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Normal families of meromorphic mappings of several complex variables into $\mathbb{C}P^n$ for moving hypersurfaces

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Abstract. We prove some normality criteria for families of meromorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$ under a condition on the inverse images of moving hypersurfaces.

1. Introduction. Classically, a family $\mathcal{F}$ of meromorphic functions defined on a domain D of the complex plane $\mathbb{C}$ is said to be *normal* on D if every sequence of functions of $\mathcal{F}$ has a subsequence which converges uniformly on every compact subset of D with respect to the spherical metric to a function meromorphic or identically ∞ on D.

The concept of normal families of meromorphic functions in several complex variables was first introduced by H. Rutishauser and W. Stoll. In 1974, H. Fujimoto introduced the notion of a meromorphically normal family into the complex projective space.

Let f be a meromorphic mapping of a domain D in $\mathbb{C}^m$ into $\mathbb{C}P^n$. Then for each $a \in D$, f has a reduced representation $\widetilde{f} = (f_0, \ldots, f_n)$ on a neighborhood U of a in D, which means that each f_i is a holomorphic function on U and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $I(f) := \{z \mid f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2.

Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbb{C}^m$ into a compact complex manifold X. Then $\mathcal{F}$ is said to be a *normal family* on D if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into X.

A sequence $\{f_k\}_{k=1}^{\infty}$ of meromorphic mappings of a domain D in $\mathbb{C}^m$ into $\mathbb{C}P^n$ is said to *converge meromorphically* on D to a meromorphic mapping f of D into $\mathbb{C}P^n$ if, for any $z \in D$, each f_k has a reduced representation $\widetilde{f}_k = (f_{k0}, \ldots, f_{kn})$ on some fixed neighborhood U of z such that $\{f_{ki}\}_{k=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function f_i

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$(0 \le i \le n)$ on U with the property that $(f_0, \ldots, f_n)$ is a representation of f in U.

A family $\mathcal{F}$ of meromorphic mappings of a domain D in $\mathbb{C}^m$ into $\mathbb{C}P^n$ is said to be *meromorphically normal* on D if any sequence in $\mathcal{F}$ has a meromorphically convergent subsequence on D.

Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on D. Let Q be a homogeneous polynomial in $\mathcal{H}_D[x_0, \ldots, x_n]$ of degree $d \ge 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by substituting a specific point $z \in D$ into the coefficients of Q. We define a *moving hypersurface* in $\mathbb{C}P^n$ to be any homogeneous polynomial $Q \in \mathcal{H}_D[x_0, \ldots, x_n]$ such that the coefficients of Q have no common zero point. We say that moving hypersurfaces $\{Q_j\}_{j=1}^q$ $(q \ge n+1)$ in $\mathbb{C}P^n$ are *in general position* (respectively *in pointwise general position* on a subset $\Omega \subset \mathbb{C}^m$) if for some $z \in \mathbb{C}^m$ (respectively for all $z \in \Omega$) and for any $1 \le j_0 < \cdots < j_n \le q$ the system of equations

$$Q_{j_i}(z)(w_0, \ldots, w_n) = 0, \quad 0 \le i \le n,$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$.

Let F be a nonzero holomorphic function on a connected open neighborhood D in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $D^\alpha F := \partial^{|\alpha|} F / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}$. For each $a \in D$, the number $\nu_F(a) = \max\{p \mid D^\alpha F(a) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}$ is said to be the *zero-multiplicity* of F at a. Set

$$|\nu_F| = \overline{\{z \mid \nu_F(z) \neq 0\}}.$$

Let f be a meromorphic mapping of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$. For each moving hypersurface Q in $\mathbb{C}P^n$, we define the divisor $\nu(f, Q)$ on D as follows: For each $a \in D$, let $\widetilde{f} = (f_0, \ldots, f_n)$ be a reduced representation of f in a neighborhood U of a, and put $\nu(f, Q)(a) := \nu_{Q(\widetilde{f})}(a)$, where $Q(\widetilde{f}) := Q(f_0, \ldots, f_n)$. Sometimes we identify $f^{-1}(Q)$ with the divisor $\nu(f, Q)$. We say that f *intersects* Q *on* D *with multiplicity at least* k if $\nu(f, Q)(z) \ge k$ for all $z \in \operatorname{supp} \nu(f, Q)$.

In 1974, H. Fujimoto proved the following result.

THEOREM 1.1. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^{2n+1}$ be hyperplanes in $\mathbb{C}P^n$ in general position such that for each $f \in \mathcal{F}$, $f(D) \not\subset H_j$ $(j = 1, \ldots, 2n+1)$, and for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(H_j) \cap K$ $(j = 1, \ldots, 2n+1)$ counting multiplicities are uniformly bounded above for all f in $\mathcal{F}$. Then $\mathcal{F}$ is a meromorphically normal family on D.*

In 2005, Tu and Li [13] extended the above theorem to the case of moving hyperplanes as follows:

THEOREM 1.2. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^q$ be q $(\geq 2n+1)$ moving hyperplanes in $\mathbb{C}P^n$ in pointwise general position on D such that each f in $\mathcal{F}$ intersects H_j on D with multiplicity at least m_j $(j = 1, \ldots, q)$, where $m_1, \ldots, m_q$ are fixed positive integers or $+\infty$, with $\sum_{j=1}^q 1/m_j < (q-n-1)/n$. Then $\mathcal{F}$ is a normal family on D.*

