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On the Green function on a certain class of hyperconvex domains

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Abstract. We study the behavior of the pluricomplex Green function on a bounded hyperconvex domain D that admits a smooth plurisubharmonic exhaustion function ψ such that $1/|\psi|$ is integrable near the boundary of D, and moreover satisfies the estimate $|\psi| \leq C\exp(-C'(\log(1/\delta_D))^\alpha)$ at points close enough to the boundary with constants $C, C' > 0$ and $0 < \alpha < 1$. Furthermore, we obtain a Hopf lemma for such a function ψ. Finally, we prove a lower bound on the Bergman distance on D.

1. Introduction. In 1985 M. Klimek introduced the pluricomplex Green function of a bounded domain $D \subset \mathbb{C}^n$. It is defined by

$$\mathscr{G}_D(z,w) = \sup\{u(z) \mid u : D \to [-\infty, 0),\ u \in \mathrm{PSH}(D),\\ u(z) - \log|z-w| \text{ is bounded from above near } w\},$$

where $\mathrm{PSH}(D)$ denotes the family of plurisubharmonic functions on D. In [18] and [9] the important properties of the Green function were established and also its relationship to the complex Monge–Ampère equation was clarified.

The Green function is a powerful tool for investigations in Bergman theory, when one wants to construct good holomorphic square-integrable functions by means of the $\overline{\partial}$-technique with weights (see for example [5, 12, 15, 17]). On a hyperconvex domain it is known from [9, 18] that $\mathscr{G}_D(z,w) \to 0$ as z tends to the boundary and w is kept fixed. A domain $D \subset \mathbb{C}^n$ is called *hyperconvex* ([21]) if it admits a bounded continuous plurisubharmonic exhaustion function $\psi : D \to (-1, 0)$.

When using the Green function as a weight, one needs, however, information on the sublevel sets of the Green function $\mathscr{G}_D(\cdot, w)$ as w tends to the boundary; more precisely, it is desirable to describe, in terms of the boundary distance of the pole w, where the sets $\{\mathscr{G}_D(\cdot, w) < -1\}$ are situated.

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This is a difficult question, since for $n > 1$ the pluricomplex Green function is no longer symmetric (see [2]). Recently, some progress has been made in this direction: see for example [5, 12, 16].

Carlehed, Cegrell and Wikström [8] obtained a first result on the behavior of the Green function $\mathscr{G}_D(\cdot, w)$ as w approaches a boundary point w_0: If $(w_j)_j$ is a sequence of points in D that tends towards w_0, then there exists a pluripolar set E such that

$$\limsup_{j\to\infty} \mathscr{G}_D(z, w_j) = 0 \quad \text{for } z \in D \setminus E.$$

We prove that under a mild additional condition on D (which is considerably weaker than those from [12] and [16]), the set E is empty and the lim sup is in fact a limit. We assume in Sections 2 through 6 that $n > 1$. The case $n = 1$ will be discussed in Section 7.

Theorem 1.1. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ with $n \geq 2$ that admits a plurisubharmonic smooth exhaustion function $\psi : D \to [-1, 0)$ with the following two properties*:

(1) *There is a positive measurable function h such that $h^{-1/(n-1)}$ is integrable over D (with respect to the Lebesgue measure) and*
$$(dd^c\psi)^n \geq h(dd^c|z|^2)^n.$$
(2) *There are constants $1 > \alpha > 0$ and $\widehat{C}_1, \widehat{C}_2 > 0$ such that*
$$\psi \geq -\widehat{C}_1 \exp\biggl(-\widehat{C}_2\biggl(\log\frac{1}{\delta_D}\biggr)^{\alpha}\biggr)$$
on $D \cap \{\delta_D < 1\}$, where δ_D denotes the boundary distance function on D.

Then there are constants $\widetilde{C}, \delta_0 > 0$ such that for any compact subset $K \subset D$ and $w \in D \setminus K$ with $\delta_D(w) \leq \min\{\delta_0, \delta_D(K)/4\}$,

$$\sup_{z\in K} |\mathscr{G}_D(z, w)| \leq \widetilde{C}\biggl(\frac{|\psi(w)|^{1/3n}}{\delta_D(K)^{2n+3}} + \delta_D(w)\biggr).$$

A function with property (1) of the above theorem exists on a general hyperconvex domain, as follows from a result of [6]:

Theorem 1.2. *Let D be a bounded hyperconvex domain. Given a continuous function f on ∂D that extends to a plurisubharmonic function on D, and a continuous function $F : \overline{D} \to [0, \infty)$, there exists a uniquely determined continuous function $u = u_{(f,F)}$ on $\overline{D}$ that is plurisubharmonic on D and such that $u = f$ on ∂D, and $(dd^c u)^n = F dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ on D.*

The uniquely determined solution φ_D from Theorem 1.2 which corresponds to the data $f = 0, F = 1$ will of course satisfy the requirement (1).

With slight changes the method used to prove Theorem 1.1 yields

THEOREM 1.3. *Let $n \geq 2$. Suppose that $D \subset\subset \mathbb{C}^n$ is hyperconvex and admits a smooth plurisubharmonic exhaustion function $\psi : D \to (-1,0)$ that satisfies with some constant $C_1 > 0$ the estimate*

$$(3)\quad \frac{1}{C_1}\left(\log\frac{1}{\delta_D}\right)^{-M} \leq |\psi| \leq C_1\left(\log\frac{1}{\delta_D}\right)^{-N}$$

on $D \cap \{\delta_D < 1\}$ with exponents $M, N > 2(n+1)$ such that $N \leq M \leq -1 + N^2/4n$. Then there exist constants $\widetilde{C}, \delta_* > 0$ such that for any compact set $K \subset D$ and any $w \in D \setminus K$ with $\delta_D(w) \leq \min\{\delta_*, \delta_D(K)/4\}$,

$$\sup_{z\in K} |\mathscr{G}_D(z,w)| \leq \widetilde{C}\left(\frac{|\psi(w)|^{\gamma(n-1)/n}}{\delta_D(K)^{2n}} + |\psi(w)|^{\beta_1(n-1)/n}\right),$$

where $\beta = \frac{1}{2}(N(n-1)/n^2 - M' + 1/N')$, $N' = N(1-1/n)$, $M' = M(1-1/n)$, $\gamma = n\beta/2N(n-1)$, and $\beta_1 = \min\{\beta, 1/n - \gamma\}$. Note that $\beta \geq N/48n$ for $n \geq 2$.

Under the hypotheses of Theorem 1.1 a lower bound on the Bergman distance can be obtained (using the idea of [14]):

THEOREM 1.4. *Let D and ψ be as in Theorem 1.1. Then there exists a constant $C_* > 0$ such that for any fixed $P \in D$,*

$$d_D^{\mathrm{B}}(Q,P) \geq C_* \log\log\log\frac{1}{|\psi(Q)|}$$

whenever $\delta_D(Q) \ll 1$.

Note that no Hölder condition on the exhaustion function ψ is needed.

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2. Lower bounds on the Demailly regularization of a plurisubharmonic function

Some notations. Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain and V a negative plurisubharmonic function on D. Following the method of Demailly [10] we regularize V. For this let $m > 0$ be a positive number and $\mathcal{H}_{2mV}(D)$ the Hilbert space of holomorphic functions in D such that the weighted L^2-norm

$$\|f\|_{2mV} := \Big(\int_D |f|^2 e^{-2mV}\, d^{2n}z\Big)^{1/2}$$

is finite. We denote by $K_{D,2mV}$ the Bergman kernel associated to $\mathcal{H}_{2mV}$. Then the function

$$V_m(z) := \frac{1}{2m} \log K_{D,2mV}(z,z)$$

is plurisubharmonic, and we have

$$V_m(z) := \frac{1}{2m} \sup\{\log |f(z)|^2 \mid f \in \mathcal{H}_{2mV},\ \|f\|_{2mV} \le 1\}.$$

It is shown in Proposition 3.1 of [10] that, with suitable constants C_1, C_2, for any $z \in D$ and any $0 < r < \delta_D(z)$ one has

$$V(z) - \frac{1}{C_1 m} \le V_m(z) \le \sup_{x \in B(z,r)} V(x) + \frac{1}{m} \log \frac{C_2}{r^n}, \tag{2.1}$$

and moreover V_m tends to V pointwise and in the L^1_{loc}-topology on D as $m \to \infty$.

For $z^0 \in D$ and $r \in (0, \delta_D(z^0))$, and a measurable function V on D, we denote by $M[V, z^0, r]$ the spherical mean of V over $\partial B(z^0, r)$ and by $A[V, z^0, r]$ the average of V over the full ball $B(z^0, r)$.

For any subharmonic function V on D one has

$$A[V, z^0, r] \le M[V, z^0, r].$$

Given $z^0 \in D$, we need a lower estimate for $V_m(z^0)$ in terms of the mass of the ball $B(z^0, r)$ with respect to the measure ΔV. Then we can apply the result to the case where V is the Green function with a pole at $w \in D$. This together with an estimate of Błocki will help control the value $V_m(z^0)$ as w tends to a boundary point.

We begin with a comparison lemma for the weighted Bergman kernel.

LEMMA 2.1. *Let* $0 < r < \delta_D(z^0)/2$. *Then*:

(a) *We have*

$$K_{2mV}(z^0, z^0) \ge C_D \frac{1}{A[e^{-2mV}, z^0, r]},$$

where $C_D > 0$ *depends only on the diameter of* D.

(b) *The regularization* V_m *can be estimated from below by*

$$V_m(z^0) \ge -\frac{\log C_D}{2m} - \frac{1}{2m} \log(A[e^{-2mV}, z^0, r]). \tag{2.2}$$

Proof. We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ with $\chi(s) = 1$ for $s \le 1/4$ and $\chi(s) = 0$ for $s > 9/16$. Then the $(0,1)$-form

$$v = \bar\partial\left(\chi\left(\frac{|z - z^0|^2}{r^2}\right)\right) = \chi'\left(\frac{|z - z^0|^2}{r^2}\right) \frac{\bar\partial |z - z^0|^2}{r^2}$$

is smooth and $\bar{\partial}$-closed on D. We will solve a suitable $\bar{\partial}$-problem for these data on D with weight function

$$\varphi(z) := 2mV(z) + 2n \log|z - z^0|$$

Our plan is to use the L^2-technique developed in [20, 3] (see also [13]). The relevant tool for solving $\bar{\partial}$ will be the following slight modification of Lemma 2.2 from [17]:

LEMMA 2.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with a C^2-smooth boundary. Suppose that on $\bar{\Omega}$ we are given two smooth functions $\widetilde{\varphi}, \widetilde{\eta}$, where $\widetilde{\eta} > 0$, whose Levi forms $\mathscr{L}_{\widetilde{\varphi}}$ and $\mathscr{L}_{\widetilde{\eta}}$ satisfy*

$$\widetilde{\eta}\mathscr{L}_{\widetilde{\varphi}} - \mathscr{L}_{\widetilde{\eta}} \geq Q + \frac{1}{4}\frac{\partial\widetilde{\eta} \otimes \overline{\partial\widetilde{\eta}}}{\widetilde{\eta}^2}$$

with some positive hermitian form Q on Ω. Then, given a smooth $\bar{\partial}$-closed $(0,1)$-form $v = v_1 d\bar{z}_1 + \cdots + v_n d\bar{z}_n$ on Ω such that

$$\mathscr{I}_{Q,\widetilde{\varphi}}(v) := \int_{\Omega} |v|_Q^2 e^{-\widetilde{\varphi}}\, d^{2n}z < \infty,$$

one can solve the equation $\bar{\partial}(\sqrt{\widetilde{\eta} + \widetilde{\eta}^2}\, u) = v$ with a smooth function u on Ω such that

$$\int_{\Omega} |u|^2 e^{-\widetilde{\varphi}}\, d^{2n}z \leq 20\mathscr{I}_{Q,\widetilde{\varphi}}(v).$$

Here $|v|_Q^2$ denotes the square of the length of v with respect to the form Q: If $(Q_{a\bar{b}})_{a,b=1}^n$ is the coefficient matrix of Q and $(Q^{a\bar{b}})_{a,b=1}^n$ is its inverse matrix, then $|v|_Q^2 = \sum_{a,b=1}^n Q^{a\bar{b}} v_a \bar{v}_b$.

Let $(D^t)_{r>t>0}$ be an exhaustion for D by smooth bounded pseudoconvex domains D^t such that $B(z^0, 2r) \subset\subset D^t$ for each t.

CASE 1: V is continuous. On each D^t we can choose a regularization V^t of V such that $V \leq V^t \leq V + 1/2m$ on D^t. Then in the above lemma we choose $\Omega := D^t$ and

$$\widetilde{\varphi}_t(z) = |z|^2 + 2mV^t(z) + n\log(t^2 + |z - z^0|^2).$$

Next we put

$$\widetilde{\eta}_t = \eta_t + \log\eta_t,$$

where

$$\eta_t(z) := -\log\frac{r^2 + t^2 + |z - z^0|^2}{8eR_D^2}$$

and R_D is the diameter of D. This function has values > 1, and $-\eta_t$ and

$-\widetilde{\eta}_t$ are plurisubharmonic on D^t. We estimate

$$\widetilde{\eta}_t \mathscr{L}_{\widetilde{\varphi}_t} - \mathscr{L}_{\widetilde{\eta}_t} \geq \widetilde{\eta}_t \mathscr{L}_{|z|^2} - \left(1 + \frac{1}{\eta_t}\right) \mathscr{L}_{\eta_t} + \frac{\partial \widetilde{\eta}_t \otimes \overline{\partial \widetilde{\eta}_t}}{(1+\eta_t)^2}$$
$$\geq \widetilde{\eta}_t \mathscr{L}_{|z|^2} - \mathscr{L}_{\eta_t} + \frac{\partial \widetilde{\eta}_t \otimes \overline{\partial \widetilde{\eta}_t}}{4\widetilde{\eta}_t^2},$$

because $-\mathscr{L}_{\eta_t}$ is positive and $1 + \eta_t \leq 2\eta_t \leq 2\widetilde{\eta}_t$. We will choose $Q := \widetilde{\eta}_t \mathscr{L}_{|z|^2} - \mathscr{L}_{\eta_t}$.

