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A general Dirichlet problem for the complex Monge–Ampère operator

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Abstract. We study a general Dirichlet problem for the complex Monge–Ampère operator, with maximal plurisubharmonic functions as boundary data.

1. Introduction. In classical potential theory, the Riesz representation theorem says that every negative subharmonic function can be written as a sum of a Green potential and a harmonic function. The smallest harmonic majorant of the potential is zero and the harmonic function is determined by its behaviour near the boundary. So it is natural to say that the harmonic part is the boundary value of the subharmonic function.

The purpose of this paper is to formulate and study a pluripotential analogue of the classical setting.

In pluripotential theory, the reminiscence of the Riesz representation theorem is inequality $(*)$ below. Here the harmonic functions are replaced by the so called maximal functions, already considered by Bremermann in [6]. We refer to the books [15] and [16] for background and references.

We will find that important results from classical potential theory carry over to our setting but we will also note some significant differences.

In this paper we study a particular class of plurisubharmonic functions, the class $\mathcal{E}$, defined in [11]. The reason for this choice is that the complex Monge–Ampère operator $(dd^c u)^n$ is well-defined in this class and the maximal functions u in $\mathcal{E}$ are precisely the functions with $(dd^c u)^n = 0$.

Subclasses of functions in $\mathcal{E}$ with continuous boundary values have been studied in [1] and [8]. The case of upper semicontinuous boundary values was considered in [14]. Here, we allow any bounded maximal function as boundary value when we show existence and uniqueness in the Dirichlet problem for the complex Monge–Ampère operator.

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In the notation of Section 2, we formulate one of our main results, Theorem 4.1:

Suppose $\mu = (dd^c v)^n$ where $v \in \mathcal{N}^a$. For every $H \in \mathcal{M}^\infty$ there is a uniquely determined function $\varphi \in \mathcal{N}^a(H)$ with $(dd^c\varphi)^n = \mu$.

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2. The boundary values. We start by recalling some notations. Denote by $\mathrm{PSH}(\Omega)$ the plurisubharmonic functions on $\Omega \subset \mathbb{C}^n$ and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions. A set $\Omega \subset \mathbb{C}^n$ is said to be a *hyperconvex domain* if it is open, bounded, connected and if there exists $\varphi \in \mathrm{PSH}^-(\Omega)$ such that $\{z \in \Omega;\ \varphi(z) < -c\} \subset\subset \Omega$ for all $c > 0$. Such a function is called an *exhaustion function* for Ω.

Throughout this paper, we let Ω denote a hyperconvex domain.

We define the classes of plurisubharmonic functions, studied in this paper. Details can be found in [11], where among other things it is proved that the functions in these classes have well-defined Monge–Ampère measures. We set

$$\begin{aligned}\mathcal{E}_0 &= \mathcal{E}_0(\Omega)\\ &= \Big\{\varphi \in \mathrm{PSH}^- \cap L^\infty(\Omega);\ \lim_{z\to\xi}\varphi(z) = 0,\ \forall \xi \in \partial\Omega,\ \int_\Omega (dd^c\varphi)^n < +\infty\Big\},\\ \mathcal{F} &= \mathcal{F}(\Omega)\\ &= \Big\{u \in \mathrm{PSH}^-(\Omega);\ \exists\, u_j \in \mathcal{E}_0(\Omega),\ u_j \searrow u,\ \sup_j \int_\Omega (dd^c u_j)^n < +\infty\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{E}(\Omega) = \Big\{&u \in \mathrm{PSH}^-(\Omega);\ \forall z_0 \in \Omega,\ \exists\, \omega,\ \text{a neighbourhood of } z_0,\\ &\exists\, h_j \in \mathcal{E}_0(\Omega),\ h_j \searrow u \text{ on } \omega,\ \sup_j \int_\Omega (dd^c h_j)^n < +\infty\Big\}.\end{aligned}$$

We define $\mathcal{F}^a$ to be the class of functions u in $\mathcal{F}$ such that $(dd^c u)^n$ vanishes on all pluripolar sets. The class $\mathcal{E}^a$ is defined similarly.

It can be proved that every $u \in \mathcal{E}(\Omega)$ is locally in $\mathcal{F}(\Omega)$: for every $u \in \mathcal{E}(\Omega)$ and every ω, open and relatively compact in Ω, there is a $u_\omega \in \mathcal{F}(\Omega)$ with $u \le u_\omega$ with equality on ω.

In [11] it was proved that for every $u \in \mathrm{PSH}^-(\Omega)$ there are $u_j \in \mathcal{E}_0(\Omega)$ with $u_j \searrow u$ as $j \to +\infty$. This, together with integration by parts, shows that $\mathcal{F}$ is closed in the following sense: if $u_j \in \mathcal{F}(\Omega)$, $u_j \searrow$ and $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$, then $\lim_{j\to+\infty} u_j \in \mathcal{F}(\Omega)$.

We now come to the boundary values.

Let Ω_j be a fundamental sequence of strictly pseudoconvex subdomains of Ω. Let $u \in \mathcal{E}$ be given and put

$$u^j = \sup\{\varphi \in \mathrm{PSH}(\Omega);\ \varphi|_{C\Omega_j} \le u|_{C\Omega_j}\}.$$

Then $u \le u^j \le u^{j+1}$, hence each u^j is in $\mathcal{E}$, and so is $\widetilde{u} = (\lim_{j\to+\infty} u^j)^*$, the smallest upper semicontinuous majorant of $\lim u^j$.

Note that the definition of $\widetilde{u}$ is independent of the sequence Ω_j, that $(dd^c\widetilde{u})^n = 0$, and that if u is continuous up to the boundary then so is $\widetilde{u}$. For if f is continuous on the boundary of Ω and if there is a function $\varphi \in \mathrm{PSH}(\Omega)$ with $\lim_{z\to\xi} \varphi(z) = f$ for all $\xi \in \partial\Omega$, then by a theorem of Walsh [18], $\widetilde{\varphi} \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$.

We define

$$\mathcal{M} = \{u \in \mathcal{E};\, (dd^c u)^n = 0\}, \qquad \mathcal{M}^\infty = \mathcal{M} \cap L^\infty, \qquad \mathcal{N} = \{u \in \mathcal{E}; \widetilde{u} = 0\},$$

and let $\mathcal{N}^a$ be the class of functions u in $\mathcal{N}$ such that $(dd^c u)^n$ vanishes on all pluripolar sets.

Note that $\mathcal{E}_0 \subset \mathcal{F}^a \subset \mathcal{F} \subset \mathcal{N} \subset \mathcal{E}$. Also, the classes $\mathcal{E}_p$, $p > 0$, defined in [8], are subsets of $\mathcal{N}^a$.