THEOREM 1.3. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^{2n+1}$ be moving hyperplanes in $\mathbb{C}P^n$ in pointwise general position on D such that for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(H_j) \cap K$ $(j = 1, \ldots, 2n+1)$ counting multiplicities are uniformly bounded above for all f in $\mathcal{F}$. Then $\mathcal{F}$ is a meromorphically normal family on D.*

The following question arises naturally: *Are there normality criteria for families of meromorphic mappings, involving hypersurfaces?*

It seems to us that the difficulty of this case comes from the fact that we do not have the Second Main Theorem in value distribution theory for hypersurfaces and truncated multiplicities. In this paper we will give some normality criteria for families of meromorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$, involving moving hypersurfaces. Our first aim is to generalize the above results to this case. Furthermore, we also obtain an improvement concerning counting multiplicities (Theorem 1.4). The second aim is to find some normality criteria for the case of few moving hypersurfaces (Theorems 1.5–1.6). We note that so far, all results about normality criteria for families of meromorphic mappings into $\mathbb{C}P^n$ have been restricted to the case where the number of hyperplanes q is at least $2n+1$.

In order to prove Theorems 1.5 and 1.6, we need some results of value distribution theory of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$, involving hypersurfaces. But the Second Main Theorems as in [8] (for fixed hypersurfaces) or as in [1] (for moving hypersurfaces) which are the best results available at present seem not to be sufficient for our purpose. In order to overcome this difficulty we establish, for the special situation of the hypersurfaces in these theorems, a Second Main Theorem for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and multiplicities truncated by n.

Our main results are as follows:

THEOREM 1.4. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$, and let $Q_1, \ldots, Q_q$ $(q \geq 2n+1)$ be q moving hypersurfaces in $\mathbb{C}P^n$ in general position such that:*

(i) *For any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap K$ $(1 \leq j \leq n+1)$ counting multiplicities are uniformly bounded above for all f in $\mathcal{F}$.*

(ii) *There exists a thin analytic subset S of D such that for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap (K \setminus S)$ $(n+2 \le j \le q)$ regardless of multiplicities are uniformly bounded above for all f in $\mathcal{F}$.*

Then $\mathcal{F}$ is a meromorphically normal family on D.

THEOREM 1.5. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$, and let $Q_0, \dots, Q_n$ be $n+1$ moving hypersurfaces in $\mathbb{C}P^n$ in pointwise general position on $\overline{D}$ of common degree $d \ge 1$. Define moving hypersurfaces $L_1, \dots, L_n$ in $\mathbb{C}P^n$ by*

$$L_i = \sum_{j=0}^{n} a_{ij} Q_j^p,$$

where p is a fixed positive integer $(p > n(n+1))$ and a_{ij} $(0 \le i, j \le n)$ are holomorphic functions on $\mathbb{C}^m$ such that for any square submatrix A of $(a_{ij})_{0 \le i,j \le n}$, $\det A \ne 0$ on $\overline{D}$. Assume that each f in $\mathcal{F}$ intersects L_i on D with multiplicity at least m_i, where $m_1, \dots, m_n$ are fixed positive integers or ∞, with

$$\sum_{i=1}^{n} \frac{1}{m_i} < \frac{p - n(n+1)}{np}.$$

Then $\mathcal{F}$ is a normal family.

THEOREM 1.6. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$, and let $Q_0, \dots, Q_n$ be $n+1$ moving hypersurfaces in $\mathbb{C}P^n$ in general position of common degree $d \ge 1$. Define moving hypersurfaces $L_1, \dots, L_n$ in $\mathbb{C}P^n$ by*

$$L_i = \sum_{j=0}^{n} a_{ij} Q_j^p,$$

where p is a fixed positive integer $(p > n(n+1))$ and a_{ij} $(0 \le i, j \le n)$ are holomorphic functions on D such that for any square submatrix A of $(a_{ij})_{0 \le i,j \le n}$, $\det A \not\equiv 0$. Assume that for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(L_i) \cap K$ (for all $1 \le i \le n$) and $f^{-1}(Q_{i_0}) \cap K$ (for some $i_0 \in \{0, \dots, n\}$) counting multiplicities are uniformly bounded above for all f in $\mathcal{F}$. Then $\mathcal{F}$ is a meromorphically normal family.

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2. Notations

2.1. For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, we set $\|z\| = (\sum_{j=1}^{n} |z_j|^2)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^m \mid \|z\| < r\}, \qquad S(r) = \{z \in \mathbb{C}^m \mid \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \mathcal{V} = (dd^c\|z\|^2)^{m-1}, \quad \sigma = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For each positive integer p (or $+\infty$), we define the counting function of F (where multiplicities are truncated by p) by

$$N_F^{[p]}(r) := \int_1^r \frac{n_F^{[p]}(t)}{t^{2m-1}}\, dt \qquad (1 < r < +\infty),$$

where

$$n_F^{[p]}(t) = \begin{cases} \displaystyle\int_{|\nu_F| \cap B(t)} \min\{v_F, p\} \cdot \mathcal{V} & \text{for } m \geq 2, \\ \displaystyle\sum_{|z| \leq t} \min\{\nu_F(z), p\} & \text{for } m = 1. \end{cases}$$

