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Some monotonicity and limit results for the regularised incomplete gamma function

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Abstract. Letting $P(u,x)$ denote the regularised incomplete gamma function, it is shown that for each $\alpha \geq 0$, $P(x,x+\alpha)$ decreases as x increases on the positive real semi-axis, and $P(x,x+\alpha)$ converges to $1/2$ as x tends to infinity. The statistical significance of these results is explored.

1. Introduction. Euler's *gamma function*

$$\Gamma(u) \triangleq \int_0^\infty t^{u-1}e^{-t}\,dt \qquad (u>0)$$

plays an important role in many areas of mathematics and has been widely studied. The *incomplete gamma function* and its *complement*

$$\left.\begin{aligned}\gamma(u,x) &\triangleq \int_0^x t^{u-1}e^{-t}\,dt\\ \Gamma(u,x) &\triangleq \int_x^\infty t^{u-1}e^{-t}\,dt\end{aligned}\right\} \qquad (u>0,\ x\geq 0),$$

and the *regularised incomplete gamma function* and its *complement*

$$\left.\begin{aligned}P(u,x) &\triangleq \frac{\gamma(u,x)}{\Gamma(u)}\\ Q(u,x) &\triangleq 1-P(u,x)\end{aligned}\right\} \qquad (u>0,\ x\geq 0)$$

also appear in many different contexts and applications. An extended and highly readable overview on the incomplete gamma function and the related functions can be found in [2]. For a sample of more recent work, see [3].

The aim of this paper is to prove that for each $\alpha \geq 0$, (i) $P(x,x+\alpha)$ decreases as x increases on the positive real semi-axis; and (ii) $P(x,x+\alpha)$ tends to $1/2$ as $x\to\infty$.

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The original motivation for these results comes from estimation theory. Suppose that the outcome of a chance experiment is described by a real-valued random variable X with mean m and variance σ^2. In the event that m and σ^2 are unknown, these values can be estimated based on several repetitions of the experiment. If the outcomes of n repetitions are represented by a sequence $X_1, \ldots, X_n$ of n independent copies of X, then a natural estimate of m is the *sample mean*

$$\overline{X}_n \triangleq \frac{1}{n} \sum_{k=1}^{n} X_k$$

and a natural estimate of σ^2 is the *sample variance*

$$S_n^2 \triangleq \frac{1}{n} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2.$$

Sometimes the sample variance is defined as

$$S_n'^2 \triangleq \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2.$$

The advantage of adopting the latter expression is that it specifies a *mean-unbiased* estimator of σ^2—the expected value of $S_n'^2$ is equal to σ^2.

Assume henceforth that X is normally distributed. The random variable

$$Y_n \triangleq nS_n^2/\sigma^2 = (n-1)S_n'^2/\sigma^2 = \frac{1}{\sigma^2} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2$$

then has a chi-square distribution with $n-1$ degrees of freedom [14, Chapter 8, §45, Theorem 1] and its cumulative distribution function is given by

$$\begin{aligned} \mathsf{P}(Y_n \le x) &= \frac{1}{2^{(n-1)/2} \Gamma\big(\frac{1}{2}(n-1)\big)} \int_0^x t^{(n-1)/2-1} e^{-t/2}\, dt \\ &= P\big(\tfrac{1}{2}(n-1), \tfrac{1}{2}x\big) \qquad (x \ge 0), \end{aligned}$$

with $\mathsf{P}(A)$ denoting the probability of the event A. Furthermore, in accordance with a result of van der Vaart [11], $S_n'^2$ is a *negatively median-biased* estimator of σ^2 in the sense that

$$\mathsf{P}(S_n'^2 \le \sigma^2) > \frac{1}{2} \tag{1}$$

for each n. Starting from the identities

$$\mathsf{P}(S_n'^2 \le \sigma^2) = \mathsf{P}((n-1)S_n'^2/\sigma^2 \le n-1) = P\big(\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1)\big), \tag{2}$$

van der Vaart derived inequality (1) from a more general inequality that he had established, namely,

$$P(x, x) > \frac{1}{2} \tag{3}$$

for each $x > 0$.