We say that $u \in \mathcal{E}$ has *boundary value* $\widetilde{u}$ if there is a function $\psi \in \mathcal{N}$ such that

$$(*) \qquad \widetilde{u} \ge u \ge \widetilde{u} + \psi.$$

It follows from [4] or [10] that "$\widetilde{u}$ is the smallest maximal plurisubharmonic function above u", so in particular $\mathcal{M} = \{u \in \mathcal{E};\, \widetilde{u} = u\}$.

It was shown in [11] that if μ is any positive measure that vanishes on all pluripolar sets and $\mu(\Omega) < +\infty$, then there is a uniquely determined function $\varphi \in \mathcal{F}^a$ such that $(dd^c\varphi)^n = \mu$. We write $\varphi = U(\mu, 0)$, a notation which will be generalized in Sections 3 and 4.

We define $\mathcal{F}(H)$ $(= \mathcal{F}(H,\Omega))$ for $H \in \mathcal{M}$ to be the class of plurisubharmonic functions u such that

$$H \ge u \ge H + \psi, \qquad \psi \in \mathcal{F}.$$

In particular, $H = \widetilde{u}$. We define $\mathcal{E}_0(H)$ etc. similarly.

A problem is now to decide which functions in $\mathcal{E}$ have boundary values. Some particular cases were studied in [1] and [8]. We do not know if every function in $\mathcal{E}$ has a boundary value, but we have the following theorem.

THEOREM 2.1. *Suppose $u \in \mathcal{E}$ with $\int_\Omega (dd^c u)^n < +\infty$. Then $u \in \mathcal{F}(\widetilde{u})$ and $u \ge \psi + \widetilde{u}$ for some $\psi \in \mathcal{F}$ with $\int_\Omega (dd^c\psi)^n \le \int_\Omega (dd^c u)^n$.*

Proof. Choose $u_j \in \mathcal{E}_0 \cap C(\overline{\Omega})$ decreasing to u and Ω_j a fundamental sequence of Ω. Then for each j there is $s_j > s_{j-1}$ such that for $s \ge s_j$,

$$\int_\Omega \chi_{\Omega_j} (dd^c u_s)^n \le \int_\Omega (dd^c u)^n + 1/j.$$

Now, $u_s \ge U(\chi_{\Omega_j}(dd^c u_s)^n, 0) + u_s^j$ by the comparison principle (see Section 3), so if $t \ge s \ge s_j$, then

$$u_s \ge u_t \ge U(\chi_{\Omega_j}(dd^c u_t)^n, 0) + u_t^j.$$

In particular,

$$u_s \geq (\sup_{t\geq s} U(\chi_{\Omega_j}(dd^c u_t)^n, 0))^* + u^j$$

and

$$u \geq \lim_{s\to+\infty} (\sup_{t\geq s} U(\chi_{\Omega_j}(dd^c u_t)^n, 0))^* + u^j = \psi_j + u^j.$$

Now, ψ_j is a decreasing sequence of functions in $\mathcal{F}$ and $\int_\Omega (dd^c\psi_j)^n \leq \int_\Omega (dd^c u)^n + 1/j$, so $\psi = \lim \psi_j \in \mathcal{F}$ since $\mathcal{F}$ is closed, as noted above. Since u^j increases a.e. to $\widetilde{u}$ as $j \to +\infty$, we find that $u \geq \psi + \widetilde{u}$ and $\int_\Omega (dd^c\psi)^n \leq \int_\Omega (dd^c u)^n$, which completes the proof. ■

REMARK. The condition $\int_\Omega (dd^c u)^n < +\infty$ is not necessary for u to be in $\mathcal{F}(\widetilde{u})$. For $\Omega =$ the bidisc, an example of a function $u \in \mathcal{F}(\widetilde{u})$ with $\int_\Omega (dd^c u)^n = +\infty$ is given in [13]. See also Example 2.4 in [2].

THEOREM 2.2. *For every $u \in \mathcal{E}$, there is a sequence $u_s \in \mathcal{E}_0(\widetilde{u})$ such that u_s decreases to u as $s \to +\infty$.*

Proof. Choose $v_j \in \mathcal{E}_0 \cap C(\overline{\Omega})$ decreasing to u. Then $v_s \geq v_s + \widetilde{u}$. Define $u_s = \max(u, v_s + \widetilde{u})$. Then $\widetilde{u} \geq u_s \geq v_s + \widetilde{u}$, so $u_s \in \mathcal{E}_0(\widetilde{u})$. Also (u_s) is a decreasing sequence and $\lim u_s = u$, which proves the theorem. ■

3. The Dirichlet problem in $\mathcal{F}(f)$ for $f \in \mathcal{M}(\Omega) \cap C(\overline{\Omega})$. In this section, we consider the case when the maximal function f is continuous up to the boundary.

We begin with the comparison principle for $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(W)$, where W is a domain in $\mathbb{C}^n$.

THEOREM 3.1. *If $u, v \in \mathrm{PSH} \cap L^\infty(W)$ and $u \geq v$ near ∂W, then $\int_{\{u<v\}} (dd^c u)^n \geq \int_{\{u<v\}} (dd^c v)^n$.*

If $u, v \in \mathrm{PSH} \cap L^\infty(W), \lim_{z\to\zeta}(u(z) - v(z)) = 0$ for all $\zeta \in \partial W$ and $(dd^c u)^n \leq (dd^c v)^n$ on W, then $u \geq v$ on W.

These statements were proved in [3]. (See also [7], [8].)

We recall the (relative) capacity introduced by Bedford and Taylor [3]: for $E \subset W$, $\mathrm{cap}(E)$ is defined as

$$\mathrm{cap}(E) = \sup\Big\{ \int_E (dd^c u)^n;\ -1 \leq u \leq 0,\ u \in \mathrm{PSH}(W)\Big\}.$$

The next lemma should have been included in [11].

LEMMA 3.2. *Assume $u_j^p, u^p \in \mathcal{E}(\Omega), u_j^p \geq u^p$, $1 \leq p \leq n$, $h \in \mathrm{PSH}^- \cap L^\infty(\Omega)$ and u_j^p tends weakly to u^p. Then $h(dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n)$ tends weakly to $h(dd^c u^1 \wedge \cdots \wedge dd^c u^n)$ as $j \to +\infty$.*

Proof. Since every function in $\mathcal{E}$ is locally the restriction of a function in $\mathcal{F}$, we can assume that all the functions are in $\mathcal{F}$. Choose $g_j^p \searrow u^p$, $g_j^p \in \mathcal{E}_0$, and put $v_j^p = (\sup_{s\geq j}(g_j^p, u_s^p))^*$. It follows from Proposition 5.1 and Corollary 5.2 in [11] that $\int h(dd^c v_j^1 \wedge \cdots \wedge dd^c v_j^n)$ tends to $\int h(dd^c u^1 \wedge \cdots \wedge dd^c u^n)$ as $j \to +\infty$ and that $h(dd^c v_j^1 \wedge \cdots \wedge dd^c v_j^n)$ tends weakly to $h(dd^c u^1 \wedge \cdots \wedge dd^c u^n)$ as $j \to +\infty$. Integration by parts gives

$$\int h(dd^c v_j^1 \wedge \cdots \wedge dd^c v_j^n) \geq \int h(dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n) \geq \int h(dd^c u^1 \wedge \cdots \wedge dd^c u^n).$$