2.2. Let f be a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. For fixed homogeneous coordinates $(w_0 : \dots : w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $\widetilde{f} = (f_0, \dots, f_n)$ of f. Set $\|f\| = \max\{|f_0|, \dots, |f_n|\}$. The *characteristic function* of f is defined by

$$T_f(r) := \int_{S(r)} \log \|f\|\, \sigma - \int_{S(1)} \log \|f\|\, \sigma, \qquad 1 < r < +\infty.$$

2.3. We state the First and Second Main Theorems in value distribution theory:

First Main Theorem. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and Q be a homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$ of degree $d \geq 1$ such that $Q(f) \not\equiv 0$. Then*

$$N_{Q(f)}(r) \leq dT_f(r) + O(1) \quad \textit{for all } r > 1.$$

Second Main Theorem (Classical version). *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1, \dots, H_q$ $(q \geq n+1)$ be hyperplanes in $\mathbb{C}P^n$ in general position. Then*

$$(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N_{(f,H_j)}^{[n]}(r) + o(T_f(r))$$

for all r except for a subset E of $(1, +\infty)$ of finite Lebesgue measure.

3. Proof of Theorem 1.4. In order to prove Theorem 1.4, we need some preparations.

DEFINITION 3.1 ([10]). Let M be a locally compact Hausdorff space. A point p of M is called a *limit point* of a sequence $\{E_k\}_{k=1}^{\infty}$ of closed subsets of M if there exist a positive integer k_0 and points $p_k \in E_k$ $(k \geq k_0)$ such that $p = \lim p_k$. A point of M is called a *cluster point* of $\{E_k\}_{k=1}^{\infty}$ if it is a limit point of some subsequence of $\{E_k\}_{k=1}^{\infty}$. If the set E of limit points coincides with the set of cluster points, $\{E_k\}_{k=1}^{\infty}$ is said to *converge to* E and we write $\lim E_k = E$.

LEMMA 3.1 ([10, Proposition 4.11]). *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of pure $(m-1)$-dimensional analytic subsets of a domain D in $\mathbb{C}^m$. Assume that the $2(m-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \dots)$ are bounded above for any fixed compact subset K of D, and $\{N_i\}_{i=1}^{\infty}$ converges to N as a sequence of closed subsets of D. Then N is either empty or a pure $(m-1)$-dimensional analytic subset of D.*

LEMMA 3.2 ([10, Proposition 4.12]). *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of pure $(m-1)$-dimensional analytic subsets of a domain D in $\mathbb{C}^m$. Assume that the $2(m-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \dots)$ are bounded above for any fixed compact subset K of D. Then $\{N_i\}$ is normal in the sense of the convergence of closed subsets in D.*

DEFINITION 3.2 ([13, Definition 4.4]). Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of nonnegative divisors on a domain D in $\mathbb{C}^m$. It is said to *converge* to a nonnegative divisor ν on D if any $a \in D$ has a neighborhood U such that there exist nonzero holomorphic functions h and h_i on U with $\nu_i = \nu_{h_i}$ and $\nu = \nu_h$ on U such that $\{h_i\}_{i=1}^{\infty}$ converges to h uniformly on compact subsets of U.

LEMMA 3.3 ([10, Theorem 2.24]). *A sequence $\{\nu_i\}_{i=1}^{\infty}$ of nonnegative divisors on a domain D in $\mathbb{C}^m$ is normal in the sense of convergence of divisors on D if and only if the $2(m-1)$-dimensional Lebesgue areas of $\nu_i \cap E$ $(i = 1, 2, \dots)$ counting multiplicities are bounded above for any compact subset E of D.*

LEMMA 3.4 ([3, Proposition 3.5]). *Let $\{f_i\}$ be a sequence of meromorphic mappings of a domain D in $\mathbb{C}^m$ into $\mathbb{C}P^n$ and let S be a thin analytic subset of D. Suppose that $\{f_i\}$ meromorphically converges on $D \setminus S$ to a meromorphic mapping f of $D \setminus S$ into $\mathbb{C}P^n$. If there exists a hyperplane H in $\mathbb{C}P^n$ such that $f(D \setminus S) \not\subset H$ and $\{\nu(f_i, H)\}$ is a convergent sequence of divisors on D, then $\{f_i\}$ is meromorphically convergent on D.*

DEFINITION 3.3 ([12, Definition 2.2]). Let X, Y be complex spaces and $\mathcal{F} \subset \mathrm{Hol}(X, Y)$.

(i) We say that a sequence $\{f_j\} \subset \mathcal{F}$ is *compactly divergent* if for every compact set $K \subset X$ and for every compact set $L \subset Y$ there is a number $j_0(K, L)$ such that $f_j(K) \cap L = \emptyset$ for all $j \geq j_0(K, L)$.

(ii) The family $\mathcal{F}$ is *not compactly divergent* if $\mathcal{F}$ contains no compactly divergent subsequences.

