ANNALES
POLONICI MATHEMATICI
96.1 (2009)

Weighted composition operators between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces

by Elke Wolf (Paderborn)

Abstract. Let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps. They induce a weighted composition operator ψC_ϕ acting between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces. Under some assumptions on the weights we give a necessary as well as a sufficient condition for such an operator to be bounded resp. compact.

1. Introduction. In this note we consider an analytic self-map ϕ of the open unit disk $\mathbb{D}$ as well as an analytic map ψ on $\mathbb{D}$. These maps induce a weighted composition operator $\psi C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto \psi(f \circ \phi)$, where $H(\mathbb{D})$ denotes the set of all holomorphic functions on $\mathbb{D}$. Furthermore, let v and w be strictly positive continuous and bounded functions (*weights*) on $\mathbb{D}$. We are interested in weighted composition operators ψC_ϕ acting between weighted Banach spaces of holomorphic functions

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and the weighted Bloch type spaces B_w of functions $f \in H(\mathbb{D})$ satisfying $\|f\|_{B_w} := \sup_{z\in\mathbb{D}} w(z)|f'(z)| < \infty$. Provided we identify functions that differ by a constant, $\|.\|_{B_w}$ becomes a norm and B_w a Banach space. Composition operators and weighted composition operators acting between various spaces of analytic functions have been investigated by several authors (see e.g. [13], [7], [11], [2], [4], [3], [8], [12]). In [13] and [12] weighted composition operators between weighted Bloch type spaces resp. between the space H^∞ of bounded analytic functions on $\mathbb{D}$ and the Bloch space have been studied.

In this article we want to give necessary and sufficient conditions for a weighted composition operator acting between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces to be bounded resp.

2000 *Mathematics Subject Classification*: 47B33, 47B38.
Key words and phrases: weighted composition operators, weighted Bloch type spaces.

DOI: 10.4064/ap96-1-3

compact. These conditions are given in terms of the weights as well as the analytic functions ϕ and ψ involved.

2. Notation and auxiliary results. For notation on composition operators we refer the reader to the monographs [5] and [14]. In order to give results concerning weighted spaces of analytic functions such as weighted Banach spaces of holomorphic functions or weighted Bloch type spaces we need the so called *associated weights*. For a weight v we can define the associated weight as follows:

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}},$$

where δ_z denotes the point evaluation at z. By [1] the associated weight $\tilde{v}$ is continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = 1/\tilde{v}(z)$.

For a better understanding let us recall some auxiliary results:

THEOREM 1 (Harutyunyan–Lusky, [6, Theorem 2.1]). *Let v and w be radial weights which are continuously differentiable with respect to $|z|$ with $\lim_{|z|\to 1} v(z) = \lim_{|z|\to 1} w(z) = 0$ and such that H_w^∞ is isomorphic to l_∞. If $\limsup_{r\to 1}(-w'(r)/v(r)) < \infty$, then $D : H_v^\infty \to H_w^\infty$, $f \mapsto f'$, is bounded.*

For conditions ensuring that H_w^∞ is isomorphic to l_∞ we refer the reader to [10] and [6]. By [6] we know that the following weights have the desired properties:

$$w(z) = (1-|z|)^\alpha, \quad \alpha > 0, \qquad w(z) = e^{-1/(1-|z|)}, \qquad z \in \mathbb{D}.$$

For the study of the compactness of the operator ψC_ϕ we need the following result.

PROPOSITION 2 (Cowen–MacCluer, [5, Proposition 3.11]). *Let X and Y be H_v^∞ or B_w. Then $\psi C_\phi : X \to Y$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\psi C_\phi f_n \to 0$ in Y.*

3. Main result. We consider weights v of the following type: Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing and strictly positive on $[0,1[$. Moreover, we assume that ν is decreasing on $[0,1[$ and satisfies $\lim_{r\to 1} \nu(r) = 0$. Then we define the corresponding weight v by $v(z) = \nu(|z|^2)$ for every $z \in \mathbb{D}$. Furthermore, we suppose that ν' is bounded on $\mathbb{D}$.

