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Geometry of Markov systems and codimension one foliations

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Abstract. We show that the theory of graph directed Markov systems can be used to study exceptional minimal sets of some foliated manifolds. A C^1 smooth embedding of a contracting or parabolic Markov system into the holonomy pseudogroup of a codimension one foliation allows us to describe in detail the h-dimensional Hausdorff and packing measures of the intersection of a complete transversal with exceptional minimal sets.

1. Introduction. Cantwell and Conlon [4] observed that there exists a special class of pseudogroups, called Markov pseudogroups, which are semiconjugate to subshifts of finite type. Markov pseudogroups appear in a natural way in the theory of foliations as the holonomy pseudogroups of some closed, transversally oriented, C^2 foliated manifolds of codimension one. A detailed introduction to foliation theory can be found in [3]. For the convenience of the reader we shall recall some definitions.

Given a topological space X, denote by $\mathrm{Homeo}(X)$ the family of all homeomorphisms between open subsets of X. If $g \in \mathrm{Homeo}(X)$, then D_g is its domain and $R_g = g(D_g)$ is its range.

Definition 1. Let M be a Riemannian manifold. A C^r *pseudogroup* Γ on M is a collection of C^r diffeomorphisms $h : D_h \to R_h$ between open subsets D_h and R_h of M such that:

(1) If $g, h \in \Gamma$ then $g \circ f : f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$ is in Γ.
(2) If $h \in \Gamma$, then $h^{-1} \in \Gamma$.
(3) $\mathrm{id}_M \in \Gamma$.

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(4) If $h \in \Gamma$ and $W \subset D_h$ is an open subset, then $h|_W \in \Gamma$.
(5) If $h : D_h \to R_h$ is a C^r diffeomorphism between open subsets of M and if, for each $p \in D_h$, there exists a neighborhood N of p in D_h such that $h|_N \in \Gamma$, then $h \in \Gamma$.

For any set $G \subset \mathrm{Homeo}(M)$ which satisfies the condition

$$\bigcup_{g\in G} \{D_g \cup R_g : g \in G\} = M,$$

there exists a unique smallest (in the sense of inclusion) pseudogroup $\Gamma(G)$ which contains G. Notice that $\gamma \in \Gamma(G)$ if and only if $\gamma \in \mathrm{Homeo}(M)$ and for any $x \in D_\gamma$ there exist maps $g_1, \ldots, g_k \in G$, exponents $e_1, \ldots, e_k \in \{-1, 1\}$ and an open neighbourhood U of x in M such that

$$U \subset D_\gamma \quad \text{and} \quad \gamma|_U = g_1^{e_1} \circ \cdots \circ g_k^{e_k}|_U.$$

The pseudogroup $\Gamma(G)$ is said to be *generated by* G. If the set G is finite, we say that $\Gamma(G)$ is *finitely generated.*

Definition 2 (following [18]). A finite system $S = \{h_1, \ldots, h_m\} \subset \mathrm{Homeo}(M)$, $h_j : D_j \to R_j$, together with nonempty compact sets $K_j \subset R_j$, is called a *Markov system* if

(1) $R_i \cap R_j = \emptyset$ when $i \neq j$,
(2) either $K_i \subset Q_j$ or $K_i \cap D_j = \emptyset$,

where $Q_j = h_j^{-1}(K_j)$. If S is a Markov system and $\bigcup_{i=1}^m (D_{h_i} \cup R_{h_i}) = M$, then $\Gamma(S)$ is called a *Markov pseudogroup.*

Notice that Markov pseudogroups are generated by maps $h_i, h_j \in S$ such that either $D_{h_i \circ h_j} = D_{h_i}$ or $D_{h_i \circ h_j} = \emptyset$. Therefore, the following definition is useful:

Definition 3. For any Markov system $S = \{h_1, \ldots, h_m\}$ one defines its *transition matrix* $P = [p_{ij}]_{i,j=1}^m$ as follows:

$$p_{ij} \in \{0, 1\} \quad \text{and} \quad p_{ij} = 1 \text{ iff } K_j \subset Q_i.$$

The *Markov invariant set* Z_0 is defined as $Z_0 = Z \setminus \mathrm{int}(Z)$, where