Therefore

$$\begin{aligned}\lim_{j\to+\infty} \int h(dd^c v_j^1 \wedge \cdots \wedge dd^c v_j^n) &= \lim_{j\to+\infty} \int h(dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n) \\ &= \int h(dd^c u^1 \wedge \cdots \wedge dd^c u^n)\end{aligned}$$

and

$$\lim_{j\to+\infty} (dd^c v_j^1 \wedge \cdots \wedge dd^c v_j^n) = \lim_{j\to+\infty} (dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n) = dd^c u^1 \wedge \cdots \wedge dd^c u^n.$$

Hence

$$\lim_{j\to+\infty} h(dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n) \leq h(dd^c u^1 \wedge \cdots \wedge dd^c u^n),$$

and since both measures have the same total mass, they are equal. ■

LEMMA 3.3. *Assume that Ω is hyperconvex, $w \in \mathcal{F}(\Omega), -1 \leq h \in$ PSH$^-(\Omega)$ and $u, v \in$ PSH. If $\mathcal{F} \ni w_j \searrow w$ as $j \to \infty$ and $\int_E (1+h)(dd^c w_j)^n$ is uniformly small when* cap(E) *is small then*

$$\begin{aligned}\int_{\{u<v\}} (1+h)(dd^c w)^n &\leq \varliminf_j \int_{\{u<v\}} (1+h)(dd^c w_j)^n \\ &\leq \varlimsup_j \int_{\{u\leq v\}} (1+h)(dd^c w_j)^n \leq \int_{\{u\leq v\}} (1+h)(dd^c w)^n.\end{aligned}$$

Proof. Since, by Proposition 5.1 in [11],

$$\int_\Omega (dd^c w)^n = \lim_{j\to+\infty} \int_\Omega (dd^c w_j)^n$$

and $h(dd^c w_j)^n$ tends weakly to $h(dd^c w)^n$ as $j \to +\infty$ it follows that $(1+h)(dd^c w_j)^n$ tends weakly to $(1+h)(dd^c w)^n$ as $j \to +\infty$.

Let $\delta > 0$ be given. Since, by [3], u and v are quasicontinuous, there is an open set O_δ with $\sup_j \int_{O_\delta} (1+h)(dd^c w_j)^n < \delta$ and there are two continuous functions $\widetilde{u}$ and $\widetilde{v}$ such that $\{u \neq \widetilde{u}\} \cup \{v \neq \widetilde{v}\} \subset O_\delta$. Therefore

$$\begin{aligned}\{u < v\} &\subset \{\widetilde{u} < \widetilde{v}\} \cup O_\delta \subset \{u < v\} \cup O_\delta, \\ \{u \leq v\} &\subset \{\widetilde{u} \leq \widetilde{v}\} \cup O_\delta \subset \{u \leq v\} \cup O_\delta\end{aligned}$$

and

$$\int_{\{u<v\}} (1+h)(dd^c w)^n \leq \varliminf \int_{\{\widetilde{u}<\widetilde{v}\}\cup O_\delta} (1+h)(dd^c w_j)^n \leq \varliminf \int_{\{u<v\}} (1+h)(dd^c w_j)^n + 2\delta,$$

which proves the first inequality of the statement of Lemma 3.3. Moreover,

$$\overline{\lim_j} \int_{\{u\le v\}} (1+h)(dd^c w_j)^n \le \overline{\lim_j} \int_{\{\tilde u\le \tilde v\}} (1+h)(dd^c w_j)^n + \delta$$

$$\le \int_{\{\tilde u\le \tilde v\}} (1+h)(dd^c w)^n + \delta \le \int_{\{u\le v\}} (1+h)(dd^c w)^n + 2\delta.$$

This completes the proof of the lemma. ∎

LEMMA 3.4. *Suppose* $\omega \in \mathcal{E}$. *Then*

$$(dd^c\omega)^n = f(dd^c\psi)^n + \nu,$$

where $\psi \in \mathcal{E}_0$, $f \in L^1_{\mathrm{loc}}((dd^c\psi)^n)$ *and* ν *is carried by* $\{\omega = -\infty\}$. *Moreover, if* $m < s$, *then*

$$\chi_{\{\omega\ge -m\}}(dd^c \max(\omega,-s))^n = f_m f(dd^c\psi)^n,$$

where $0 \le f_m \le 1$.

Proof. The first statement is Theorem 5.11 in [11]. To prove the second statement we can, as in the proof of Lemma 3.2, assume that $\omega \in \mathcal{F}$. By Lemma 5.4 in [8],

$$\chi_{\{\omega> -m\}}(dd^c \max(\omega,-s))^n$$

is independent of s if $s > m$. Hence

$$\chi_{\{\omega> -m\}}(dd^c \max(\omega,-s))^n \le \chi_{\{\omega\ge -m\}} f(dd^c\psi)^n,$$

so

$$\chi_{\{\omega> -m\}}(dd^c \max(\omega,-s))^n = f_m f(dd^c\psi)^n,$$

where $0 \le f_m \le 1$ and $f_m \nearrow 1$. ∎

LEMMA 3.5. *Suppose* $\omega \in \mathcal{F}$ *and* $u, v \in \mathrm{PSH}(\Omega)$. *Then*

$$\int_{\{u<v\}\cap\{\omega>-\infty\}} (dd^c\omega)^n \le \underline{\lim}_{j\to+\infty} \int_{\{u<v\}} (dd^c\omega_j)^n,$$

where $\omega_j = \max(\omega, -j)$.