LEMMA 3.5 ([12, Theorem 2.5]). *Let D be a domain in $\mathbb{C}^m$ and M be a complex Hermitian space. Let $\mathcal{F} \subset \mathrm{Hol}(D, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{p_j\} \subset D$ with $p_j \to p_0 \in D$, $\{f_j\} \subset \mathcal{F}$, $\varrho_j \subset \mathbb{R}$ with $\varrho_j > 0$ and $\varrho_j \to 0$ such that the functions $g_j(z) := f_j(p_j + \varrho_j z)$, $z \in \mathbb{C}^m$ satisfy one of the following two assertions*:

(i) *The sequence $\{g_j\}$ is compactly divergent on $\mathbb{C}^m$.*

(ii) *The sequence $\{g_j\}$ converges uniformly on compact subsets of $\mathbb{C}^m$ to a nonconstant holomorphic map $g : \mathbb{C}^m \to M$.*

Proof of Theorem 1.4. Without loss of generality, we may assume that D is a polydisc in $\mathbb{C}^m$, $D = \Delta^m$. By replacing Q_i by $Q_i^{d_i}$ where d_i is a suitable positive integer for all $i = 1, \ldots, q$, we may assume that all the Q_i $(i = 1, \ldots, q)$ have the same degree d.

Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ be an arbitrary sequence. By Lemma 3.2, there exists a subsequence (again denoted by $\{f_k\}_{k=1}^{\infty}$) such that

$$\lim_{k\to\infty} f_k^{-1}(Q_i) = S_i \quad (i = 1, \ldots, q) \tag{3.1}$$

as a sequence of closed subsets of D, where S_i $(i = 1, \ldots, q)$ are either empty or pure $(m-1)$-dimensional analytic sets of D.

Set

$$\mathcal{T}_d := \big\{(i_0, \ldots, i_n) \in \mathbb{N}_0^{n+1} \mid i_0 + \cdots + i_n = d\big\}.$$

Assume that

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I \quad (j = 1, \ldots, q),$$

where $a_{jI} \in \mathcal{H}_D$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$ for $x = (x_0, \ldots, x_n)$ and $I = (i_0, \ldots, i_n)$.

Let $T = (\ldots, t_{kI}, \ldots)$ $(k \in \{1, \ldots, q\}, I \in \mathcal{T}_d)$ be a family of variables. Set

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{Z}[T, x], \quad j = 1, \ldots, q.$$

For each subset $L \subset \{1, \ldots, q\}$ with $|L| = n+1$, let $\widetilde{R}_L \in \mathbb{Z}[T]$ be the resultant of $\widetilde{Q}_j$ $(j \in L)$. Since $\{Q_j\}_{j\in L}$ are in general position, $\widetilde{R}_L(\ldots, a_{kI}, \ldots) \not\equiv 0$. Set $\widetilde{S} := \{z \in D \mid \widetilde{R}_L(\ldots, a_{kI}, \ldots) = 0$ for some $L \subset \{1, \ldots, q\}$ with $|L| = n+1\}$. Let $E := \bigcup_{i=1}^{q} S_i \cup S \cup \widetilde{S}$. Then E is a thin analytic subset of D. For any fixed point $z_0 \in D \setminus E$, there exist a relatively compact Stein neighborhood U_{z_0} of z_0 in $D \setminus E$ and a positive integer k_0 such that for all $k \geq k_0$,

$$f_k^{-1}(Q_i) \cap U_{z_0} = \emptyset. \tag{3.2}$$

Since $Q_i(\widetilde{f}_k) \neq 0$ on U_{z_0}, we deduce that $U_{z_0} \cap I(f_k) = \emptyset$ $(k \geq k_0)$. Then $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty} \subset \mathrm{Hol}(U_{z_0}, \mathbb{C}P^m)$. We now prove that $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}. Indeed, otherwise by Lemma 3.5 there exist a subsequence (again denoted by $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$) and $p_0 \in U_{z_0}$, $\{p_k\}_{k\geq k_0}^{\infty} \subset U_{z_0}$ with $p_k \to p_0$, $\{\varrho_k\} \subset (0, +\infty)$ with $\varrho_j \to 0$ such that the sequence of holomorphic maps

$$g_k(z) := f_k(p_k + \varrho_k z) : \varDelta_{r_k}^m \to \mathbb{C}P^n, \quad k \geq k_0 \ (r_k \to \infty)$$

converges uniformly on compact subsets of $\mathbb{C}^m$ to a nonconstant holomorphic map $g : \mathbb{C}^m \to \mathbb{C}P^n$. Then there exist reduced representations $\widetilde{g}_j = (g_{j0}, \dots, g_{jn})$ of g_j $(j \geq k_0)$ and a representation $\widetilde{g} = (g_0, \dots, g_n)$ of g such that $\{\widetilde{g}_j\}$ converges uniformly on compact subsets of $\mathbb{C}^m$ to $\widetilde{g}$. This implies that $Q_j(p_k + \varrho_k z)(\widetilde{g}_k(z))$ converges uniformly on compact subsets of $\mathbb{C}^m$ to $Q_j(p_0)(\widetilde{g}(z))$. By (3.2) and Hurwitz's theorem, for each $j \in \{1, \dots, q\}$ we have

(i) $Q_j(p_0)(\widetilde{g}) \neq 0$ on $\mathbb{C}^m$, i.e. $g(\mathbb{C}^m) \cap Q_j(p_0) = \emptyset$, or
(ii) $Q_j(p_0)(\widetilde{g}) \equiv 0$ on $\mathbb{C}^m$, i.e. $g(\mathbb{C}^m) \subset Q_j(p_0)$

(we identify the polynomial $Q_j(p_0) \in \mathbb{C}[x_0, \dots, x_n]$ with the hypersurface in $\mathbb{C}P^n$ defined by $Q_j(p_0)$).

