$$\psi_{jk} \leq \varphi_{jk} \quad \text{on } \Omega_1.$$

Thus, $\psi_j \le u_j$ on Ω_1. For each $j \in \mathbb{N}$ define $v_j = (\sup_{l \ge j} \psi_l)^*$. By construction we have $v_j \in \mathcal{F}(\Omega_2, G)$ and

$$v_j \le u_j \quad \text{on } \Omega_1, \tag{2.5}$$

and $v_j \ge \psi_j$ on Ω_2, and therefore

$$\int_{\Omega_2} (dd^c v_j)^n \le \int_{\Omega_2} (dd^c \psi_j)^n = \int_{\Omega_1} (dd^c u_j)^n,$$

hence

$$\sup_j \int_{\Omega_2} (dd^c v_j)^n \le \sup_j \int_{\Omega_1} (dd^c u_j)^n < \infty. \tag{2.6}$$

Thus, $(\lim_{j\to\infty} v_j) \in \mathcal{F}(\Omega_2, G)$, by Proposition 2.3. Let $v = \lim_{j\to\infty} v_j$. Then (2.5) implies that $v \le u$ on Ω_1 and by (2.6) and Proposition 2.2(b) we have

$$\int_{\Omega_2} (dd^c v)^n \le \int_{\Omega_1} (dd^c u)^n,$$

which completes the proof in this case.

Now assume that $u \in \mathcal{F}(\Omega_1, F)$ is such that

$$\int_{\Omega_1} (dd^c u)^n = \infty. \tag{2.7}$$

Then it suffices to construct v in $\mathcal{F}(\Omega_2, G)$ such that $v \le u$ on Ω_1. By definition there exists $u' \in \mathcal{F}(\Omega_1, 0)$ such that

$$F \ge u \ge u' + F.$$

From the first part of the proof there exists $v' \in \mathcal{F}(\Omega_2, 0)$ such that $v' \le u'$ on Ω_1. Now let $v = v' + G$. Then $v \in \mathcal{F}(\Omega_2, G)$ and (1.2) yields

$$u \ge u' + F \ge v' + G = v$$

on Ω_1. Thus, the proof is complete. ∎

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