Proof. For $E \subset \Omega$ we write $h_E = \sup\{\varphi \in \mathrm{PSH}^-;\ -1 \le \varphi,\ \varphi = -1 \text{ on } E\}$. Let $h_r = h_{\{\omega<-r\}}$, where $r > 0$ is given. It follows from Lemma 3.4 that $\int_E (1+h_r)(dd^c\omega_p)^n$ is uniformly small when $\mathrm{cap}(E)$ is small. By Lemma 3.3, we have

$$\int_{\{u<v\}} (1+h_r)(dd^c\omega)^n \le \underline{\lim}_p \int_{\{u<v\}} (1+h_r)(dd^c\omega_p)^n \le \underline{\lim}_p \int_{\{u<v\}} (dd^c\omega_p)^n.$$

Therefore,

$$\int_{\{u<v\}\cap\{\omega>-\infty\}} (dd^c\omega)^n = \lim_{r\to+\infty} \int_{\{u<v\}} (1+h_r)(dd^c\omega)^n \le \underline{\lim}_j \int_{\{u<v\}} (dd^c\omega_j)^n.$$

This proves Lemma 3.5. ∎

COROLLARY 3.6. *If $u \in \mathcal{F}$ and $v \in \mathcal{E}$, then*

$$\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}\cup\{u=-\infty\}} (dd^c u)^n.$$

Proof. In the case $u, v \in \mathcal{F} \cap L^\infty$, the inequality follows from Lemma 4.4 in [8], but let us give a complete proof. Let

$$u_j, v_j \in \mathcal{E}_0, \quad u_j \searrow u, \quad v_j \searrow v, \quad j \to +\infty.$$

Using Theorem 3.1 and Lemma 3.3 we have, for $\varepsilon > 0$,

$$\int_{\{u_k+\varepsilon<v\}} (dd^c v)^n \le \varliminf_j \int_{\{u_j+\varepsilon<v_j\}} (dd^c u_j)^n \le \varlimsup_j \int_{\{u+\varepsilon\le v_k\}} (dd^c u_j)^n$$
$$\le \int_{\{u+\varepsilon\le v_k\}} (dd^c u)^n.$$

If we let k tend to $+\infty$ and ε tend to 0 we get the desired conclusion.

Let now $u_j = \max(u, -j)$ and $v_j = \max(v, -j)$. Then, by the above,

$$\int_{\{u_j+\varepsilon<v_j\}} (dd^c v_j)^n \le \int_{\{u_j+\varepsilon<v_j\}} (dd^c u_j)^n.$$

For every fixed k Lemma 3.5 gives

$$\varliminf_j \int_{\{u_j+\varepsilon<v_j\}} (dd^c v_j)^n \ge \varliminf_j \int_{\{u_k+\varepsilon<v\}} (dd^c v_j)^n \ge \int_{\{u_k+\varepsilon<v\}} (dd^c v)^n$$

and

$$\varliminf_j \int_{\{v_j+\varepsilon<u_j\}} (dd^c u_j)^n \ge \int_{\{v_k+\varepsilon<u\}} (dd^c u)^n.$$

Moreover,

$$\int_{\{u_j+\varepsilon<v_j\}} (dd^c u_j)^n = \int_\Omega (dd^c u_j)^n - \int_{\{v_j<u_j+\varepsilon\}} (dd^c u_j)^n - \int_{\{v_j=u_j+\varepsilon\}} (dd^c u_j)^n,$$

where we can assume $\varepsilon > 0$ is chosen so that

$$\int_{\{v_j=u_j+\varepsilon\}} (dd^c u_j)^n = 0, \quad \forall j \ge 1.$$

Then

$$\varlimsup_j \int_{\{u_j+\varepsilon<v_j\}} (dd^c u_j)^n = \int_\Omega (dd^c u)^n - \varliminf_j \int_{\{v_j<u_j+\varepsilon\}} (dd^c u_j)^n$$
$$\le \int_\Omega (dd^c u)^n - \int_{\{v<u+\varepsilon\}} (dd^c u)^n = \int_{\{u+\varepsilon\le v\}} (dd^c u)^n$$
$$\le \int_{\{u<v\}} (dd^c u)^n + \int_{\{u=v=-\infty\}} (dd^c u)^n.$$

Combining these estimates and letting $\varepsilon \to 0$ we get

$$\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}} (dd^c u)^n + \int_{\{u=v=-\infty\}} (dd^c u)^n.$$

If $v \in \mathcal{E}$ only, put $v_j = \sup\{g \in \mathrm{PSH}^-(\Omega);\ g|_{\Omega_j} \le v|_{\Omega_j}\}$. Then $v_j \in \mathcal{F}$ and

$$\int_{\{u<v_j\}} (dd^c v_j)^n \le \int_{\{u<v_j\}} (dd^c u)^n + \int_{\{u=v_j=-\infty\}} (dd^c u)^n$$

so for $j > k$,

$$\begin{aligned}\int_{\{u<v_j\}\cap\Omega_k} (dd^c v)^n &= \int_{\{u<v_j\}\cap\Omega_k} (dd^c v_j)^n \le \int_{\{u<v_j\}} (dd^c v_j)^n \\ &\le \int_{\{u<v_j\}} (dd^c u)^n + \int_{\{u=v_j=-\infty\}} (dd^c u)^n.\end{aligned}$$

To obtain the desired inequality, we let first j and then k tend to $+\infty$. ∎

The comparison principle is not true in general in $\mathcal{F}$ (cf. [19, Example 3.4]. However, we have

THEOREM 3.7. *Suppose* $u \in \mathcal{F}^a$, $v \in \mathcal{E}$ *and* $(dd^c u)^n \le (dd^c v)^n$. *Then* $v \le u$ *on* Ω.

Proof. Via Corollary 3.6, the proof of Theorem 3.1 can be adapted to this case. A different proof was given in [11]. ∎

We now extend the comparison principle to the classes $\mathcal{F}^a(f)$, $f \in \mathcal{M}(\Omega) \cap C(\overline{\Omega})$. As a consequence, we have the following generalization of Theorem 3.7.

THEOREM 3.8. *Suppose* $u \in \mathcal{F}^a(f)$ *and* $v \in \mathcal{F}(g)$, *where* $0 \ge f \ge g$ *are boundary values of two plurisubharmonic functions on* Ω, *continuous on the closure of* Ω. *If* $(dd^c u)^n \le (dd^c v)^n$, *then* $v \le u$ *on* Ω.

THEOREM 3.9. *Suppose* μ *is a positive measure which vanishes on all pluripolar subsets of* Ω *and with bounded total mass. Then there is a uniquely determined* $u \in \mathcal{F}^a(f)$ *with* $(dd^c u)^n = \mu$.

Proof. The case $f = 0$ is Lemma 5.14 in [11] and the general case was proved in [1], where methods from [11] were used. The uniqueness can also be proved using Theorem 3.10 below. ∎

We define $U(\mu, f)$ to be the function determined in Theorem 3.9.

Throughout the rest of this section we assume that $0 \ge f \ge g$ are boundary values of two maximal plurisubharmonic functions on Ω, continuous on the closure of Ω.

THEOREM 3.10. *If $u \in \mathcal{F}^a(f)$, then there exists an increasing sequence (r_j) of functions in $\mathcal{F}$, tending to 0 a.e. such that*

$$\int_{\{u+\varepsilon<v+r_j\}} (dd^c v)^n \le \int_{\{u+\varepsilon<v+r_j\}} (dd^c u)^n$$

for every $v \in \mathcal{F}(g)$ and every $\varepsilon > 0$.