Denote by I the set of all indices $j \in \{1, \dots, q\}$ with $g(\mathbb{C}^m) \subset Q_j(p_0)$. Set $X := \bigcap_{j \in I} Q_j(p_0)$ if $I \neq \emptyset$, and $X := \mathbb{C}P^n$ if $I = \emptyset$. Since $\mathbb{C}^m$ is irreducible, there exists an irreducible component Z of X such that $g(\mathbb{C}^m) \subset Z \setminus \bigcup_{i \notin I} Q_j(p_0)$. Since $p_0 \in U_{z_0}$, we see that $\{Q_j(p_0)\}_{j=1}^{q}$ are in general position in $\mathbb{C}P^n$. This implies that $\{Q_j(p_0) \cap Z\}_{j \notin I}$ are in general position in Z. Furthermore, it is easy to see that $\#(\{1, \dots, q\} \setminus I) \geq 2 \dim Z + 1$, since $q \geq 2n + 1$. From these facts, by Corollary 1.4 in [7], we infer that $Z \setminus \bigcup_{i \notin I} Q_i(p_0)$ is hyperbolic. Hence, g is constant. This is a contradiction, hence $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}.

By the usual diagonal argument, we can find a subsequence (again not relabeled) which converges uniformly on compact subsets of $D \setminus E$ to a holomorphic map f. Since $\{Q_j\}_{j=1}^{n+1}$ are in general position, there exists a fixed index j_0 $(1 \leq j_0 \leq n+1)$ such that $Q_{j_0}(\widetilde{f}) \not\equiv 0$ on $D \setminus E$. We define meromorphic mappings $\{F_k\}_{k=1}^{\infty}$ of D into $\mathbb{C}P^{n+1}$ as follows: for any $z \in D$, if f_k has a reduced representation $\widetilde{f}_k = (f_{k0}, \dots, f_{kn})$ on a neighborhood $U_z \subset D$ then F_k has a reduced representation $\widetilde{F}_k = (f_{k0}^d, \dots, f_{kn}^d, Q_{j_0}(\widetilde{f}_k))$

on U_z. Let H_i $(i = 0, \dots, n)$ be hyperplanes in $\mathbb{C}P^n$ defined by

$$H_i = \{(w_0 : \cdots : w_n) \mid w_i = 0\}$$

and let $\overline{H}_i$ $(i = 0, \dots, n+1)$ be hyperplanes in $\mathbb{C}P^{n+1}$ defined by

$$\overline{H}_i = \{(w_0 : \cdots : w_{n+1}) \mid w_i = 0\}.$$

It is easy to see that $\{F_k\}$ converges uniformly on compact subsets of $D \setminus E$ to a holomorphic map F of $D \setminus E$ into $\mathbb{C}P^{n+1}$, and if f has a reduced representation $\widetilde{f} = (f_0, \dots, f_n)$ on an open subset $U \subset D$ then F has the reduced representation $\widetilde{F} = (f_0^d, \dots, f_n^d, Q_{j_0}(\widetilde{f}))$ on U. Since f is holomorphic on $D \setminus E$, there exists i_0 $(0 \le i_0 \le n)$ such that $H_{i_0}(\widetilde{f}) \not\equiv 0$ on $D \setminus E$, and hence $\overline{H}_{i_0}(\widetilde{F}) \not\equiv 0$ on $D \setminus E$. Then there exists $k_0 > 0$ such that $H_{i_0}(\widetilde{f}_k) \not\equiv 0$, $\overline{H}_{i_0}(\widetilde{F}_k) \not\equiv 0$ on $D \setminus E$ for all $k \ge k_0$.

Since $Q_{j_0}(\widetilde{f}) \not\equiv 0$ on $D \setminus E$, we have $\overline{H}_{n+1}(\widetilde{F}) \not\equiv 0$ on $D \setminus E$. By (3.1) and Lemma 3.3, we may assume that $F_k^{-1}(\overline{H}_{n+1})$ converges in the sense of convergence of divisors on D to a divisor (note that $1 \le j_0 \le n+1$). By Lemma 3.4, $\{F_k\}$ is meromorphically convergent on D. This implies that $\{F_k^{-1}(\overline{H}_{i_0})\}_{k \ge k_0}$ converges, and hence $\{f_k^{-1}(H_{i_0})\}_{k \ge k_0}$ converges in the sense of convergence of divisors on D. By Lemma 3.4 again, $\{f_k\}_{k \ge k_0}$ is meromorphically convergent on D. Therefore $\{f_k\}$ has a meromorphically convergent subsequence on D. Thus $\mathcal{F}$ is a meromorphically normal family on D. ∎

4. Proof of Theorem 1.5. As usual, by the notation "|| P" we mean that the assertion P holds for all $r \in (1, +\infty)$ excluding a subset E of $(1, +\infty)$ of finite Lebesgue measure.