Our next aim is to estimate $|v|_Q$. On the support of v, which is contained in $B(z^0, r) \setminus B(z^0, r/2)$, we have

$$Q = \widetilde{\eta}_t \mathscr{L}_{|z|^2} - \mathscr{L}_{\eta_t} \geq \widetilde{\eta}_t \mathscr{L}_{|z|^2} + \frac{r^2 + t^2}{(r^2 + t^2 + |z - z^0|^2)^2} \mathscr{L}_{|z-z^0|^2}$$
$$\geq \mathscr{L}_{|z|^2} + \frac{|z - z^0|^2}{(r^2 + t^2 + |z - z^0|^2)^2} \mathscr{L}_{|z-z^0|^2}$$
$$\geq \mathscr{L}_{|z|^2} + \frac{\partial(|z - z^0|^2) \otimes \overline{\partial(|z - z^0|^2)}}{(r^2 + t^2 + |z - z^0|^2)^2}$$
$$\geq \widetilde{Q} := \mathscr{L}_{|z|^2} + \frac{\partial(|z - z^0|^2)}{3r^2} \otimes \overline{\frac{\partial(|z - z^0|^2)}{3r^2}},$$

which implies

$$|v|_Q^2 \leq |v|_{\widetilde{Q}}^2 \leq \xi_r \, (\sup |\chi'|)^2 \left| \frac{\overline{\partial}|z - z^0|^2}{r^2} \right|_{\widetilde{Q}} = 9\xi_r (\sup |\chi'|)^2 \, \frac{|z - z^0|^2}{r^4 + |z - z^0|^2}$$
$$\leq 9\xi_r (\sup |\chi'|)^2,$$

where ξ_r is the characteristic function of $B(z^0, r) \setminus B(z^0, r/2)$. On the support of v we also have

$$\widetilde{\varphi}_t(z) \geq 2mV(z) + 2n \log(r/2).$$

This yields

$$\mathscr{I}_{Q, \widetilde{\varphi}_t}(v) \leq 9(\sup |\chi'|)^2 \, \frac{4^n}{r^{2n}} \int\limits_{B(z^0, r)} e^{-2mV} \, d^{2n}z = c_n A[e^{-2mV}, z^0, r],$$

with $c_n = 9 \cdot 4^n (\sup |\chi'|)^2$ times the volume of the unit ball. By Lemma 2.2 we obtain a smooth solution u_t to the equation $\overline{\partial}(\sqrt{\widetilde{\eta}_t + \widetilde{\eta}_t^2} u_t) = v$ such that

$$\int\limits_{D^t} |u_t|^2 e^{-\widetilde{\varphi}_t} \, d^{2n}z \leq 20 c_n A[e^{-2mV}, z^0, r].$$

Next we observe that

$$(\widetilde{\eta}_t + \widetilde{\eta}_t^2) e^{-2mV} e^{\widetilde{\varphi}_t} = (\widetilde{\eta}_t + \widetilde{\eta}_t^2) e^{|z|^2 + 2m(V^t - V)} (t^2 + |z - z^0|^2)^n$$
$$\leq 4e^{\widetilde{R}_D^2 + 1} \max_{0 < x \leq (2R_D)^2} \left(x^n \left(\log \frac{8eR_D^2}{r^2 + x} + \left(\log \frac{8eR_D^2}{r^2 + x} \right)^2 \right) \right) \leq C'_D$$

with

$$\widetilde{R}_D := \max_{z\in D}|z|, \qquad C'_D := 8e^{\widetilde{R}_D^2+1}(2R_D)^n \max_{0<\xi\le 4}\left(\xi^n\left(\log\frac{4e}{\xi}\right)^2\right).$$

The function

$$f_t(z) = \chi\left(\frac{|z-z^0|^2}{r^2}\right) - \sqrt{\widetilde{\eta}_t + \widetilde{\eta}_t^2}\, u_t(z)$$

now becomes an element of $\mathcal{H}_{2mV}(D^t)$, with norm

$$\|f_t\|_{2mV,D^t} \le (\gamma_n R_D^n + 5\sqrt{C'_D})\sqrt{A[e^{-2mV}, z^0, r]},$$

where $\gamma_n := \sqrt{\text{volume of the unit ball}}$. To see this, note that

$$\left\|\chi\left(\frac{|z-z^0|^2}{r^2}\right)\right\|^2_{2mV,D^t} \le \gamma_n^2 r^{2n} A[e^{-2mV}, z^0, r]$$

and $r \le R_D$.

By a standard weak-star limit argument (similar to [17]) we find a holomorphic function $\widetilde{f}$ on D of the form

$$\widetilde{f}(z) = \chi\left(\frac{|z-z^0|^2}{r^2}\right) - \sqrt{\widetilde{\eta} + \widetilde{\eta}^2}\, u(z),$$

where

$$\widetilde{\eta} = \eta + \log\eta, \qquad \eta = -\log\frac{r^2 + |z-z^0|^2}{8eR_D^2},$$

that satisfies

$$\|\widetilde{f}\|_{2mV,D} \le (\gamma_n R_D^n + \sqrt{C'_D})\sqrt{A[e^{-2mV}, z^0, r]}.$$

Moreover,

$$\int_D |u|^2 e^{-\varphi}\, d^{2n}z \le 20 c_n A[e^{-2mV}, z^0, r]$$

with $\varphi = |z|^2 + 2mV(z) + 2n\log|z-z^0|$. This gives $u(z^0) = 0$. The function $\widetilde{f}/\|\widetilde{f}\|_{2mV}$ is a candidate for $K_{2mV}(z^0)$. So we obtain

$$K_{2mV}(z^0, z^0) \ge \frac{|\widetilde{f}(z^0)|^2}{\|\widetilde{f}\|^2_{2mV}} \ge \frac{C_D}{A[e^{-2mV}, z^0, r]}$$

with $C_D := (\gamma_n R_D^n + 5\sqrt{C'_D})^{-2}$. From this the first claim follows immediately.

Case 2: V is arbitrary. First we fix a number $s \ll 1$ and consider the Demailly regularizations of the functions V^t, taken over the domain D^s. We will denote them by $(V^t)_{m,D^s}$. Here $t < s$. In explicit form, $(V^t)_{m,D^s} = (2m)^{-1}\log K_{2mV^t,D^s}$, where

$$K_{2mV^t,D^s}(z) = \sup\{|f_t(z)|^2 \mid f_t \in \mathcal{H}_{2mV^t}(D^s),\ \|f_t\|_{2mV^t} \le 1\}$$

on D^s. Our first claim is: There exists a null sequence $(t_k)_k$ such that

$K_{2mV^t,D^s}(z^0) \to K_{2mV,D^s}(z^0) := \sup\{|f(z)|^2 \mid f \in \mathcal{H}_{2mV}(D^s),\ \|f\|_{2mV} \le 1\}$.

To see this we choose for each $t < s$ a function $f_{t,s} \in \mathcal{H}_{2mV^t}(D^s)$ with $\|f_t\|_{2mV^t} \le 1$ such that

$$K_{2mV^t,D^s}(z^0) = |f_{t,s}(z^0)|^2.$$

Then the Alaoglu–Bourbaki theorem can be applied to the functions $\widehat{f}_{t,s} := f_{t,s}e^{-mV^t}$, which belong to $L^2(D^s)$. We have $\|\widehat{f}_{t,s}\|_{L^2(D^s)} \le 1$. We can choose a null sequence $(t_k)_k$ and a function $\widehat{f}_{0,s} \in L^2(D^s)$, having norm ≤ 1 in $L^2(D^s)$, such that $\widehat{f}_{t_k,s} \to \widehat{f}_{0,s}$ in the weak-star topology in $L^2(D^s)$. The function $f_{0,s} := \widehat{f}_{0,s}e^{-mV}$ is square-integrable with respect to the weighted Lebesgue measure $e^{-2mV}d^{2n}z$ over D^s with $\|f_{0,s}\|_{2mV} \le 1$.

We claim that $f_{0,s}$ is even holomorphic. Since V and V^{t_k} are negative, the functions $e^{mV}, e^{mV^{t_k}}$ are bounded. Therefore, the sequences $(f_{t_k,s})_k$ and $(\widehat{f}_{t_k,s}e^{mV})_k$ tend to $f_{0,s}$ in the weak-star topology of $L^2(D^s)$. If now β denotes an arbitrary test form of bidegree $(0,1)$, and ϑ is the formal adjoint of $\overline{\partial}$ (in $L^2(D^s)$), then

$$\begin{aligned}\langle f_{0,s}, \vartheta\beta\rangle &= \lim_{k\to\infty}\langle \widehat{f}_{t_k,s}e^{mV}, \vartheta\beta\rangle = \lim_{k\to\infty}\langle f_{t_k,s}, e^{m(V-V^{t_k})}\vartheta\beta\rangle \\ &= \lim_{k\to\infty}(\langle f_{t_k,s}, \vartheta\beta\rangle + \langle f_{t_k,s}, (e^{m(V-V^{t_k})}-1)\vartheta\beta\rangle) \\ &= \lim_{k\to\infty}\langle f_{t_k,s}, (e^{m(V-V^{t_k})}-1)\vartheta\beta\rangle.\end{aligned}$$

But

$$\begin{aligned}|\langle f_{t_k,s}, (e^{m(V-V^{t_k})}-1)\vartheta\beta\rangle| &= |\langle \widehat{f}_{t_k,s}, e^{mV^{t_k}}(e^{m(V-V^{t_k})}-1)\vartheta\beta\rangle| \\ &\le \|\widehat{f}_{t_k,s}\|_{L^2(D^s)}\|e^{mV^{t_k}}(e^{m(V-V^{t_k})}-1)\vartheta\beta\|_{L^2(D^s)} \\ &\le \|(e^{m(V-V^{t_k})}-1)\vartheta\beta\|_{L^2(D^s)} \to 0\end{aligned}$$

as $k \to \infty$. This proves that $f_{0,s} \in \mathcal{H}_{2mV}(D^s)$. It also follows that the functions $f_{t_k,s}$ tend pointwise to $f_{0,s}$. Hence we obtain the desired lower bound:

$$\begin{aligned}K_{2mV,D^s}(z^0) = |f_{0,s}(z^0)|^2 = \lim_{k\to\infty}|f_{t_k,s}(z^0)|^2 \ge C_{D^s}\lim_{k\to\infty}\frac{1}{A[e^{-2mV^{t_k}}, z^0, r]} \\ \ge C_{D^s}\frac{1}{A[e^{-2mV}, z^0, r]} \ge C_D\frac{1}{A[e^{-2mV}, z^0, r]}.\end{aligned}$$

Finally, we apply a similar weak-star limit argument to the functions

$$f^*_{0,s}(z) := \begin{cases} f_{0,s}e^{-mV}(z) & \text{for } z \in D^s, \\ 0 & \text{for } z \in D \setminus D^s \end{cases}$$

to obtain a function $f_0 \in \mathcal{H}_{2mV}(D)$ with norm $\|f_0\|_{2mV} \le 1$ such that

$$|f_0(z^0)|^2 \ge \frac{C_D}{A[e^{-2mV}, z^0, r]},$$

proving part (a) of the lemma.

(b) The second assertion follows by taking the logarithm on both sides in (a). ∎

We next estimate the integral

$$\mathscr{I}_m(r) := \int\limits_{B(z^0,r)} e^{-2mV}\, d^{2n}z$$

for $r \in (0, \delta_D(z^0)/16)$. Let μ_V be the measure defined by ΔV.

LEMMA 2.3. *Let z^0 and $0 < \varepsilon < \delta_D(z^0)/4$ and $0 < r < \varepsilon/4$ as in the preceding lemma. If*

$$m \le \frac{(2\varepsilon)^{2n-2}}{16(n+1)e\mu_V(B(z^0, 2\varepsilon))},$$

then

$$\mathscr{I}_m(r) \le C_n \omega_{2n} \varepsilon^{2n} \exp\biggl(\frac{9m}{8c_n}\, \frac{\mu_V(B(0,\varepsilon))}{\varepsilon^{2n-2}} - 4^n \cdot 2mM[V, 0, \varepsilon]\biggr)$$

with some unimportant constant $C_n > 0$. Here ω_{2n} is the area of the unit sphere in $\mathbb{C}^n$.