We now give some examples of weights of this type:

(i) If $\nu(z) = (1-z)^\alpha$, $\alpha \ge 1$, then the corresponding weight is the so-called standard weight $v(z) = (1-|z|^2)^\alpha$.
(ii) If $\nu(z) = e^{-1/(1-z)^\alpha}$, $\alpha \ge 1$, then $v(z) = e^{-1/(1-|z|^2)^\alpha}$.
(iii) If $\nu(z) = \sin(1-z)$, then $v(z) = \sin(1-|z|^2)$.

Fix a point $p \in \mathbb{D}$. We introduce a function

$$v_p(z) := \nu(\overline{\phi(p)}z) \qquad \text{for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, the function v_p is also holomorphic on $\mathbb{D}$. Furthermore, $v_p(\phi(p)) = \nu(|\phi(p)|^2) = v(\phi(p))$ and $v_p'(z) = \overline{\phi(p)}\nu'(\overline{\phi(p)}z)$ for every $z \in \mathbb{D}$, i.e. $v_p'(\phi(p)) = \overline{\phi(p)}\nu'(|\phi(p)|^2)$.

Proposition 3. *Let w be a weight and v be a weight as described at the beginning of this section. Let $\psi \in H(\mathbb{D})$ and ϕ an analytic self-map of $\mathbb{D}$. If $\psi C_\phi : H_v^\infty \to B_w$ is bounded, then:*

(a) $$\sup_{z\in\mathbb{D}} |\psi'(z)| \frac{w(z)}{(v(\phi(z)))^{1/2}} < \infty,$$

(b) $$\sup_{z\in\mathbb{D}} \frac{w(z)|\nu'(|\phi(z)|^2)|}{v(\phi(z))} |\psi(z)\phi'(z)\phi(z)| < \infty.$$

Proof. In order to show condition (a) we set

$$f_p(z) := \left(\frac{2}{v_p(z)} - \frac{v_p(\phi(p))}{v_p(z)^2}\right)^{1/2}.$$

Then

$$\|f_p\|_v = \sup_{z\in\mathbb{D}} \left| v(z)^2 \frac{2}{v_p(z)} - v(z)^2 \frac{v_p(\phi(p))}{v_p(z)^2}\right|^{1/2} \le (3M)^{1/2}$$

where $M = \sup_{z\in\mathbb{D}} v(z)$ and therefore the constant does not depend on the choice of p. Thus, $f_p \in H_v^\infty$ and

$$f_p'(z) = \left(-\frac{v_p'(z)}{v_p(z)^2} + \frac{v_p'(z)v_p(\phi(p))}{v_p(z)^3}\right)\left(\frac{2}{v_p(z)} - \frac{v_p(\phi(p))}{v_p(z)^2}\right)^{-1/2}.$$

We get $f_p(\phi(p)) = 1/v(\phi(p))^{1/2}$ and $f_p'(\phi(p)) = 0$. Now,

$$\begin{aligned} |\psi'(p)| \frac{w(p)}{v(\phi(p))^{1/2}} &= w(p)|\psi'(p)f_p(\phi(p)) + \psi(p)\phi'(p)f_p'(\phi(p))| \\ &\le \|\psi C_\phi\| \, \|f_p\|_v < \infty. \end{aligned}$$

Thus, (a) follows.