Lemma 4.1 ([11, Corollary 1]). *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping. Let $\{H_i\}_{i=1}^q$ $(q \ge 2n+1)$ be fixed hyperplanes in $\mathbb{C}P^n$ in general position such that $H_i(\widetilde{f}) \not\equiv 0$ $(1 \le i \le q)$. Then*

$$\Big\| \quad \frac{q}{2n+1} T_f(r) \le \sum_{i=1}^{q} N^{[n]}_{H_i(\widetilde{f})}(r) + o(T_f(r)).$$

Lemma 4.2. *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping. Let $Q_0, \dots, Q_n$ be $n+1$ homogeneous polynomials of $\mathbb{C}[x_0, \dots, x_n]$ of common degree $d \ge 1$. Assume that the hypersurfaces defined by $Q_0, \dots, Q_n$ in $\mathbb{C}P^n$ have no common point. Define homogeneous polynomials $L_1, \dots, L_n$ by*

$$L_i = \sum_{j=0}^{n} \lambda_{ij} Q_j^p,$$

where λ_{ij} are constants such that all submatrices of $(\lambda_{ij})_{1 \le i \le n,\, 0 \le j \le n}$ are nonsingular and p is a positive integer $(p > n(n+1))$. Denote by F the meromorphic mapping $F = (Q_0(\widetilde{f}) : \cdots : Q_n(\widetilde{f})) : \mathbb{C}^m \to \mathbb{C}P^n$. Then

(i) $T_F(r) = dT_f(r) + O(1)$.

(ii) $\Big\|\ T_f(r) \le \dfrac{1}{(p-n(n+1))d} \displaystyle\sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n]}_{L_i(\widetilde{f})}(r) + o(T_f(r))$.

Proof. Let $\widetilde{f} = (f_0, \dots, f_n)$ be a reduced representation of f. It is clear that $(Q_0(f), \dots, Q_n(f))$ is a reduced representation of F. Then we have

$$\|F\| = \max\{|Q_i(f)| \mid i = 0, \dots, n\} \le c_1\|f\|^d, \tag{4.1}$$

where c_1 is a positive constant.

Since the hypersurfaces defined by $\{Q_i\}_{i=0}^n$ in $\mathbb{C}P^n$ have no common point, by Hilbert's Nullstellensatz there exists a positive integer $s \ge d$ such that

$$x_j^s = \sum_{i=0}^n R_{ij}Q_i, \quad j \in \{0, \dots, n\}, \tag{4.2}$$

where R_{ij} $(i, j \in \{0, \dots, n\})$ are zero or homogeneous polynomials with degree $s - d$. Then

$$f_j^s = \sum_{i=0}^n R_{ij}(f)Q_i(f) \quad \text{for all } j \in \{0, \dots, n\}.$$

Thus, there is a positive constant c_2 such that

$$|f_j|^s \le \sum_{i=0}^n |R_{ij}(f)| \cdot \max_{i=0,\dots,n} |Q_i(f)| \le c_2\|f\|^{s-d} \cdot \max_{i=0,\dots,n} |Q_i(f)|$$

for all $j \in \{0, \dots, n\}$. This implies that

$$\|f\|^d \le c_2\|F\|. \tag{4.3}$$

From (4.1) and (4.3) we deduce (i).

Let $F : \mathbb{C}^m \to \mathbb{C}P^n$ be the meromorphic mapping which has a reduced representation

$$\widetilde{F} = (Q_0^p(\widetilde{f}), \dots, Q_n^p(\widetilde{f})).$$

By (i) we have

$$pdT_f(r) = T_F(r) + O(1), \text{ where } d = \deg Q_i. \tag{4.4}$$

We define hyperplanes $\{H_i\}_{i=0}^{2n}$ in $\mathbb{C}P^n$ by

$$H_i := \Big\{ \sum_{j=0}^n a_{ij}w_j = 0 \Big\} \quad (i = 0, \dots, 2n),$$

where

$$a_{ij} = \begin{cases} 0 & \text{if } i \le n,\ i \ne j, \\ 1 & \text{if } i \le n,\ i = j, \\ \lambda_{(i-n)j} & \text{if } i \ge n+1. \end{cases}$$

Since all submatrices of (λ_{ij}) are nonsingular, the hyperplanes $\{H_i\}_{i=0}^{2n}$ are in general position. It is easy to see that $H_j(\widetilde{F}) = Q_j^p(\widetilde{f})$ for all $j \in \{0, \ldots, n\}$, and $H_j(\widetilde{F}) = L_{j-n}(f)$ for all $j \in \{n+1, \ldots, 2n\}$. We assume that $H_{j_i}(\widetilde{F}) \equiv 0$ with $i \in \{0, \ldots, k\}$ and $H_{j_i}(\widetilde{F}) \not\equiv 0$ with $i \in \{k+1, \ldots, 2n\}$, where $\{j_0, \ldots, j_{2n}\} = \{0, \ldots, 2n\}$. Since $\{H_i\}_{i=0}^{2n}$ are in general position, it follows that $k \leq n-1$.