In light of the above, one may wonder whether S_n^2 is also negatively median-biased. Noting, in analogy to (2), that

$$\mathsf{P}(S_n^2 \le \sigma^2) = \mathsf{P}(nS_n^2/\sigma^2 \le n) = P\big(\tfrac{1}{2}(n-1), \tfrac{1}{2}n\big), \tag{4}$$

one may ask, more generally, whether

$$P\left(x, x + \frac{1}{2}\right) > \frac{1}{2} \tag{5}$$

holds for each $x > 0$. It turns out that the answer to both these questions is in the affirmative.

Indeed, the monotonicity and limit properties of the functions $x \mapsto P(x, x+\alpha)$, $\alpha \ge 0$, that will be established below immediately imply that

$$P(x, x+\alpha) > \frac{1}{2}$$

for each $\alpha \ge 0$ and each $x > 0$. This inequality subsumes (3) and (5) as special cases corresponding to $\alpha = 0$ and $\alpha = 1/2$.

But perhaps a more significant consequence of the afore-mentioned properties of the functions $x \mapsto P(x, x+\alpha)$, $\alpha \ge 0$, one that relies on relations (2) and (4), is that the sequences $\{\mathsf{P}(S_n^2 \le \sigma^2)\}_{n=1}^\infty$ and $\{\mathsf{P}(S_n'^2 \le \sigma^2)\}_{n=1}^\infty$ decrease and have the common limit $1/2$. Thus, while always non-zero, the negative median bias in S_n^2 and in $S_n'^2$, measured by $\mathsf{P}(S_n^2 \le \sigma^2) - 1/2$ and $\mathsf{P}(S_n'^2 \le \sigma^2) - 1/2$, respectively, systematically decreases as n, the number of samples, mounts, reaching in limit the value zero.

2. Monotonicity result. We first establish the following.

THEOREM 1. *For each $\alpha \ge 0$, the function $x \mapsto P(x, x+\alpha)$ is decreasing on $(0, \infty)$.*

Proof. Fix $\alpha \ge 0$ arbitrarily. For each $x > 0$, represent

$$Q(x, x+\alpha) = \frac{1}{\Gamma(x)} \int_{x+\alpha}^{\infty} t^{x-1} e^{-t}\, dt$$

as

$$Q(x, x+\alpha) = f_1(x) f_2(x),$$

where

$$f_1(x) \triangleq \frac{x^{x-1/2} e^{-x}}{\Gamma(x)},$$

$$f_2(x) \triangleq x^{1/2-x} e^{x} \int_{x+\alpha}^{\infty} t^{x-1} e^{-t}\, dt.$$

The result of the theorem will be established once we show that both f_1 and f_2 are increasing.

That f_1 is increasing is a well-known fact and a special case of more general results (cf. [1, Theorem 2], [7, Theorem 1]). In what follows, we give a self-contained proof of the monotonicity property of f_1. We start with Binet's formula [13, p. 249]

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tx}}{t}\, dt,$$

which implies that

$$\ln f_1(x) = -\frac{1}{2} \ln(2\pi) - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tx}}{t}\, dt. \tag{6}$$

Now, as we shall see shortly, the function

$$g(t) \triangleq \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \quad (t > 0)$$

is positive, and, for each $t > 0$, the function $x \mapsto e^{-tx}$ monotonically decreases. This immediately implies the desired monotonicity result for f_1.

That $g(t)$ is positive for each $t > 0$ can be seen as follows. Using the Maclaurin series expansion of $t \mapsto e^t$, we find

$$\frac{1}{e^t - 1} - \frac{1}{t} = -\frac{e^t - 1 - t}{t(e^t - 1)} = -\frac{\frac{1}{2}t^2 + o(t^2)}{t^2 + o(t^2)} \to -\frac{1}{2} \quad \text{as } t \to 0,$$

so $\lim_{t\to 0} g(t) = 0$. The proof of the assertion will be complete once we show that g is increasing. Now

$$g'(t) = \frac{1}{t^2} - \frac{e^t}{(e^t - 1)^2} = \frac{(e^t - 1)^2 - t^2 e^t}{t^2 (e^t - 1)^2}.$$

The numerator of the rightmost term is equal to zero when $t = 0$ and its derivative

$$2(e^t - 1)e^t - 2te^t - t^2 e^t = 2e^t \left(e^t - 1 - t - \frac{t^2}{2}\right)$$

is positive, implying that both the numerator and $g'(t)$ are positive for $t > 0$. Thus $g(t)$ is indeed increasing for $t > 0$.