Proof. Since $u \in \mathcal{F}^a(f)$ and $v \in \mathcal{F}(g)$, we can find $\varphi \in \mathcal{F}^a$ and $\nu \in \mathcal{F}$ such that $f \ge u \ge f + \varphi$ and $g \ge v \ge g + \nu$. It follows from results in [11] that $(dd^c\varphi)^n = p(dd^c\psi)^n$, where $\psi \in \mathcal{E}_0$. It follows from Theorem 3.7 that $\varphi \ge u_s + r_s$, where $u_s = U(\min(p,s)(dd^c\psi)^n, 0)$ is the unique solution to the Dirichlet problem $u_s \in \mathcal{F}, (dd^c u_s)^n = \min(p,s)(dd^c\psi)^n$, which exists by Lemma 5.14 in [11]. It follows that $u_s \in \mathcal{E}_0$ since $u_s \ge s^{1/n}\psi$. The function r_s is defined as

$$r_s = U(\chi_{\{p\ge s\}} p(dd^c\psi)^n, 0).$$

We see that r_s increases to 0 a.e. and

$$\begin{aligned}\{u+\varepsilon<v+r_s\} \subset \{\varphi+f+\varepsilon<v+r_s\} \subset \{u_s+r_s+f+\varepsilon<v+r_s\} \\ \subset \{u_s+\varepsilon<0\} \subset\subset \Omega, \quad \forall s.\end{aligned}$$

Let $R \in \mathcal{E}_0$ be any continuous exhaustion function for Ω such that $R < g$ near the closure of $\{u_s+\varepsilon<0\}$. Then $\max(u, \varphi + \max(U(0,f),R)) \in \mathcal{F}^a$, $\max(v, \nu + \max(U(0,g),R)) \in \mathcal{F}$, and

$$u = \max(u, \varphi + \max(U(0,f),R)), \qquad v = \max(v, \nu + \max(U(0,g),R))$$

near the closure of $\{u_s+\varepsilon<0\}$. The conclusion of the theorem now follows from Corollary 3.6. ∎

We finish this section by generalizing Theorem 3.7. We will need a result by Błocki [5], which in our setting is: If $u \in \mathrm{PSH}^- \cap L^\infty$ and $v \in \mathcal{F}$ then

$$\int (-v)^n (dd^c u)^n \le n!(\sup -u)^{n-1} \int -u(dd^c v)^n.$$

We first generalize our notation $U(\mu, 0)$. Let $\mu = (dd^c v)^n$ for some $v \in \mathcal{E}^a$. Define $U(\mu,0)$ to be $\lim_j U(\chi_{\Omega_j}\mu, 0)$. Then $U(\mu,0) \in \mathcal{E}^a$ and $U(\mu,0) \ge v$.

THEOREM 3.11. *Suppose $u \in \mathcal{N}^a$. Then $u = U((dd^c u)^n, 0)$.*

Proof. It follows from Theorem 3.7 that $u \le U((dd^c u)^n, 0)$, so it remains to prove the opposite inequality. Let $t \in C_0^\infty, 0 \le t \le 1$, and put

$$u_{ts} = U(t(dd^c \max(u,-s))^n, 0).$$

Then $u_{ts} \ge \max(u,-s) \ge u_{ts} + u^k$ when $\Omega_k \subset \{t=1\}$. We claim that $\overline{\lim}\, u_{ts} \ge U((dd^c u)^n, 0)$. If we prove this, then $\max(u,-s) \ge U((dd^c u)^n, 0) + u^k$ and therefore $u \ge U((dd^c u)^n, 0)$ since u^k tends to 0 as $k \to +\infty$.

Now, for $s > q$,

$$u_{ts} = U(t(dd^c \max(u, -s))^n, 0)$$
$$\geq U(\chi_{\{u>-q\}} t(dd^c \max(u, -s))^n, 0) + U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s))^n, 0)$$

and by Lemma 3.4,

$$u_{ts} = U(t(dd^c \max(u, -s))^n, 0)$$
$$\geq U(\chi_{\{u>-q\}} t(dd^c u)^n, 0) + U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s))^n, 0)$$

and

$$u_{ts} = U(t(dd^c \max(u, -s))^n, 0)$$
$$\geq U(t(dd^c u)^n, 0) + U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s))^n, 0).$$

It remains to prove that $U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s))^n, 0)$ is close to zero. Let $\psi \in \mathcal{E}_0, -1 \leq \psi < 0$. Then by [5],

$$\int (-U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s)^n, 0))^n (dd^c \psi)^n$$
$$\leq n! \int \chi_{\{u\leq -q\}} t(dd^c \max(u, -s))^n$$
$$\leq n! \int -\max(u/q, -1) t(dd^c \max(u, -s))^n.$$

By Lemma 3.2, $-\max(u/q, -1)(dd^c \max(u, -s))^n$ tends weakly to $-\max(u/q, -1)(dd^c u)^n$ as $s \to \infty$. Thus

$$\limsup_{s\to+\infty} \int (-U(\chi_{\{u\leq -q\}} t(dd^c \max(u, -s), 0)^n)^n (dd^c \psi)^n$$
$$\leq n! \int -\max(u/q, -1) t(dd^c u)^n.$$

Since the right hand side tends to 0 as $q \to \infty$, the proof is complete. ∎

Theorem 3.11 together with Theorem 3.7 gives

THEOREM 3.12. *Suppose $u \in \mathcal{N}^a, v \in \mathcal{E}$ and $(dd^c u)^n \leq (dd^c v)^n$. Then $v \leq u$ on Ω.*

Theorem 3.11 together with Corollary 3.6 gives

COROLLARY 3.13. *If $u \in \mathcal{N}^a$ and $v \in \mathcal{E}$, then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

COROLLARY 3.14. *If $u \in \mathcal{N}^a$ and $u \leq v \in \mathcal{E}$, then $v \in \mathcal{N}^a$.*

Proof. It is no restriction to assume that $v \in \mathcal{F}$. We know that $u = U((dd^c u)^n, 0)$ and we write

$$u_j = U(\chi_{\Omega_j}(dd^c u)^n, 0), \qquad e_j = U((1 - \chi_{\Omega_j})(dd^c u)^n, 0).$$

Then $v \geq u \geq u_j + e_j$ so $v \geq u_j + \max(v, e_j)$. Let P be a given compact pluripolar set and let $h \in \mathcal{E}_0$ with $h \leq -1$ near P. Given $\varepsilon > 0$, choose j so that $\int h(dd^c \max(v, e_j))^n > -\varepsilon$. We get

$$\int h(dd^c v)^n = \int v dd^c h \wedge (dd^c v)^{n-1} \ge \int (u_j + \max(v, e_j)) dd^c h \wedge (dd^c v)^{n-1}$$
$$\ge \int u_j dd^c h \wedge (dd^c v)^{n-1} - \varepsilon = \int h dd^c u_j \wedge (dd^c v)^{n-1} - \varepsilon.$$

Thus, $\int t(dd^c v)^n \ge \int t dd^c u_j \wedge (dd^c v)^{n-1} - \varepsilon$ for every $t \in \mathcal{E}_0$ with $t \ge h$. Since P is pluripolar, we can choose $0 \ge h_p \ge h$ with $h_p = -1$ on P so that h_p increases to 0 outside a pluripolar set. It follows that $(dd^c v)^n$ vanishes at P, and the proof is complete. ∎

We have already noted that the comparison principle fails to hold in $\mathcal{F}$. However, the following identity theorem does hold.