Case 1: $k = n-1$. Then F is constant. By (4.4), we have

$$T_f(r) = \frac{1}{pd} T_F(r) = 0.$$

Case 2: $k < n-1$. Let $G : \mathbb{C}^m \to \mathbb{C}P^{n-k-1}$ be the meromorphic mapping which has a reduced representation

$$\widetilde{G} = (H_{j_{k+1}}(\widetilde{F}), \ldots, H_{j_n}(\widetilde{F})).$$

Since $\{H_i\}_{i=0}^{2n}$ are in general position and $H_{j_i}(\widetilde{F}) \equiv 0$ for $i \in \{0, \ldots, k\}$, we have

$$T_G(r) = T_F(r) + O(1). \tag{4.5}$$

We define hyperplanes $\{\widetilde{H}_i\}_{i=k+1}^{2n}$ in $\mathbb{C}P^{n-k-1}$ by

$$\widetilde{H}_i = \Big\{ \sum_{j=0}^{n-k-1} b_{(i+k+1)j} w_j = 0 \Big\}, \qquad i = k+1, \ldots, 2n, \tag{4.6}$$

where the b_{ij} are constants satisfying

$$\begin{pmatrix} a_{j_i 0} \\ \vdots \\ a_{j_i n} \end{pmatrix} = \begin{pmatrix} a_{j_0 0} & \cdots & a_{j_n 0} \\ \vdots & \vdots & \vdots \\ a_{j_0 n} & \cdots & a_{j_n n} \end{pmatrix} \begin{pmatrix} b_{i0} \\ \vdots \\ b_{in} \end{pmatrix}.$$

Then $\{\widetilde{H}_i\}_{i=k+1}^{2n}$ are in general position.

We note that $2n-k \geq 2(n-k-1)+1$, so by Lemma 4.1 and by the First Main Theorem, we have

$$\begin{aligned} \Big\| \quad T_G(r) &\leq \frac{2n-k}{2(n-k-1)+1} T_G(r) \leq \sum_{i=k+1}^{2n} N^{[n-k-1]}_{\widetilde{H}_i(\widetilde{G})}(r) + o(T_G(r)) \\ &= \sum_{i=k+1}^{2n} N^{[n-k-1]}_{H_{j_i}(\widetilde{F})}(r) + o(T_G(r)) \\ &\leq \sum_{Q_i(\widetilde{f}) \not\equiv 0} N^{[n-k-1])}_{Q_i^p(\widetilde{f})}(r) + \sum_{L_i(\widetilde{f}) \not\equiv 0} N^{[n-k-1]}_{L_i(\widetilde{f})}(r) + o(T_G(r)) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{Q_i(\widetilde{f})\not\equiv 0} \frac{n-k-1}{p} N_{Q_i^p(\widetilde{f})}(r) + \sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n-k-1]}_{L_i(\widetilde{f})}(r) + o(T_G(r))\\
&\le \frac{n(n+1)}{p} T_F(r) + \sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n]}_{L_i(\widetilde{f})}(r) + o(T_G(r))\\
&\le \frac{n(n+1)}{p} T_G(r) + \sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n]}_{L_i(\widetilde{f})}(r) + o(T_F(r)).
\end{aligned}
$$

Then

$$
\Big\| \quad T_G(r) \le \frac{p}{p-n(n+1)} \sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n]}_{L_i(\widetilde{f})}(r) + o(T_G(r)). \tag{4.7}
$$

By (4.4) and (4.5) and (4.7), we have

$$
\begin{aligned}
\Big\| \quad T_f(r) &= \frac{1}{pd} T_F(r) + O(1) = \frac{1}{pd} T_G(r) + O(1)\\
&\le \frac{1}{(p-n(n+1))d} \sum_{L_i(\widetilde{f})\not\equiv 0} N^{[n]}_{L_i(\widetilde{f})}(r) + o(T_f(r)). \ \blacksquare
\end{aligned}
$$

4.3. *Proof of Theorem 1.5.* Without loss of generality, we may assume that D is a polydisc in $\mathbb{C}^m$, $D = \Delta^m$. Suppose that $\mathcal{F}$ is not normal on D. Then by Lemma 3.5 there exist a subsequence denoted by $\{f_k\}_{k=1}^{\infty}$ and $p_0 \in D$, $\{p_k\}_{k\ge 1}^{\infty} \subset D$ with $p_k \to p_0$, $\{\varrho_k\} \subset (0,+\infty)$ with $\varrho_j \to 0$ such that the sequence of holomorphic maps

$$
g_k(z) := f_k(p_k + \varrho_k z) : \Delta^m_{r_k} \to \mathbb{C}P^n, \qquad k \ge k_0 \ (r_k \nearrow \infty)
$$

converges uniformly on compact subsets of $\mathbb{C}^m$ to a nonconstant holomorphic map $g : \mathbb{C}^m \to \mathbb{C}P^n$.