The positivity of g can alternatively be deduced from the representation

$$\frac{g(t)}{t} = \sum_{n=1}^\infty \frac{2}{t^2 + 4n^2\pi^2} \quad (t > 0)$$

(cf. [9, p. 64]). We also mention that the positivity of g can be viewed as part of a more general result concerning the Maclaurin series expansion of $t \mapsto t/(e^t - 1)$ (cf. [5, Theorem 3]).

We now pass to proving that f_2 is increasing. Setting $t = xw$, we obtain

$$\int_{x+\alpha}^{\infty} t^{x-1}e^{-t}\,dt = x^x \int_{1+\alpha/x}^{\infty} w^{x-1}e^{-xw}\,dw = x^x e^{-x} \int_{1+\alpha/x}^{\infty} e^{-xv(w)}\,\frac{dw}{w}, \tag{7}$$

where

$$v(w) \triangleq w - \ln w - 1.$$

It is readily verified that the function $w \mapsto v(w)$ is increasing on $[1,\infty)$ with image $[0,\infty)$. Let $t \mapsto w(t)$ be its inverse, which, of course, is an increasing function from $[0,\infty)$ onto $[1,\infty)$. For each $x > 0$, let

$$t_x \triangleq \frac{\alpha}{x} - \ln\left(1 + \frac{\alpha}{x}\right).$$

Clearly, t_x is non-negative, with $t_x = 0$ when $\alpha = 0$, and, as

$$v\left(1 + \frac{\alpha}{x}\right) = t_x,$$

we have

$$w(t_x) = 1 + \frac{\alpha}{x}.$$

In an independent step, note that differentiating the relation

$$w(t) - \ln w(t) - 1 = t$$

leads to

$$w'(t) = \frac{w(t)}{w(t) - 1} \tag{8}$$

for $t > 0$. Now, the change of variable $w = w(t)$ and the subsequent change $t = s/x$ in the rightmost integral of (7) with use of (8) in between yield

$$\begin{aligned}\int_{1+\alpha/x}^{\infty} e^{-xv(w)}\,\frac{dw}{w} &= \int_{t_x}^{\infty} e^{-xt}\,\frac{w'(t)}{w(t)}\,dt = \int_{t_x}^{\infty} e^{-xt}\,\frac{dt}{w(t)-1}\\ &= x^{-1}\int_{xt_x}^{\infty} e^{-s}\,\frac{ds}{w\left(\frac{s}{x}\right) - 1}.\end{aligned}$$

Hence

$$f_2(x) = x^{-1/2}\int_{xt_x}^{\infty} e^{-s}\,\frac{ds}{w\left(\frac{s}{x}\right) - 1},$$

or, equivalently,

$$f_2(x) = \int_0^{\infty} 1_{(xt_x,\infty)}(s)\,h\left(\frac{s}{x}\right)s^{-1/2}e^{-s}\,ds, \tag{9}$$

where 1_E denotes the characteristic function of the set E and

$$h(t) \triangleq \frac{t^{1/2}}{w(t)-1} \qquad (t>0). \tag{10}$$

We shall next show that

(i) the function h is decreasing on $(0,\infty)$;
(ii) the function $x \mapsto x t_x$ is non-increasing on $(0,\infty)$.

This will imply that, for each $s>0$, the function $x \mapsto h(s/x)$ is increasing on $(0,\infty)$ and the function $x \mapsto 1_{(xt_x,\infty)}(s)$ is non-decreasing on $(0,\infty)$. The increasing monotonicity of f_2 will then follow on account of (9).