Proof. Let us assume that $z^0 = 0$. We choose a number $\varepsilon \in (4r, \delta_D(0))$. By the Riesz representation theorem we can write

$$V(z) = P^*_\varepsilon(z) + h_\varepsilon(z) \quad \text{for } z \in B(0,\varepsilon),$$

where P^*_ε is the Green potential,

$$P^*_\varepsilon(z) = \int\limits_{|\zeta|<\varepsilon} G_\varepsilon(z,\zeta)\, d\mu_V(\zeta),$$

and $G_\varepsilon(z,\zeta)$ denotes the (real) Green function of $B(0,\varepsilon)$. Further, h_ε is the smallest harmonic majorant for V on $B(0,\varepsilon)$ and is given by the Poisson integral of V,

$$h_\varepsilon(z) = \frac{1}{\omega_{2n}\varepsilon} \int\limits_{|\zeta|=\varepsilon} \frac{\varepsilon^2 - |z|^2}{|z-\zeta|^{2n}}\, V(\zeta)\, dS(\zeta).$$

The Green function for $n \ge 2$ is defined by

$$G_\varepsilon(z,\zeta) = \begin{cases} E(z-\zeta) - E((z^* - \zeta)|z|/\varepsilon) & \text{if } z \ne \zeta,\ z \ne 0,\\ E(\zeta) + \dfrac{1}{(2n-2)\omega_{2n}}\, \dfrac{1}{\varepsilon^{2n-2}} & \text{if } z \ne \zeta,\ z = 0,\\ -\infty & \text{if } z = \zeta. \end{cases}$$

Here $z^* := \varepsilon^2 z/|z|^2$ and E is the fundamental solution to the Laplacian,

$$E(x) := \frac{-1}{(2n-2)\omega_{2n}} \frac{1}{|x|^{2n-2}}.$$

First we apply Harnack's inequality to the harmonic function h_ε. Note that $h_\varepsilon \le 0$, because $V < 0$. For $|z| < r$ we have

$$\begin{aligned} h_\varepsilon(z) &\ge \frac{1+|z|/\varepsilon}{(1-|z|/\varepsilon)^{2n-1}} h_\varepsilon(0) \ge \frac{1+r/\varepsilon}{(1-r/\varepsilon)^{2n-1}} h_\varepsilon(0) \\ &= \frac{1+r/\varepsilon}{(1-r/\varepsilon)^{2n-1}} M[V,0,\varepsilon] \ge 4^n M[V,0,\varepsilon]. \end{aligned}$$

Furthermore, $G_\varepsilon(z,\zeta) \ge E(z-\zeta)$. Therefore, if we put

$$P_\varepsilon(z) := \frac{-1}{(2n-2)\omega_{2n}} \int_{|\zeta|<\varepsilon} \frac{1}{|z-\zeta|^{2n-2}} \, d\mu_V(\zeta),$$

we obtain $P_\varepsilon^*(z) \ge P_\varepsilon(z)$, and altogether

$$e^{-2mV(z)} \le e^{-4^n \cdot 2mM[V,0,\varepsilon]} e^{-2mP_\varepsilon(z)}.$$

This gives us

$$\mathscr{J}_m(r) \le e^{-4^n\cdot 2mM[V,0,\varepsilon]} \int_{|z|<r} e^{-2mP_\varepsilon(z)} \, d^{2n}z. \tag{2.3}$$

Next we transform the term $P_\varepsilon(z)$ using ideas of [22] (see for instance p. 475). For $s \in (\varepsilon/3, \varepsilon/2)$ and $x \in \mathbb{C}^n$ with $|x| = s$ and $z \in B(0,r)$ we write

$$\begin{aligned} |P_\varepsilon(z) - P_\varepsilon(x)| &\le |x-z| \int_0^1 |\nabla P_\varepsilon(z + t(x-z))| \, dt \\ &\le (r+s) \int_0^1 |\nabla P_\varepsilon(z + t(x-z))| \, dt. \end{aligned}$$

Integrating over the sphere $\{|x| = s\}$ we find (note that $r \le \varepsilon/4 \le 3s/4 \le s$)

$$\begin{aligned} &\int_{|x|=s} P_\varepsilon(x) \, dS(x) - I(s,z) \\ &\qquad \le \omega_{2n} s^{2n-1} P_\varepsilon(z) \le \int_{|x|=s} P_\varepsilon(x) \, dS(x) + I(s,z), \end{aligned} \tag{2.4}$$

where

$$I(s,z) := 2s \int_{|x|=s} \int_0^1 |\nabla P_\varepsilon(z + t(x-z))| \, dt \, dS(x).$$

To estimate the integral $I(s,z)$, we parametrize the positive hemisphere $M_s^+ := \{|\xi| = s \mid \xi_{2n} > 0\}$ by $\phi(\alpha) := (\alpha, \sqrt{s^2 - |\alpha|^2})$. Let $\psi(t,\alpha) := z + t(\phi(\alpha) - z)$ for $(t,\alpha) \in (0,1) \times B_{2n-1}(0,s)$. This defines an injective mapping from $(0,1)\times B_{2n-1}(0,s)$ into $B_{2n}(0,2s)$. Its Jacobian determinant is

$$\det J_{\psi}(t,\alpha) = -\frac{t^{2n-1}}{\sqrt{s^2-|\alpha|^2}}\,(s^2-\langle\phi(\alpha),z\rangle)$$

(where $\langle\cdot,\cdot\rangle$ denotes the euclidean inner product). Since $r<3s/4$, we obtain

$$|\det J_{\psi}(t,\alpha)| \ge \frac{t^{2n-1}(\,s^2-|\phi(\alpha)||z|)}{\sqrt{s^2-|\alpha|^2}} \ge \frac{t^{2n-1}s^2}{4\sqrt{s^2-|\alpha|^2}}.$$

Then

$$\begin{aligned}&\int_{M_s^+} |\nabla P_\varepsilon(z+t(x-z))|\,dS(x)\\ &\qquad= \int_{B_{2n-1}(0,s)} |\nabla P_\varepsilon(z+t(\phi(\alpha)-z))|\,\frac{s}{\sqrt{s^2-|\alpha|^2}}\,d^{2n-1}\alpha\\ &\qquad\le \frac{4}{t^{2n-1}s}\int_{B_{2n-1}(0,s)} |\nabla P_\varepsilon(\psi(t,\alpha))|\,|\det J_{\psi}(t,\alpha)|\,d^{2n-1}\alpha.\end{aligned}$$

Now we observe that

$$|\psi(t,\alpha)-z| = t|\phi(\alpha)-z| \le t(s+|z|) \le 2st,$$

which implies

$$\frac{4}{t^{2n-1}s} \le 2^{2n+1}s^{2n-2}\,\frac{1}{|\psi(t,\alpha)-z|^{2n-1}}.$$

We obtain

$$\begin{aligned}&\int_0^1\int_{M_s^+} |\nabla P_\varepsilon(z+t(x-z))|\,dS(x)\,dt\\ &\qquad\le 4^n s^{2n-2}\int_{(0,1)\times B_{2n-1}(0,s)} \frac{|\nabla P_\varepsilon(\psi(t,\alpha))|\det J_{\psi}(t,\alpha)}{|\psi(t,\alpha)-z|^{2n-1}}\,d^{2n-1}\alpha\,dt\\ &\qquad\le 4^n s^{2n-2}\int_{B_{2n}(0,2s)} \frac{|\nabla P_\varepsilon(\zeta)|}{|z-\zeta|^{2n-1}}\,d^{2n}\zeta.\end{aligned}$$

A corresponding estimate holds for $\int_0^1\int_{M_s^-} |\nabla P_\varepsilon(z+t(x-z))|\,dS(x)\,dt$, where $M_s^- := \{\xi \mid -\xi\in M_s^+\}$.

This proves that (note that $2s\le\varepsilon$)

$$I(s,z) \le 4^{n+1}s^{2n-1}\int_{B_{2n}(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|z-\zeta|^{2n-1}}\,d^{2n}\zeta.$$

We now integrate over all $s\in(\varepsilon/3,\varepsilon/2)$ and divide by ε^{2n}. This gives, in conjunction with (2.4),

$$(2.5)\quad \frac{1}{c_n\varepsilon^{2n}}\int\limits_{\varepsilon/3\le|x|\le\varepsilon/2} P_\varepsilon(x)\,d^{2n}x-\mathscr{K}_\varepsilon(z) \le P_\varepsilon(z)\le\frac{1}{c_n\varepsilon^{2n}}\int\limits_{\varepsilon/3\le|x|\le\varepsilon/2} P_\varepsilon(x)d^{2n}x+\mathscr{K}_\varepsilon(z),$$

where $c_n:=\frac{\omega_{2n}}{2n}((1/2)^{2n}-(1/3)^{2n})$ and

$$\mathscr{K}_\varepsilon(z)=\frac{4^{n+1}}{\omega_{2n}}\int\limits_{B_{2n}(0,\varepsilon)}\frac{|\nabla P_\varepsilon(\zeta)|}{|z-\zeta|^{2n-1}}\,d^{2n}\zeta.$$

We estimate

$$\begin{aligned}\int\limits_{\varepsilon/3\le|x|\le\varepsilon/2} P_\varepsilon(x)\,d^{2n}x&=-\int\limits_{\varepsilon/3\le|x|\le\varepsilon/2}\left(\frac{1}{(2n-2)\omega_{2n}}\int\limits_{|\zeta|<\varepsilon}\frac{d\mu_V(\zeta)}{|x-\zeta|^{2n-2}}\right)d^{2n}x\\&=-\int\limits_{|\zeta|<\varepsilon}\left(\frac{1}{(2n-2)\omega_{2n}}\int\limits_{\varepsilon/3\le|x|\le\varepsilon/2}\frac{d^{2n}x}{|x-\zeta|^{2n-2}}\right)d\mu_V(\zeta)\\&\ge-\int\limits_{|\zeta|<\varepsilon}\left(\frac{1}{(2n-2)\omega_{2n}}\int\limits_{|x-\zeta|\le3\varepsilon/2}\frac{d^{2n}x}{|x-\zeta|^{2n-2}}\right)d\mu_V(\zeta)\\&\ge-\frac{9}{16}\,\varepsilon^2\mu_V(B(0,\varepsilon))\end{aligned}$$

for $\varepsilon<1/2$. Thus we obtain from (2.5), since $P_\varepsilon\le0$,

$$-\frac{9}{16c_n\varepsilon^{2n-2}}\,\mu_V(B(0,\varepsilon))-\mathscr{K}_\varepsilon(z)\le P_\varepsilon(z)\le\mathscr{K}_\varepsilon(z)$$

and

$$(2.6)\quad \int\limits_{|z|<r} e^{-2mP_\varepsilon(z)}\,d^{2n}z\le\exp\!\left(\frac{9\cdot4m}{16c_n\varepsilon^{2n-2}}\,\mu_V(B(0,\varepsilon))\right)\int\limits_{|z|<r} e^{2m\mathscr{K}_\varepsilon(z)}\,d^{2n}z.$$

We only have to estimate the integral

$$(2.7)\quad \int\limits_{|z|<r} e^{2m\mathscr{K}_\varepsilon(z)}\,d^{2n}z=\frac{1}{2n}\,\omega_{2n}r^{2n}+2m\int\limits_{|z|<r}\mathscr{K}_\varepsilon(z)\,d^{2n}z +\sum_{q=2}^{\infty}\frac{(2m)^q}{q!}\,\|\mathscr{K}_\varepsilon\|^q_{L^q(B(0,r))}.$$

First,

$$\int_{B_{2n}(0,\varepsilon)} |\nabla P_\varepsilon(\zeta)|\, d^{2n}\zeta \le \frac{1}{\omega_{2n}} \int_{B_{2n}(0,\varepsilon)} \Bigg(\int_{B_{2n}(0,\varepsilon)} \frac{d\mu_V(y)}{|\zeta - y|^{2n-1}} \Bigg) d^{2n}\zeta$$
$$= \frac{1}{\omega_{2n}} \int_{B_{2n}(0,\varepsilon)} \Bigg(\int_{B_{2n}(0,\varepsilon)} \frac{d^{2n}\zeta}{|\zeta - y|^{2n-1}} \Bigg) d\mu_V(y) \le (2\varepsilon)\mu_V(B(0,\varepsilon)). \tag{2.8}$$

By Fubini's theorem we get

$$2m \int_{|z|<r} \mathscr{K}_\varepsilon(z)\, d^{2n}z = 2m\, \frac{4^{n+1}}{\omega_{2n}} \int_{|z|<r} \int_{B_{2n}(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|z-\zeta|^{2n-1}}\, d^{2n}\zeta\, d^{2n}z$$
$$= 2m\, \frac{4^{n+1}}{\omega_{2n}} \int_{B_{2n}(0,\varepsilon)} |\nabla P_\varepsilon(\zeta)| \Bigg(\int_{|z|<r} \frac{1}{|z-\zeta|^{2n-1}}\, d^{2n}z \Bigg) d^{2n}\zeta$$
$$\le (2m)(2\varepsilon)4^{n+1} \int_{B_{2n}(0,\varepsilon)} |\nabla P_\varepsilon(\zeta)|\, d^{2n}\zeta \le 4^{n+1}\cdot 2m \cdot (2\varepsilon)^2 \mu_V(B(0,\varepsilon)),$$

using (2.8). For $q \ge 2$ we estimate the norms $\|\mathscr{K}_\varepsilon\|^q_{L^q(B(0,r))}$ by means of Hölder's inequality (see also the proof of Theorem 1 in [22, p. 476]). We use the formula

$$\|\mathscr{K}_\varepsilon\|_{L^q(B(0,r))} = \sup_{F\in L^p(B(0,r)),\, \|F\|_{L^p(B(0,r))}=1} \Bigg| \int_{B(0,r)} F(x)\mathscr{K}_\varepsilon(x)\, d^{2n}x \Bigg|$$

(where $p = q/(q-1)$). Let $F \in L^p(B(0,r))$ be normalized. Then we get

$$\Bigg| \int_{B(0,r)} F(x)\mathscr{K}_\varepsilon(x)\, d^{2n}x \Bigg| \le \int_{B(0,r)\times B(0,\varepsilon)} f(x,\zeta) g(x,\zeta)\, d^{2n}x\, d^{2n}\zeta$$

with

$$f(x,\zeta) = \bigg(\frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1/q}} \bigg)^{1/q}, \qquad g(x,\zeta) = F(x) \bigg(\frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}} \bigg)^{1/p}.$$

The L^q-norm of f is estimated by

$$\|f\|^q_{L^q(B(0,r)\times B(0,\varepsilon))} = \int_{B(0,r)\times B(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1/q}}\, d^{2n}x\, d^{2n}\zeta$$
$$= \int_{B(0,r)} \bigg(\frac{d^{2n}x}{|x-\zeta|^{2n-1/q}} \bigg) \int_{B(0,\varepsilon)} |\nabla P_\varepsilon(\zeta)|\, d^{2n}\zeta$$
$$\le \omega_{2n} q (r+\varepsilon)^{1/q} \int_{B(0,\varepsilon)} |\nabla P_\varepsilon(\zeta)|\, d^{2n}\zeta \le 4\omega_{2n} q \varepsilon^{1+1/q} \mu_V(B(0,\varepsilon)),$$

again by (2.8).