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{g\in S_n} K_g, \qquad S_n = \{h_{i_1} \circ \cdots \circ h_{i_n} : i_1, \ldots, i_n \le m\},$$
$$K_g = g(Q_{i_n}) \quad \text{when } g = h_{i_1} \circ \cdots \circ h_{i_n}.$$

Examples of Markov pseudogroups and their minimal sets abound in the literature on foliation theory; let us mention only a few papers: [16], [8], [9], [10]. From our point of view, the importance of Markov pseudogroups for foliation theory stems from the result of Cantwell and Conlon [5] which states that any one-dimensional Markov pseudogroup can be realized as the holonomy pseudogroup of some foliated manifold. A more precise formulation of this result and a detailed proof is due to Walczak [18]:

THEOREM 1 (Thm. 1.4.8 in [18]). *If Γ is a Markov pseudogroup on a circle such that its Markov invariant set Z_0 contains a Γ-invariant minimal set C, then there exist a closed foliated* 3*-manifold* (M, F), $\dim F = 2$, *an exceptional minimal set $E \subset M$, a complete transversal T and a homeomorphism $h : E \cap T \to C$ which conjugates $\Gamma|_C$ to $H|_{E\cap T}$, where H is the holonomy pseudogroup of F acting on T.*

The reader can find more results and a list of open problems on Markov pseudogroups in [1]. Another realization of a Markov pseudogroup, obtained from a hyperbolic Markov system, as the holonomy pseudogroup of a codimension one foliation on a compact 3-manifold, was provided by Biś, Hurder and Shive [2] in their construction of generalized Hirsch foliations. Several months after that paper was written, similar results were obtained in [7].

2. Contracting and parabolic Markov systems. Suppose that X is a 1-dimensional smooth compact manifold, not necessarily connected and possibly with boundary. Then the distances induced by any two Riemannian metrics on X compatible with its smooth structure are bi-Lipschitz equivalent. Therefore, Hausdorff measures calculated with respect to two such metrics either simultaneously vanish, are positive and finite, or are simultaneously equal to infinity. Consequently, if one of these measures is finite and positive then its Radon–Nikodym derivative with respect to the other is uniformly separated from zero and infinity. In particular, the Hausdorff dimension of any subset of X is the same, regardless of which metric compatible with the smooth structure of X is taken to calculate it. Therefore, we do not explicitly single out any such metric. However, we emphasize that the Hausdorff dimension depends on the smooth structure of X, and in particular it is not a topological concept.

Let I be a countable set and let $\mathcal{S} = \{h_j : j \in I\}$ be a Markov system in the sense of Definition 2 such that all D_j's and R_j's are proper subarcs of X. Suppose further that all homeomorphisms h_j have $C^{1+\varepsilon}$ extensions to $\overline{D}_j$, the closures of their domains. We call the Markov system $\mathcal{S}$ *contracting* provided that

$$(*)\qquad s = \sup\{\|h'_{ij}\|_\infty : F_{ij} = 1\} < 1,$$

where $F_{ij} = 1$ if $K_i \subseteq Q_j = h_j^{-1}(K_j)$, and $F_{ij} = 0$ otherwise, and $h_{ij} = h_j|_{K_i}$. The associated Markov pseudogroup is also called contracting. We want to associate to $\mathcal{S}$ a (conformal) graph directed Markov system $\widehat{S}$ in the sense of [12]. Indeed, take $V = \{1, \ldots, m\}$ to be the set of vertices, and

$E = \{(i,j) : F_{ij} = 1\}$ to be the set of edges. Define the incidence matrices $A : E \times E \to \{0,1\}$ by the formula

$$A_{(i,j)(k,l)} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{if } i \neq l. \end{cases}$$

Put further

$$\varphi_{(i,j)} = h_{ij}$$

for all $(i,j) \in E$. Then $\widehat{S} = \{\varphi_e\}_{e\in E}$ is our graph directed Markov system. In order to meet all the formal conditions from [13] we extend all the maps φ_e, $e \in E$, in a $C^{1+\varepsilon}$ fashion to some open intervals $\Delta_j \supset K_j$ such that all the components of $\Delta_j \setminus K_j$, $j = 1, \ldots, m$, have the same length and $|\varphi'_{(i,j)}(x)| \le s$ for all $(i,j) \in E$ and all $x \in \Delta_i$. It is easy to notice that the limit set of the graph directed Markov system $\widehat{S}$ is equal to $Z_0 = Z$ (this equality being a consequence of $(*)$), the Markov invariant set of $\mathcal{S}$ introduced in Definition 3.