To prove (i), it suffices to show that the function

$$h_1(t) \triangleq h^{-2}(t) = \frac{(w(t)-1)^2}{t} \qquad (t>0)$$

is increasing. To this end, define

$$h_2(t) \triangleq \frac{1}{2}\,(w(t)-1)^2 - tw(t) \qquad (t \geq 0).$$

In view of (8),

$$h_2'(t) = (w-1)w' - w - tw' = -tw' = -\frac{tw}{w-1} < 0,$$

so h_2 is decreasing. Since $h_2(0)=0$, it follows that $h_2(t)<0$ for each $t>0$. The latter result can be reformulated as

$$2 - \frac{(w-1)^2}{tw} > 0 \tag{11}$$

for each $t>0$. Now, in view of (8),

$$\begin{aligned} h_1'(t) &= \frac{2(w-1)w'}{t} - \frac{(w-1)^2}{t^2} = \frac{2w}{t} - \frac{(w-1)^2}{t^2} \\ &= \frac{w}{t}\left[2 - \frac{(w-1)^2}{tw}\right]. \end{aligned}$$

This together with (11) yields $h_1'(t)>0$ for each $t>0$, showing that h_1 is increasing.

To establish (ii), note that the derivative of $x \mapsto x t_x$ at $x>0$ is equal to

$$\frac{\alpha}{x+\alpha} - \ln\left(1+\frac{\alpha}{x}\right). \tag{12}$$

By the mean-value theorem,

$$\ln\left(1+\frac{\alpha}{x}\right) = \ln(x+\alpha) - \ln x = \frac{\alpha}{\xi}$$

for some ξ with $x \leq \xi \leq x+\alpha$. It is now obvious that expression (12) is non-positive, yielding the desired result. ∎

3. Limit result. We now prove the following.

THEOREM 2. *For each* $\alpha \geq 0$, $\lim_{x\to\infty} P(x, x+\alpha) = 1/2$.

Proof. Continuing with the notation from the proof of Theorem 1, we first calculate separately $\lim_{x\to\infty} f_1(x)$ and $\lim_{x\to\infty} f_2(x)$.

Using (6) and the fact that the integrand in (6) tends decreasingly to zero as x increases to infinity, we infer from Levi's monotone convergence theorem that

$$\lim_{x\to\infty} \ln f_1(x) = -\frac{1}{2}\ln(2\pi),$$

whence

$$\lim_{x\to\infty} f_1(x) = \frac{1}{\sqrt{2\pi}}. \tag{13}$$

This latter result can also be deduced from the well-known asymptotic expansion for the logarithm of the gamma function (see e.g. [9, p. 62]).

To determine the other limit, first note that

$$\lim_{x\to\infty} xt_x = \alpha - \lim_{x\to\infty} x\ln\left(1+\frac{\alpha}{x}\right) = 0.$$

As the function $x \mapsto xt_x$ is non-increasing on $(0,\infty)$, we see that, for each $s>0$, $1_{(xt_x,\infty)}(s)$ non-decreasingly tends to 1 as x increases to infinity. Next, note that by de l'Hôpital's rule and (8),

$$\lim_{t\to 0}\frac{(w(t)-1)^2}{t} = \lim_{t\to 0} 2(w(t)-1)w'(t) = \lim_{t\to 0} 2w(t) = 2.$$

As h (defined in (10)) is decreasing on $(0,\infty)$, we deduce that, for each $s>0$, $x \mapsto h(s/x)$ increasingly tends to $2^{-1/2}$ as x increases to infinity. Thus, for each $s>0$, the integrand in (9) non-decreasingly tends to $2^{-1/2}s^{-1/2}e^{-s}$ as x increases to infinity. An application of Levi's monotone convergence theorem now reveals that

$$\lim_{x\to\infty} f_2(x) = 2^{-1/2}\int_0^\infty s^{-1/2}e^{-s}\,ds,$$

which jointly with

$$\int_0^\infty s^{-1/2}e^{-s}\,ds = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$$