THEOREM 3.15. *If $u, v \in \mathcal{F}(\Omega)$, $(dd^c u)^n = (dd^c v)^n$ and $u \le v$ then $u = v$.*

Proof. By [12], there is a strictly plurisubharmonic exhaustion function $\psi \in \mathcal{E}_0 \cap C^\infty(\Omega)$ for Ω. We would like to show that

$$\int d(u-v) \wedge d^c(u-v) \wedge (dd^c\psi)^{n-1} = 0.$$

It is easy to see that

$$0 = \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge dd^c\psi, \qquad a+b = n-2.$$

Assume

$$0 = \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^p,$$
$$a+b = n-1-p.$$

Then for $a+b = n-2-p$ we have

$$\begin{aligned}
0 &\le \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^{p+1} \\
&= \int -(u-v) dd^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^{p+1} \\
&= \int -\psi (dd^c(u-v))^2 \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^p \\
&= \int d\psi \wedge d^c(u-v) \wedge dd^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^p \\
&\le \Big| \int d\psi \wedge d^c(u-v) \wedge dd^c u \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^p \Big| \\
&\quad + \Big| \int d\psi \wedge d^c(u-v) \wedge dd^c v \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c\psi)^p \Big| \\
&\le \Big[\int d\psi \wedge d^c\psi \wedge (dd^c u)^{a+1} \wedge (dd^c v)^b \wedge (dd^c\psi)^p \\
&\quad \times \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{a+1} \wedge (dd^c v)^b \wedge (dd^c\psi)^p \Big]^{1/2} \\
&\quad + \Big[\int d\psi \wedge d^c\psi \wedge (dd^c u)^a \wedge (dd^c v)^{b+1} \wedge (dd^c\psi)^p \\
&\quad \times \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^{b+1} \wedge (dd^c\psi)^p \Big]^{1/2} = 0. \ \blacksquare
\end{aligned}$$

Remark. A different proof is given in [17].

4. The Dirichlet problem in $\mathcal{N}^a(H)$ for $H \in \mathcal{M}^\infty$. In this section, we prove the following theorem.

Theorem 4.1. *Suppose $\mu = (dd^c v)^n$, where $v \in \mathcal{N}^a$. For every $H \in \mathcal{M}^\infty$ there is a uniquely determined function $\varphi \in \mathcal{N}^a(H)$ with $(dd^c\varphi)^n = \mu$.*

We extend our definition from Section 3 and define $U(\mu, H)$ to be this function. In the previous section, we treated the case when $v \in \mathcal{F}^a$ and $H \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$, $(dd^c H)^n = 0$. To proceed to the general case, we need a lemma.

Lemma 4.2. *Suppose $H \in \mathcal{M}^\infty$, $\psi \in \mathcal{E}_0(\Omega)$, and $\mathrm{supp}\,(dd^c\psi)^n \subset\subset \Omega$. Then there is a $u \in \mathcal{E}_0(H)$ such that $(dd^c u)^n = (dd^c\psi)^n$.*

Proof. Put $\mu = (dd^c\psi)^n$, let $\mathrm{supp}\,\mu \subset\subset \Omega_0 \subset\subset \cdots \subset\subset \Omega_j \subset\subset \Omega$ be a fundamental sequence of strictly pseudoconvex subsets of Ω, and let $H_k, \psi_k \in \mathcal{E}_0 \cap C(\overline{\Omega})$ with H_k decreasing to H and ψ_k to ψ as $k \to +\infty$. Solve for $u_{j,k} \in \mathcal{F}(\mu, (\psi_k + H_k)|_{\{\partial\Omega_j\}})$ on Ω_j. Define $h_{j,k}$ to be equal to $\max(u_{j,k}, \psi_k + H_k^j)$ on $\overline{\Omega}_j$ and $\psi_k + H_k^j$ on $C\overline{\Omega}_j$. Then $h_{j,k} \in \mathrm{PSH}^-(\Omega)$, $H_k^j \geq h_{j,k} \geq \psi_k + H_k^j$ and obviously $h_{j,k+1} \leq h_{j,k}$ on Ω, $u_{j,k+1} \leq u_{j,k}$ on Ω_j, so $\lim_{k\to+\infty} h_{j,k} = h_j$ satisfies $H \geq h_j \geq \psi + H$. Define $u_j = \lim_{k\to+\infty} u_{j,k}$ on Ω_j.

Claim 1. *$u_j = h_j$ on Ω_j.*

Indeed, it is clear that $h_j \geq u_j$. But on $\partial\Omega_j$ we have $\overline{\lim}_{z\to\xi}(\psi + H) \leq u_{j,k} = \psi_k + H_k$, and on Ω_j, $(dd^c(\psi+H))^n \geq (dd^c u_{j,k})^n = \mu$. Hence $u_j \geq \psi + H$ on Ω_j, which proves the claim. In particular, $(dd^c h_j)^n|_{\Omega_j} = \mu$.

Claim 2. *$h_{j+1} \geq h_j$ on Ω.*

Indeed, on $C\Omega_{j+1}$, $h_{j+1} = h_j = H + \psi$. On $\Omega_{j+1}, h_{j+1} = u_{j+1}$ by Claim 1, so

- $(dd^c h_{j+1})^n = \mu$ on Ω_{j+1} and $(dd^c h_j)^n \geq \mu$ on Ω_{j+1},
- on $\Omega_{j+1} \cap C\Omega_j$, $h_{j+1} = u_{j+1} \geq \psi + H = h_j$.

Hence $h_{j+1} \geq h_j$ on Ω_{j+1}, so $h_{j+1} \geq h_j$ on Ω.

Put $u = (\lim_{j\to+\infty} h_j)^*$. Then $(dd^c u)^n = \mu$ and $H \geq u \geq H + \psi$, which concludes the proof of the lemma. ∎

Note that if $\psi_1 \leq \psi_2$ are as in Lemma 4.2 with $(dd^c\psi_2)^n \leq (dd^c\psi_1)^n$, then $u_1 \leq u_2$ for the solutions obtained in the lemma.

We now turn to the existence part of the proof of Theorem 4.1.