Assume that the incidence matrix A is finitely primitive, meaning that there exists a finite set Λ of A-admissible words of the same length such that for any two elements a and b of E there exists $\gamma \in \Lambda$ such that the word $a\gamma b$ is A-admissible. Let $h = \mathrm{HD}(Z)$ be the Hausdorff dimension of the invariant set Z.

Packing measures are counterparts of Hausdorff measures; both are used in measuring fractals. They are defined for subsets of finite-dimensional Euclidean space. One can find detailed definitions and properties of those measures in [11], [6] or [12].

A more general set-up of finite conformal graph directed Markov systems with primitive incidence matrix was investigated by Mauldin and Urbański in [12], where they showed (Theorem 4.2.11 in [12]) that there exists $C > 1$ such that

$$C^{-1} \le \frac{m(B(x,r))}{r^h} \le C$$

for any x of Markov invariant set Z_0 and small radius r. Here h is the unique zero of the pressure function and m denotes the unique h-conformal measure.

The following theorems are direct consequences of Theorem 4.2.11 in [12]:

Theorem 2. *If $\mathcal{S}$ is a contracting Markov system, then $0 < h < 1$, $H_h(Z) < \infty$ and $P_h(Z) > 0$, where H_h denotes the h-dimensional Hausdorff measure and P_h denotes the h-dimensional packing measure. If S is finite, then in addition $H_h(Z) > 0$ and $P_h(Z) < \infty$. Furthermore, the measures $H_h|_Z$ and $P_h|_Z$ are then equal up to a multiplicative constant.*

To be more precise, Theorems 4.5.10, 4.5.1 and 4.5.2 of [12] yield the first part of Theorem 2.

From now on, we assume that our contracting Markov system is finite; $I = \{1, \ldots, m\}$.

THEOREM 3. *If* $\mathcal{S}$ *is a finite contracting Markov system*, *then there exists a constant* $c \geq 1$ *such that for all* $r \in (0, 1]$ *and all* $z \in Z$,

$$c^{-1} \leq \frac{H_h(B(z, r))}{r^h} \leq c.$$

THEOREM 4. *If* $\mathcal{S}$ *is a contracting Markov system*, *then* $\mathrm{BD}(Z) = \mathrm{PD}(Z) = \mathrm{HD}(Z)$, *where* $\mathrm{BD}(Z)$ *and* $\mathrm{PD}(Z)$ *are respectively the box counting and packing dimensions of* Z.

Proof. The proof coincides with that of Theorem 7.6.7 in [15]. ∎

Now, replace in the above considerations condition $(*)$ by the following. For all $i, j \in \{1, \ldots, m\}$ with $A_{ij} = 1$ and all $x \in K_i$,

$$|h'_{ij}(x)| \leq 1,$$

and if $|h'_{ij}(x)| = 1$, then $h_{ij}(x) = x$. Such a point x is called *parabolic*. The set Ω of parabolic points is assumed to be nonempty and $K_i \cap \Omega$ contains at most one point, for all $i \in I$; in particular the set Ω is finite. Assume also that the maps h_{ij} are C^2 with nonvanishing second derivative at parabolic points; more generally, assume that condition (1.5) from [17] is satisfied. Call any such system $\mathcal{S}$ *parabolic Markov*. Then Theorems 2, 3 and 4 take on the following form:

THEOREM 5. *If* $\mathcal{S}$ *is a parabolic Markov system and* $0 < h < 1$, *then the* h*-dimensional Hausdorff measure of* Z *vanishes whereas the* h*-dimensional packing measure is positive and finite.*

Proof. It is known that $0 < P_h(Z) < \infty$ and $H_h(Z) < \infty$. Moreover, using Theorem 6.4 of [17] we get $H_h(Z) = 0$ if and only if $h < 1$. ∎

THEOREM 6. *Suppose that* $\mathcal{S}$ *is a parabolic Markov system. Then for any* $z \in Z$ *off a set of* P_h *measure zero*, *we have*

$$\liminf_{r \to 0} \frac{P_h(B(z, r))}{r^h} \in (0, +\infty] \quad \textit{and} \quad \limsup_{r \to 0} \frac{P_h(B(z, r))}{r^h} = +\infty,$$

where P_h *denotes the* h*-dimensional packing measure on* Z.