For the proof of (b) we fix $p \in \mathbb{D}$ and construct a function $v_p(z)$ as above. Now we put

$$g_p(z) := \frac{v_p(\phi(p))}{v_p(z)} - \left(\frac{v_p(\phi(p))}{v_p(z)}\right)^{1/2}.$$

Hence $\|g_p\|_v \leq 2M$ and we get

$$g_p'(z) = -\frac{v_p(\phi(p))v_p'(z)}{v_p(z)^2} + \frac{1}{2}\,\frac{v_p(\phi(p))^{1/2}v_p'(z)}{v_p(z)^{3/2}}.$$

Thus, we obtain

$$g_p(\phi(p)) = 0 \quad \text{and} \quad g_p'(\phi(p)) = -\frac{1}{2}\,\frac{\overline{\phi(p)}\nu'(|\phi(p)|^2)}{v_p(\phi(p))}.$$

Finally,

$$\begin{aligned}\frac{1}{2}\,|\phi(p)|\,\frac{w(p)|\nu'(|\phi(p)|^2)|}{v(\phi(p))}\,|\psi(p)\phi'(p)|&\\ &= w(p)|\psi'(p)g_p(\phi(p)) + \psi(p)\phi'(p)g_p'(\phi(p))|\\ &\leq \|\psi C_\phi\|\,\|g_p\|_v < \infty.\end{aligned}$$

The claim follows. ■

Proposition 4. *Let v and w be weights. If*

(a) *there is a weight u such that*

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{u(\phi(z))}\,|\psi(z)\phi'(z)| < \infty$$

and the operator $D : H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is bounded,

(b) $\sup_{z\in\mathbb{D}} |\psi'(z)|w(z)/v(\phi(z)) < \infty$,

then $\psi C_\phi : H_v^\infty \to B_w$ is bounded.

Proof. Let $f \in H_v^\infty$. We have

$$\begin{aligned}&\sup_{z\in\mathbb{D}} w(z)|(\psi C_\phi f)'(z)|\\ &\quad\leq \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f(\phi(z))| + \sup_{z\in\mathbb{D}} w(z)|f'(\phi(z))\phi'(z)\psi(z)|\\ &\quad\leq \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}\,|\psi'(z)|\,\|f\|_v + \sup_{z\in\mathbb{D}} \frac{w(z)}{u(\phi(z))}\,|\phi'(z)\psi(z)|u(\phi(z))|f'(\phi(z))|\\ &\quad\leq \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}\,|\psi'(z)|\,\|f\|_v + \sup_{z\in\mathbb{D}} \frac{w(z)}{u(\phi(z))}\,|\phi'(z)\psi(z)|\,\|f'\|_u\\ &\quad\leq \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}\,|\psi'(z)|\,\|f\|_v + \sup_{z\in\mathbb{D}} \frac{w(z)}{u(\phi(z))}\,|\phi'(z)\psi(z)|\,\|D\|\,\|f\|_v,\end{aligned}$$

and the claim follows. ■

Proposition 5. *Let w be a weight and v be a weight as described at the beginning of this section. Let $\psi \in H(\mathbb{D})$ and ϕ an analytic self-map of $\mathbb{D}$.*

If $\psi C_\phi : H_v^\infty \to B_w$ is compact, then:

(a) $$\limsup_{|\phi(z)|\to 1} |\psi'(z)| \frac{w(z)}{v(\phi(z))^{1/2}} = 0,$$

(b) $$\limsup_{|\phi(z)|\to 1} \frac{w(z)|\nu'(|\phi(z)|^2)|}{v(\phi(z))} |\psi(z)\phi'(z)\phi(z)| = 0.$$

Proof. Consider a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Defining functions v_{z_n} as in the proof of Proposition 3 we set

$$f_n(z) := v_{z_n}(\phi(z_n))^{1/6}\left(\frac{3}{2}\,\frac{1}{v_{z_n}(z)^2} - \frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)^3}\right)^{1/3} \quad \text{for } z \in \mathbb{D}.$$