We next consider the L^p-norm of g. Let

$$u(t) := \int_{B(x,t)} |\nabla P_\varepsilon(\zeta)|\, d^{2n}\zeta.$$

First we observe that

$$\begin{aligned}
\int_{B(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}}\, d^{2n}\zeta &\le \int_{B(x,2\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}}\, d^{2n}\zeta \\
&= \int_0^{2\varepsilon} \left(\int_{|\zeta-x|=t} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}}\, dS(\zeta) \right) dt \\
&= \int_0^{2\varepsilon} \frac{1}{t^{2n-1-1/q}}\, u'(t)\, dt \\
&= \frac{u(2\varepsilon)}{(2\varepsilon)^{2n-1-1/q}} + \left(2n-1-\frac{1}{q}\right) \int_0^{2\varepsilon} \frac{u(t)}{t^{2n-1/q}}\, dt.
\end{aligned}$$

Now we note that

$$u(t) \le 2t\mu_V(B(0,t)) \quad \text{and} \quad \frac{\mu_V(B(0,t))}{t^{2n-2}} \le \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}.$$

The second inequality follows from the fact that the function

$$t \mapsto \mu_V(B(0,t))/t^{2n-2}$$

is increasing (see [19, pp. 72–73]). Hence

$$\int_0^{2\varepsilon} \frac{u(t)}{t^{2n-1/q}}\, dt \le 2\, \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}} \int_0^{2\varepsilon} \frac{dt}{t^{1-1/q}} = 2q\varepsilon^{1/q}\, \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}.$$

This implies

$$\int_{B(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}}\, d^{2n}\zeta \le 2(2n+2)q\varepsilon^{1/q}\, \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}.$$

Then

$$\begin{aligned}
\|g\|^p_{L^p(B(0,r)\times B(0,\varepsilon))} &= \int_{B(0,r)} |F(x)|^p \left(\int_{B(0,\varepsilon)} \frac{|\nabla P_\varepsilon(\zeta)|}{|x-\zeta|^{2n-1-1/q}}\, d^{2n}\zeta \right) d^{2n}x \\
&\le 2(2n+2)q\varepsilon^{1/q}\, \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}} \int_{B(0,r)} |F(x)|^p\, d^{2n}x \\
&= 2(2n+2)q\varepsilon^{1/q}\, \frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}.
\end{aligned}$$

By Hölder's inequality we find (using $1/p + 1/q = 1$) that

$$\Big|\int\limits_{B(0,r)} F(x)\mathscr{K}_\varepsilon(x)\,d^{2n}x\Big| \le (4\omega_{2n}q\varepsilon^{1+1/q}\mu_V(B(0,\varepsilon)))^{1/q}$$
$$\times\bigg(2(2n+2)q\varepsilon^{1/q}\,\frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}\bigg)^{1/p}$$
$$\le (2n+2)(2\omega_{2n})^{1/q}\cdot 2q\mu_V(B(0,2\varepsilon))(2\varepsilon)^{2-2n/p}$$
$$= (2n+2)(2\omega_{2n})^{1/q}\cdot 2q\mu_V(B(0,2\varepsilon))(2\varepsilon)^{2-2n+2n/q}.$$

This implies

$$\|\mathscr{K}_\varepsilon\|^q_{L^q(B(0,r))} \le 2\omega_{2n}(2q)^q\bigg((2n+2)\,\frac{\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}\bigg)^q(2\varepsilon)^{2n}.$$

Substituting this into (2.7) we obtain

$$\int\limits_{|z|<r} e^{2m\mathscr{K}_\varepsilon(z)}\,d^{2n}z$$
$$\le \frac{1}{2n}\,\omega_{2n}r^{2n} + 4^{2n+2}\omega_{2n}(2\varepsilon)^{2n}\sum_{q=1}^{\infty}\frac{(2q)^q}{q!}\bigg(\frac{8(n+1)m\mu_V(B(0,2\varepsilon))}{(2\varepsilon)^{2n-2}}\bigg)^q$$
$$\le C_n\omega_{2n}\varepsilon^{2n},$$

if we choose

$$m \le \frac{(2\varepsilon)^{2n-2}}{16(n+1)e\mu_V(B(0,2\varepsilon))}$$

in order to make the above series converge.

In conjunction with (2.3) we get

$$\mathscr{J}_m(r) \le e^{-4^n\cdot 2mM[V,0,\varepsilon]}\int\limits_{|z|<r} e^{-2mP_\varepsilon(z)}\,d^{2n}z$$
$$\le \exp\bigg(\frac{9m}{8c_n}\,\frac{\mu_V(B(0,\varepsilon))}{\varepsilon^{2n-2}} - 4^n\cdot 2mM[V,0,\varepsilon]\bigg)\int\limits_{|z|<r} e^{2m\mathscr{K}_\varepsilon(z)}\,d^{2n}z$$
$$\le C_n\omega_{2n}\varepsilon^{2n}\exp\bigg(\frac{9m}{8c_n}\,\frac{\mu_V(B(0,\varepsilon))}{\varepsilon^{2n-2}} - 4^n\cdot 2mM[V,0,\varepsilon]\bigg). \;\blacksquare$$

This will give a lower bound for the regularization V_m:

LEMMA 2.4. *Let D be as above and $z^0 \in D$. Let $\varepsilon < \delta_D(z^0)/4$. If*

$$m \le \frac{(2\varepsilon)^{2n-2}}{16(n+1)e\mu_V(B(z^0,2\varepsilon))},$$

then

$$V_m(z^0) \ge -\frac{\log(C_D/C_n)}{2m} - \frac{9}{16c_n}\,\frac{\mu_V(B(z^0,\varepsilon))}{\varepsilon^{2n-2}} + 4^nA[V,z^0,\varepsilon].$$

Proof. We combine Lemmas 2.1 and 2.3 with $r = \varepsilon/8$:

$$\begin{aligned} V_m(z^0) &\geq -\frac{\log C_D}{2m} - \frac{1}{2m}\log(A[e^{-2mV}, z^0, \varepsilon/8]) \\ &= -\frac{\log C_D}{2m} - \frac{1}{2m}\log\left(\frac{2n8^{2n}}{\omega_{2n}\varepsilon^{2n}}\,\mathscr{I}_m(\varepsilon/8)\right) \\ &\geq -\frac{\log(2nC_D/C_n)}{2m} - \frac{9}{16c_n}\,\frac{\mu_V(B(z^0,\varepsilon))}{\varepsilon^{2n-2}} + 4^n M[V, z^0, \varepsilon]. \end{aligned}$$

This gives the desired estimate, since $M[V, z^0, \varepsilon] \geq A[V, z^0, \varepsilon]$. ∎

3. Application to the pluricomplex Green function. We start with a technical lemma:

LEMMA 3.1. *Assume that D is bounded and hyperconvex and $\psi \in \mathrm{PSH}(D)$ is continuous, negative and satisfies $\|\psi\|_\infty = 1$ and condition* (1) *of Main Theorem* 1.1. *Let $w, z^0 \in D$. Then, for any $0 < r < \frac{4}{5}\delta_D(z^0)$,*

$$\int\limits_{B(z^0,6r/5)} |\mathscr{G}_D(x,w)|\,d^{2n}x \leq 2\pi(n!)^{1/n}|\psi(w)|^{1/n} I\left(\frac{6}{5}r, h\right), \tag{3.1}$$

and

$$A[|\mathscr{G}_D(\cdot,w)|, z^0, r] + \frac{\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0,r))}{r^{2n-2}} \leq \frac{C_n^*}{r^{2n}}\,|\psi(w)|^{1/n} I\left(\frac{6}{5}\,r, h\right) \tag{3.2}$$

with some constant C_n^* that depends only on n. Here we define, for $0 < \varrho < \delta_D(z^0)$,

$$I(\varrho, h) := \Big(\int\limits_{B(z^0,\varrho)} h^{-1/(n-1)}\,d^{2n}x\Big)^{1-1/n}.$$

Proof. We apply an idea from [7]. The key tool is an estimate obtained in [5]: Given an arbitrary bounded domain D' and negative locally bounded plurisubharmonic functions $u, v_1, \ldots, v_n$ on D' such that $\lim_{z\to q} u(z) = 0$ for any $q \in \partial D'$, we have

$$\int\limits_{D'} |u|^n dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq n!\|v_1\|_\infty \cdot \ldots \cdot \|v_{n-1}\|_\infty \int\limits_{D'} |v_n|(dd^c u)^n. \tag{3.3}$$

For an arbitrary number $L > 1$ we put

$$u_L := \max\{\mathscr{G}_D(\cdot,w), -L\}.$$

By the Hölder inequality we have

$$(3.4)\qquad \int\limits_{B(z^0,6r/5)} |u_L|\,d^{2n}x = \int\limits_{B(z^0,6r/5)} |u_L|\,h^{1/n}\,\frac{1}{h^{1/n}}\,d^{2n}x$$
$$\le \Big(\int\limits_{B(z^0,6r/5)} |u_L|^n h\,d^{2n}x\Big)^{1/n}\Big(\int\limits_{B(z^0,6r/5)} \frac{1}{h^{1/(n-1)}}\,d^{2n}x\Big)^{1-1/n}.$$

We see that

$$\int\limits_{B(z^0,6r/5)} |u_L|^n h\,d^{2n}x \le \int\limits_{B(z^0,6r/5)} |u_L|^n\,(dd^c\psi)^n.$$

Now we can apply (3.3) for $D' := D$ and $v_1 = \cdots = v_n = \psi$ to obtain (because of $\|\psi\|_\infty = 1$)

$$(3.5)\qquad \int\limits_{B(z^0,6r/5)} |u_L|^n\,h\,d^{2n}x \le \int\limits_{D} |u_L|^n (dd^c\psi)^n$$
$$\le n!\int\limits_{D} |\psi|(dd^c \max\{\mathscr{G}_D(\cdot,w),-L\})^n.$$

We want to let L tend to infinity. This is allowed, since the well-known convergence theorem of Bedford–Taylor gives that the currents

$$T_L := (dd^c \max\{\mathscr{G}_D(\cdot,w),-L\})^n$$

tend weakly to $(2\pi)^n$ times the Dirac measure with center w as $L\to\infty$. But, since all of them have the same total mass (namely $(2\pi)^n$), we may apply Satz 45.7 of [1]. This gives us, in conjunction with the Beppo-Levi theorem, on letting $L\to\infty$ in (3.5),

$$\int\limits_{B(z^0,6r/5)} |\mathscr{G}_D(x,w)|^n h\,d^{2n}x \le n!\int\limits_{D} |\psi|(dd^c\mathscr{G}_D(\cdot,w))^n = (2\pi)^n n!|\psi(w)|,$$

and because of (3.4),

$$\int\limits_{B(z^0,6r/5)} |\mathscr{G}_D(x,w)|\,d^{2n}x \le ((2\pi)^n n!|\psi(w)|)^{1/n}\Big(\int\limits_{B(z^0,6r/5)} h^{-1/(n-1)}\,d^{2n}x\Big)^{1-1/n}.$$

From this we get (3.1).

Next we prove (3.2). This time we use (3.3) with $D' := B(z^0,6r/5)$ and

$$u(z) = \log\frac{1+|z-z^0|^2/4r^2}{2},$$

and $v_1(z) = \cdots = v_{n-1}(z) = |z-z^0|^2 - 4r^2$, $v_{n,L}(z) = \max\{\mathscr{G}_D(z,w),-L\}$ for a number $L>0$. Now (3.3) applies and gives, because of $|u| \ge \log(8/5)$

on $B(z^0, r)$,

$$\left(\log \frac{8}{5}\right)^n \mu_{v_{n,L}}(B(z^0, r))$$
$$\le \int_{B(z^0,r)} |u(z)|^n \, d\mu_{v_{n,L}}(z) = \int_{B(z^0,r)} |u(z)|^n \, dd^c v_1 \wedge \cdots \wedge dd^c v_n$$
$$\le \int_{B(z^0,6r/5)} |u(z)|^n \, dd^c v_1 \wedge \cdots \wedge dd^c v_n \le n!(4r^2)^{n-1} \int_{B(z^0,6r/5)} |v_{n,L}| \, (dd^c u)^n$$
$$\le c_n \frac{1}{r^2} \int_{B(z^0,6r/5)} \min\{|\mathscr{G}_D(x, w)|, L\} \, d^{2n}x$$

with some constant c_n. We let L tend to infinity. By Beppo-Levi's theorem we obtain

$$\frac{\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0, r))}{r^{2n-2}} \le \frac{c_n}{r^{2n}(\log(8/5))^n} \int_{B(z^0,6r/5)} |\mathscr{G}_D(x, w)| \, d^{2n}x.$$

From this and (3.1) the lemma follows. ■

We apply the above lemmas 2.4 and 3.1 to the case $V = \mathscr{G}_D(\cdot, w)$ for $w \in D$ and find

LEMMA 3.2. *Assume that D is hyperconvex and $\psi \in \mathrm{PSH}(D)$ is continuous and satisfies $\|\psi\|_\infty = 1$ and condition* (1) *from Main Theorem* 1.1. *Let C_n^* be the constant from Lemma* 3.1. *Then, for any point $z^0 \in D$ and any $0 < \varepsilon < \delta_D(z_0)/4$,*

$$(\mathscr{G}_D(\cdot, w))_m(z_0) \ge -\frac{1}{2m} \log \frac{2nC_D}{C_n} - \widetilde{C}_n C_n^* I_0 \frac{|\psi(w)|^{1/n}}{\varepsilon^{2n}},$$

provided that

$$m \le C_n^{**} \frac{\varepsilon^{2n}}{|\psi(w)|^{1/n}}$$

with the constants $C_n^{**} := 1/16e(n+1)C_n^* I_0$, $\widetilde{C}_n := (9/16c_n + 4^n)$, and $I_0 := \int_D h^{-1/(n-1)}(x) \, d^{2n}x$. The constant c_n was defined after formula (2.5).