yields

$$\lim_{x\to\infty} f_2(x) = \sqrt{\frac{\pi}{2}}.$$

Finally, the last equality together with (13) leads to

$$\lim_{x\to\infty} P(x,x+\alpha) = 1 - \lim_{x\to\infty} Q(x,x+\alpha) = 1 - \lim_{x\to\infty} f_1(x)f_2(x) = \frac{1}{2},$$

establishing the theorem. ∎

4. Related work. We conclude with a few comments about related results reported in the literature.

Van der Vaart [11] established that for each $x > 0$ the sequence $\{P(x+n, x+n)\}_{n=1}^{\infty}$ decreases and has limit $1/2$. Inequality (3) is one consequence of this result. Another, based on (2), is that the sequence $\{\mathsf{P}(S'^2_{2n+m} \le \sigma^2)\}_{n=1}^{\infty}$ decreases when $m = 0$ and $m = 1$; the objects involved here are the same as in the Introduction. Note that van der Vaart's result is insufficient to infer that the sequence $\{\mathsf{P}(S'^2_n \le \sigma^2)\}_{n=1}^{\infty}$ decreases. However, as was already alluded to earlier, this latter result follows immediately from our Theorem 1.

Vietoris [12] proved that the sequence $\{P(n,n)\}_{n=1}^{\infty}$ decreases and the sequence $\{P(n,n-1)\}_{n=1}^{\infty}$ increases, with $1/2$ being the common limit of both sequences.

Van de Lune [10] and, independently, Temme [8] proved that the function $x \mapsto P(x, x-1)$ increases to $1/2$ on $[1,\infty)$.

Merkle [6] asserted that the function $x \mapsto P(x,x)$ is decreasing on $(0,\infty)$, but his argument to validate the statement is incorrect. Merkle represents $P(x,x)$ as $P(x,x) = p_1(x)p_2(x)$, where $p_1(x) \triangleq x^{x-1}e^{-x}/\Gamma(x)$ and $p_2(x) \triangleq \gamma(x,x)x^{1-x}e^{x}$, and claims that both p_1 and p_2 are decreasing. But while the first function is decreasing [4], the second is not. Figure 1 illustrates the

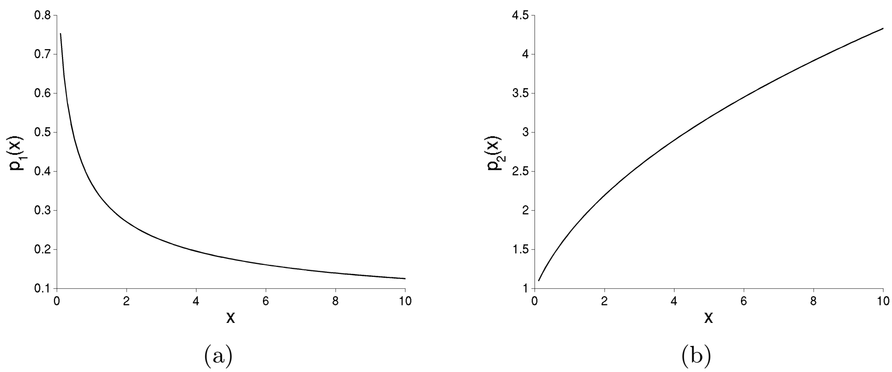

Fig. 1. Contrasting behaviours of p_1 and p_2: (a) graph of p_1; (b) graph of p_2.

```
x = linspace(0,10);                           % 100 equally
                                              % spaced values
                                              % between 0 and 10
y = x.^(x-1).*exp(-x)./gamma(x);              % definition of p1
z = gammainc(x,x)./y;                         % definition of p2
plot(x,y); xlabel('x'); ylabel('p_1(x)');     % graph of p1
plot(x,z); xlabel('x'); ylabel('p_2(x)');     % graph of p2
```

Fig. 2. Basic MATLAB code to generate graphs of p_1 and p_2.

different behaviours of the two functions. A basic MATLAB code to generate the relevant graphs is given in Figure 2.

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