Using [11], we find that

$$(dd^c v)^n = \mu = f(dd^c g)^n, \quad g \in \mathcal{E}_0(\Omega).$$

Consider $\psi_k = U(\chi_{\Omega_k} \inf(f,k)(dd^c g)^n, 0) \in \mathcal{E}_0(\Omega)$ and by Lemma 4.2 find $u_k \in \mathcal{E}_0(H)$ such that $(dd^c u_k)^n = \chi_{\Omega_k} \inf(f,k)(dd^c g)^n$.

As already noted, (u_k) is a decreasing sequence and $H \ge u_k \ge \psi_k + H \ge v + H$, so $\lim u_k = u \in \mathcal{N}^a$ and $(dd^c u)^n = \mu$.

This establishes the existence of a solution. If Ω is a so-called B-regular domain, the proof of the existence part can be simplified. The uniqueness will follow from Theorem 4.4 below.

THEOREM 4.3. *Assume* $u \in \mathcal{N}^a(F)$, $F - \varepsilon \ge v \in \mathcal{E}$, $F \in \mathcal{M}^\infty$ *and* $\varepsilon > 0$. *Then there is an increasing sequence* (r_j) *of functions in* $\mathcal{N}^a$ *such that* $\lim_{j\to+\infty} r_j = 0$ *(a.e.) and*

$$\int_{\{u<v+r_j\}} (dd^c v)^n \le \int_{\{u<v+r_j\}} (dd^c u)^n, \qquad \forall j \in \mathbb{N}.$$

Proof. Let $\varphi \in \mathcal{N}^a$ be such that $F \ge u \ge F + \varphi$, as in the definition for u. By [11], we know that $(dd^c\varphi)^n = P(dd^c\psi)^n$ for some $\psi \in \mathcal{E}_0(\Omega)$ and $P \in L^1_{\mathrm{loc}}((dd^c\psi)^n)$. Then, by Corollary 3.12, $\varphi \ge u_s + r_s$, where

$$u_s = U(\chi_{\Omega_s} \min(P,s)(dd^c\psi)^n, 0), \qquad r_s = U(\chi_{\{P\ge s\}} P(dd^c\psi)^n, 0),$$

$u_s \in \mathcal{E}_0$, $r_s \in \mathcal{N}^a$ and r_s increases to 0 as $s \to +\infty$. We have

$$\{u < v + r_s\} \subset \{\varphi + F < F - \varepsilon + r_s\} \subset \{u_s + r_s + \varepsilon < r_s\}$$
$$\subset \{u_s + \varepsilon < 0\} \subset\subset \Omega, \qquad \forall s.$$

Let $R \in \mathcal{E}_0$ be any continuous exhaustion function for Ω such that $R < F - \varepsilon$ near the closure of $\{u_s + \varepsilon < 0\}$. Then, by Corollary 3.14, $w_1 = \max(u, \varphi + \max(F,R)) \in \mathcal{N}^a$, $w_2 = v + r_s \in \mathcal{E}$ and $w_1 = u$, $w_2 = v + r_s$ near the closure of $\{u_s + \varepsilon < 0\}$.

Also, $u \ge v + r_s$ on the complement of $\{u_s + \varepsilon < 0\}$, so $w_1 \ge w_2$ on the complement of $\{u_s + \varepsilon < 0\}$. Then, by Corollary 3.13,

$$\int_{\{w_1<w_2\}} (dd^c w_2)^n \le \int_{\{w_1<w_2\}} (dd^c w_1)^n,$$

so

$$\int_{\{u<v+r_s\}} (dd^c(v + r_s))^n \le \int_{\{u<v+r_s\}} (dd^c u)^n,$$

which completes the proof. ∎

THEOREM 4.4. *Assume* $u \in \mathcal{N}^a(F)$ *and* $F \ge v \in \mathcal{E}$ *where* $F \in \mathcal{M}^\infty$. *If* $(dd^c u)^n \le (dd^c v)^n$ *then* $u \ge v$ *on* Ω.

Proof. We can assume that $F - \varepsilon \ge v$ for some $\varepsilon > 0$. Let (r_j) be a sequence as in Theorem 4.3. If there is a point $z_0 \in \Omega$ with $u(z_0) < v(z_0) + r_j(z_0)$, then there is a constant $\eta > 0$ such that $u(z_0) < \eta T(z_0) + v(z_0) + r_j(z_0)$, where $T \in \mathcal{E}_0, (dd^c T)^n = dV =$ Lebesgue measure near z_0.

By Theorem 4.3 with $u = u, v = \eta T + v$ we get

$$\int_{\{u<\eta T+v+r_j\}} (dd^c(\eta T + v))^n \le \int_{\{u<\eta T+v+r_j\}} (dd^c u)^n \le \int_{\{u<\eta T+v+r_j\}} (dd^c v)^n.$$

Hence

$$\int_{\{u<\eta T+v+r_j\}} (dd^c v)^n + \eta^n \int_{\{u<\eta T+v+r_j\}} (dd^c T)^n \le \int_{\{u<\eta T+v+r_j\}} (dd^c v)^n.$$

so $\int_{\{u<\eta T+v+r_j\}}(dd^c T)^n = 0$, which shows that $u \ge \eta T+v+r_j$ in a neighbourhood of z_0, contrary to the assumption that $u(z_0) < \eta T(z_0)+v(z_0)+r_j(z_0)$. ∎

5. Some examples and remarks. In this section, we discuss some examples.

In classical potential theory, a positive measure μ is the Laplacian of a negative subharmonic function if and only if $\int \psi\, d\mu > -\infty$ for some negative subharmonic function ψ. The corresponding statement for plurisubharmonic functions is not true as Example 5.3 shows. But we have:

Proposition 5.1. *Suppose $\mathcal{E}_0(\Omega) \ni u_j \searrow u$ as $j \to +\infty$ and there is a $\psi \in \mathcal{E}_0(\Omega)$ with $\psi \not\equiv 0$ and $\sup_j \int_\Omega -\psi(dd^c u_j)^n < +\infty$. Then $u \in \mathcal{E}$.*

Proof. Let $\omega \subset\subset \Omega$ be given and consider $u_{j\omega} = \sup\{\varphi \in \mathrm{PSH}^-(\Omega);\ \varphi|_\omega \le u_j|_\omega\}$. Then $u_{j\omega} \in \mathcal{E}_0(\Omega)$ and $u_{j\omega} \searrow u$ on ω as $j \to +\infty$. Integration by parts gives

$$\sup_j \int -\psi(dd^c u_{j\omega})^n \le \sup_j \int -\psi(dd^c u_j)^n < +\infty$$

and since $\operatorname{supp}(dd^c u_{j\omega})^n \subset \overline{\omega}$ and

$$\sup_j \int_\Omega (dd^c u_{j\omega})^n \le (\inf_\omega -\psi)^{-1} \sup_j \int -\psi(dd^c u_{j\omega})^n,$$

it follows that $\lim_{j\to+\infty} u_{j\omega} \in \mathcal{F}$, so $u \in \mathcal{E}$. ∎