Proof. By Lemma 2.4 we have

$$(\mathscr{G}_D(\cdot, w))_m(z^0) \ge -\frac{1}{2m} \log \frac{2nC_D}{C_n} - \left(4^n |A[\mathscr{G}_D(\cdot, w), z^0, \varepsilon]| + \frac{9}{16c_n} \frac{\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0, \varepsilon))}{\varepsilon^{2n-2}}\right),$$

provided that

$$m \le \frac{(2\varepsilon)^{2n-2}}{16e(n+1)\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0, 2\varepsilon))}.$$

Now, estimate (3.2) of Lemma 3.1 with $r=\varepsilon$ gives us

$$4^n|A[\mathscr{G}_D(\cdot,w),z^0,\varepsilon]|+\frac{9}{16c_n}\,\frac{\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0,\varepsilon))}{\varepsilon^{2n-2}}\le \widetilde{C}_nC_n^*\,\frac{|\psi(w)|^{1/n}}{(2\varepsilon)^{2n}}\,I_0.$$

In particular,

$$\frac{(2\varepsilon)^{2n-2}}{16e(n+1)\mu_{\mathscr{G}_D(\cdot,w)}(B(z^0,2\varepsilon))}\ge\frac{1}{16e(n+1)C_n^*I_0}\,\frac{\varepsilon^{2n}}{|\psi(w)|^{1/n}}.$$

This proves the lemma. ∎

We want to apply the above results to the localization of the sublevel sets of the regularizations of $\mathscr{G}_D(\cdot,w)$.

A first step in this direction is

Lemma 3.3. *Let $D\subset\subset\mathbb{C}^n$ be hyperconvex and ψ as in Lemma 3.2. There exists a constant $r_*>0$, depending only on n, such that for $z^0\in D$ and $0<r<\min\{r_*,\delta_D(z^0)\}$,*

$$\sup_{x\in B(z^0,r)}\mathscr{G}_D(x,w)\ge-\frac{4^{2n}}{C_n^{**}\delta_D(z^0)^{2n}}\,|\psi(w)|^{1/n}\log\frac{C_2}{r^n}$$

with the constants C_2 from (2.1) *and C_n^{**} from Lemma* 3.2.

Proof. As in (2.1), for any $l>0$ we have

$$(\mathscr{G}_D(\cdot,w))_l(z^0)\le -M(r)+\frac{1}{l}\log\frac{C_2}{r^n},$$

where $M(r):=|\sup_{x\in B(z^0,r)}\mathscr{G}_D(x,w)|$. The number C_2 in (2.1) depends only on the dimension n. We assume $r<\sqrt[n]{C_2}$. Then the number

$$l:=\frac{2\log(C_2/r^n)}{M(r)}.$$

is positive. This gives

$$(\mathscr{G}_D(\cdot,w))_l(z^0)\le-\frac{1}{2}\,M(r). \tag{3.6}$$

Further we put

$$\varepsilon:=\left(\frac{l}{C_n^{**}}\,|\psi(w)|^{1/n}\right)^{1/2n},$$

where C_n^{**} is as in Lemma 3.2. Suppose that $\varepsilon<\delta_D(z^0)/4$. Then Lemma 3.2 applies with $m=l$ and we obtain

$$(\mathscr{G}_D(\cdot,w))_l(z^0)\ge-\frac{1}{2l}\log\frac{2nC_D}{C_n}-\widetilde{C}_nC_n^*I_0\,\frac{|\psi(w)|^{1/n}}{\varepsilon^{2n}}=-\frac{C_n'}{l} \tag{3.7}$$

with

$$C_n':=\log\frac{2nC_D}{C_n}+C_n^{**}\widetilde{C}_nC_n^*I_0.$$

Combining this with (3.6) we find

$$-\frac{1}{2}M(r) \ge -\frac{C_n'}{l} = -C_n'\,\frac{M(r)}{2\log(C_2/r^n)},$$

Hence we would have, for $0 < r < r_*$,

$$1 \le \frac{C_n'}{\log(C_2/r^n)} \le \frac{C_n'}{\log(C_2/r_*^n)} < 1,$$

provided we choose r_* suitably; the latter is possible uniformly in z^0, w. This contradiction implies

$$\frac{\log(C_2/r^n)}{C_n^{**}M(r)}\,|\psi(w)|^{1/n} = \frac{l}{2C_n^{**}}\,|\psi(w)|^{1/n} = 2\varepsilon^{2n} \ge (\delta_D(z^0)/4)^{2n}.$$

This means that

$$M(r) \le \left(\frac{4}{\delta_D(z^0)}\right)^{2n} \frac{\log(C_2/r^n)}{C_n^{**}}\,|\psi(w)|^{1/n},$$

which proves the lemma. ■

The same method allows a growth estimate in the spirit of the Hopf lemma:

LEMMA 3.4. *Suppose that $\psi : D \to [-1, 0)$ is a smooth plurisubharmonic exhaustion function for D with property* (1) *of Main Theorem* 1.1. *Then there exists a constant $\gamma_1 > 0$ such that*

$$\psi \le -\gamma_1 \delta_D^{2n^2}.$$

Proof. Let $w \in D$. We choose

$$\varepsilon := \left(\frac{3n}{C_n^{**}}\right)^{1/2n} |\psi(w)|^{1/2n^2},$$

where C_n^{**} is the constant from Lemma 3.2. For any point $z \in D$ such that $\delta_D(z) > 4\varepsilon$, Lemma 3.2 applies with $m = 3n$. Hence

$$\begin{aligned}(\mathscr{G}_D(\cdot, w))_m(z) &\ge -\frac{1}{6n}\log\frac{2nC_D}{C_n} - C_n'\,\frac{|\psi(w)|^{1/n}}{\varepsilon^{2n}}\\ &= -\frac{1}{6n}\log\frac{2nC_D}{C_n} - \frac{C_n'C_n^{**}}{3n}.\end{aligned}$$

Since $m > 2n$, the function $(\mathscr{G}_D(\cdot, w))_m$ has a pole at w, which implies

$$\delta_D(w) \le 4\varepsilon = 4\left(\frac{3nC_n'}{C_n^{**}}\right)^{1/2n} |\psi(w)|^{1/2n^2}.$$

From this the claim follows with $\gamma_1 := 4^{-2n^2}(C_n^{**}/3nC_n')^n$. ■

LEMMA 3.5. *If D and ψ are as in the preceding lemma, then*

$$\mathscr{G}_D(z,w) \geq \frac{\log(2R_D/\delta_D(w))}{\inf_{B(w,\delta_D(w)/2)}|\psi|}\cdot\psi(z) \tag{a}$$

and in particular

$$\mathscr{G}_D(z,w) \geq \gamma_2\,\frac{\psi(z)}{\delta_D(w)^{2n^2+1}} \tag{b}$$

for any $z\in D$ such that $|z-w|\geq\delta_D(w)/2$, where $\gamma_2 := 2^{2n^2+1}R_D/\gamma_1$.

Proof. We proceed as in the proof of Lemma 2.4 of [12]. Let v be the following function:

$$v(x) = \begin{cases} \max\left\{C\psi(x), \log\dfrac{|x-w|}{R_D}\right\} & \text{if } |x-w|\geq\delta_D(w)/2,\\ \log\dfrac{|x-w|}{R_D} & \text{if } |x-w|\leq\delta_D(w)/2. \end{cases}$$

The lemma will be proved if v is well-defined. For this the constant $C>0$ must be chosen in such a way that

$$C\psi(x)\leq\log\frac{\delta_D(w)}{2R_D}\quad\text{on } \partial B(w,\delta_D(w)/2). \tag{$*$}$$

This estimate is satisfied, if we choose

$$C\geq\frac{\log(2R_D/\delta_D(w))}{\inf_{B(w,\delta_D(w)/2)}|\psi|}.$$

This proves (a).

For the proof of (b) we observe that from Lemma 3.4 we know $\psi(x)\leq -\gamma_1\delta_D(x)^{2n^2}$ for all $x\in D$. Since $\delta_D(x)\geq\delta_D(w)/2$ on $\partial B(w,\delta_D(w)/2)$, this gives

$$|\psi(x)|\geq 2^{-2n^2}\gamma_1\delta_D(w)^{2n^2}\quad\text{on } \partial B(w,\delta_D(w)/2).$$

Hence we choose

$$C := \frac{2^{2n^2+1}R_D}{\gamma_1\delta_D(w)^{2n^2+1}},\qquad \gamma_2 := \frac{2^{2n^2+1}R_D}{\gamma_1}.$$

In both cases (a) and (b), on $D\setminus B(w,\delta_D(w)/2)$ we get

$$\mathscr{G}_D(z,w)\geq v(z)\geq C\psi(z)=\gamma_2\,\frac{\psi(z)}{\delta_D(w)^{2n^2+1}}.$$

4. On the boundary behavior of the pluricomplex Green function. First we make two general observations.

LEMMA 4.1. *Let D be a bounded domain in $\mathbb{C}^n$. For any $m > 1$ and $w, z \in D$,*

$$K_{2m\mathscr{G}_D(\cdot,w)}(z,z) \geq \left(\frac{|z-w|}{n\,R_D}\right)^{2m+2} K_D(z,z)$$

and hence

$$(\mathscr{G}_D(\cdot,w))_m(z) \geq \left(1+\frac{1}{m}\right)\log\frac{|z-w|}{n\,R_D} + \frac{1}{2m}\log K_D(z,z).$$

Proof. Given $z \in D$ we choose j such that $|z_j - w_j| \geq |z-w|/n$. For $m > 1$ let $[m]$ denote the integer part of m. If now $f \in H^2(D)$ is arbitrary with $\|f\| = 1$, then the function

$$f_j(x) := \left(\frac{x_j - w_j}{R_D}\right)^{[m]+1} f(x)$$

belongs to $\mathcal{H}_{2m\mathscr{G}_D(\cdot,w)}(D)$, since (note that $\mathscr{G}_D(x,w) \geq \log\frac{|x-w|}{R_D}$)

$$\begin{aligned}|f_j(x)|^2 e^{-2m\mathscr{G}_D(x,w)} &\leq |f_j(x)|^2\left(\frac{R_D}{|x-w|}\right)^{2m}\\ &\leq \left(\frac{|x-w|}{R_D}\right)^{2([m]+1)-2m} |f(x)|^2 \leq |f(x)|^2.\end{aligned}$$

This implies (if we choose $f = K_D(\cdot,z)/\sqrt{K_D(z,z)}$)

$$\begin{aligned}K_{2m\mathscr{G}_D(\cdot,w)}(z,z) &\geq \frac{|f_j(z)|^2}{\|f_j\|^2_{2m\mathscr{G}_D(\cdot,w)}} \geq |f_j(z)|^2\\ &\geq \left(\frac{|z-w|}{nR_D}\right)^{2m+2} K_D(z,z).\end{aligned}$$

Taking log on both sides and dividing by $2m$ we obtain the lemma. ∎

Next we estimate the modulus of continuity of the Demailly regularization $(\mathscr{G}_D(\cdot,w))_m$ as follows:

LEMMA 4.2. *Suppose that $D_1 \subset\subset \mathbb{C}^n$ is any bounded domain. Then there is a constant $C_0' > 0$ such that, for any $m > 0$ and $z^0, w \in D_1$ with $\delta_{D_1}(z^0) \geq 4\delta_{D_1}(w)$,*

$$|(\mathscr{G}_{D_1}(\cdot,w))_m(z_*) - (\mathscr{G}_{D_1}(\cdot,w))_m(z^0)| \leq C_0'\,\frac{1}{m}\left(\frac{2nR_{D_1}}{\delta_D(z^0)}\right)^{m+1}\frac{|z_* - z^0|}{\delta_{D_1}(z^0)^{n+1}}$$

whenever $z_* \in B(z^0, \delta_{D_1}(z^0)/8)$.