For any fixed $z_0 \in \mathbb{C}^m$, there exists k_0 such that $z_0 \in \Delta^m_{r_k}$ for all $k > k_0$. By the convergence of $\{g_k\}_{k>k_0}$, there exist reduced representations $\widetilde{g}_k = (g_{k0},\dots,g_{kn})$ of g_k $(k > k_0)$ on $\Delta^m_{r_k}$ and a representation $\widetilde{g} = (g_0,\dots,g_n)$ of g on $\Delta^m_{r_k}$ such that $\{g_{ki}\}$ converges uniformly on compact subsets of $\Delta^m_{r_k}$ to g_{ki}. This implies that $Q_j(p_k + \varrho_k z)(\widetilde{g}_k(z))$ and $L_j(p_k + \varrho_k z)(\widetilde{g}_k(z))$ converge uniformly on compact subsets of $\Delta^m_{r_k}$ to $Q_j(p_0)(\widetilde{g}(z))$ and $L_j(p_0)(\widetilde{g}(z))$ respectively. On the other hand, each f in $\mathcal{F}$ intersects L_i on D with multiplicity at least m_i. So, by Hurwitz's theorem, either $\widetilde{L}_j(p_0)(\widetilde{g}) \equiv 0$ or all zero points of $\widetilde{L}_j(p_0)(\widetilde{g})$ have multiplicity at least m_j $(j = 1,\dots,n)$. Thus, by Lemma 4.2,

$$
\Big\| \quad T_g(r) \le \frac{1}{(p-n(n+1))d} \sum_{L_j(p_0)(\widetilde{g})\not\equiv 0} N^{[n]}_{L_j(p_0)(\widetilde{g})}(r) + o(T_g(r))
$$

$$\leq \frac{n}{(p-n(n+1))d} \sum_{L_j(p_0)(\widetilde{g}) \not\equiv 0} \frac{1}{m_i} N_{L_j(p_0)(\widetilde{g})}(r) + o(T_g(r)).$$

On the other hand, by the First Main Theorem,

$$N_{L_j(p_0)(\widetilde{g})}(r) \leq pdT_g(r) + o(T_g(r)).$$

Thus, we get

$$\Big\| \quad T_g(r) \leq \frac{np}{p-n(n+1)} \sum_{i=1}^{n} \frac{1}{m_i} T_g(r) + o(T_g(r)).$$

Letting $r \to +\infty$, we obtain

$$\sum_{i=1}^{n} \frac{1}{m_i} \geq \frac{p-n(n+1)}{np}.$$

This is impossible. Hence $\mathcal{F}$ is normal on D. ∎

5. Proof of Theorem 1.6. Without loss of generality, we may assume D is a polydisc in $\mathbb{C}^m$, $D = \Delta^m$.

Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ be an arbitrary sequence. By Lemma 3.2, we can find a subsequence (not relabeled) such that

$$(5.1) \qquad \lim_{k\to\infty} f_k^{-1}(L_i) = S_i \quad (i = 1, \ldots, n) \quad \text{and} \quad \lim_{k\to\infty} f_k^{-1}(Q_{i_0}) = S_0$$

as sequences of closed subsets of D, where S_i $(i = 0, \ldots, n)$ are either empty or pure $(m-1)$-dimensional analytic sets of D. Because all submatrices of $(a_{ij})_{0\leq i\leq n,\, 1\leq j\leq n}$ are nonsingular, $\{Q_0, \ldots, Q_n, L_1, \ldots, L_n\}$ are in general position. Clearly, there exists an analytic set S of codimension at least 1 such that $\{Q_0, \ldots, Q_n, L_1, \ldots, L_n\}$ are in pointwise general position on $D\setminus S$ and all submatrices of $(a_{ij})_{0\leq i\leq n,\, 1\leq j\leq n}$ are pointwise nonsingular on $D\setminus S$. Set $E := \bigcup_{i=0}^{n} S_i \cup S \cup \bigcup_{k=1}^{\infty} I(f_k)$. Then E is a thin analytic subset of D.

For any $z_0 \in D \setminus E$, by (5.1) there exist an open ball $B(z_0, r_0) \subset D \setminus E$ and a positive integer k_0 such that $L_i(\widetilde{f_k})$ $(i = 1, \ldots, n)$ and $Q_{i_0}(\widetilde{f_k})$ have no zero point on $B(z_0, r_0)$ for all $k > k_0$. By Theorem 1.5, $\{f_k\}_{k>k_0}$ is a holomorphically normal family on $B(z_0, r_0)$. Hence, $\{f_k\}$ has a subsequence which converges uniformly on compact subsets of $D \setminus E$ to a holomorphic map.

By the usual diagonal argument, we can find a subsequence (again not relabeled) which converges uniformly on compact subsets of $D \setminus E$ to a holomorphic map f. We denote by L_{n+1} the moving hypersurface Q_{i_0}. Because $\{L_i\}_{i=1}^{n+1}$ are in pointwise general position on $D \setminus E$, there exists $j_0 \in \{1, \ldots, n+1\}$ such that $L_{j_0}(\widetilde{f}) \not\equiv 0$ on $D \setminus E$. We define the meromorphic mappings $\{F_k\}_{k=1}^{\infty}$ of D into $\mathbb{C}P^{n+1}$ as follows: for any $z \in D$, if f_k has a reduced representation $\widetilde{f_k} = (f_{k0}, \ldots, f_{kn})$ on a neighborhood $U_z \subset D$ then F_k has a reduced representation $\widetilde{F_k} = (f_{k0}^d, \ldots, f_{kn}^d, L_{j_0}(\widetilde{f_k}))$ on U_z. By the same argument of the proof of Theorem 1.4, $\{F_k\}_{k=1}^{\infty}$ is a meromorphically convergent sequence on D and $\{f_k\}_{k=1}^{\infty}$ has a meromorphically convergent subsequence on D.

Thus, $\mathcal{F}$ is a meromorphically normal family on D. ∎

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