Proof. For the h-conformal measure m there exists a constant $c > 0$ such that $P_h = cm$, therefore by Theorem 4.2.10 of [12] we get the first relation. Notice that the set Y of those points whose ω-limit set does not contain any parabolic point, has m measure equal to zero. Therefore the second formula we claim in our theorem holds for all $z \in Z \setminus Y$ because of Theorem 6.2 from [17]. ∎

THEOREM 7. *If* $\mathcal{S}$ *is a parabolic Markov system*, *then* $\mathrm{BD}(Z) = \mathrm{PD}(Z) = \mathrm{HD}(Z)$.

3. Contracting and parabolic Markov systems versus codimension one foliations. Denote the unit disc, the unit circle, a circle and an open ball in the complex plane respectively by:

$$D = \{w \in \mathbb{C} : |w| \le 1\}, \quad S(z,r) = \{w \in \mathbb{C} : |w-z| = r\},$$
$$S^1 = \{w \in \mathbb{C} : |w| = 1\}, \quad B(z,r) = \{w \in \mathbb{C} : |w-z| \le r\}.$$

Choose an integer $n > 1$ and an analytic embedding $\varphi : S^1 \to S^1 \times D$ so that its homotopy class is equal to ng, where g is a generator of the fundamental group of the solid torus.

Now we recall the construction of a generalized Hirsch foliation in codimension one, which was presented in detail in Section 2 of [2], in the following way. Choose a nonzero interior point $z_0 \in D$ (such that $0 < |z_0| < 1$) and $\varepsilon > 0$ such that $0 < 2\varepsilon < \min\{|z_0|, 1 - |z_0|\}$. Now define the n-punctured disc

$$P = D \setminus (B(z_0,\varepsilon) \cup B(z_1,\varepsilon) \cup \cdots \cup B(z_{n-1},\varepsilon)),$$

where $z_m = \varrho^m z_0$ for any $0 \le m < n$ and $\varrho = e^{2\pi i/n}$.

The analytic 3-manifold N_1 with boundary is defined as the quotient of $\mathbb{R} \times P$ by the equivalence relation $\sim$ that identifies the points (x,z) and $(x+1, \varrho z)$. Notice that N_1 is diffeomorphic to the solid torus $S^1 \times D$ with an open tubular neighborhood of $\varphi(S^1)$ removed. Remember that the embedding $\varphi : S^1 \to S^1 \times D$ winds n times around the core. The boundary of N_1 consists of two disjoint tori, $\partial N_1 = \partial^+ N_1 \cup \partial^- N_1$, where

$$\partial^+ N_1 = (\mathbb{R} \times S^1)/\sim, \quad \partial^- N_1 = (\mathbb{R} \times ((S(z_0,\varepsilon) \cup S(z_1,\varepsilon) \cup \cdots \cup S(z_{n-1},\varepsilon)))/\sim.$$

The manifold N_1 admits a foliation $\mathcal{F}_{N_1} = \{\{x\} \times P : x \in [0,1)\}$ by compact 2-manifolds with boundary. Notice that the intersection of the leaves of $\mathcal{F}_{N_1}$ with the boundary tori consists of circles, therefore each boundary torus is foliated by circles. Gluing the boundary $\partial^+ N_1$ with the boundary $\partial^- N_1$ by a properly chosen diffeomorphism $f : \partial^+ N_1 \to \partial^- N_1$, which maps the foliations of the boundary tori to one another, we get a foliated manifold N with foliation $\mathcal{F}$. To construct such a diffeomorphism f choose an immersion $H : S^1 \to S^1$ of degree n. Notice that the choice of H is equivalent to the choice of a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + 1$. So, $H = h \pmod 1$.