Proof. Fix $l \in \{1,\dots,n\}$. Then we have

$$\left|\frac{\partial}{\partial z_l}(\mathscr{G}_{D_1}(\cdot,w))_m(z)\right| = \frac{1}{2m}\,\frac{\left|\frac{\partial}{\partial z_l}K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)\right|}{K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)}$$
$$\le \frac{1}{2m}\,\frac{\left|\frac{\partial^2}{\partial z_l\partial \overline{z}_l}K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)\right|^{1/2}}{\sqrt{K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)}}.$$

The second inequality is due to the logarithmic plurisubharmonicity of $K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)$. From Bergman theory we know that

$$\frac{\partial^2}{\partial z_l\partial \overline{z}_l}K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z) = \sup_{f\in\mathcal{H}(2m\mathscr{G}_{D_1}(\cdot,w)),\,\|f\|_{2m\mathscr{G}_{D_1}(\cdot,w)}=1}\left|\frac{\partial f}{\partial z_l}(z)\right|^2,$$

from which it follows that

$$\left|\frac{\partial}{\partial z_l}(\mathscr{G}_{D_1}(\cdot,w))_m(z)\right| \le \frac{1}{2m}\,\frac{\sup_{f\in\mathcal{H}(2m\mathscr{G}_{D_1}(\cdot,w)),\,\|f\|_{2m\mathscr{G}_{D_1}(\cdot,w)}=1}\left|\frac{\partial f}{\partial z_l}(z)\right|}{\sqrt{K_{2m\mathscr{G}_{D_1}}(z,w)}}.$$

Now, any function $f \in \mathcal{H}(2m\mathscr{G}_{D_1}(\cdot,w))$ belongs to $H^2(D_1)$. Hence

$$\left|\frac{\partial f}{\partial z_l}(z)\right| \le c_n\,\frac{\|f\|_{L^2(D_1)}}{\delta_{D_1}(z)^{n+1}} \le c_n\,\frac{\|f\|_{2m\mathscr{G}_{D_1}(\cdot,w)}}{\delta_{D_1}(z)^{n+1}}$$

for any $z \in D_1$. Furthermore,

$$K_{2m\mathscr{G}_{D_1}(z,w)}(z,z) \ge \left(\frac{|z-w|}{n\,R_{D_1}}\right)^{2m+2} K_{D_1}(z,z) \ge \frac{1}{\operatorname{vol}(D_1)}\left(\frac{|z-w|}{n\,R_{D_1}}\right)^{2m+2}.$$

This will give

$$\left|\frac{\partial}{\partial z_l}(\mathscr{G}_{D_1}(\cdot,w))_m(z)\right| \le \frac{1}{2m}\,\frac{\sup_{f\in\mathcal{H}(2m\mathscr{G}_{D_1}(\cdot,w)),\,\|f\|_{2m\mathscr{G}_{D_1}(\cdot,w)}=1}\left|\frac{\partial f}{\partial z_l}(z)\right|}{\sqrt{K_{2m\mathscr{G}_{D_1}(\cdot,w)}(z,z)}}$$
$$\le \frac{\sqrt{\operatorname{vol}(D_1)}}{2m}\left(\frac{nR_{D_1}}{|z-w|}\right)^{m+1}\delta_{D_1}(z)^{-(n+1)}$$
$$\le \frac{\sqrt{\operatorname{vol}(D_1)}}{2m}\left(\frac{2nR_{D_1}}{\delta_{D_1}(z^0)}\right)^{m+1}\delta_{D_1}(z)^{-(n+1)}$$

on $B(z^0,\delta_{D_1}(z^0)/8)$. From this the assertion follows by the mean value theorem. ∎

In the next step we prove a quantitative result on upper semicontinuity of the Green function.

LEMMA 4.3. *Assume that D and ψ are as in Theorem 1.1 and γ_2 is the constant from Lemma 3.5. Then there are constants $\delta_0, \widetilde{C}_1, \widetilde{C}_2 > 0$ such* *that, for any* $z^0, w \in D$ *with* $\delta_D(w) < \min\{\delta_0, \delta_D(z)/4\}$ *and* $\mathscr{G}_D(z^0, w) \leq -\gamma_2 \delta_D(w)$, *we have*

$$\mathscr{G}_D(z_*, w) \leq \mathscr{G}_D(z^0, w) + \widetilde{C}_1 \delta_D(w) + \widetilde{C}_2 \frac{|\psi(w)|^{1/3n}}{\delta_D(z^0)^{n+3}}$$

whenever $|z_* - z^0| \leq \exp(-1/|\psi(w)|^{1/2n})$.

Proof. Let $m > n$. We define the function

$$v(x) := \begin{cases} \mathscr{G}_D(x, w) & \text{if } \mathscr{G}_D(x, w) \geq -\gamma_2 \delta_D(w), \\ \max\{\mathscr{G}_D(x, w), \widetilde{v}_w(x)\} & \text{if } \mathscr{G}_D(x, w) \leq -\gamma_2 \delta_D(w). \end{cases}$$

Let C_2 be the constant from the proof of Lemma 3.3. We define the function $\widetilde{v}_w$ by

$$\widetilde{v}_w(x) := (\mathscr{G}_D(\cdot, w))_m(x) - \frac{1}{m} \log \frac{C_2}{r_w^n} - \gamma_2 \delta_D(w).$$

If the radius r_w is less than or equal to the boundary distance of the set $S := \{x \in d \mid \mathscr{G}_D(x, w) = -\gamma_2 \delta_D(w)\}$, the function v is well-defined and negative. We have

$$v \leq (1 - n/m) \mathscr{G}_D(\cdot, w).$$

Next we want to estimate the boundary distance of S from below, using the growth condition on ψ.

Let $x \in S$ with $\delta_D(x) < 1$. If $|x - w| \leq \delta_D(w)/2$, we get $\delta_D(x) \geq \delta_D(w)/2$, hence

$$\log \frac{1}{\delta_D(x)} \leq \log \frac{2}{\delta_D(w)}.$$

If $|x - w| \geq \delta_D(w)/2$, we find, because of Lemma 3.5,

$$-\gamma_2 \delta_D(w) = \mathscr{G}_D(x, w) \geq \gamma_2 \frac{\psi(x)}{\delta_D(w)^{2n^2+1}},$$

hence

$$\delta_D(w)^{2n^2+2} \leq |\psi(x)| \leq \widehat{C}_1 \exp\left(-\widehat{C}_2 \left(\log \frac{1}{\delta_D(x)}\right)^{\alpha}\right).$$

For $w \in D$ with $\delta_D(w) < \delta_0 := \min\{1/2, \widehat{C}_1^{-1/(2n^2+2)}, \widehat{C}_1^{1/(2n^2+2)}\}$ this leads to

$$\log \frac{1}{\delta_D(x)} \leq \left(\frac{1}{\widehat{C}_2} \log \frac{\widehat{C}_1}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha},$$

and hence

$$\delta_D(x) \geq \exp\left(-\left(\frac{1}{\widehat{C}_2} \log \frac{\widehat{C}_1}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha}\right).$$

This estimate holds trivially for $x \in S$ with $\delta_D(x) \geq 1$.

Hence we may choose

$$r_w = \min\left\{\delta_D(w)/2, \exp\left(-\left(\frac{1}{\widehat{C}_2}\log\frac{\widehat{C}_1}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha}\right)\right\}.$$

We want to estimate the term $\log(C_2/r_w^n)$, which appeared in the definition of $\widetilde{v}_w$, when $\delta_D(w) < \delta_0$.

From $|\psi(w)| \le \widehat{C}_1 \exp(-\widehat{C}_2 \log(1/\delta_D(w))^\alpha)$ we obtain

$$\log\frac{1}{\delta_D(w)} \le \left(\frac{1}{\widehat{C}_2}\log\frac{\widehat{C}_1}{|\psi(w)|}\right)^{1/\alpha}.$$

Assume now that $r_w = \delta_D(w)/2$. Then

$$\begin{aligned}\log\frac{C_2}{r_w^n} &= \log C_2 + n\log\frac{2}{\delta_D(w)}\\ &\le \log C_2 + 2n\log\frac{1}{\delta_D(w)}\\ &\le \log C_2 + 2n\left(\frac{1}{\widehat{C}_2}\log\frac{\widehat{C}_2}{|\psi(w)|}\right)^{1/\alpha}.\end{aligned}$$

If

$$r_w = \exp\left(-\left(\frac{1}{\widehat{C}_2}\log\frac{\widehat{C}_1}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha}\right)$$

we get

$$\begin{aligned}\log\frac{C_2}{r_w^n} &= \log C_2 + n\log\frac{1}{r_w}\\ &= \log C_2 + n\left(\frac{1}{\widehat{C}_2}\log\frac{\widehat{C}_2}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha}\\ &\le \log C_2 + n\left(\frac{2}{\widehat{C}_2}\right)^{1/\alpha}\left(\log\frac{1}{\delta_D(w)^{2n^2+2}}\right)^{1/\alpha}\\ &\le \log C_2 + n\left(\frac{2n^2+2}{\widehat{C}_2}\right)^{2/\alpha^2}\left(\log\frac{\widehat{C}_2}{|\psi(w)|}\right)^{1/\alpha^2}.\end{aligned}$$

Let $z^0, w \in D$ be as in the hypothesis of the lemma. If $z_* \in D$ is close enough to z^0, we can estimate, for any $m > 0$,

$$\begin{aligned}(4.1)\qquad \mathscr{G}_D(z_*, w) &\le (\mathscr{G}_D(\cdot, w))_m(z_*) + \frac{1}{C_1\, m} \quad \text{(by (2.1))}\\ &= (\mathscr{G}_D(\cdot, w))_m(z^0)\\ &\quad + (\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0) + \frac{1}{C_1\, m}.\end{aligned}$$

But $\mathscr{G}_D(z^0,w) \le -\gamma_2\delta_D(w)$, hence

$$(4.2)\qquad (\mathscr{G}_D(\cdot,w))_m(z^0) \le \widetilde{v}_w(z^0) + \frac{1}{m}\log\frac{C_2}{r_w^n} + \gamma_2\delta_D(w)$$
$$\le v(z^0) + \frac{1}{m}\log\frac{C_2}{r_w^n} + \gamma_2\delta_D(w)$$
$$\le \left(1-\frac{n}{m}\right)\mathscr{G}_D(z^0,w) + \frac{1}{m}\log\frac{C_2}{r_w^n} + \gamma_2\delta_D(w)$$
$$\le \left(1-\frac{n}{m}\right)\mathscr{G}_D(z^0,w) + \frac{\log C_2}{m} + \gamma_2\delta_D(w)$$
$$+\frac{n}{m}\left(\frac{2n^2+2}{\widehat{C}_2}\right)^{2/\alpha^2}\left(\log\frac{\widehat{C}_1}{|\psi(w)|}\right)^{1/\alpha^2}$$

and

$$|(\mathscr{G}_D(\cdot,w))_m(z_*) - (\mathscr{G}_D(\cdot,w))_m(z^0)| \le C_0'\,\frac{1}{m}\left(\frac{2nR_D}{\delta_D(z^0)}\right)^{m+1}\frac{|z_*-z^0|}{\delta_D(z^0)^{n+1}}$$

if $|z_*-z^0| < \delta_D(z^0)/8$. We choose

$$m := \frac{1}{\log(2nR_D/\delta_D(z^0))}\,\frac{1}{|\psi(w)|^{1/2n}}.$$

Then for $z_*\in D$ such that $|z_*-z^0| \le \exp(-1/|\psi(w)|^{1/2n})$ we estimate

$$C_0'\,\frac{1}{m}\left(\frac{2nR_D}{\delta_D(z^0)}\right)^{m+1}\frac{|z_*-z^0|}{\delta_D(z^0)^{n+1}} \le C_0'\,\frac{1}{m}\exp\left(\frac{1}{|\psi(w)|^{1/2n}}\right)\frac{|z_*-z^0|}{\delta_D(z^0)^{n+1}}$$
$$\le C_0'\,\frac{1}{m}\,\frac{1}{\delta_D(z^0)^{n+1}}.$$

For this we note that, after shrinking δ_0 if necessary, for $\delta_D(w)\le\delta_D(z^0)/4$ we have

$$\exp\left(-\frac{1}{|\psi(w)|^{1/2n}}\right) \le \delta_D(w)/2 \le \delta_D(z^0)/8.$$

We combine (4.1) and (4.2) and the estimate

$$-\frac{n}{m}\,\mathscr{G}_D(z^0,w) \le \frac{n}{m}\log\frac{R_D}{|z^0-w|} \le \frac{n}{m}\log\frac{2R_D}{\delta_D(z^0)}.$$

This in conjunction with our choice of m gives, for $z^0, w\in D$ as in the hypothesis of the lemma and $\delta_D(w)\le\delta_0$,

$$\mathscr{G}_D(z_*,w) \le \mathscr{G}_D(z^0,w) + C_3\,\frac{\log(2nR_D/\delta_D(z^0))}{\delta_D(z^0)^{n+2}}\,|\psi(w)|^{1/2n}(\log(1/|\psi(w)|))^{1/\alpha^2}$$
$$+\gamma_2\delta_D(w).$$

From this we obtain the lemma. ∎

Proof of Theorem 1.1. Let $\delta_0 > 0$ be as in the preceding lemma. Let $K \subset D$ be compact; without loss of generality, let $\delta_D(K) < 1$. Assume that $w \in D$ is such that $\delta_D(w) \le \min\{\delta_0, \delta_D(K)/4\}$ and $A := \sup_{z\in K} |\mathscr{G}_D(z,w)|$. Then there exists a $z^0 \in K$ such that $\mathscr{G}_D(z^0,w) = -A$. We put $r := \exp(-1/|\psi(w)|^{1/2n})$. Assume that $A \ge \gamma_2\delta_D(w)$. Then, combining Lemmas 3.3 and 4.3, we obtain

$$\begin{aligned}-A = \mathscr{G}_D(z^0,w) &\ge -\frac{\widetilde{C}_n}{\delta_D(z^0)^{2n}}\,|\psi(w)|^{1/n}\log\frac{C_2}{r^n} - \widetilde{C}_1\delta_D(w) - \widetilde{C}_2\,\frac{|\psi(w)|^{1/3n}}{\delta_D(z^0)^{n+3}}\\ &\ge -C^*\,\frac{|\psi(w)|^{1/3n}}{\delta_D(z^0)^{2n+3}} - \widetilde{C}_1\delta_D(w)\\ &\ge -C^*\,\frac{|\psi(w)|^{1/3n}}{\delta_D(K)^{2n+3}} - \widetilde{C}_1\delta_D(w)\end{aligned}$$

with some universal constant C^*. The case $A \le \gamma_2\delta_D(w)$ is trivial. This implies the theorem. ∎

5. Estimation of the Bergman distance. We are now going to prove Theorem 1.4. The proof is based on the methods developed in [14].

Using Theorem 1.1 we localize the sublevel sets of the Green function.

LEMMA 5.1. *Under the hypotheses of Theorem* 1.4, *there are constants* $0 < C_2' \le 1 \le C_1', C_3'$ *such that for any* $w \in D$ *sufficiently close to* ∂D,

$$\{z \in D \mid \mathscr{G}_D(z,w) \le -1\}$$
$$\subset \left\{z \in D \,\middle|\, C_2'\exp\left(-C_3'\left(\log\frac{1}{|\psi(w)|}\right)^{1/\alpha^2}\right) \le \delta_D(z) \le C_1'|\psi(w)|^{1/3n(2n+1)}\right\}.$$

Proof. Suppose $z \in D$ and $\mathscr{G}_D(z,w) \le -1$.