Proposition 5.2. *If μ is a positive measure which vanishes on all pluripolar sets and if there is a $\psi \in \mathcal{E}$ with $\psi \not\equiv 0$ and $\int \psi\, d\mu > -\infty$, then $U(\mu, 0) \in \mathcal{N}^a$ and $(dd^c U(\mu, 0))^n = \mu$.*

Proof. By [11], we can assume that $-1 \le \psi \in \mathcal{E}_0$. Then, by Błocki's inequality [5],

$$\int (-U((1-\chi_{\Omega_j})\mu, 0))^n (dd^c\psi)^n \le n! \int -\psi(1-\chi_{\Omega_j})\, d\mu.$$

By Lebesgue's monotone convergence theorem, the right hand side tends to zero as j tends to $+\infty$. ∎

Example 5.3. We construct a function $W \in \mathcal{N} \cap L^\infty(\Omega)$ with the property that $\int \psi(dd^c W)^n = -\infty$ for all $\psi \in \mathrm{PSH}^-$, $\psi \not\equiv 0$.

Put $u_j = \max(j^2 \log|z|, -1/j^2) \in \mathcal{E}_0(B)$, $B =$ the unit ball in $\mathbb{C}^n$. Then

$$\int_B \max(\log|z|, -1)(dd^c u_j)^n$$
$$= j^{2n} \int_B \max(\log|z|, -1)(dd^c \max(\log|z|, -1/j^4))^n = j^{2n}(2\pi)^n \frac{-1}{j^4}.$$

Thus if $w_p = \sum_{j=1}^p u_j$ then

$$\int_B \max(\log|z|, -1)(dd^c w_p)^n \le -p \to -\infty, \qquad p \to +\infty,$$

but $W = \lim_{p\to+\infty} w_p \in \mathcal{E}$ since $W \in \mathrm{PSH}^-(B) \cap L^\infty$.

Finally, $\widetilde{W} \ge \sum_{j=k}^\infty u_j$ for all k. Since the right hand side tends to zero as k tends to ∞, it follows that $\widetilde{W} = 0$.

Remark. If $n > 1$, there is no $0 \not\equiv u \in \mathcal{F}(B)$ with $-(-u)^{1/n} \in \mathcal{F}$. Indeed, suppose u exists and use W from Example 5.3. Then, by Błocki's inequality,

$$\int_B (-u)(dd^c W)^n = \int_B ((-u)^{1/n})^n (dd^c W)^n$$
$$\le n!(\sup -W)^n \int_B (dd^c - (-u)^{1/n})^n < +\infty,$$

which is a contradiction, since $u \le \alpha \max(\log|z|, -1) < 0$ for some $\alpha > 0$.

Example 5.4. We will now construct a sequence of functions $\sum_{j=1}^m \varphi_j \in \mathcal{E}_0(B)$ such that $\sum_{j=1}^m \varphi_j \searrow -\infty$ as $m \to +\infty$ and

$$\sup_m \int_B (\log|z|^2)^2 \Big(dd^c \sum_{j=1}^m \varphi_j\Big)^2 < +\infty.$$

Put

$$a_j = \frac{1}{j^{1/2}}, \qquad b_j = \frac{1}{j^{1/2}\log j}, \qquad j \ge 2.$$

Then $\sum_{j=2}^\infty a_j^2 = +\infty$, $\sum_{j=2}^\infty a_j b_j = +\infty$ but $\sum_{j=2}^\infty b_j^2 < +\infty$. Also $a_j \sum_{k=2}^j a_k \le 6$ for all $j \ge 2$. Define

$$\varphi_j = \frac{a_j}{2\pi} \max(\log|z|, \log(1 - b_j)), \qquad j \ge 2.$$

Then

$$dd^c\varphi_j \wedge dd^c\varphi_k = \begin{cases} a_j^2 \sigma_{1-b_j}, & j = k, \\ a_j a_k \sigma_{\max(1-b_j, 1-b_k)}, & j \ne k. \end{cases}$$

Here, σ_r denotes the normalized Lebesgue measure on the sphere with radius r. Now

$$\int(\log|z|^2)^2\Big(dd^c\sum_{j=2}^m\varphi_j\Big)^2=\sum_{j,k=2}^m\int(\log|z|^2)^2(dd^c\varphi_j\wedge dd^c\varphi_k)$$
$$\le 2\sum_{j=2}^m\sum_{k=2}^j\int(\log|z|^2)^2(dd^c\varphi_j\wedge dd^c\varphi_k)$$
$$\le 2\sum_{j=2}^m(\log 1-b_j)^2a_j\sum_{k=1}^j a_k\le 12\sum(\log 1-b_j)^2<+\infty.$$

Remark. To conclude that a sequence $\varphi_j\in\mathcal{E}_0$ decreases to a plurisubharmonic function $\not\equiv-\infty$, it is not enough to know that $(dd^c\varphi_j)^n$ is weak*-convergent.

Remark. If $\psi,\varphi_1,\varphi_2\in\mathcal{E}_0(\Omega)$ with $\varphi_1\ge\varphi_2$ then

$$\int-\psi(dd^c\varphi_1)^n\le\int-\psi(dd^c\varphi_2)^n,$$

which is a very useful inequality.

This cannot be generalized to higher powers of ψ. For $n=2$, there is no constant c such that

$$\int(-\psi)^2(dd^c\varphi_1)^2\le c\int(-\psi)^2(dd^c\varphi_2)^2,\qquad\forall\psi,\varphi_1,\varphi_2\in\mathcal{E}_0(\Omega),\varphi_1\ge\varphi_2.$$

For assume there is such a constant. Let $0>v\in\mathrm{PSH}^-$ be any function and consider $h_m=\max(v,\sum_{j=2}^m\varphi_j)$, where $(\varphi_j)_{j=1}^\infty$ is the sequence of functions in Example 5.4. Then we would have

$$\lim_{m\to+\infty}\int(-\log|z|^2)^2(dd^ch_m)^2\le c\lim_{m\to+\infty}\int(\log|z|^2)^2\Big(dd^c\sum_{j=1}^m\varphi_j\Big)^2<+\infty,$$

so it would follow that $v=\lim h_m\in\mathcal{E}$, which is not true in general (see for instance [9]).

Remark. Let μ be a weak limit of $(dd^c\sum_{j=1}^m\varphi_j)^n$ where $\sum_{j=1}^m\varphi_j\in\mathcal{E}_0(B)$ is the sequence of functions constructed in Example 5.4. Then there is no function $u\in\mathcal{E}$ with $(dd^cu)^2=\mu$. For, by Theorem 3.7, $u\le\sum_{j=1}^m\varphi_j$ for every j, which forces u to be $-\infty$ everywhere.

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