LEMMA 1 ([2, p. 76–77]). *For any diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + n$, the map $\tilde{f} : \mathbb{R} \times D \to \mathbb{R} \times D$ defined by the formula*

$$\tilde{f}(x,z) = (h(x), z_1 + \varepsilon z e^{2\pi x i/n})$$

induces a map $f : \partial^+ N_1 \to \partial^- N_1$.

Finally, define

$$N = N_1/\sim_f,$$

where $\sim_f$ identifies the points (x, z) and $f(x, z)$. Then leaves of $\mathcal{F}_{N_1} \cap \partial^+ N_1$ are mapped to leaves $\mathcal{F}_{N_1} \cap \partial^- N_1$, which implies that N has a foliation $\mathcal{F}_N$ whose leaves are n-punctured discs.

The foliation $\mathcal{F}_N$ on N is called a *generalized Hirsch foliation.*

The foliation $\mathcal{F}_N$ on N admits a complete transversal $T : S^1 \to N$. Observe that the foliation $\mathcal{F}_{N_1}$ on N_1 is defined by a fibration, therefore $\mathcal{F}_{N_1}$ has no holonomy. So, all the holonomy of $\mathcal{F}_N$ is introduced by the identification of the outer boundary $\partial^+ N_1$ with the inner boundary $\partial^- N_1$ via the diffeomorphism f.

The immersion $H : S^1 \to S^1$ of degree n induces an equivalence relation on S^1: two points $x, y \in S^1$ are said to be in the same "grand orbit" of H if there exist positive integers k and l such that $H^k(x) = H^l(y)$ (cf. Milnor [14]). The grand orbit of a point x is denoted by $O(x)$.

Recall that a subset $K \subset S^1$ is said to be H-*invariant* if for all $x \in K$ the grand orbit $O(x)$ is contained in K.

Definition 4. An invariant set K is called *minimal* if it is closed and for all $x \in K$ the H-orbit $O(x)$ is dense in K. A minimal set K is *exceptional* if it is nowhere dense and is not finite.

Our first, obvious application to the theory of foliations is the following.

Theorem 8. *Suppose $\mathcal{F}$ is a smooth codimension* 1 *foliation on a Riemannian manifold M and T is a complete transversal for $\mathcal{F}$. If the holonomy pseudogroup of $\mathcal{F}$ acting on T is generated by a contracting* (*resp. parabolic*) *Markov system*, *then there is an exceptional minimal set E for $\mathcal{F}$ such that Theorems* 2–4 (5–7) *are true with the set Z replaced by $E \cap T$.*

Theorem 9. *If $\mathcal{S} = \{h_1, \ldots, h_m\}$ is either a contracting or parabolic C^r Markov system on the circle S^1, $r \geq 1$, such that all maps h_i are defined on the closed interval $I_0 \subset S^1$, then there exists a generalized Hirsch foliation $(N, \mathcal{F})$ with* $\operatorname{codim} \mathcal{F} = 1$, *an exceptional minimal set $E \subseteq N$, a complete transversal T and a C^r diffeomorphism $f : E \cap T \to J_{\mathcal{G}}$* (*the Markov invariant set of the pseudogroup $\mathcal{G}$ generated by the Markov system $\mathcal{S}$*).

Proof. Take a contracting or parabolic C^r Markov system $\mathcal{S} = \{h_1, \ldots, h_m\}$ and closed intervals $I_j = [a_j, b_j]$, $j = 0, 1, \ldots, m$, such that

(1) $h_i : I_0 \to I_i \subset I_0$,
(2) $I_i \cap I_j = \emptyset$ for $i \neq j$,
(3) $|h_i'(x)| \leq 1$ for any $x \in I_0$ and the equality $|h_i'(x_0)| = 1$ holds for at most one point x_0 of I_i.

We may assume that $I_0 = [0, c]$, where $c < 1$. Denote the unique fixed point of h_i by x_i^*. Let $a_0 = \min\{x_i^* : 1 \leq i \leq m\}$ and $b_0 = \max\{x_i^* : 1 \leq i \leq m\}$.