CASE 1: $\delta_D(w)/4 \le \delta_D(z) \le 4\delta_D(w)$. Then, because

$$C_2'\exp\left(-C_3'\left(\log\frac{1}{|\psi(w)|}\right)^{1/\alpha}\right) \le \delta_D(w)$$

and $\delta_D(w) \le \gamma_1^{-1/2n^2}|\psi(w)|^{1/2n^2} \le \gamma_1^{-1/2n^2}|\psi(w)|^{1/3n(2n+1)}$ (for w close to ∂D), as follows from our hypothesis on the growth of $|\psi|$ and from Lemma 3.4, respectively, we obtain

$$C_2'\exp\left(-C_3'\left(\log\frac{1}{|\psi(w)|}\right)^{1/\alpha}\right) \le \delta_D(z) \le 4\gamma_1^{-1/2n^2}|\psi(w)|^{1/3n(2n+1)}.$$

CASE 2: $\delta_D(z) \ge 4\delta_D(w)$. First, Theorem 1.1 with $K = \{z\}$ gives $\delta_D(z) \le \widetilde{C}^{1/(2n+1)}|\psi(w)|^{1/3n(2n+1)}$. For the lower bound on $\delta_D(z)$ we note that for z with $\delta_D(z) < 1$ Lemma 3.5 implies, in conjunction with condition

(2) of Main Theorem 1.1,

$$1 \leq |\mathscr{G}_D(z,w)| \leq \gamma_2 \frac{|\psi(z)|}{\delta_D(w)^{2n^2+1}} \leq \gamma_2 \widehat{C}_1 \frac{\exp(-\widehat{C}_2 |\log \delta_D(z)|^\alpha)}{\delta_D(w)^{2n^2+1}}.$$

This implies

$$\delta_D(z) \geq \exp\biggl(-\widehat{C}_2^{-1/\alpha}\biggl(\log \frac{\gamma_2}{\widehat{C}_1 \delta_D(w)^{2n^2+1}}\biggr)^{1/\alpha}\biggr).$$

We apply the growth condition on ψ, this time at the point w, to find

$$\log \frac{1}{\delta_D(w)} \leq \biggl(\frac{1}{\widehat{C}_2} \log \frac{\widehat{C}_1}{|\psi(w)|}\biggr)^{1/\alpha}.$$

Combining this with the preceding estimate we obtain

$$\delta_D(z) \geq C_2' \exp\biggl(-C_3'\biggl(\log \frac{1}{|\psi(w)|}\biggr)^{1/\alpha^2}\biggr)$$

with a suitable constant $C_2' \leq 1$. If $\delta_D(z) \geq 1$, this estimate holds trivially.

CASE 3: $\delta_D(z) \leq \delta_D(w)/4$. Again we have $|z-w| \geq \delta_D(w)/2$, hence the lower bound on $\delta_D(z)$ follows as in Case 2. The upper bound on $\delta_D(z)$ is obtained as in Case 1. ∎

The above lemma enables us to estimate the Bergman distance between two points that have different boundary distances.

LEMMA 5.2. *In the situation of Theorem* 1.4 *there exists a constant* $c_0 > 0$ *such that for* $A, B \in D$ *one has* $d_D^{\mathrm{B}}(A,B) \geq c_0$ *provided that*

$$C_2' \exp\biggl(-C_3'\biggl(\log \frac{1}{|\psi(A)|}\biggr)^{1/\alpha^2}\biggr) \geq C_1' |\psi(B)|^{1/3n(2n+1)},$$

where C_1', C_2', C_3' *are as in Lemma* 5.1.

Proof. Under the hypothesis of the lemma, the sublevel sets $\{\mathscr{G}_D(\cdot,A) < -1\}$ and $\{\mathscr{G}_D(\cdot,B) < -1\}$ are disjoint. Hence by Theorem 4.4 of [4],

$$d_D^{\mathrm{B}}(A,B) \geq c_0 := \frac{\pi}{2} - \arctan\biggl(1 + \frac{4e^n}{\eta_n}\biggr),$$

with $\eta_n = \int_n^\infty \frac{dx}{xe^x}$. ∎

Proof of Theorem 1.4. Let $0 < \tau_0 < e^{-e}$ be so small that $\sup |\psi| \geq 2\tau_0$ and

$$(\dagger) \quad C_2' \exp\biggl(-C_3'\biggl(\log \frac{1}{x}\biggr)^{1/\alpha^2}\biggr) \leq \frac{1}{2} C_1' x^{1/3n(2n+1)} \quad \text{for all } 0 < x \leq \tau_0.$$

We first consider a smooth curve $\mathit{\Gamma} : [0, L] \to D$ such that $|\psi(\mathit{\Gamma}(0))| = \tau_0$ and $|\psi(\mathit{\Gamma}(L))| < \tau_0$. Furthermore, we suppose that

$$(\dagger\dagger) \qquad |\psi(\mathit{\Gamma}(L))|^{1/3n(2n+1)} < \min\left\{\tau_0^{1/3n(2n+1)}, \frac{C_2'}{C_1'}\, e^{C_3'|\log \tau_0|^{1/\alpha^2}}\right\}.$$

We put

$$\varphi(t) := C_2' \exp\left(-C_3'\left(\log \frac{1}{|\psi(\mathit{\Gamma}(t))|}\right)^{1/\alpha^2}\right).$$

Then

$$C_1'|\psi(\mathit{\Gamma}(L))|^{1/3n(2n+1)} < \varphi(0) < C_1'|\psi(\mathit{\Gamma}(0))|^{1/3n(2n+1)}.$$

The left inequality comes from (††) and the right from (†). Hence we find $t_1 \in (0, L)$ such that

$$C_1'|\psi(\mathit{\Gamma}(t_1))|^{1/3n(2n+1)} = \varphi(0).$$

We choose a sequence $0 = t_0 < t_1 < \cdots < t_\nu < L$ of maximal length such that

$$(5.1) \qquad \varphi(t_s) = C_1'|\psi(\mathit{\Gamma}(t_{s+1}))|^{1/3n(2n+1)}, \quad 0 \le s < \nu.$$

Then we obtain

$$(5.2) \qquad \text{Bergman length}(\mathit{\Gamma}) \ge \sum_{s=0}^{\nu-1} \text{Bergman length}(\mathit{\Gamma}|[t_s, t_{s+1}]) \ge c_0\nu$$

with c_0 as in Lemma 5.2.

Next we estimate ν from below. By (5.1) we have

$$\log \frac{1}{|\psi(\mathit{\Gamma}(t_{s+1}))|} = 3n(2n+1) \log \frac{C_1'}{\varphi(t_s)}.$$

Also,

$$\log \frac{1}{|\psi(\mathit{\Gamma}(t))|} = \left(\frac{1}{C_3'} \log \frac{C_2'}{\varphi(t)}\right)^{\alpha^2},$$

which gives

$$(5.3) \qquad \left(\frac{1}{C_3'} \log \frac{C_2'}{\varphi(t_{s+1})}\right)^{\alpha^2} = 3n(2n+1) \log \frac{C_1'}{\varphi(t_s)}.$$

We write

$$a_s := \frac{1}{C_3'} \log \frac{C_1'}{\varphi(t_s)}$$

and deduce from (5.3) that

$$(-C_4' + a_{s+1})^{\alpha^2} = 3n(2n+1)C_3' a_s$$

with $C_4' := \frac{1}{C_3'} \log \frac{C_1'}{C_2'}$. Note that $C_4' > 0$. In this way we obtain the recursive formula

$$a_{s+1} = C_5' a_s^{1/\alpha^2} + C_4'$$

with $C_5' := (3n(2n+1)C_3')^{1/\alpha^2}$. Hence the sequence $(a_s)_s$ increases, and

$$a_{s+1} \le C_6' a_s^{1/\alpha^2},$$

where the constant $C_6' := 1 + C_5' + C_4' a_0^{-1/\alpha^2}$ depends only on τ_0.

By induction on l we get

$$a_l \le C_6'^{\,1+\alpha^{-2}+\cdots+\alpha^{-2(l-1)}} a_0^{\alpha^{-2l}}.$$

Note that

$$a_0 = \frac{1}{C_3'} \log \frac{C_1'}{\varphi(0)} \ge \frac{1}{C_3'} \log \frac{C_1'}{C_1' \tau_0^{1/3n(2n+1)}} \ge 2$$

after shrinking τ_0 again. Also, $a_s > 1$ for all $s \ge 0$.

We choose $l = \nu$ and take logarithms on both sides:

$$\begin{aligned}\log a_\nu &\le (1 + \alpha^{-2} + \cdots + \alpha^{-2(\nu-1)}) \log C_6' + \alpha^{-2\nu} \log a_0 \\ &= \frac{\alpha^{-2\nu} - 1}{\alpha^{-2} - 1} \log C_6' + \alpha^{-2\nu} \log a_0 \le C_7' \alpha^{-2\nu}\end{aligned}$$

with $C_7' := \frac{\alpha^2(1-\alpha^{2\nu})}{1-\alpha^2} \log C_6' + \log a_0$. Finally, we get

$$\nu \ge C_8' \log\log a_\nu - \frac{\log C_7'}{\log(1/\alpha)}, \tag{5.4}$$

with $C_8' := \frac{1}{2\log(1/\alpha)}$.

We now estimate a_ν from below as follows: Assume that

$$\varphi(t_\nu) > C_1' |\psi(\Gamma(L))|^{1/3n(2n+1)}.$$

Then we would obtain

$$C_1' |\psi(\Gamma(L))|^{1/3n(2n+1)} < \varphi(t_\nu) < C_1' |\psi(\Gamma(t_\nu))|^{1/3n(2n+1)},$$

the right inequality being implied by (†). In particular, we could choose a number $t_{\nu+1} \in (t_\nu, L)$ such that

$$C_1' |\psi(\Gamma(t_{\nu+1}))|^{1/3n(2n+1)} = \varphi(t_\nu),$$

which is (5.1) for $s = \nu$, a contradiction to the maximality of ν.

Thus we have $\varphi(t_\nu) \le C_1' |\psi(\Gamma(L))|^{1/3n(2n+1)}$ and consequently

$$a_\nu = \frac{1}{C_3'} \log \frac{C_1'}{\varphi(t_\nu)} \ge \frac{1}{3n(2n+1)C_3'} \log \frac{1}{|\psi(\Gamma(L))|}.$$

Combining this with (5.4) and (5.2) we obtain

$$\text{Bergman length}(\Gamma) \ge \log\log\log(1/|\psi(\Gamma(L))|) - C_9.$$

The right-hand side is well-defined, since $|\psi(\Gamma(L))| < \tau_0 < e^{-e}$.

Finally, we fix $P \in D$ such that $|\psi(P)| > \tau_0$, and $Q \in D$ close enough to the boundary of D such that

$$C_1|\psi(Q)|^{1/3n(2n+1)} \leq C_2' \exp\left(-C_3'\left(\log \frac{1}{\tau_0}\right)^{1/\alpha^2}\right).$$

Then there exists a geodesic $\widetilde{\Gamma}$ in the Bergman metric with length $d_D^{\mathrm{B}}(P, Q)$. It contains a piece $\Gamma : [0, L] \to D$ such that $\Gamma(L) = Q$, to which the preceding considerations apply. Its length satisfies

$$d_D^{\mathrm{B}}(P, Q) \geq \log\log\log(1/|\psi(\Gamma(L))|) - C_9 = \log\log\log(1/|\psi(Q)|) - C_9.$$

This gives the desired result. ∎

6. Supplementary remarks and proof of Theorem 1.3. We want to discuss condition (1) of Main Theorem 1.1. A negative plurisubharmonic function ψ with property (2) of that theorem whose reciprocal is integrable induces a function ψ_1 that has both properties (1) and (2). We will prove this as follows.

LEMMA 6.1. *Assume that $\psi : D \to [-1, 0)$ is plurisubharmonic and continuous. Then the following statements are equivalent*:

(a) *There exists an exponent $\eta > 0$ such that $|\psi|^{-\eta}$ is integrable over D.*
(b) *There exists an exponent N and a constant γ_* such that $\psi \leq -\gamma_* \delta_D^N$.*

Proof. The implication (b)⇒(a) is clear.

For the proof of the reverse implication we use Lemma 6.2 below. If $\eta > 0$ is sufficiently small, then for a suitable choice of $L > 0$ we can apply Lemma 3.4 to the function

$$\psi_1(z) := -(-\psi(z)e^{-L|z|^2})^\eta,$$

since

$$(dd^{\mathrm{c}}\psi_1)^n \geq (\eta L/2)^n |\psi_1|^n (dd^{\mathrm{c}}|z|^2)^n.$$

Now ψ_1, and hence also ψ, satisfies an estimate of the form $\psi \leq -\gamma_* \delta_D^N$. ∎

We complete the proof of the above lemma by proving

LEMMA 6.2. *Let ψ be a continuous negative plurisubharmonic function on a domain $D_1 \subset\subset \mathbb{C}^n$. Then, given a number $\eta \in (0, 1)$, one can choose $L > 0$ in such a way that the function $\psi_1(z) := -(-\psi(z)e^{-L|z|^2})^\eta$ is also plurisubharmonic, and satisfies*

$$(dd^{\mathrm{c}}\psi_1)^n \geq (\eta L/2)^n |\psi_1|^n (dd^{\mathrm{c}}|z|^2)^n.$$

Proof. Because the desired estimate is meant in the sense of distributions, it is enough to show it over an arbitrary subdomain $D' \subset\subset D$.