Then

$$\begin{aligned}\|f_n\|_v &= \sup_{z\in\mathbb{D}} v_{z_n}(\phi(z_n))^{1/6}\left|\frac{3}{2}\,\frac{v(z)^3}{v_{z_n}(z)^2} - \frac{v(z)^3 v_{z_n}(\phi(z_n))}{v_{z_n}(z)^3}\right|^{1/3} \\ &\le M^{1/6}\left(\frac{5}{2}\,M\right)^{1/3}\end{aligned}$$

for every $n \in \mathbb{N}$, where $M := \sup_{z\in\mathbb{D}} v(z)$. Thus, $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in H_v^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Moreover,

$$\begin{aligned}f_n'(z) &= v_{z_n}(\phi(z_n))^{1/6}\left(\frac{3}{2}\,\frac{1}{v_{z_n}(z)^2} - \frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)^3}\right)^{-2/3} \\ &\quad\times\left(-\frac{v_{z_n}'(z)}{v_{z_n}(z)^3} + \frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)^4}\,v_{z_n}'(z)\right)\end{aligned}$$

for every $n \in \mathbb{N}$. By Proposition 2, the fact that ψC_ϕ is compact yields

$$\|\psi C_\phi f_n\|_{B_w} \to 0 \quad \text{as } n \to \infty.$$

Finally,

$$\|\psi C_\phi f_n\|_{B_w} \ge w(z_n)\left|\frac{\psi'(z_n)}{v(\phi(z_n))^{1/2}}\right|.$$

Thus, (a) follows.

Consider now

$$g_n(z) := \frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)} - \left(\frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)}\right)^{1/2}.$$

Then

$$\|g_n\|_v = \sup_{z\in\mathbb{D}} v(z)\left|\frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)} - \left(\frac{v_{z_n}(\phi(z_n))}{v_{z_n}(z)}\right)^{1/2}\right| \le 2M$$

for every $n \in \mathbb{N}$. Thus $(g_n)_n$ is a bounded sequence in H_v^∞ and $g_n \to 0$ uniformly on every compact subset of $\mathbb{D}$. Moreover,

$$g_n(\phi(z_n)) = 0 \quad \text{and} \quad g_n'(\phi(z_n)) = -\frac{1}{2}\,\frac{v_{z_n}'(\phi(z_n))}{v_{z_n}(\phi(z_n))}.$$

Since ψC_ϕ is compact, by Proposition 2 we have $\|\psi C_\phi g_n\|_{B_w} \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}\|\psi C_\phi g_n\|_{B_w} &= \sup_{z\in\mathbb{D}} w(z)|(\psi C_\phi g_n)'(z)| \\ &\geq w(z_n)|\psi'(z_n)g_n(\phi(z_n)) + \psi(z_n)\phi'(z_n)g_n'(\phi(z_n)| \\ &\geq \frac{1}{2}\, w(z_n)|\psi(z_n)\phi'(z_n)\phi(z_n)|\,\frac{|\nu'(|\phi(z_n)|^2)|}{v(\phi(z_n))}.\end{aligned}$$

Finally,

$$\limsup_{|\phi(z)|\to 1} w(z)|\psi(z)|\,|\phi'(z)|\,|\phi(z)|\,\frac{|\nu'(|\phi(z)|^2)|}{v(\phi(z))} = 0. \ \blacksquare$$

PROPOSITION 6. *Let v and w be weights. If*

(a) *there is a weight u such that*

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)}{u(\phi(z))}|\psi(z)\phi'(z)| = 0$$

and the operator $D : H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is bounded,

(b) $\limsup_{|\phi(z)|\to 1} |\psi'(z)|w(z)/v(\phi(z)) = 0$,

then $\psi C_\phi : H_v^\infty \to B_w$ is compact.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in H_v^∞ with $\|f_n\|_v \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By the assumption, for any $\varepsilon > 0$ there is $0 < \delta < 1$ such that $\delta < |\phi(z)| < 1$ implies

$$\frac{w(z)}{v(\phi(z))}\,|\psi'(z)| < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{w(z)}{u(\phi(z))}\,|\psi(z)\phi'(z)| < \frac{\varepsilon}{2\|D\|}.$$