Without losing generality, we can assume for $1 \le i \le m$ that $I_i = [a_i, b_i]$ and

$$0 = a_0 = a_1 < b_1 < a_2 < \cdots < b_{m-1} < a_m < b_m = b_0.$$

Following Example 6.1 in [2] we define a C^r diffeomorphism $h : [0,1] \to [0,m]$ in the following way:

(1) $h|_{[a_i,b_i]} = (i-1) + h_i$ for any $1 \le i \le m-1$;
(2) $h|_{[b_i,a_{i+1}]} = f_i$, where $f_i : [b_i, a_{i+1}] \to [i-1+c, i]$ is a C^r diffeomorphism chosen so that h is C^r at the points b_i and a_{i+1}, $1 \le i \le m-1$;
(3) $h|_{[c,1]} = f_m$, where the C^r diffeomorphism $f_m : [c,1] \to [c,1]$ satisfies
   (a) f_m has a unique attracting fixed point at $x_0 = (1+c)/2$,
   (b) $f_m(c) = c$ and $f_m(1) = 1$,
   (c) $f_m|_{(c,1)}$ is a contraction of the open interval $(c,1)$ to the attracting fixed point x_0,
   (d) h is C^r at the points $b_m = c$ and 1.

Let $H : S^1 \to S^1$ be the immersion of degree m, defined by $H = h \pmod 1$, and define the open set $U \subset S^1$ to be the union of the H-orbits of the open interval $(c,1)$. Putting $K = S^1 \setminus U$ we find that $K \subset I_1 \cup \cdots \cup I_m$. Modifying slightly the proof of Lemma 2.1 in [2] we get

LEMMA 2. *Let $H : S^1 \to S^1$ be the immersion of degree m, defined by $H = h \pmod 1$. Then there exists a unique minimal set $J_{\mathcal{G}} \subset S^1$ with respect to H. Moreover, $J_{\mathcal{G}} = K$.*

Gluing the outer boundary $\partial^+ N_1$ to the inner boundary $\partial^- N_1$ via the diffeomorphism h, we obtain a three-dimensional manifold N. The foliation $\mathcal{F}_N$ on N admits a complete transversal. It can be constructed by the embedding

$$\widehat{t} : \mathbb{R} \to \mathbb{R} \times P,$$

where $\widehat{t}(x) = (x, 0)$. Notice that

$$\widehat{t}(x+1) = (x+1, 0) \sim (x, \varrho 0) = (x, 0) = \widehat{t}(x).$$

Passing to the quotient manifold we get $t : S^1 \to N_1$. By our construction we deduce that $L \cap t(S^1) \neq \emptyset$ for any leaf $L \in \mathcal{F}_{N_1}$. Therefore, after gluing outer and inner boundaries we get a complete transversal $T : S^1 \to N$. The construction of the foliation $\mathcal{F}_N$ on N implies that for the exceptional minimal set E of $\mathcal{F}_N$ we have

$$E \cap T(S^1) = K = J_{\mathcal{G}},$$

which completes the proof. ∎

COROLLARY 1. *With the assumptions and terminology of Theorem 9, Theorems 2–7 remain true with the set Z replaced by $E \cap T$.*

Similarly, making the construction in the proof of Theorem 1.4.8 in [18] C^1-smooth, we get the following

THEOREM 10. *If $\mathcal{G}$ is a contracting (resp. parabolic) Markov pseudogroup on a circle, then there exist a closed foliated 3-manifold $(M, \mathcal{F})$, $\dim \mathcal{F} = 2$, an exceptional minimal set $E \subset M$, a complete transversal T and a C^1 diffeomorphism $f : E \cap T \to J_{\mathcal{G}}$ (the Markov invariant set of $\mathcal{G}$) which conjugates $\mathcal{G}|_{J_{\mathcal{G}}}$ to $H|_{E \cap T}$, where H is the holonomy pseudogroup of F acting on T. Consequently, Theorems 2–4 (resp. 5–7) remain true with the set Z replaced by $E \cap T$.*

Now, if E is an exceptional minimal set for a codimension 1 foliation $\mathcal{F}$ and T is a complete transversal for $\mathcal{F}$, then E is locally diffeomorphic to the Cartesian product of $E \cap T$ and an interval. Consequently, Theorems 8 and 9 remain true with $E \cap T$ replaced by "sufficiently small" open subsets of E. The dimension h then equals $\mathrm{HD}(E) = \mathrm{HD}(E \cap T) + 1$.

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