First we assume that ψ is of class C^∞. We repeat the computation from [11] to find

$$\mathscr{L}_{\psi_1}(z;X) = \eta(-\psi(z)e^{-L|z|^2})^{\eta-1}e^{-L|z|^2}\biggl(\mathscr{L}_\psi(z;X) + L|\psi(z)|\,|X|^2 - 2L\eta\,\mathrm{Re}\langle\partial\psi(z),X\rangle\langle z,X\rangle + \frac{1-\eta}{-\psi(z)}\,|\langle\partial\psi(z),X\rangle|^2 - \eta L^2|\langle z,X\rangle|^2\biggr).$$

But

$$-2L\eta\,\mathrm{Re}\langle\partial\psi(z),X\rangle\langle z,X\rangle + \frac{1-\eta}{-\psi(z)}\,|\langle\partial\psi(z),X\rangle|^2 \ge -\frac{\eta^2L^2}{1-\eta}\,|\psi(z)|\,|\langle z,X\rangle|^2.$$

This gives

$$\mathscr{L}_{\psi_1}(z;X) = \eta(-\psi(z)e^{-L|z|^2})^{\eta-1}e^{-L|z|^2} \times \biggl(\mathscr{L}_\psi(z;X) + L|\psi(z)|\biggl(|X|^2 - \frac{\eta L}{1-\eta}\,|\langle z,X\rangle|^2\biggr)\biggr).$$

Given η we let L be so small that $\eta L|z|^2/(1-\eta) < 1/2$ throughout D. Then

$$\mathscr{L}_{\psi_1}(z;X) \ge \eta(-\psi(z)e^{-L|z|^2})^{\eta-1}e^{-L|z|^2}\biggl(\mathscr{L}_\psi(z;X) + \frac{L}{2}\,|\psi(z)|\,|X|^2\biggr) \ge \frac{\eta L}{2}\,|\psi_1(z)|\,|X|^2.$$

Taking the determinants we obtain

$$(dd^{\mathrm{c}}\psi_1)^n \ge (\eta L/2)^n|\psi_1|^n(dd^{\mathrm{c}}|z|^2)^n.$$

If ψ is not necessarily smooth, we approximate ψ from above on D' by a sequence $(v_j)_j$ which decreases to ψ and apply the first part of the proof. The numbers η, L do not depend on the v_j's or on D'. Then we get

$$(dd^{\mathrm{c}} - (-v_j\,e^{-L|z|^2})^\eta)^n \ge (\eta L/2)^n(-v_j\,e^{-L|z|^2})^{n\eta}(dd^{\mathrm{c}}|z|^2)^n.$$

Now Bedford–Taylor's approximation theorem for the Monge–Ampère operator gives the claim. ∎

This enables us to replace (roughly speaking) condition (1) of Main Theorem 1.1 by the condition that $1/|\psi|$ is integrable over D.

Corollary. *Assume that the continuous plurisubharmonic function* ψ *is negative on* D *and* $1/|\psi|$ *is integrable over* D. *Then* $\psi_1 := -(-\psi)^{1-1/n}$ *is also negative and plurisubharmonic*, *and*

$$(dd^{\mathrm{c}}\psi_1)^n \ge \gamma_3|\psi|^{n-1}(dd^{\mathrm{c}}|z|^2)^n.$$

If, *furthermore*, ψ *is smooth and satisfies also condition* (2) *of Main Theorem* 1.1, *then* ψ_1 *satisfies both conditions* (1) *and* (2) *of that theorem.*

Proof of Theorem 1.3. For a suitable choice of the constant $L > 0$ we have seen that

$$\psi_1(z) := -(-\psi(z)e^{-L|z|^2})^{1-1/n}$$

satisfies condition (1) of Theorem 1.1 and also condition (3) of Theorem 1.3, but with M replaced by $M' := (1-1/n)M$ and N by $N' := (1-1/n)N$. Let $z^0 \in K$. From Lemma 3.3 we have, for any $0 < r < \min\{r_*, \delta_D(z^0)\}$,

$$\sup_{x\in B(z^0,r)} \mathscr{G}_D(x,w) \geq -\frac{\widetilde{C}_n}{\delta_D(z^0)^{2n}} |\psi_1(w)|^{1/n} \log \frac{C_2}{r^n}. \tag{6.1}$$

First we use part (a) of Lemma 3.5, applied to ψ_1: By condition (3) we find

$$\frac{\log(2R_D/\delta_D(w))}{\inf_{B(w,\delta_D(w)/2)} |\psi_1|} \leq C_{13}|\psi_1(w)|^{-(M'+1)/N'}$$

and hence

$$\mathscr{G}_D(z,w) \geq C_{13}\,\frac{\psi_1(z)}{|\psi_1(w)|^{(M'+1)/N'}},$$

provided that $|z-w| \geq \delta_D(w)/2$. We want to modify the proof of Lemma 4.3 in order to estimate the left-hand side from above for those $z^0 \in D$ for which $\mathscr{G}_D(z^0,w) \leq -C_{13}|\psi_1(w)|^\beta$.

For this purpose we define $S := \{x \in D \mid \mathscr{G}_D(x,w) < -C_{13}|\psi_1(w)|^\beta\}$ and

$$v(x) := \begin{cases} \mathscr{G}_D(x,w) & \text{for } x \in D\setminus S,\\ \max\{\mathscr{G}_D(x,w), \widetilde{v}_w(x)\} & \text{for } x \in S,\end{cases}$$

where r_w is the distance of ∂S to the boundary of D, and $\widetilde{v}_w$ is defined by

$$\widetilde{v}_w(x) := (\mathscr{G}_D(\cdot,w))_m(x) - \frac{1}{m}\log\frac{C_2}{r_w^n} - C_{13}|\psi_1(w)|^\beta,$$

where C_2 is the constant from (2.1). Then v is plurisubharmonic, and $v \leq (1-n/m)\mathscr{G}_D(\cdot,w)$. We want to estimate $\log(1/r_w)$ from above. For this purpose let $x \in \partial S$ with $\delta_D(x) < 1$. If $|x-w| \leq \delta_D(w)/2$ we have $\delta_D(x) \geq \delta_D(w)/2$, and hence (since $\delta_D(w) < 1/2$)

$$\log\frac{1}{\delta_D(x)} \leq \log\frac{2}{\delta_D(w)} \leq 2\log\frac{1}{\delta_D(w)} \leq 2\left(\frac{C_2}{|\psi(w)|}\right)^{1/N}.$$

Now assume that $|x-w| \geq \delta_D(w)/2$. Then

$$-C_{13}|\psi_1(w)|^\beta = \mathscr{G}_D(x,w) \geq -C_{13}\,\frac{|\psi_1(x)|}{|\psi_1(w)|^{(M'+1)/N'}}.$$

This implies

$$\log\frac{1}{\delta_D(x)} \leq C_{14}|\psi_1(w)|^{-\frac{n}{N(n-1)}(\beta+\frac{M'+1}{N'})}.$$

Trivially, this holds if $x \in S$ and $\delta_D(x) \geq 1$. In particular, if $\delta_D(w) \leq \delta_* \ll 1$, then

$$\log \frac{C_2}{r_w^n} \leq C_{15}|\psi_1(w)|^{-1/n+2\gamma},$$

because

$$-\frac{n}{N(n-1)}\left(\beta + \frac{M'+1}{N'}\right) = -\frac{1}{n} + 2\gamma.$$

Let $K \subset D$ be a compact set and $z^0 \in K$ and $w \in D \setminus K$ with $\delta_D(w) \leq \min\{\delta_*, \delta_D(K)/4\}$. For $z_* \in B(z^0, \delta_D(z^0)/2)$ we can estimate $\mathscr{G}_D(z_*, w)$ in terms of $\mathscr{G}_D(z^0, w)$ and $|\psi(w)|$. Let us assume that $\mathscr{G}_D(z^0, w) \leq -C_{13}|\psi(w)|^\beta$. As in the proof of Lemma 4.3, we have

$$\begin{aligned}
\mathscr{G}_D(z_*, w) &\leq (\mathscr{G}_D(\cdot, w))_m(z_*) + \frac{1}{C_1 m} \\
&= (\mathscr{G}_D(\cdot, w))_m(z^0) + \frac{1}{C_1 m} + (\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0) \\
&\leq v(z^0) + \frac{1}{m}\log\frac{C_2}{r_w^n} + C_{13}|\psi_1(w)|^\beta + \frac{1}{C_1 m} \\
&\quad + (\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0) \\
&\leq \left(1 - \frac{n}{m}\right)\mathscr{G}_D(z^0, w) + \frac{1}{m}\log\frac{C_2}{r_w^n} + C_{13}|\psi_1(w)|^\beta + \frac{1}{C_1 m} \\
&\quad + (\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0) \\
&\leq \mathscr{G}_D(z^0, w) + \frac{n}{m}\log\frac{R_D}{|z_* - w|} + \frac{C_{15}}{m}|\psi_1(w)|^{-1/n+2\gamma} + C_{13}|\psi_1(w)|^\beta \\
&\quad + \frac{1}{C_1 m} + (\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0).
\end{aligned}$$

We note that

$$\log\frac{R_D}{|z_* - w|} \leq \log\frac{R_D}{|z^0 - w| - |z_* - z^0|} \leq \log\frac{8R_D}{5\delta_D(z^0)}.$$

Next choose

$$m := \frac{1}{\log(2nR_D/\delta_D(z^0))}\,|\psi_1(w)|^{-1/n+\gamma}.$$

From Lemma 4.2 we get, as in the proof of Lemma 4.3,

$$\begin{aligned}
&|(\mathscr{G}_D(\cdot, w))_m(z_*) - (\mathscr{G}_D(\cdot, w))_m(z^0)| \\
&\qquad\leq C_0'\frac{1}{m}\left(\frac{2nR_D}{\delta_D(z^0)}\right)^{m+1}\frac{|z_* - z^0|}{\delta_D(z^0)^{n+1}} \\
&\qquad\leq C_{16}|\psi_1(w)|^{1/n-\gamma}\frac{|z_* - z^0|}{\delta_D(z^0)^{n+2}}\exp(|\psi_1(w)|^{-1/n+\gamma}).
\end{aligned}$$

For $z_* \in B(z^0, \delta_D(z^0)/8)$ (after possibly shrinking δ_*) this yields

$$\mathscr{G}_D(z_*, w) \le \mathscr{G}_D(z^0, w) + C_{17}\, \frac{(1 + r\exp(|\psi_1(w)|^{-1/n+\gamma}))}{\delta_D(z^0)^{n+2}}\, |\psi_1(w)|^{1/n-\gamma} + C'_{17}|\psi_1(w)|^{\beta_1}$$

with $\beta_1 = \min\{\beta, 1/n - \gamma\}$. Now we choose the radius r from (6.1) as $r = \exp(-|\psi_1(w)|^{-1/n+\gamma})$. Inserting this into (6.1) we find

$$\mathscr{G}_D(z^0, w) + C_{18}\, \frac{|\psi_1(w)|^{1/n-\gamma}}{\delta_D(z^0)^{n+2}} + C'_{17}|\psi_1(w)|^{\beta_1} \ge \sup_{x \in B(z^0, r)} \mathscr{G}_D(x, w) \ge -\frac{\widetilde{C}_n}{\delta_D(z^0)^{2n}}\, |\psi_1(w)|^{1/n} \log \frac{C_2}{r^n} \ge -\frac{C_{19}\, |\psi_1(w)|^{\gamma}}{\delta_D(z^0)^{2n}}$$

if $\delta_D(w) \le \delta_* \ll 1$.

This proves (because of $\gamma < 1/2n$)

$$|\mathscr{G}_D(z^0, w)| \le \frac{C_{20}|\psi_1(w)|^{\gamma}}{\delta_D(z^0)^{2n}} + C'_{17}|\psi_1(w)|^{\beta_1}$$

whenever $z^0 \in K$ and $\mathscr{G}_D(z^0, w) \le -C_{13}|\psi_1(w)|^{\beta}$. (Again, we tacitly supposed that $\delta_D(K) < 1$, which is allowed.) For those z^0 for which $\mathscr{G}_D(z^0, w) \ge -C_{13}|\psi_1(w)|^{\beta}$ there is nothing to be done.

The proof of Theorem 1.3 is complete. ∎

7. The case $n = 1$. In the one-dimensional case things are much easier, since the pluricomplex Green function equals the classical one and, in particular, it is symmetric.

THEOREM 7.1. *Let $D \subset \mathbb{C}$ be a bounded hyperconvex domain and φ_D be as in the introduction. Then*

$$\mathscr{G}_D(z, w) \ge -\frac{1}{2} \log\biggl(1 + 4\, \frac{|\varphi_D(w)|}{|z - w|^2}\biggr).$$

Proof. Let $z, w \in D$ be different points. Then the function

$$\phi(x) := \varphi_D(x) - \frac{1}{4}\, |x - w|^2$$

is subharmonic on D and negative. Also (since $\varphi_D < 0$)

$$\frac{|x - w|^2}{-4\phi(x)} < 1.$$

As the left-hand side is logarithmically subharmonic, the function

$$\frac{1}{2} \log \frac{|x - w|^2}{-4\phi(x)} = \log|x - w| - \frac{1}{2} \log(-4\phi(x))$$

is a candidate for $\mathscr{G}_D(\cdot, w)$, and for $x = z$ we obtain

$$\mathscr{G}_D(z,w) \ge -\frac{1}{2}\log\frac{-4\phi(z)}{|z-w|^2} = -\frac{1}{2}\log\biggl(1 + 4\,\frac{|\varphi_D(z)|}{|z-w|^2}\biggr).$$

By symmetry,

$$\mathscr{G}_D(z,w) = \mathscr{G}_D(w,z) \ge -\frac{1}{2}\log\biggl(1 + 4\,\frac{|\varphi_D(w)|}{|z-w|^2}\biggr). \quad \blacksquare$$

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