Then

$$\begin{aligned}\|\psi C_\phi f_n\|_{B_w} &= \sup_{z\in\mathbb{D}} w(z)|(\psi C_\phi f_n)'(z)| \\ &\leq \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f_n(\phi(z))| + \sup_{z\in\mathbb{D}} w(z)|\psi(z)\phi'(z)f_n'(\phi(z))| \\ &\leq \sup_{\{z;\,|\phi(z)|\leq\delta\}} w(z)|\psi'(z)f_n(\phi(z))| \\ &\quad + \sup_{\{z;\,|\phi(z)|\leq\delta\}} w(z)|\psi(z)\phi'(z)f_n'(\phi(z))| + \varepsilon\end{aligned}$$

$$\leq \sup_{\{z;\,|\phi(z)|\leq\delta\}} \frac{w(z)}{v(\phi(z))}\,|\psi'(z)| \sup_{\{z;\,|\phi(z)|\leq\delta\}} v(\phi(z))|f_n(\phi(z))|$$
$$+ \sup_{\{z;\,|\phi(z)|\leq\delta\}} \frac{w(z)}{v(\phi(z))}\,|\psi(z)\phi'(z)|\,\|D\|$$
$$\times \sup_{\{z;\,|\phi(z)|\leq\delta\}} v(\phi(z))|f_n(\phi(z))|.$$

The claim follows. ∎

EXAMPLES 7. (a) Set $w(z) = (1-|z|)^4$, $u(z) = (1-|z|)^2$, $v(z) = (1-|z|)^3$ and $\phi(z) = (z+1)/2$ and $\psi(z) = 1-z$ for every $z \in \mathbb{D}$. Then easy calculations show that the corresponding weighted composition operator $\psi C_\phi : H_v^\infty \to B_w$ is bounded and even compact.

(b) For $\psi(z) = 1-z$, $\phi(z) = (z+1)/2$, $v(z) = (1-|z|^2)^2$ and $w(z) = 1-|z|^2$ for every $z \in \mathbb{D}$ the operator $\psi C_\phi : H_v^\infty \to B_w$ is not bounded.

References

[1] K. D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), 137–168.

[2] J. Bonet, P. Domański, M. Lindström and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A 64 (1998), 101–118.

[3] J. Bonet, M. Lindström and E. Wolf, *Differences of composition operators between weighted Banach spaces of holomorphic functions*, ibid. 84 (2008), 9–20.

[4] M. D. Contreras and A. G. Hernández-Díaz, *Weighted composition operators in weighted Banach spaces of analytic functions*, ibid. 69 (2000), 41–60.

[5] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[6] A. Harutyunyan and W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. 184 (2008), 233–247.

[7] T. Hosokawa, K. Izuchi and S. Ohno, *Topological structure of the space of weighted composition operators on H^∞*, Integral Equations Operator Theory 53 (2005), 509–526.

[8] M. Lindström and E. Wolf, *Essential norm of the difference of weighted composition operators*, Monatsh. Math. 153 (2008), 133–143.

[9] W. Lusky, *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc. 51 (1995), 309–320.

[10] —, *On the isomorphism classes of weighted spaces of harmonic and holomorphic functions*, Studia Math. 175 (2006), 19–45.

[11] B. MacCluer, S. Ohno and R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integral Equations Operator Theory 40 (2001), 481–494.

[12] S. Ohno, *Weighted composition operators between H^∞ and the Bloch space*, Taiwanese J. Math. 5 (2001), 555–563.

[13] S. Ohno, K. Stroethoff and R. Zhao, *Weighted composition operators between Bloch type spaces*, Rocky Mountain J. Math. 33 (2003), 191–215.

[14] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.

Mathematical Institute
University of Paderborn
D-33095 Paderborn, Germany
E-mail: lichte@math.uni-paderborn.de

Received 17.9.2008
and in final form 26.11.2008 (1922)